

Research Article

On the Independent Coloring of Graphs with Applications to the Independence Number of Cartesian Product Graphs

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Let G be a graph with $V = V(G)$. A nonempty subset S of V is called an independent set of G if no two distinct vertices in S are adjacent. The union of a class $\{S: S \text{ is an independent set of } G\}$ and $\{\emptyset\}$ is denoted by $\mathcal{I}(G)$. For a graph H , a function $f: V \rightarrow \mathcal{I}(H)$ is called an H -independent coloring of G (or simply called an H -coloring) if $f(x) \cap f(y) = \emptyset$ for any adjacent vertices $x, y \in V$ and $f(V)$ is a class of disjoint sets. Let $\alpha(H, G)$ denote the maximum cardinality of the set $\{\sum_{x \in V} |f(x)|: f \text{ is an } H\text{-coloring of } G\}$. In this paper, we obtain basic properties of an H -coloring of G and find $\alpha(H, G)$ of some families of graphs G and H . Furthermore, we apply them to determine the independence number of the Cartesian product of a complete graph K_n and a graph G and prove that $\alpha(K_n \square G) = \alpha(K_n, G)$.

1. Introduction

In graph theory, the study of graph invariants is a worldwide research related to a property of graphs that depends on the abstract structure. This invariant property can be formalized as a function from graphs to a class of values such that any two isomorphic graphs have the same value. Two well-known graph invariants or parameters are the independence number and the chromatic number of graphs which indicate the maximum size and the minimum size of independent sets and color sets of graphs, respectively. Their applications can be found in several fields such as computer science, engineering, and optimization problems. Many researchers study those parameters in various ways and sometime combine their concepts together to model certain new structures. For example, in 2011, Arumugam et al. [1] investigated maximal independent sets in minimum colorings. In 2012, Wu and Hao [2] presented an effective approach to coloring large graphs by using a preprocessing method to extract large independent sets from the graphs. In 2016, Samanta et al. [3] introduced a new concept of coloring

of fuzzy graphs to color world political map mentioning the strength of relationship among the countries. Later in 2020, Mahapatra et al. [4] proposed the edge coloring of fuzzy graphs to solve job oriented web sites and traffic light problems. Recently, in 2022, Brešar and Štesl [5] presented the independence coloring game of graphs and proved that the independence game chromatic number of a tree can be arbitrarily large. In this paper, we introduce a new parameter of graphs, called the independent coloring, which is motivated from the combination of the independence and coloring concepts. Moreover, prominent properties of the independent coloring are provided. Furthermore, we apply the result for studying the independence number of Cartesian product graphs. More information about other operations and product of graphs can be found in [6–9].

2. Preliminaries and Notations

Throughout this paper, all graphs are considered to be finite and simple. Let $G = (V, E)$ be a graph of order $|V|$. Two vertices u and v are said to be adjacent, if $uv \in E$. For

a nonempty subset S of V , the induced subgraph of G induced by S , denoted by $G[S]$, is a subgraph of G provided that if $u, v \in S$ and $uv \in E$, then $uv \in E(G[S])$, as well. A subgraph H of G is said to be spanning if $V(H) = V(G)$. We remark that a graph without edges is called an empty graph. For a non-negative integer n , we denote K_n, C_n, P_n, W_n , and S_n to be a complete graph of order $n \geq 1$, a cycle of length $n \geq 0$, a path of length $n \geq 0$, a wheel of order $n \geq 4$, and a star of order $n \geq 2$, respectively. For other graph terminologies and notations, we refer the reader to [10].

A set S of vertices of G is said to be an independent set of G if no two distinct vertices in S are adjacent. The maximum cardinality of an independent set of G is called the independence number of G and is denoted by $\alpha(G)$. If S is an independent set such that $|S| = \alpha(G)$, we say that S is an α -set of G . For a positive integer n , an n -coloring of a graph G means a surjection from $V(G)$ to the set $\{1, 2, \dots, n\}$ with $f(u) \neq f(v)$ for every adjacent vertices u, v in G . The chromatic number $\chi(G)$ of G is defined to be the minimum of i over all i -colorings of G , and we denote a $\chi(G)$ -coloring of G by χ -coloring of G .

We now introduce a graph parameter. For a graph G , we denote by $\mathcal{I}(G)$ the class $\{S: S \text{ which is an independent set of } G\} \cup \{\emptyset\}$. Let G and H be graphs with $V = V(G)$. A function $f: V \rightarrow \mathcal{I}(H)$ is called an H -independent coloring of G (or simply called an H -coloring) if $f(x) \cap f(y) = \emptyset$ for any adjacent vertices $x, y \in V$ and $f(V)$ is a class of disjoint sets. An H -coloring f of G is said to be trivial when $f(x) = \emptyset$ for any vertex $x \in V$. In addition, we denote by $\alpha(H, G)$ the maximum cardinality of

the set $\{\sum_{x \in V} |f(x)|: f \text{ is an } H\text{-coloring of } G\}$. If f is an H -coloring of G such that $\sum_{x \in V} |f(x)| = \alpha(H, G)$, we say that f is an α - H -coloring of G . It is not hard to see that $\alpha(H, G) \geq 1$. For a trivial case $H = K_1$, we note that $\alpha(H, G)$ is actually the independence number $\alpha(G)$ of G .

3. Independent Coloring of Graphs

We start this section by presenting the following two useful lemmas which are referred in the proofs of other results.

Lemma 1. *Let G and H be graphs and let f be an H -coloring of G with $f(V(G)) = \{D_1, D_2, \dots, D_m\}$, where $1 \leq m \leq |V(H)| + 1$. Then, the following statements hold:*

- (1) $f^{-1}(\{D\})$ is an independent set of G for each $D \in f(V(G)) \setminus \{\emptyset\}$.
- (2) $f^{-1}(\{D_i\})$ and $f^{-1}(\{D_j\})$ are disjoint for $i \neq j$.
- (3) $\sum_{x \in V(G)} |f(x)| = \sum_{i=1}^m |D_i| |f^{-1}(\{D_i\})|$.

Proof. Let $D \in f(V(G)) \setminus \{\emptyset\}$. If there are adjacent vertices $x, y \in f^{-1}(\{D\})$, then $\emptyset = f(x) \cap f(y) = D \neq \emptyset$, a contradiction. Hence, (1) holds. Next, we show that (2) holds. It is clear for $f(V(G)) = \{\emptyset\}$. Then, we may assume that $f(V(G)) \neq \{\emptyset\}$. Let $D_i, D_j \in f(V(G))$ with $i \neq j$. If there is a vertex $v \in f^{-1}(\{D_i\}) \cap f^{-1}(\{D_j\})$, then $D_i = f(v) = D_j$. Therefore, $\emptyset = D_i \cap D_j = D_i = D_j$ and this implies $f(V(G)) = \{\emptyset\}$, a contradiction. Hence, (2) holds. And this leads to

$$\begin{aligned} \sum_{x \in V(G)} |f(x)| &= \sum_{x \in f^{-1}(\{D_1\})} |D_1| + \sum_{x \in f^{-1}(\{D_2\})} |D_2| + \dots + \sum_{x \in f^{-1}(\{D_m\})} |D_m| \\ &= |D_1| \left(\sum_{x \in f^{-1}(\{D_1\})} 1 \right) + |D_2| \left(\sum_{x \in f^{-1}(\{D_2\})} 1 \right) + \dots + |D_m| \left(\sum_{x \in f^{-1}(\{D_m\})} 1 \right) \\ &= |D_1| |f^{-1}(\{D_1\})| + |D_2| |f^{-1}(\{D_2\})| + \dots + |D_m| |f^{-1}(\{D_m\})| \\ &= \sum_{i=1}^m |D_i| |f^{-1}(\{D_i\})|, \end{aligned} \quad (1)$$

and hence (3) holds. \square

Lemma 2. *Let G and H be graphs and let f be an H -coloring of G . If $g: V(H) \rightarrow \mathcal{I}(G)$ is a function defined by*

$$g(x) = \begin{cases} f^{-1}(\{D\}), & \text{if } x \in D \text{ for some } D \in f(V(G)), \\ \emptyset, & \text{otherwise,} \end{cases} \quad (2)$$

for all $x \in V(H)$, then the following statements hold:

- (1) g is a G -coloring of H .
- (2) $\sum_{x \in V(G)} |f(x)| = \sum_{x \in V(H)} |g(x)|$.

Proof. It is clear, by Lemma 1, that (1) holds. Let $f(V(G)) = \{D_1, D_2, \dots, D_m\}$ where $1 \leq m \leq |V(H)| + 1$. Again by Lemma 1, we obtain that

$$\begin{aligned} \sum_{x \in V(G)} |f(x)| &= \sum_{i=1}^m |D_i| |f^{-1}(\{D_i\})| = \sum_{i=1}^m |f^{-1}(\{D_i\})| |D_i| \\ &= \sum_{i=1}^m |f^{-1}(\{D_i\})| |g^{-1}(f^{-1}(\{D_i\}))| \\ &= \sum_{x \in V(H)} |g(x)|, \end{aligned} \quad (3)$$

as required.

We note that graphs G and H in $\alpha(H, G)$ can be switched as in the following result. \square

Proposition 3. Let G and H be graphs. Then,

$$\alpha(H, G) = \alpha(G, H). \quad (4)$$

Proof. Let f be an α - H -coloring of G and g be a G -coloring of H defined in Lemma 2. Then, by Lemma 2, $\alpha(H, G) = \sum_{x \in V(G)} |f(x)| = \sum_{x \in V(H)} |g(x)| \leq \alpha(G, H)$. Similarly, $\alpha(G, H) \leq \alpha(H, G)$. Hence, the equality follows. \square

Proposition 4. Let G , H_1 , and H_2 be graphs. Then, $\alpha(H_1, G) \geq \alpha(H_2, G)$ if H_1 is a spanning subgraph of H_2 .

Proof. Let f be an α - H_2 -coloring of G . Define $g: V(G) \rightarrow \mathcal{I}(H_1)$ by $g(x) = f(x)$ for all $x \in V(G)$. Clearly, g is an H_1 -coloring of G . Hence, $\alpha(H_1, G) \geq \sum_{x \in V(G)} |g(x)| = \sum_{x \in V(G)} |f(x)| = \alpha(H_2, G)$.

We now give some basic properties of an α - H -coloring of G which are useful for describing the lower bound and the upper bound of $\alpha(H, G)$. \square

Lemma 5. Let G and H be graphs and f be an α - H -coloring of G .

- (1) If $\emptyset \in f(V(G))$, then $f(V(G)) \setminus \{\emptyset\}$ is a partition of $V(H)$.
- (2) If $\emptyset \notin f(V(G))$ and $V(H) \neq \cup f(V(G))$, then $\alpha(H - \cup f(V(G))) \leq \min_{P \in f(V(G))} |P|$.

Proof. Assume that $\emptyset \in f(V(G))$. Let $w \in V(G)$ with $f(w) = \emptyset$. We show that $V(H) \setminus \cup f(V(G)) = \emptyset$. Suppose, to the contrary, that there is a vertex $v \in V(H) \setminus \cup f(V(G))$. Define $g: V(G) \rightarrow \mathcal{I}(H)$ by

$$g(u) = \begin{cases} \{v\}, & \text{if } u = w, \\ f(u), & \text{otherwise,} \end{cases} \quad (5)$$

for all $u \in V(G)$. Obviously, g is an H -coloring of G . However,

$$\begin{aligned} \alpha(H, G) &= \sum_{x \in V(G)} |f(x)| = |f(w)| + \sum_{x \in V(G) \setminus \{w\}} |f(x)| = |\emptyset| + \sum_{x \in V(G) \setminus \{w\}} |g(x)| \\ &= \sum_{x \in V(G) \setminus \{w\}} |g(x)| < |\{v\}| + \sum_{x \in V(G) \setminus \{w\}} |g(x)| \\ &= |g(w)| + \sum_{x \in V(G) \setminus \{w\}} |g(x)| = \sum_{x \in V(G)} |g(x)|, \end{aligned} \quad (6)$$

which is impossible. Therefore, $V(H) = \cup f(V(G)) = \cup (f(V(G)) \setminus \{\emptyset\})$. Hence, $f(V(G)) \setminus \{\emptyset\}$ is a partition of $V(H)$.

Next, suppose that $\emptyset \notin f(V(G))$ and $V(H) \setminus \cup f(V(G)) \neq \emptyset$. Let S be an α -set of $H - \cup f(V(G))$ and $T \in f(V(G))$ with $|T| = \min_{P \in f(V(G))} |P|$. If

$\alpha(H - \cup f(V(G))) > \min_{P \in f(V(G))} |P|$, then define $g: V(G) \rightarrow \mathcal{I}(H)$ by

$$g(x) = \begin{cases} S, & \text{if } f(x) = T, \\ f(x), & \text{otherwise,} \end{cases} \quad (7)$$

for all $x \in V(G)$. It is easy to see that g is an H -coloring of G . However,

$$\begin{aligned} \alpha(H, G) &= \sum_{x \in V(G)} |f(x)| = \sum_{x \in f^{-1}(\{T\})} |f(x)| + \sum_{x \in V(G) \setminus f^{-1}(\{T\})} |f(x)| \\ &\leq \sum_{x \in f^{-1}(\{T\})} |T| + \sum_{x \in V(G) \setminus f^{-1}(\{T\})} |f(x)| < \sum_{x \in f^{-1}(\{T\})} |S| + \sum_{x \in V(G) \setminus f^{-1}(\{T\})} |f(x)| \\ &= \sum_{x \in g^{-1}(\{S\})} |S| + \sum_{x \in V(G) \setminus g^{-1}(\{S\})} |g(x)| = \sum_{x \in g^{-1}(\{S\})} |g(x)| + \sum_{x \in V(G) \setminus g^{-1}(\{S\})} |g(x)| \\ &= \sum_{x \in V(G)} |g(x)|, \end{aligned} \quad (8)$$

which is impossible. Hence, $\alpha(H - \cup f(V(G))) \leq \min_{P \in f(V(G))} |P|$.

We obtain the lower and upper bounds for $\alpha(H, G)$ in terms of independence numbers and order of graphs. \square

Theorem 6. Let G and H be graphs. Then,

$$\alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\} \leq \alpha(H, G) \leq \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}. \quad (9)$$

Proof. Let A be an α -set of G and B be an α -set of H . Without loss of generality, we can assume that $|V(G)| - |A| \geq |V(H)| - |B|$. We consider the following two cases. \square

Case 1. $V(H) \setminus B = \emptyset$.

Define $f: V(G) \rightarrow \mathcal{J}(H)$ by

$$f(x) = \begin{cases} B, & \text{if } x \in A, \\ \emptyset, & \text{otherwise,} \end{cases} \quad (10)$$

for all $x \in V(G)$. Clearly, f is an H -coloring of G . Thus,

$$\begin{aligned} \alpha(H, G) &\geq \sum_{x \in V(G)} |f(x)| = |A||B| = \alpha(G)\alpha(H) + 0 = \alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), 0\} \\ &= \alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\}. \end{aligned} \quad (11)$$

Case 2. $V(H) \setminus B \neq \emptyset$.

Let $v \in V(H) \setminus B$. Define $f: V(G) \rightarrow \mathcal{J}(H)$ by

$$f(x) = \begin{cases} B, & \text{if } x \in A, \\ \{v\}, & \text{if } x \in V(G) \setminus A, \\ \emptyset, & \text{otherwise,} \end{cases} \quad (12)$$

for all $x \in V(G)$. Clearly, f is an H -coloring of G . Thus,

$$\begin{aligned} \alpha(H, G) &\geq \sum_{x \in V(G)} |f(x)| = |A||B| + |v||V(G) \setminus A| = |A||B| + (|V(G)| - |A|) \\ &= \alpha(G)\alpha(H) + |V(G)| - \alpha(G) \geq \alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\}. \end{aligned} \quad (13)$$

Next, let f be an α - H -coloring of G and g be an α - G -coloring of H with $f(V(G)) = \{D_1, D_2, \dots, D_m\}$ and

$g(V(H)) = \{D'_1, D'_2, \dots, D'_m\}$, where $1 \leq m \leq |V(H)| + 1$ and $1 \leq m' \leq |V(G)| + 1$. By Lemma 1, it follows that

$$\alpha(H, G) = \sum_{x \in V(G)} |f(x)| = \sum_{i=1}^m |D_i| |f^{-1}(\{D_i\})| \leq \sum_{i=1}^m |D_i| |\alpha(G) = \alpha(G) \sum_{i=1}^m |D_i| \leq \alpha(G)|V(H)|, \quad (14)$$

and similarly,

$$\alpha(H, G) = \alpha(G, H) \leq \alpha(H)|V(G)|. \quad (15)$$

Therefore, $\alpha(H, G) \leq \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$. Consequently, the result follows.

To establish the sharpness of the lower bound stated in Theorem 6, consider an empty graph H . Obviously, $\alpha(H, G) = \alpha(G)|V(H)| + 0 = \alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\}$. Next, we investigate the sharpness of the upper bound also stated in Theorem 6. Let $G \cong C_{2m}$ and $H \cong C_{2n}$, where $m, n \in \mathbb{N}$ with $m, n > 1$. We see that $\alpha(H, G) = 2mn = \min\{m(2n), n(2m)\} = \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$.

Lemma 9 (see [10]). Let G be any graph. Then, $|V(G)| \leq \chi(G)\alpha(G)$.

Proof. Let f be a χ -coloring of G . Then, $|V(G)| = \sum_{i=1}^{\chi(G)} |f^{-1}(\{i\})| \leq \sum_{i=1}^{\chi(G)} \alpha(G) = \chi(G)\alpha(G)$. \square

Corollary 10. Let G and H be graphs. Then,

$$\alpha(G)\alpha(H) \leq \alpha(H, G) \leq \min\{\chi(G), \chi(H)\}\alpha(G)\alpha(H). \quad (16)$$

Proof. By Theorem 6 and Lemma 9, we can conclude that

$$\begin{aligned}\alpha(G)\alpha(H) &\leq \alpha(H, G) \leq \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\} \\ &= \min\{\alpha(G)\chi(H)\alpha(H), \alpha(G)\alpha(H)\chi(G)\} = \min\{\chi(G), \chi(H)\}\alpha(G)\alpha(H).\end{aligned}\quad (17)$$

The Nordhaus–Gaddum bound is a sharp lower or upper bound on the sum or product of a parameter of a graph and its complement. By deriving of Corollary 10, we obtain sharp bounds for $\alpha(G)\alpha(\overline{G})$ in terms of $\alpha(G, \overline{G})$, $\chi(G)$, and $\chi(\overline{G})$. Moreover, the graph families attaining the bounds in Proposition 3 attain these bounds also. \square

Corollary 11. For every graph G ,

$$\frac{\alpha(G, \overline{G})}{\min\{\chi(G), \chi(\overline{G})\}} \leq \alpha(G)\alpha(\overline{G}) \leq \alpha(G, \overline{G}). \quad (18)$$

Now, we focus on a G –coloring of G . Before that, we need the following lemma.

Lemma 12. For a positive integer n , let $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$ be real numbers. Then,

$$\sum_{i=1}^n x_i y_i \leq \max \left\{ \sum_{i=1}^n x_i^2, \sum_{i=1}^n y_i^2 \right\}. \quad (19)$$

Proof. For $1 \leq i \leq n$, we obviously have $2x_i y_i \leq x_i^2 + y_i^2$. Consequently,

$$\begin{aligned}2 \sum_{i=1}^n x_i y_i &\leq \sum_{i=1}^n (x_i^2 + y_i^2) = \sum_{i=1}^n x_i^2 + \sum_{i=1}^n y_i^2 \\ &\leq \max \left\{ \sum_{i=1}^n x_i^2, \sum_{i=1}^n y_i^2 \right\} + \max \left\{ \sum_{i=1}^n x_i^2, \sum_{i=1}^n y_i^2 \right\} \\ &= 2 \max \left\{ \sum_{i=1}^n x_i^2, \sum_{i=1}^n y_i^2 \right\}.\end{aligned}\quad (20)$$

Therefore, $\sum_{i=1}^n x_i y_i \leq \max \left\{ \sum_{i=1}^n x_i^2, \sum_{i=1}^n y_i^2 \right\}$.

The following theorem shows that $\alpha(G, G)$ is a sum of squares of positive integers. \square

Theorem 13. For any graph G , we have

$$\alpha(G, G) = \max \left\{ \sum_{P \in \mathcal{P}} |P|^2 : \mathcal{P} \text{ is a class of disjoint independent sets of } G \right\}. \quad (21)$$

Proof. Let $M = \max \left\{ \sum_{P \in \mathcal{P}} |P|^2 : \mathcal{P} \text{ is a class of disjoint independent sets of } G \right\}$. Furthermore, let $\mathcal{P}' = \{D_1, D_2, \dots, D_m\}$ be a class of disjoint independent sets of G such that $\sum_{P \in \mathcal{P}'} |P|^2 = M$, where m is a positive integer. We first show that $\alpha(G, G) \geq M$. Define $g: V(G) \rightarrow \mathcal{I}(G)$ by

$$g(x) = \begin{cases} P, & \text{if } x \in P \text{ for some } P \in \mathcal{P}', \\ \emptyset, & \text{otherwise,} \end{cases} \quad (22)$$

for all $x \in V(G)$. Clearly, g is a G –coloring of G . And thus, by Lemma 1,

$$\alpha(G, G) \geq \sum_{x \in V(G)} |g(x)| = \sum_{i=1}^m |D_i| |g^{-1}(\{D_i\})| = \sum_{i=1}^m |D_i| |D_i| = \sum_{i=1}^m |D_i|^2 = \sum_{P \in \mathcal{P}'} |P|^2 = M. \quad (23)$$

Now, we show that $\alpha(G, G) \leq M$. Let f be an α – G –coloring of G . Furthermore, let $f(V(G)) = \{D_1, D_2, \dots, D_m\}$ be such that

$|D_m| \leq |D_{m-1}| \leq \dots \leq |D_1|$, where m is a positive integer. By Lemmas 1 and 12, we obtain that

$$\alpha(G, G) = \sum_{x \in V(G)} |f(x)| = \sum_{i=1}^m |D_i| |f^{-1}(\{D_i\})| \leq \max \left\{ \sum_{i=1}^m |D_i|^2, \sum_{i=1}^m |f^{-1}(\{D_i\})|^2 \right\}. \quad (24)$$

Case 1. $D_m = \emptyset$.

Clearly, $m \geq 2$. Since both of $\{D_1, D_2, \dots, D_{m-1}\}$ and $\{f^{-1}(\{D_1\}), f^{-1}(\{D_2\}), \dots, f^{-1}(\{D_m\})\}$ are classes of disjoint independent sets of G , we obtain that

$$\alpha(G, G) \leq \max \left\{ \sum_{i=1}^m |D_i|^2, \sum_{i=1}^m |f^{-1}(\{D_i\})|^2 \right\} = \max \left\{ \sum_{i=1}^{m-1} |D_i|^2, \sum_{i=1}^m |f^{-1}(\{D_i\})|^2 \right\} \leq M. \quad (25)$$

Case 2. $D_m \neq \emptyset$.

Since both of $\{D_1, D_2, \dots, D_m\}$ and $\{f^{-1}(\{D_1\}), f^{-1}(\{D_2\}), \dots, f^{-1}(\{D_m\})\}$ are classes of disjoint independent sets of G , we get that

$$\alpha(G, G) \leq \max \left\{ \sum_{i=1}^m |D_i|^2, \sum_{i=1}^m |f^{-1}(\{D_i\})|^2 \right\} \leq M. \quad (26)$$

Hence, the equality holds.

Next, we consider the parameter in case one of two graphs is complete. \square

Theorem 14. For any graph G and a positive integer n , let $M = \max \{ \sum_{P \in \mathcal{P}} |P| : \mathcal{P} \text{ be a class of } m \text{ disjoint independent sets of } G \text{ with } m \leq n \}$. Then,

$$\alpha(K_n, G) = \begin{cases} |V(G)|, & \text{if } \chi(G) \leq n, \\ M, & \text{otherwise.} \end{cases} \quad (27)$$

Hence, $\alpha(K_n, G) = |V(G)|$, as required.

Case 2. $\chi(G) > n$.

Let f be a G -coloring of K_n .

Claim $\emptyset \notin f(V(K_n))$.

Proof. Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$.

Case 1. $\chi(G) \leq n$.

Since $\chi(G) \leq n$, we have $\chi(G)\alpha(G) \leq n\alpha(G)$. By Lemma 9, we get $|V(G)| \leq \chi(G)\alpha(G) \leq n\alpha(G)$. Therefore, by Theorem 6, $\alpha(K_n, G) \leq \min\{\alpha(G)|V(K_n)|, \alpha(K_n)|V(G)|\} = \min\{n\alpha(G), |V(G)|\} = |V(G)|$. Next, let f be a χ -coloring of G and $\mathcal{P} = \{f^{-1}(\{x\}) : x \in f(V(G))\} = \{B_1, B_2, \dots, B_k\}$, where $k = \chi(G) \leq n$. Define $g: V(G) \rightarrow \mathcal{P}(K_n)$ by

$$g(x) = \{v_i\} \text{ if } x \in B_i \text{ for some } i \in \{1, 2, \dots, k\}, \quad (28)$$

for all $x \in V(G)$. It is clear that g is a K_n -coloring of G . This implies, by Lemma 1, that

$$\alpha(K_n, G) \geq \sum_{x \in V(G)} |g(x)| = \sum_{i=1}^k |\{v_i\}| |g^{-1}(\{\{v_i\}\})| = \sum_{i=1}^k |\{v_i\}| |B_i| = \sum_{i=1}^k |B_i| = |V(G)|. \quad (29)$$

Suppose, to the contrary, that $\emptyset \in f(V(K_n))$. By Lemma 5, $f(V(K_n)) \setminus \{\emptyset\}$ is a partition of $V(G)$. Thus,

$$n < \chi(G) \leq |f(V(K_n)) \setminus \{\emptyset\}| \leq |V(K_n) \setminus f^{-1}(\{\emptyset\})| < |V(K_n)| = n, \quad (30)$$

a contradiction. This proves our claim. Since every independent set of K_n is singleton and by Lemma 1, we get $f(V(K_n))$ which is a class of n disjoint independent sets of

G . Let $f(V(K_n)) = \{B_1, B_2, \dots, B_n\}$. It follows, by Lemma 1, that

$$\alpha(K_n, G) = \alpha(G, K_n) = \sum_{x \in V(K_n)} |f(x)| = \sum_{i=1}^n |B_i| |f^{-1}(\{B_i\})| = \sum_{i=1}^n |B_i| \leq M. \quad (31)$$

Next, we show that $\alpha(K_n, G) \geq M$. Let $\mathcal{P}' = \{B_1, B_2, \dots, B_k\}$ be a class of k disjoint independent sets of G with $k \leq n$ such that $\sum_{i=1}^k |B_i| = M$. Define $g: V(G) \longrightarrow \mathcal{F}(K_n)$ by

$$g(x) = \{v_i\} \text{ if } x \in B_i \text{ for some } i \in \{1, 2, \dots, k\}, \quad (32)$$

for all $v \in V(G)$. Clearly, g is a K_n -coloring of G . This implies, by Lemma 1, that $\alpha(K_n, G) \geq \sum_{x \in V(G)} |g(x)| = \sum_{i=1}^k |\{v_i\}| |g^{-1}(\{v_i\})| = \sum_{i=1}^k |\{v_i\}| |B_i| = \sum_{i=1}^k |B_i| = M$. Hence, $\alpha(K_n, G) = M$, as required.

The following corollaries present the prominent results of $\alpha(H, G)$ in which G and H are elements of some basic families of graphs. \square

Corollary 15. *Let G be a graph and $n \geq 2$ be a positive integer. If G is bipartite, then*

$$\alpha(K_n, G) = |V(G)|. \quad (33)$$

Corollary 16. *Let G be a graph and n be a positive integer greater than 1. Then, the following statements hold:*

- (1) $\alpha(P_{2m}, K_n) = 2m + 1$ for all non-negative integers m .
- (2) $\alpha(P_{2m+1}, K_n) = 2m + 2$ for all non-negative integers m .
- (3) $\alpha(C_{2m}, K_n) = 2m + 2$ for all positive integers $m \geq 2$.
- (4) $\alpha(S_m, K_n) = 2m + 2$ for all positive integers $m \geq 2$.

Corollary 17. *For positive integers m and n , we have*

$$\alpha(K_m, K_n) = \min\{m, n\}. \quad (34)$$

Proof. Without loss of generality, let $m \leq n$. For an injection $f: V(K_m) \longrightarrow \mathcal{F}(K_n)$, we see that f is a K_n -coloring of K_m , since every independent set of K_n is singleton. This implies that

$$\alpha(K_m, K_n) = \alpha(K_n, K_m) \geq \sum_{x \in V(K_m)} |f(x)| = \sum_{x \in V(K_m)} 1 = |V(K_m)| = m = \min\{m, n\}. \quad (35)$$

Hence, the result follows by Theorem 6. \square

Corollary 18. *For positive integers m and n with $n \geq 2$, we have*

$$\alpha(C_{2m+1}, K_n) = \begin{cases} 2m, & \text{if } n = 2, \\ 2m + 1, & \text{if } n \geq 3. \end{cases} \quad (36)$$

Proof. Let $f: V(C_{2m+1}) \longrightarrow \{1, 2, 3\}$ be a 3-coloring of C_{2m+1} such that $|f^{-1}(\{1\})| = |f^{-1}(\{2\})| = m$ and $|f^{-1}(\{3\})| = 1$.

Case 1. $n = 2$.

Let $V(K_n) = \{v_1, v_2\}$. Then, we define a function $g: V(C_{2m+1}) \longrightarrow \mathcal{F}(K_n)$ by

$$g(x) = \begin{cases} \{v_1\}, & \text{if } x \in f^{-1}(\{1\}), \\ \{v_2\}, & \text{if } x \in f^{-1}(\{2\}), \\ \emptyset, & \text{if } x \in f^{-1}(\{3\}). \end{cases} \quad (37)$$

for all $x \in V(C_{2m+1})$. It is precise that g is a K_n -coloring of C_{2m+1} and so

$$\begin{aligned} \alpha(C_{2m+1}, K_n) &= \alpha(K_n, C_{2m+1}) \geq \sum_{x \in V(C_{2m+1})} |g(x)| \\ &= |\{v_1\}| |f^{-1}(\{1\})| + |\{v_2\}| |f^{-1}(\{2\})| \\ &\quad + |\emptyset| |f^{-1}(\{3\})| = 2m. \end{aligned} \quad (38)$$

Therefore, by Theorem 6, the result follows.

Case 2. $n \geq 3$.

Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$. Define a function $g: V(C_{2m+1}) \longrightarrow \mathcal{F}(K_n)$ by

$$g(x) = \begin{cases} \{v_1\}, & \text{if } x \in f^{-1}(\{1\}), \\ \{v_2\}, & \text{if } x \in f^{-1}(\{2\}), \\ \{v_3\}, & \text{if } x \in f^{-1}(\{3\}), \end{cases} \quad (39)$$

for all $x \in V(C_{2m+1})$. It is easy to show that g is a K_n -coloring of C_{2m+1} , since every independent set of K_n is singleton. This implies that

$$\begin{aligned} \alpha(C_{2m+1}, K_n) &= \alpha(K_n, C_{2m+1}) \geq \sum_{x \in V(C_{2m+1})} |g(x)| \\ &= |\{v_1\}| |f^{-1}(\{1\})| + |\{v_2\}| |f^{-1}(\{2\})| + |\{v_3\}| |f^{-1}(\{3\})| = 2m + 1. \end{aligned} \quad (40)$$

Hence, the result follows by Theorem 6. \square

Corollary 19. For positive integers m, n with $m, n \geq 2$, we have

$$\alpha(W_{2m}, K_n) = \begin{cases} 2m-2, & \text{if } n=2, \\ 2m-1, & \text{if } n=3, \\ 2m, & \text{if } n \geq 4. \end{cases} \quad (41)$$

Proof. Let $f: V(W_{2m}) \rightarrow \{1, 2, 3, 4\}$ be a 4-coloring of W_{2m} such that $|f^{-1}(\{1\})| = |f^{-1}(\{2\})| = m-1$ and $|f^{-1}(\{3\})| = |f^{-1}(\{4\})| = 1$.

Case 1. $n=2$.

Let $V(K_n) = \{v_1, v_2\}$. Define a function $g: V(W_{2m}) \rightarrow \mathcal{F}(K_n)$ by

$$g(x) = \begin{cases} \{v_1\}, & \text{if } x \in f^{-1}(\{1\}), \\ \{v_2\}, & \text{if } x \in f^{-1}(\{2\}), \\ \emptyset, & \text{otherwise,} \end{cases} \quad (42)$$

for all $x \in V(W_{2m})$. Clearly, g is a K_n -coloring of W_{2m} . Then,

$$\begin{aligned} \alpha(W_{2m}, K_n) &= \alpha(K_n, W_{2m}) \geq \sum_{x \in V(W_{2m})} |g(x)| \\ &= |\{v_1\}| |f^{-1}(\{1\})| + |\{v_2\}| |f^{-1}(\{2\})| + |\emptyset| |f^{-1}(\{3\})| = 2m-2. \end{aligned} \quad (43)$$

By Theorem 6, we obtain that $\alpha(W_{2m}) = 2m-2$.

Case 2. $n=3$.

Let $V(K_n) = \{v_1, v_2, v_3\}$. Define a function $g: V(W_{2m}) \rightarrow \mathcal{F}(K_n)$ by

$$g(x) = \begin{cases} \{v_1\}, & \text{if } x \in f^{-1}(\{1\}), \\ \{v_2\}, & \text{if } x \in f^{-1}(\{2\}), \\ \{v_3\}, & \text{if } x \in f^{-1}(\{3\}), \\ \emptyset, & \text{otherwise,} \end{cases} \quad (44)$$

for all $x \in V(W_{2m})$. Clearly, g is a K_n -coloring of W_{2m} . Then,

$$\begin{aligned} \alpha(W_{2m}, K_n) &= \alpha(K_n, W_{2m}) \geq \sum_{x \in V(W_{2m})} |g(x)| \\ &= |\{v_1\}| |f^{-1}(\{1\})| + |\{v_2\}| |f^{-1}(\{2\})| + |\{v_3\}| |f^{-1}(\{3\})| + |\emptyset| |f^{-1}(\{4\})| \\ &= 2m-1. \end{aligned} \quad (45)$$

By Theorem 6, we get that $\alpha(W_{2m}) = 2m-1$.

Case 3. $n \geq 4$.

Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$. Define a function $g: V(W_{2m}) \rightarrow \mathcal{F}(K_n)$ by

$$g(x) = \begin{cases} \{v_1\}, & \text{if } x \in f^{-1}(\{1\}), \\ \{v_2\}, & \text{if } x \in f^{-1}(\{2\}), \\ \{v_3\}, & \text{if } x \in f^{-1}(\{3\}), \\ \{v_4\}, & \text{if } x \in f^{-1}(\{4\}), \\ \emptyset, & \text{otherwise,} \end{cases} \quad (46)$$

for all $x \in V(W_{2m})$. Clearly, g is a K_n -coloring of W_{2m} . Then,

$$\begin{aligned}\alpha(W_{2m}, K_n) &= \alpha(K_n, W_{2m}) \geq \sum_{x \in V(W_{2m})} |g(x)| \\ &= |\{v_1\}| |f^{-1}(\{1\})| + |\{v_2\}| |f^{-1}(\{2\})| + |\{v_3\}| |f^{-1}(\{3\})| + |\{v_4\}| |f^{-1}(\{4\})| + |\emptyset| |g^{-1}(\{\emptyset\})| \\ &= 2m.\end{aligned}\quad (47)$$

By Theorem 6, we have $\alpha(W_{2m}) = 2m$. This completes the proof. \square

Corollary 20. For positive integers m and n with $m, n \geq 2$, we have

$$\alpha(W_{2m+1}, K_n) = \begin{cases} 2m, & \text{if } n = 2, \\ 2m + 1, & \text{if } n \geq 3. \end{cases} \quad (48)$$

Proof. By applying the proof of Corollary 19, the result follows.

Actually, we can determine $\alpha(H, G)$, where G and H belong to some basic families of graphs by applying the proofs of Corollaries 15, 16, and 19. As shown in Table 1, we proved only the value $\alpha(H, G)$ in the first column. However, we determine the rest without proofs and we leave the rest to the reader as an exercise. \square

4. Applications on Cartesian Product Graphs

In this section, we apply the coloring by independent sets to the Cartesian product of a complete graph and a graph. Firstly, we provide some preparations for background.

Given two graphs G and H , many definitions exist that are known as the product of G and H . For a detailed treatment of graph products, we refer the reader to [11, 12]. The Cartesian product of G and H , denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (v_1, h_1) and (v_2, h_2) are adjacent whenever $v_1 v_2 \in E(G)$ and $h_1 = h_2$, or $v_1 = v_2$ and $h_1 h_2 \in E(H)$. There are several types of graphs defined by the Cartesian product of graphs. In particular, we focus the following types of those graphs. For a positive integer n , an n -ladder graph L_n is defined to be the Cartesian product graph $K_2 \square P_n$. An n -book graph B_n means the Cartesian product graph $K_2 \square S_{n+1}$. An n -dimensional hypercube Q_n is recursively defined to be the Cartesian product graph $K_2 \square Q_{n-1}$, where $n \geq 2$ and $Q_1 = K_2$. And we note that $|V(Q_n)| = 2^n$ and $\chi(Q_n) = 2$ which can be found in [13].

In order to properly study graph products, we need some definitions that consider the set product of sets A and B . In particular, for $S \subseteq A \times B$, we denote by $\pi_1(S) = \{a : (a, b) \in S \text{ where } b \in B\}$. Moreover, for $s \in \pi_1(S)$, we denote by $\pi_s(S) = \{b : (s, b) \in S\}$.

Determining the independence number and its variants of a graph product in terms of its factors is well studied in

graph theory. For papers concerning the graph products, we refer the reader, for example, to [14–18]. In this section, we continue the study of the graph product independence by considering the independence of the Cartesian product graphs. Namely, this section provides results regarding the independence number and gives certain valuable corollaries to the results.

Now, we characterize the independent sets of the Cartesian product of a complete graph and a graph.

Theorem 21. Let H be a graph. For a positive integer n , let S be a nonempty subset of $V(K_n \square H)$. Then, S is an independent set of $K_n \square H$ if and only if the following conditions hold:

- (1) $\pi_s(S)$ is an independent set of H for every $s \in \pi_1(S)$.
- (2) $\pi_a(S) \cap \pi_b(S) = \emptyset$ for any adjacent vertices $a, b \in \pi_1(S)$.

Proof. Let S be an independent set of $K_n \square H$.

Furthermore, let $s \in \pi_1(S)$ and $u_s, v_s \in \pi_s(S)$. If $u_s = v_s$, then $u_s v_s \notin E(H)$. So, we assume the rest that $u_s \neq v_s$. Since $(s, u_s), (s, v_s) \in S$, we have $(s, u_s)(s, v_s) \notin E(K_n \square H)$. Thus, $u_s v_s \notin E$.

Let a, b be two adjacent vertices in $\pi_1(S)$. Furthermore, let $v_a \in \pi_a(S)$ and $v_b \in \pi_b(S)$. Since $(a, v_a), (b, v_b) \in S$, we have $(a, v_a)(b, v_b) \notin E(K_n \square H)$. Thus, $v_a \neq v_b$ since $ab \in E(K_n[\pi_1(S)])$. Hence, $\pi_a(S) \cap \pi_b(S) = \emptyset$. For the converse, we assume that (1) and (2) hold. Let $(a, v_a), (b, v_b) \in S$. If $ab \in E$, then clearly $a, b \in \pi_1(S)$. We obtain that $v_a \neq v_b$ because $\pi_a(S) \cap \pi_b(S) = \emptyset$. Thus, $(a, v_a)(b, v_b) \notin E(K_n \square H)$. If $ab \notin E$, then we distinguish the following two cases.

Case 1. $a = b$.

Clearly, $v_a, v_b \in \pi_a(S)$. Thus, $v_a v_b \notin E(H)$, since $\pi_a(S)$ is an independent set of H . Therefore, $(a, v_a)(b, v_b) \notin E(K_n \square H)$.

Case 2. $a \neq b$.

It is easy to see that $(a, v_a)(b, v_b) \notin E(K_n \square H)$, since neither $ab \in E$ nor $a \neq b$. \square

Theorem 22. Let H be a graph. For a positive integer n , we have

$$\alpha(K_n \square H) = \alpha(K_n, H). \quad (49)$$

TABLE 1: The values of $\alpha(H, G)$ where G, H belong to some basic families of graphs.

	$K_n (n \geq 2)$	$P_{2n} (n \geq 0)$	$P_{2n+1} (n \geq 0)$	$C_{2n} (n \geq 2)$	$C_{2n+1} (n \geq 1)$	$W_{2n} (n \geq 2)$	$W_{2n+1} (n \geq 1)$	$S_n (n \geq 2)$
$K_m (m \geq 2)$	$\min\{m, n\}$							
$P_{2m} (m \geq 0)$	$2m + 1$	$2mn + m + n + 1$						
$P_{2m+1} (m \geq 0)$	$2m + 2$	$2mn + m + 2n + 1$	$2(mn + m + n + 1)$					
$C_{2m} (m \geq 2)$	$2m$	$2mn + m$	$2m(n + 1)$	$2mn$				
$C_{2m+1} (m \geq 1)$	$2m + 1$	$2mn + m$	$2m(n + 1)$	$2mn$	$2mn + 1$			
$W_{2m} (m \geq 2)$	$2m - 2$ if $n = 2$ $2m - 1$ if $n = 3$ $2m$ if $n \geq 4$	$2mn + m - 2n - 1$	$2(mn + m - n - 1)$	$2n(m - 1)$	$2mn - 2n + 1$	$2(mn - m - n + 2)$		
$W_{2m+1} (m \geq 1)$	$2m$ if $n = 2$ $2m + 1$ if $n \geq 3$	$2mn + m$	$2m(n + 1)$	$2mn$	$2mn + 1$	$2mn - 2m + 1$	$2mn + 1$	
$S_m (m \geq 2)$	m	$mn + m - 1$	$m(n + 1)$	mn	mn	$m(n - 1)$	mn	$mn - m - n + 2$

Proof. We first show that $\alpha(K_n \square H) \leq \alpha(K_n, H)$. Let S' be an α -set of $K_n \square H$. Define a function $f: V(K_n) \rightarrow \mathcal{P}(H)$ by

$$f(x) = \begin{cases} \pi_x(S'), & \text{if } x \in \pi_1(S'), \\ \emptyset, & \text{otherwise,} \end{cases} \quad (50)$$

for all $x \in V(K_n)$. By Theorem 21, we obtain that f is an H -coloring of K_n . Consequently,

$$\begin{aligned} \alpha(K_n \square H) = |S'| &= \sum_{x \in \pi_1(S')} |\pi_x(S')| = \sum_{x \in \pi_1(S')} |\pi_x(S')| + \sum_{x \in V(K_n) \setminus \pi_1(S')} |\emptyset| \\ &= \sum_{x \in \pi_1(S')} |f(x)| + \sum_{x \in V(K_n) \setminus \pi_1(S')} |f(x)| = \sum_{x \in V(K_n)} |f(x)| \leq \alpha(H, K_n) = \alpha(K_n, H). \end{aligned} \quad (51)$$

Next, we show that $\alpha(K_n \square H) \geq \alpha(K_n, H)$. Let f be an α - H -coloring of K_n . Furthermore, let $A = f^{-1}(\{\emptyset\})$ and $B = V(K_n) \setminus A$. We see that $f(x)$ is independent in H for every $x \in B$ and $f(y) \cap f(z) = \emptyset$ for every adjacent vertices

y, z in B . It follows from Theorem 21 that $\{x\} \times f(x)$ is an independent set of $K_n \square H$ for every $x \in B$ and so is $\bigcup_{x \in B} (\{x\} \times f(x))$. Consequently,

$$\begin{aligned} \alpha(K_n, H) = \alpha(H, K_n) &= \sum_{x \in V(K_n)} |f(x)| = \sum_{x \in A} |f(x)| + \sum_{x \in B} |f(x)|, \\ &= 0 + \sum_{x \in B} |f(x)| = \left| \bigcup_{x \in B} (\{x\} \times f(x)) \right| \leq \alpha(K_n \square H). \end{aligned} \quad (52)$$

Hence, the result follows. \square

Corollary 23. For a positive integer n , we have

$$\alpha(L_n) = n + 1. \quad (53)$$

Proof. Since $L_n = K_2 \square P_n$ and by Theorem 22, we get that $\alpha(L_n) = \alpha(K_2 \square P_n) = \alpha(K_2, P_n)$. By Corollary 15, we have $\alpha(L_n) = |V(P_n)| = n + 1$ because $\chi(P_n) = 2$. \square

Corollary 24. For a positive integer n , we have

$$\alpha(B_n) = n + 1. \quad (54)$$

Proof. Since $B_n = K_2 \square S_{n+1}$ and by Theorem 22, we obtain that $\alpha(B_n) = \alpha(K_2 \square S_{n+1}) = \alpha(K_2, S_{n+1})$. Therefore, by Corollary 15, we conclude that $\alpha(L_n) = |V(S_{n+1})| = n + 1$, since $\chi(S_{n+1}) = 2$. \square

Corollary 25 (see [13]). For a positive integer n , we have

$$\alpha(Q_n) = 2^{n-1}. \quad (55)$$

Proof. It is clear for $n = 1$. So, we assume that $n \geq 2$. Since $Q_n = K_2 \square Q_{n-1}$ and by Theorem 22, we get $\alpha(Q_n) = \alpha(K_2, Q_{n-1})$. Since $\chi(Q_{n-1}) = 2$ and by Corollary 15, we have $\alpha(Q_n) = |V(Q_{n-1})| = 2^{n-1}$. \square

5. Conclusion

The concept of the independent coloring of graphs has been introduced in this paper. Prominent properties and results of the independent coloring have been proposed. Especially, the values of $\alpha(H, G)$, where H and G are fundamental graphs, have been collected and presented in the table. Finally, the independent coloring has been applied to determine the independence number of Cartesian product graphs. Actually, the independent coloring of graphs can be considered as a generalized concept of the independence number of graphs, as well.

Data Availability

No data were used in this research.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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