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Research Article

Stationary Patterns of a Cross-Diffusion Prey-Predator Model with Holling Type II Functional Response

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In this paper, we consider positive steady-state solutions of a cross-diffusions prey-predator model with Holling type II functional response. We investigate sufficient conditions for the existence and the nonexistence of nonconstant positive steady state solutions. It is observed that nonconstant positive steady states do not exist with small cross-diffusion coefficients, and the constant positive steady state is global asymptotically stable without cross-diffusion. Furthermore, we show that if natural diffusion coefficient or cross-diffusion coefficient of the predator is large enough and other diffusion coefficients are fixed, then under some conditions, at least one nonconstant positive steady state exists.

1. Introduction

In recent years, many researchers have studied population models with cross-diffusion terms [1–9]. Let u and v represent the densities of the prey and predator, respectively. A general partial differential prey-predator system with cross-diffusion is of the form [10]

$$\begin{cases} u_t - \nabla\left(k_{11}(u,v)\nabla u + k_{12}(u,v)\nabla v\right) = \phi(u) - p(u)v, \\ v_t - \nabla\left(k_{21}(u,v)\nabla u + k_{22}(u,v)\nabla v\right) = \psi(v) + cp(u)v, \end{cases} \tag{1}$$

where K_{11}, K_{22} and K_{12}, K_{21} embody the diffusion and cross-diffusion processes, respectively, $\phi(u)$ and $\psi(v)$ represent the self-growth of the two species, and $p(u)$ is the predator functional response, see [3, 6, 8, 11, 12]. Among many possible choices of $p(u)$, the Holling type II functional response is the most commonly used in the ecological literature, which is defined by

$$p(u) = \frac{u}{1+mu}, \tag{2}$$

where m is a positive constant measuring the ability of a generic predator to kill and consume a generic prey.

We are interested in the changes of behavior of the predator-prey system with cross-diffusion and Holling type II functional response. In this work, we investigate the following predator-prey model:

$$\begin{cases} u_t - d_1\Delta[(1+d_3v)u] = u - ku^2 - \dfrac{uv}{1+mu}, & (t,x)\in(0,\infty)\times\Omega, \\ v_t - d_2\Delta[(1+d_4u)v] = -av - bv^2 + \dfrac{uv}{1+mu}, & (t,x)\in(0,\infty)\times\Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (t,x)\in(0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x)\geq 0, v(0,x) = v_0(x)\geq 0, & x\in\overline{\Omega}. \end{cases} \tag{3}$$

Here, the two unknown functions $u(t,x)$ and $v(t,x)$ represent the spatial distribution density of the prey and predator, respectively. The positive constants d_1, d_2 are the natural diffusion coefficients, nonnegative constants d_3 and d_4 are the cross-diffusion coefficients, a is the death rate of the predator, the terms $-ku^2$ and $-bv^2$ represent the self-limitation for the prey and predator, k and b are the positive constants, Ω is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, and we impose a homogeneous Neumann-type boundary condition, which implies that system (3) is a closed system and there is no flux across the boundary $\partial\Omega$. The prey u and predator v diffuse with flux.

$$\begin{aligned} J_u &= -d_1d_3u\nabla v - (d_1 + d_1d_3v)\nabla u, \\ J_v &= -d_2d_4v\nabla u - (d_2 + d_2d_4u)\nabla v. \end{aligned} \tag{4}$$

If $d_3 > 0$, for the prey u, the term $-d_1d_3u\nabla v$ of the flux is directed towards the decreasing population density of v, and the diffusion of the predator v represents the tendency of predator to move away from a large group of the preys. In a certain kind of prey-predator relationships, a great number of prey species form a huge group to protect themselves from the attack of a predator [1, 11].

Our main concern focuses on the effects of cross-diffusion on the existence and nonexistence of nonconstant positive steady state solutions of system (3), that is to say, the existence and nonexistence of nonconstant positive solutions of the following strongly coupled elliptic system.

$$\begin{cases} -d_1\Delta[(1+d_3v)u] = u - ku^2 - \dfrac{uv}{1+mu}, & x\in\Omega, \\ -d_2\Delta[(1+d_4u)v] = -av - bv^2 + \dfrac{uv}{1+mu}, & x\in\Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega. \end{cases} \tag{5}$$

We mainly discuss the effect of $d_2, d_3,$ and d_4 on positive solutions of (5) by using the integral property and homotopy invariance of the topological degree. We convert the calculation of complex eigenvalues into judging the sign of coefficients to a simple polynomial. Our results show that system (5) has no noncontant positive steady state when cross-diffusions d_3, d_4 are sufficiently small (even equal to zero), while it has at least one nonconstant positive solution if d_2 or d_4 is large enough.

If $d_3 = 0$, the version of (5) with ratio-dependent functional response [1] was studied, where the Holling type II functional response $uv/1+mu$ was replaced by $uv/v+mu$, it was observed that the cross-diffusion could create nonconstant positive steady states by using homotopy invariance of the topological degree. If $(d_3 = d_4 = 0)$, the system (5) with homogeneous Dirichlet boundary condition [12] was considered, and the existence and uniqueness of coexistence states were proved by employing bifurcation theory. Recently, strongly coupled elliptic systems with cross-diffusion terms have received increasing attention. Some researchers have focused on Lotka–Volterra models [1, 2, 4, 6, 7, 11] and the Sel'kov models [13, 14] with homogeneous Neumann boundary condition, and given the existence of nonconstant positive steady states, while others have also considered prey-predator models with Holling type II functional response and homogeneous Dirichlet boundary condition [3, 8, 9], and their main concern is the structure of positive solutions. There are some other kinds of models (refer to the above cited papers and references therein). For the system with Holling type II functional response and homogeneous Neumann boundary condition, there are only a few results.

Motivated by above-cited works, we are concerned with problem (5), which is a more difficult mathematical problem for incorporating cross-diffusion terms to both equations. This paper is organized as follows: In Section 2, the upper and lower bounds for positive solutions of (5) are estimated. Then, in Section 3, the nonexistence of the nonconstant positive solutions is proved by using the integral property, and it is observedthat the constant positive steady state is global asymptotically stable without cross-diffusion. In Section 4, the sufficient conditions for the existence of nonconstant positive solutions are obtained. In the last section, we give a conclusion for the paper.

2. A Priori Estimate

In order to discuss the effect of cross-diffusion on the existence of nonconstant positive solutions of system (5), we provide a prior estimate in this section. For convenience, we assume that the conditions $ka + ma < 1$ and $m \leq k$ always hold. We denote $f(u) = (1 - \mathrm{ku})(1 + mu)$ and

$g(u) = 1/b(u/1+mu-a)$. Then, we have that $f(u)$ is decreasing in $[0,1/k]$ and $f(0)=1$, $f(1/k)=0$, while $g(u)$ is increasing in $[0,1/k]$ and $g(0)=-a/b$. Note that $ka+ma<1$, we have $g(1/k)>0$, and the problem (5) has an unique positive constant steady-state solution, denoted by $(\widetilde{u},\widetilde{v})$, which satisfies

$$\widetilde{u} \in \left(0,\frac{1}{k}\right), (1-k\widetilde{u})(1+m\widetilde{u}) = \frac{1}{b}\left(\frac{\widetilde{u}}{1+m\widetilde{u}}-a\right), \quad \widetilde{v} = (1-k\widetilde{u})(1+m\widetilde{u}). \tag{6}$$

To obtain a priori estimate, we give a lemma first.

Lemma 1. *Let $d_{ij}\in(0,\infty), i=1,2,3,4,\ d_{ij}\longrightarrow d_i\in[0,\infty], j\longrightarrow\infty$, u^* and v^* are constants. If the positive solution (u_j,v_j) of (5) with $d_i=d_{ij}$ uniformly converge to (u^*,v^*) on $\overline{\Omega}$, then the following equalities hold:*

$$\begin{cases} 1-ku^*-\dfrac{v^*}{1+mu^*}=0, \\ -a-bv^*+\dfrac{u^*}{1+mu^*}=0. \end{cases} \tag{7}$$

And especially, if $u^*>0$ and $v^*>0$, then $(u^*,v^*)=(\widetilde{u},\widetilde{v})$. For the sake of simplicity, a,b,m,k are denoted by Λ.

Theorem 2. *Suppose that (u,v) is a positive solution of (5), D_1,D_2 are arbitrary fixed positive numbers, and $d_1,d_2\ge D_1$ and $0\le d_3,d_4\le D_2$, then there exist two positive constants $\underline{C}(\Lambda,D_1,D_2)$ and $\overline{C}(\Lambda,D_1,D_2)$ such that*

$$\underline{C}(\Lambda,D_1,D_2)<u(x),v(x)<\overline{C}(\Lambda,D_1,D_2), \quad \forall x\in\Omega. \tag{8}$$

Proof. First, we show that there exists $C=C(\Lambda,D_1,D_2)>0$ such that

$$\max_{\overline{\Omega}} u\le C\min_{\overline{\Omega}} u, \max_{\overline{\Omega}} v\le C\min_{\overline{\Omega}} v. \tag{9}$$

Let $\varphi(x)=d_1(1+d_3v)u$, $\psi(x)=d_2(1+d_4u)v$ and set $\varphi(x_1)=\max_{\Omega}\varphi$, $\psi(x_2)=\max_{\Omega}\psi$, by the maximum principle [15], we have

$$u(x_1)-ku^2(x_1)-\frac{u(x_1)v(x_1)}{1+mu(x_1)}\ge 0, -av(x_2)-bv^2(x_2)+\frac{u(x_2)v(x_2)}{1+mu(x_2)}\ge 0. \tag{10}$$

Then, $u(x_1)\le 1/k$, $v(x_1)\le 1+mu(x_1)\le 1+m/k$; thus,

$$\max_{\overline{\Omega}} u\le d_1^{-1}\max_{\overline{\Omega}}\varphi = d_1^{-1}\varphi(x_1)\le\left[1+\frac{d_3(k+m)}{k}\right]\frac{1}{k}\le\frac{1+2d_3}{k}. \tag{11}$$

By similar calculation, we have

$$v(x_2)\le\frac{1}{b}\frac{u(x_2)}{1+mu(x_2)}<\frac{1}{b}u(x_2)\le\frac{1}{b}\max_{\overline{\Omega}} u\le\left[1+\frac{d_3(k+m)}{k}\right]\frac{1}{bk}\le\frac{1+2d_3}{bk}, \tag{12}$$

and

$$\max_{\overline{\Omega}} v\le d_2^{-1}\max_{\overline{\Omega}}\psi = d_2^{-1}\psi(x_2)\le\frac{d_4(1+2d_3)^2+k(1+2d_3)}{bk^2}. \tag{13}$$

Due to (11) and (13), by the L^p theory and embedding theorem regularity [16], we assert that

$$\varphi,\psi\in C^{1+\alpha}(\overline{\Omega}). \tag{14}$$

Using the Schauder theory [16] again, we have

$$\|u,v\|_{C^{2+\alpha}}\le\overline{C}(\Lambda,D_1,D_2). \tag{15}$$

Next, we want to estimate φ,ψ. Note that φ,ψ satisfy

$$\begin{cases} -\Delta\varphi=\left(1-ku-\dfrac{v}{1+mu}\right)\dfrac{\varphi}{d_1(1+d_3v)}, & x\in\Omega, \\ -\Delta\psi=\left(-1-bv+\dfrac{u}{1+mu}\right)\dfrac{\psi}{d_2(1+d_4u)}, & x\in\Omega, \\ \dfrac{\partial\varphi}{\partial\nu}=\dfrac{\partial\psi}{\partial\nu}=0, & x\in\partial\Omega. \end{cases} \tag{16}$$

Set

$$c_1(x) = \left(1 - ku - \frac{v}{1+mu}\right)\frac{1}{d_1(1+d_3 v)},$$
$$c_2(x) = \left(-a - bv + \frac{u}{1+mu}\right)\frac{1}{d_2(1+d_4 u)}. \tag{17}$$

We have

$$\|c_1\|_\infty \le \frac{d_4(1+2d_3)^2 + k(1+2d_3) + bk^2(2+2d_3)}{d_1 bk^2}, \|c_2\|_\infty \le \frac{1+\mathrm{ma}}{d_2 m} + \frac{d_4(1+2d_3)^2 + k(1+2d_3)}{d_2 k^2}. \tag{18}$$

Applying Harnack inequality [17], we can achieve that there exists $\widetilde{C} = \widetilde{C}(\Lambda, D_1, D_2)$ such that $\max_{\overline{\Omega}}\varphi \le \widetilde{C}\min_{\overline{\Omega}}\varphi$ and $\max_{\overline{\Omega}}\psi \le \widetilde{C}\min_{\overline{\Omega}}\psi$. Then,

$$\frac{\max_{\overline{\Omega}} u}{\min_{\overline{\Omega}} u} \le \frac{\max_{\overline{\Omega}}\varphi}{\min_{\overline{\Omega}}\varphi}\frac{\max_{\overline{\Omega}}(d_1(1+d_3 v))}{\min_{\overline{\Omega}}(d_1(1+d_3 v))} \le C, \frac{\max_{\overline{\Omega}} v}{\min_{\overline{\Omega}} v} \le \frac{\max_{\overline{\Omega}}\psi}{\min_{\overline{\Omega}}\psi}\frac{\max_{\overline{\Omega}}(d_2(1+d_4 u))}{\min_{\overline{\Omega}}(d_2(1+d_4 u))} \le C, \tag{19}$$

where $C = \max\{1 + D_2\|u\|_\infty, 1 + D_2\|v\|_\infty\}\widetilde{C}$. From those argument, (9) holds.

Now, we prove (8) by contradiction. Suppose there exist a sequence $\{d_{1j}, d_{2j}, d_{3j}, d_{4j}\}_{j=1}^{\infty}$ which satisfies $d_{1j}, d_{2j} \ge D_1, 0 \le d_{3j}, d_{4j} \le D_2$ and a sequence of corresponding positive solutions (u_j, v_j) of (5) with $(d_1, d_2, d_3, d_4) = (d_{1j}, d_{2j}, d_{3j}, d_{4j})$, such that

$$\min_{\overline{\Omega}} u_j \longrightarrow 0, or \ \min_{\overline{\Omega}} v_j \longrightarrow 0, as\ j \longrightarrow \infty. \tag{20}$$

Note that $d_{1j}, d_{2j} \ge D_1$ and $0 \le d_{3j}, d_{4j} \le D_2$, and there are subsequences, denoted by themselves, satisfying $d_{ij} \longrightarrow d_i \in [D_1, \infty]$ for $i = 1, 2$ and $d_{ij} \longrightarrow d_i \in [0, D_2]$ for $i = 3, 4$. Taking into account of (15), we may assume that $(u_j, v_j) \longrightarrow (u, v)$ in $[C^{2+\alpha}(\overline{\Omega})]^2$. Obviously, u, v are nonnegative, satisfying the estimate (15) and we obtain

$$\min_{\overline{\Omega}} u = 0, \text{or } \min_{\overline{\Omega}} v = 0. \tag{21}$$

Moreover, if $d_1, d_2 < \infty$, then we obtain (u, v) satisfies (5). If $d_1 = \infty$, note that (u_j, v_j) satisfies (15), and then, (u, v) satisfies $\Delta(1 + d_3 v)u = 0$ in Ω and $\partial u/\partial \nu = 0$ on $\partial\Omega$. Thus, $(1 + d_3 v)u \ge 0$ is a constant. Similar conclusion holds for d_2.

We next give contradictions for each possible case.

Step 1. The case $d_1, d_2 < \infty$.

If $\min_{\overline{\Omega}} u = 0$, it is followed that $u = 0$ on $\overline{\Omega}$ by (9). In that case, v satisfies

$$\begin{cases} -d_2\Delta v = -av - bv^2, & x \in \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{22}$$

By using the strong maximum principle and Hopf boundary lemma [16], we have $v = 0$ on $\overline{\Omega}$. Therefore, $(u_j, v_j) \longrightarrow (0, 0)$, which is contradictory to lemma 1. So, $\min_{\overline{\Omega}} u > 0$.

Similarly, if $\min_{\overline{\Omega}} v = 0$, then $v = 0$ on $\overline{\Omega}$. And u satisfies

$$\begin{cases} -d_1\Delta u = u - ku^2, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{23}$$

We can see that $u = 1/k$. So, we see that $(u_j, v_j) \longrightarrow (1/k, 0)$. Note that $ma + ka < 1$, and it is contradictory to lemma 1. So, $\min_{\overline{\Omega}} v > 0$.

Step 2. The cases $d_1 = \infty$ or $d_2 = \infty$.

Note that (u_j, v_j) satisfies

$$\int_\Omega \left(u_j - ku_j^2\right)dx = \int_\Omega \frac{u_j v_j}{1 + mu_j}\mathrm{d}x,$$
$$\int_\Omega \left(av_j + bv_j^2\right)\mathrm{d}x = \int_\Omega \frac{u_j v_j}{1 + mu_j}\mathrm{d}x. \tag{24}$$

We obtain that

$$\int_\Omega \left(u - ku^2\right)dx = \int_\Omega \frac{uv}{1 + mu}dx,$$
$$\int_\Omega \left(av + bv^2\right)dx = \int_\Omega \frac{uv}{1 + mu}dx. \tag{25}$$

(1): $d_1 = \infty$. We can see that $(1 + d_3 v)u = C_1$ is a nonnegative constant. If $C_1 = 0$, then $u = 0$. Taking into account of (25), we obtain $v = 0$. Hence, $(u_j, v_j) \longrightarrow (0, 0)$. This is a contradiction to lemma 1. So, $C_1 > 0$.

(1a): $d_2 < \infty$. Let

$$\widehat{v}_j = \frac{v_j}{\|v_j\|_\infty}. \quad (26)$$

Then, $u_j, v_j, \widehat{v}_j$ satisfy

$$\begin{cases} -d_2\Delta\left[\left(1+d_4u_j\right)\widehat{v}_j\right] = -a\widehat{v}_j - bv_j\widehat{v}_j + \dfrac{u_j\widehat{v}_j}{1+mu_j}, & x \in \Omega, \\ \dfrac{\partial \widehat{v}_j}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \quad (27)$$

Similarly, there exists a subsequence of $\{\widehat{v}_j\}$, denoted by itself, such that $\widehat{v}_j \longrightarrow \widehat{v}$ in $[C^{2+\alpha}(\overline{\Omega})]^2$, where $\widehat{v}$ is nonnegative and $\|\widehat{v}\|_\infty = 1$.

If $\lim_{j\longrightarrow\infty}\|v_j\|_\infty \geq \delta > 0$, then we can see that $u, v, \widehat{v}$ satisfy

$$\begin{cases} -d_2\Delta\left[\left(1+d_4u\right)\widehat{v}\right] + a\widehat{v} + bv\widehat{v} = \dfrac{u\widehat{v}}{1+mu}, & x \in \Omega, \\ \dfrac{\partial \widehat{v}}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \quad (28)$$

Because $\min_{\overline{\Omega}}\widehat{v} = 0$, it follows that $(1+d_4u)\widehat{v} = 0$; so, $\widehat{v} = 0$. This is contradictory to $\|\widehat{v}\|_\infty = 1$.

If $\lim_{j\longrightarrow\infty}\|v_j\|_\infty = 0$, then $v = 0$, $u = C_1$, and from lemma 1, C_1 satisfies

$$\begin{cases} 1 - kC_1 = 0, & x \in \Omega, \\ -a + \dfrac{C_1}{1+mC_1} = 0, & x \in \partial\Omega. \end{cases} \quad (29)$$

This contradicts $ma + ka < 1$

(1b): $d_2 = \infty$. It is obvious that $(1+d_4u)v = C_2$, here $C_2 \geq 0$. Taking into account $\min_{\overline{\Omega}}v = 0$, (11), and $0 \leq d_3, d_4 \leq D_2$, we arrive at $C_2 = 0$ and $v = 0$, so $u = C_1$ and $(u_j, v_j) \longrightarrow (C_1, 0)$. Note that ma + ka < 1, and we see that it contradicts lemma 1.

(2): $d_1 < \infty$. In this case, we claim that $u > 0$ on $\overline{\Omega}$. In truth, $u = 0$ can follow from $\min_{\overline{\Omega}}u = 0$ by (9). Note that from (25), we can see that $v = 0$. This contradicts lemma 1.

(2a): $d_2 = \infty$. Similar to case (1b), we obtain $v = 0$ and $u = C_1$. We can also obtain a contradiction by lemma 1.

The proof is complete. □

3. The Nonexistence of Nonconstant Positive Solutions

In this section, we will use the properties of integrals as in [5] to obtain Theorem 3, which describes the nonexistence of the nonconstant positive solution of the system (5) on some conditions. Especially, if $d_3 = d_4 = 0$, then we can use the method as in [18] to prove that the constant positive steady state $(\widetilde{u}, \widetilde{v})$ of the following developing system is global asymptotically stable.

$$\begin{cases} u_t - d_1\Delta u = u - ku^2 - \dfrac{uv}{1+mu}, & x \in \Omega, t > 0, \\ v_t - d_2\Delta v = -av - bv^2 + \dfrac{uv}{1+mu}, & x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(0,x) = u_0(x) \geq 0, v(0,x) = v_0(x) \geq 0, & x \in \overline{\Omega}. \end{cases} \quad (30)$$

As a consequence, problem (5) with $d_3 = d_4 = 0$ has no nonconstant positive solutions.

Theorem 3. *Let D be a positive constant, and there exist C_1 and C_2 that depend on Λ and D. The system (5) has no nonconstant positive solution when $d_1 > C_1, d_2 > C_2$ and $0 \leq d_3, d_4 \leq D$.*

Proof. Let (u, v) be a nonconstant positive solution of system (5), and $g^* = 1/|\Omega|\int_\omega g dx$. We denote

$$\begin{cases} h_1(u,v) = u - ku^2 - \dfrac{uv}{1+mu}, \\ h_2(u,v) = -av - bv^2 + \dfrac{uv}{1+mu}. \end{cases} \quad (31)$$

Multiplying $(u - u^*)$ on both sides of $h_1(u, v)$ and integrating on Ω, we obtain

$$\begin{aligned} &\int_\Omega \left(d_1(1+d_3v)|\nabla u|^2 + d_1d_3u\nabla u \cdot \nabla v\right)dx \\ &= \int_\Omega h_1(u,v)(u-u^*)dx \\ &= \int_\Omega \left[\left(h_1(u,v) - h_1(u^*,v)\right) + \left(h_1(u^*,v) - h_1(u^*,v^*)\right)\right](u-u^*)dx \\ &= \int_\Omega \left\{\left[1 - k(u+u^*) - \frac{v}{(1+mu)(1+\mathrm{mu}^*)}\right](u-u^*)^2 - \frac{u^*(u-u^*)(v-v^*)}{1+\mathrm{mu}^*}\right\}dx. \end{aligned} \quad (32)$$

Analogously, multiplying $(v-v^*)$ on both sides of $h_2(u,v)$ and integrating on Ω, we have

$$\begin{aligned}&\int_\Omega\left(d_2(1+d_4u)|\nabla v|^2+d_2d_4v\nabla u\cdot\nabla v\right)\mathrm{d}x\\&\quad=\int_\Omega\left\{\frac{v(u-u^*)(v-v^*)}{(1+mu)(1+\mathrm{mu}^*)}-\left(a+b(v+v^*)-\frac{u^*}{1+\mathrm{mu}^*}\right)(v-v^*)^2\right\}\mathrm{d}x.\end{aligned}\tag{33}$$

Adding the above two expressions, we obtain

$$\begin{aligned}&\int_\Omega\left\{d_1(1+d_3v)|\nabla u|^2+d_2(1+d_4u)|\nabla v|^2+d_1d_3u\nabla u\cdot\nabla v+d_2d_4v\nabla u\cdot\nabla v\right\}dx\\&\quad=\int_\Omega\left[1-k(u+u^*)-\frac{v}{(1+mu)(1+\mathrm{mu}^*)}\right](u-u^*)^2\\&\qquad+\left[\frac{v}{(1+mu)(1+\mathrm{mu}^*)}-\frac{u^*}{1+\mathrm{mu}^*}\right](u-u^*)(v-v^*)-\left(a+b(v+v^*)-\frac{u^*}{1+\mathrm{mu}^*}\right)(v-v^*)^2dx\\&\quad\le\int_\Omega\left[1-k(u+u^*)+\frac{u^*}{1+\mathrm{mu}^*}\right](u-u^*)^2\\&\qquad-\left[a+b(v+v^*)-\frac{2u^*}{1+\mathrm{mu}^*}-\frac{v}{(1+mu)(1+\mathrm{mu}^*)}\right](v-v^*)^2dx.\end{aligned}\tag{34}$$

Taking into account of (11) and (15), we have

$$\begin{aligned}&\left|1-k(u+u^*)+\frac{u^*}{1+\mathrm{mu}^*}\right|\le 2+2d_3+\frac{1}{m},\\&\left|a+b(v+v^*)-\frac{2u^*}{1+\mathrm{mu}^*}-\frac{v}{(1+mu)(1+\mathrm{mu}^*)}\right|\le a+\frac{2}{m}+(2b+1)\frac{d_4(1+2d_3)^2+k(1+2d_3)}{bk^2}.\end{aligned}\tag{35}$$

Denote

$$\begin{aligned}&\widetilde{C}_1=\frac{1}{\lambda_1}\left(2+2D+\frac{1}{m}\right),\\&\widetilde{C}_2=\frac{1}{\lambda_1}\left[a+\frac{2}{m}+(2b+1)\frac{D(1+2D)^2+k(1+2D)}{bk^2}\right].\end{aligned}\tag{36}$$

From Theorem 2, $\varepsilon-$Young inequality and Poincar é inequality [19], we have

$$\int_\Omega \{d_1|\nabla u|^2 + d_2|\nabla v|^2\}dx$$
$$\leq \int_\Omega \left\{\widetilde{C}_1\lambda_1(u-u^*)^2 + \widetilde{C}_2\lambda_1(v-v^*)^2 + d_1\varepsilon|\nabla u|^2 + \frac{d_3^2u^2}{4\varepsilon}|\nabla v|^2 + d_2\varepsilon|\nabla v|^2 + \frac{d_4^2v^2}{4\varepsilon}|\nabla u|^2\right\}dx \tag{37}$$
$$\leq \int_\Omega \left\{\widetilde{C}_1|\nabla u|^2 + \widetilde{C}_2|\nabla v|^2 + d_1\varepsilon|\nabla u|^2 + \frac{d_3^2u^2}{4\varepsilon}|\nabla v|^2 + d_2\varepsilon|\nabla v|^2 + \frac{d_4^2v^2}{4\varepsilon}|\nabla u|^2\right\}dx.$$

Let $\varepsilon = 1/2$, then we have

$$\int_\Omega \{d_1|\nabla u|^2 + d_2|\nabla v|^2\}dx \leq \int_\Omega \{(2\widetilde{C}_1 + d_4^2v^2)|\nabla u|^2 + (2\widetilde{C}_2 + d_3^2u^2)|\nabla v|^2\}dx. \tag{38}$$

Denote

$$C_1 = 2\widetilde{C}_1 + D^2\left[\frac{D(1+2D)^2 + k(1+2D)}{bk^2}\right]^2,$$
$$C_2 = 2\widetilde{C}_2 + D^2\left(\frac{1+2D}{k}\right)^2. \tag{39}$$

If $d_1 > C_1$ and $d_2 > C_2$, then $|\nabla u| = |\nabla v| = 0, x \in \Omega$; in other words, $u = u^*, v = v^*, x \in \Omega$. So the existence of C_1, C_2 makes that if $d_1 > C_1, d_2 > C_2$ and $0 \leq d_3, d_4 \leq D$, then the system (5) has no nonconstant positive solutions. □

Theorem 4. *If condition* $(1-4bm)\widetilde{u} < 4b$ *or* $4mb > 1$ *holds, then the constant positive steady state* $(\widetilde{u}, \widetilde{v})$ *of (30) is global asymptotically stable.*

We omit the proof because it is analogous to that of Theorem 2 in [18].

4. The Existence of Nonconstant Positive Solutions

In this section, we shall obtain nonconstant solutions for large d_4 or for large d_2 (with $d_4 > 0$).

First, we discuss the linearized system of (5) at $\widetilde{Z} = (\widetilde{u}, \widetilde{v})$. Denote $\Psi(Z) = (d_1(1+d_3v)u, d_2(1+d_4u)v)^T$; then, system (5) can be written as

$$-\Delta\Psi(Z) = G(Z), \text{with } G(Z) = \begin{pmatrix} u - ku^2 - \dfrac{uv}{1+mu} \\ -av - bv^2 + \dfrac{uv}{1+mu} \end{pmatrix}. \tag{40}$$

Applying the same method as in [18], it is obtained that

$$H(\mu) = \det\{\Psi_Z^{-1}(\widetilde{Z})\}\det\{\mu\Psi_Z(\widetilde{Z}) - G_Z(\widetilde{Z})\}. \tag{41}$$

$\det\{\Psi_Z^{-1}(\widetilde{Z})\}$ is positive, and

$$\det\{\mu\Psi_Z(\widetilde{Z}) - G_Z(\widetilde{Z})\} = C_2\mu^2 + C_1\mu + C_0 \triangleq \mathscr{C}(d_2, d_3, d_4; \mu), \tag{42}$$

where

$$C_2 = d_1d_2(1 + d_3\widetilde{v} + d_4\widetilde{u}),$$
$$C_1 = d_2\frac{2km\widetilde{u}^2 + (k-m)\widetilde{u}}{1+m\widetilde{u}} + d_2d_4\widetilde{u}\frac{3km\widetilde{u}^2 + 2(k-m)\widetilde{u} - 1}{1+m\widetilde{u}} + d_1b(\widetilde{v} + d_3\widetilde{v}^2) + \frac{d_1d_3\widetilde{u}\widetilde{v}}{(1+m\widetilde{u})^2}, \tag{43}$$
$$C_0 = b\widetilde{v}\frac{2km\widetilde{u}^2 + (k-m)\widetilde{u}}{1+m\widetilde{u}} + \frac{\widetilde{u}\widetilde{v}}{(1+m\widetilde{u})^3}.$$

Next, we study the dependence of the solution for $\mathscr{C}(d_2, d_3, d_4; \mu) = 0$ on d_4. We denote the solutions of $\mathscr{C}(d_2, d_3, d_4; \mu) = 0$ by $\widetilde{\mu}_1(d_4), \widetilde{\mu}_2(d_4)$ with $\text{Re}\widetilde{\mu}_1(d_4) \leq \text{Re}\widetilde{\mu}_2(d_4)$. Note that $C_0 > 0$, and it is obvious that $\widetilde{\mu}_1(d_4)\widetilde{\mu}_2(d_4) > 0$.

Consider the following limitations:

$$\lim_{d_4\to\infty}\frac{C_2}{d_4} = d_1d_2\widetilde{u} \triangleq a_2 > 0,$$
$$\lim_{d_4\to\infty}\frac{C_1}{d_4} = d_2\widetilde{u}\frac{3km\widetilde{u}^2 + 2(k-m)\widetilde{u} - 1}{1+m\widetilde{u}} \triangleq a_1, \tag{44}$$
$$\lim_{d_4\to\infty}\frac{C_0}{d_4} = 0.$$

That is,

$$\lim_{d_4\longrightarrow\infty}\frac{\mathscr{C}(d_2,d_3,d_4;\mu)}{d_4}=a_2\mu^2+a_1\mu=\mu[a_2\mu+a_1]. \quad (45)$$

If the condition,

$$b(2k+m)^2+4ka(2k+m)<4k, \quad (46)$$

holds, we have $f(1/2k)<g(1/2k)$, and then, $\tilde{u}<1/2k$. Moreover, we can see $a_1<0$, and hence, $C_1<0$ when d_4 is large enough. By continuity, if d_4 is large enough, then $\tilde{\mu}_1(d_4)>0$ and $\tilde{\mu}_2(d_4)>0$, satisfying

$$\lim_{d_4\longrightarrow\infty}\tilde{\mu}_1(d_4)=0,\ \lim_{d_4\longrightarrow\infty}\tilde{\mu}_2(d_4)=-\frac{a_1}{a_2}\triangleq\tilde{\mu}>0. \quad (47)$$

Analogously, we consider the dependence of $\mathscr{C}(d_2,d_3,d_4;\mu)$ on d_2; then, we derive

$$\begin{aligned}&\lim_{d_2\longrightarrow\infty}\frac{C_2}{d_2}=d_1(1+d_3\tilde{v}+d_4\tilde{v})\triangleq b_2>0,\ \lim_{d_2\longrightarrow\infty}\frac{C_0}{d_2}=0,\\&\lim_{d_2\longrightarrow\infty}\frac{C_1}{d_2}=\frac{2km\tilde{u}^2+(k-m)\tilde{u}}{1+m\tilde{u}}+d_4\tilde{u}\frac{3km\tilde{u}^2+2(k-m)\tilde{u}-1}{1+m\tilde{u}}\triangleq b_1.\end{aligned} \quad (48)$$

On the conditions (46) and

$$d_4>\frac{2km\tilde{u}+k-m}{-3km\tilde{u}^2-2(k-m)\tilde{u}+1}, \quad (49)$$

we have $b_1<0$. We can obtain similar results to (47). $\overline{\mu}_1(d_2)$ and $\overline{\mu}_2(d_2)$, and the solutions of $\mathscr{C}(d_2,d_3,d_4;\mu)=0$ are positive and real for sufficiently large d_2, satisfying

$$\lim_{d_2\longrightarrow\infty}\overline{\mu}_1(d_2)=0,\ \lim_{d_2\longrightarrow\infty}\overline{\mu}_2(d_2)=-\frac{b_2}{b_1}\triangleq\overline{\mu}>0. \quad (50)$$

Remark 5. It is obvious that if (47) or (50) holds, then the constant positive steady state $(\tilde{u},\tilde{v})$ for (1) is unstable. Noting Theorem 4, we can easily see that introducing cross-diffusion can change the asymptotic behavior of solutions to system (30).

If $\tilde{\mu}$ and $\overline{\mu}$, determined by (47) and (50) respectively, satisfy some conditions, then we can obtain the following conclusions by using homotopy invariance of the topological degree. We omit the proofs because they are analogous to them of Theorem 4.1 in [20].

Theorem 6. *Suppose condition (46) holds, there exists $\tilde{d}_4>0$; if $d_4\geq\tilde{d}_4$, $\tilde{\mu}\in(\mu_j,\mu_{j+1})$ for some $j\geq2$ and $\sum_{k=2}^{j}\dim E(\mu_k)$ is odd, then system (5) has at least one nonconstant positive solution.*

Theorem 7. *Suppose conditions (46) and (49) hold, there exists $\overline{d}_2>0$; if $d_2\geq\overline{d}_2$, $\overline{\mu}\in(\mu_j,\mu_{j+1})$ and $\sum_{k=2}^{j}\dim E(\mu_k)$ is odd, then system (5) has at least one nonconstant positive solution.*

5. Conclusion

In this paper, we investigate a cross-diffusion prey-predator model with Holling type II functional response and homogeneous Neumann boundary condition and mainly discuss the effect of d_2, d_3, and d_4 on positive solutions of (5). Furthermore, we find some interesting phenomenon of the system (5). When cross-diffusions d_3 and d_4 are small enough (even equal to zero), the system (5) has no nonconstant positive solution. When the natural diffusion d_2 or the cross-diffusion d_4 is large enough and other diffusions are fixed, the system (5) has at least one nonconstant positive solution. This shows that under certain hypotheses, cross-diffusions can create nonconstant positive steady states even though the corresponding model without cross-diffusion fails.

There are many ways that our model could be extended. It may be more realistic for the variables to have species diversity, the cross-diffusion to have different forms, and the parameters to have space-dependence and/or time-dependence. In future work, it will be of interest to explore the impact of different cross-diffusion rates and numerical simulation, as in references [6–8, 21–23]. It would be intriguing to see how cross-diffusion affects such ecosystems.

Data Availability

Data sharing is not applicable as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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