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Research Article

A Generalized Version of Polynomial Convex Functions and Some Interesting Inequalities with Applications

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In this article, we introduce a general class of convex functions and proved some of its basic properties. We establish Hermite-Hadamard type inequalities as well as fractional version of Hermite-Hadamard type inequalities by using Riemann-Liouville integral operator. At the end, some application to special means of real numbers are also given. It can be observed from the remarks given in this paper that several exiting results of ligature can be obtained immediacy from our results by taking suitable involved parameters.

1. Introduction

Due to huge applications in applied sciences, notion of convexity become important for researchers, but classical convexity is not enough to solve modern problems, so there is always a need to introduce a more general notion of convexity. In mathematics, classical convexity and concavity are two important concepts. Convexity plays a fundamental role in optimization theory, mathematical economics and engineering. Convexity of a function $\psi : J \longrightarrow \mathbb{R}$ in classical sense is defined by the following inequality:

$$\psi(\alpha x + (1-\alpha)y) \leq \alpha\psi(x) + (1-\alpha)\psi(y), \forall \alpha \in [0,1], \quad (1)$$

for every $x, y \in J$.

If the above inequality is reversed, then the function is said to be concave.

Using various techniques, the concept of convex functions has been generalized in many directions, see [1, 2]. Inequality theory become a very dynamic and attractive field of research [3, 4]. In recent years, using various notions of convex functions a wide class of integral inequalities has been derived [5, 6]. The most important and famous inequalities are Schur-type, Hermite-Hadamard-type, and Fejér-type inequalities [7, 8].

Toplu et al. in [9] established Hermite-Hadamard inequality for n-polynomial convex functions, which is given below:

Let $\psi : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be n-polynomial convex function. If a_1 and b_1 are two real numbers such that $a_1 b_1$ and $\psi \in L[a_1, b_1]$, then the following double inequality hold,

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\psi\left(\frac{a_1+b_1}{2}\right) \leq \int_{a_1}^{b_1}\psi(x)dx \leq \frac{\psi(a_1)+\psi(b_1)}{n}\sum_{s=1}^{n}\frac{s}{s+1}. \quad (2)$$

In this paper we introduce a new class of convex functions, namely (n,h)-polynomial convex functions. We drive some basic properties of (n,h)-polynomial convex function and establish Hermite-Hadamard type inequalities for (n,h)-polynomial convex function in the setting of Riemann-Liouville integral operator. Moreover some applications of established results in special means are also presented.

2. Preliminaries and Basic Results

In this section we present some preliminary material and define a new class of convex functions, which we call (n,h)-polynomial convex functions. Also we give some basic properties for this new class of convex functions. From now to onward, we consider $J = [a_1, b_1]$.

Definition 1 (see [4]). Let $n \in N$. The nonnegative function $\psi : J \subset \mathbb{R} \longrightarrow \mathbb{R}$ is n-polynomial convex function, if

$$\psi(\alpha x + (1-\alpha)y) \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-\alpha)^s]\psi(x) + \frac{1}{n}\sum_{s=1}^{n}[1-\alpha^s]\psi(y), \tag{3}$$

holds for every $x, y \in J$ and $\alpha \in [0, 1]$.

Definition 2. Let $n \in N$. A nonnegative function $\psi : J \longrightarrow \mathbb{R}$ is said to be modified n-polynomial convex function, if

$$\psi(\alpha x + (1-\alpha)y) \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-\alpha)^s]\psi(x) + \frac{1}{n}\sum_{s=1}^{n}[1-\alpha]^s\psi(y), \tag{4}$$

holds for all $x, y \in J$ and $\alpha \in [0, 1]$.

Now we, are ready to define (n,h)-polynomial convex functions.

Definition 3. Let $n \in N$. A nonnegative function $\psi : J \longrightarrow \mathbb{R}$ is said to be (n,h)-polynomial convex function, if

$$\psi(\alpha x + (1-\alpha)y) \leq \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\psi(x) + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi(y), \tag{5}$$

holds for all $x, y \in J$ and $\alpha \in [0, 1]$.

Remark 4.

(1) If we take $n = 1$ in (5), then (n,h)-polynomial convexity reduces to modified h-convexity

(2) If we take $n = 1$ and $h(\alpha) = 1 - \alpha$ in (5), then (n,h)-polynomial convexity reduces to classical convexity

(3) If we take $h = 1 - \alpha$ in (5), then (n,h)-polynomial convexity reduces to modified n-polynomial convexity

It is worth mentioning here that $(1,h)$-polynomial convexity reduces to modified h-convexity, $(1, 1)$-polynomial convexity reduces to classical convexity and $(n, 1)$-polynomial convexity reduces to modified n-polynomial convexity.

Proposition 5. *If ψ, ϕ are (n,h)-polynomial convex functions defined on J and $\beta, \gamma \in \mathbb{R}$, then $\beta\psi + \gamma\phi$ is also (n,h)-polynomial convex function on J.*

Proof. Let $\alpha \in [0, 1]$, then from the (n,h)-polynomial convexity of ψ, ϕ, we have for every $x, y \in J$,

$$\begin{aligned}
(\beta\psi + \gamma\phi)(\alpha x + (1-\alpha)y) &= \beta\psi(\alpha x + (1-\alpha)y) + \gamma\phi(\alpha x + (1-\alpha)y)\\
&\leq \beta\left(\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\psi(x) + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi(y)\right)\\
&\quad + \gamma\left(\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\phi(x) + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\phi(y)\right)\\
&\leq \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s](\beta\psi + \gamma\phi)(x)\\
&\quad + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s(\beta\psi + \gamma\phi)(y).
\end{aligned} \tag{6}$$

Hence $\beta\psi + \gamma\phi$ is a (n,h)-polynomial convex function on J. □

Proposition 6. *If ϕ be a linear function and ψ be a (n,h)-polynomial convex function on J, then $\psi \circ \phi$ is also a(n,h)-polynomial convex on J.*

Proof. Using the linearity of ϕ and (n,h)-polynomial convexity of ψ on J, we have for every $x, y \in J$ and $\alpha \in [0, 1]$,

$$\begin{aligned}
\psi \circ \phi(\alpha x + (1-\alpha)y) &= \psi(\phi(\alpha x + (1-\alpha)y))\\
&= \psi(\alpha\phi(x) + (1-\alpha)\phi(y))\\
&\leq \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\psi(\phi(x))\\
&\quad + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi(\phi(y))\\
&= \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\psi \circ \phi(x)\\
&\quad + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi \circ \phi(y).
\end{aligned} \tag{7}$$

Hence $\psi \circ \phi$ is (n,h)-polynomial convex function on J. □

Proposition 7. *If ψ and ϕ are of similarly ordered (n,h)-polynomial convex functions on J, then $\psi\phi$ is also (n,h)-polynomial convex function on J.*

Proof. Using the fact that ψ and ϕ are of similarly ordered (n,h)-polynomial convex functions on J, we have for every $x, y \in J$ and $\alpha \in [0, 1]$,

$$\begin{aligned}
&\psi(\alpha x + (1-\alpha)y)\phi(\alpha x + (1-\alpha)y)\\
&\leq \left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\psi(x) + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi(y)\right]\\
&\quad \times \left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\phi(x) + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\phi(y)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]^{2}\psi(x)\phi(x)+\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right]^{2}\psi(y)\phi(y)\\
&\quad+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}.\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}](\psi(x)\phi(y)+\psi(y)\phi(x))\\
&\leq \left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]^{2}\psi(x)\phi(x)+\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right]^{2}\psi(y)\phi(y)\\
&\quad+\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(x)\phi(x)-\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(x)\phi(x)\\
&\quad+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(y)\phi(y)-\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(y)\phi(y)\\
&\quad+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}.\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}](\psi(x)\phi(y)+\psi(y)\phi(x))\\
&\leq \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(x)\phi(x)+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(y)\phi(y)\\
&\quad-\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}.\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}](\psi(x)\phi(x)+\psi(y)\phi(y)\\
&\quad-\psi(x)\phi(y)-\psi(y)\phi(x))\leq \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(x)\phi(x)\\
&\quad+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(y)\phi(y)-\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\\
&\quad.\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]((\psi(x)-\psi(y))(\phi(x)-\phi(y)))\\
&\leq \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(x)\phi(x)+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(y)\phi(y).
\end{aligned}
\tag{8}
$$

Hence $\psi\phi$ is also (n,h)-polynomial convex function on J. □

Proposition 8. *Let $\psi_i : J \longrightarrow \mathbb{R}$, where $1 \leq i \leq m$ be nonnegative (n,h)-polynomial convex functions on J, then for $\lambda_i \geq 0$ where $1 \leq i \leq m$, the function ψ is (n,h)-polynomial convex function on J, where $\psi = \sum_{i=1}^{m}\lambda_i\psi_i$.*

Proof. For all $x, y \in J$ and $\alpha \in [0, 1]$, we have

$$
\begin{aligned}
\psi(\alpha x+(1-\alpha)y) &= \sum_{i=1}^{m}\lambda_i\psi_i(\alpha x+(1-\alpha)y)\\
&\leq \sum_{i=1}^{m}\lambda_i\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi_i(x)+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi_i(y)\right]\\
&= \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\sum_{i=1}^{m}\lambda_i\psi_i(x)+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\sum_{i=1}^{m}\lambda_i\psi_i(y)\\
&= \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(x)+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(y).
\end{aligned}
\tag{9}
$$

Hence $\psi = \sum_{i=1}^{m}\lambda_i\psi_i$ is (n,h)-polynomial convex function on J. □

3. Hermite-Hadamard Type Inequalities

In this section we will develop some Hermite Hadamard type integral inequalities for (n,h)-polynomial convex functions.

Theorem 9. *Suppose that $\psi : J \longrightarrow \mathbb{R}$ is a (n,h)-polynomial convex function on J, then the following inequality holds*

$$
\begin{aligned}
\psi\left(\frac{a_1+b_1}{2}\right) &\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx \leq \psi(a_1)+[\psi(b_1)-\psi(a_1)]\\
&\quad\cdot\frac{1}{n}\sum_{s=1}^{n}\int_0^1[h(\alpha)]^{s}d\alpha.
\end{aligned}
\tag{10}
$$

Proof. Inserting $\alpha = 1/2$ in Definition 3, we have

$$
\psi\left(\frac{x+y}{2}\right) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\psi(x)+\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\psi(y).
\tag{11}
$$

Taking $x = \alpha a_1 + (1-\alpha)b_1$ and $y = \alpha b_1 + (1-\alpha)a_1$ in above inequality, we have

$$
\begin{aligned}
\psi\left(\frac{a_1+b_1}{2}\right) &\leq \frac{1}{n}\sum_{s=1}^{n}\left[1-\left[\left(\frac{1}{2}\right)\right]^{s}\right]\psi(\alpha a_1+(1-\alpha)b_1)\\
&\quad+\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\psi(\alpha b_1+(1-\alpha)a_1).
\end{aligned}
\tag{12}
$$

Integrating the inequality (12) with respect to "α" over $[0, 1]$, we have

$$
\begin{aligned}
\psi\left(\frac{a_1+b_1}{2}\right) &\leq \frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\int_0^1\psi(\alpha a_1+(1-\alpha)b_1)\\
&\quad\cdot d\alpha+\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\int_0^1\psi(\alpha b_1+(1-\alpha)a_1)d\alpha.
\end{aligned}
\tag{13}
$$

Using the change of variable technique, $x = \alpha a_1 + (1-\alpha)b_1$ and $y = \alpha b_1 + (1-\alpha)a_1$, we have

$$
\begin{aligned}
\psi\left(\frac{a_1+b_1}{2}\right) &\leq \frac{1}{n(b_1-a_1)}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\\
&\quad\cdot\int_{a_1}^{b_1}\psi(x)dx+\frac{1}{n(b_1-a_1)}\\
&\quad\cdot\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\int_{a_1}^{b_1}\psi(x)dx\psi\left(\frac{a_1+b_1}{2}\right)\\
&\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx.
\end{aligned}
\tag{14}
$$

Which is left side of the inequality (10).

Now, for the right side of the inequality (10), we start with the following integral,

$$\int_{a_1}^{b_1}\psi(x)dx = (b_1 - a_1)\int_0^1 \psi(\alpha a_1 + (1-\alpha)b_1)d\alpha. \quad (15)$$

Since, ψ is (n,h)-polynomial convex function, so

$$\frac{1}{b_1 - a_1}\int_{a_1}^{b_1}\psi(x)dx \le \int_0^1\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\psi(a_1) + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi(b_1)\right] \cdot d\alpha \le \psi(a_1) + [\psi(b_1) - \psi(a_1)]\frac{1}{n}\sum_{s=1}^{n}\int_0^1[h(\alpha)]^s d\alpha, \quad (16)$$

which is right side of the inequality (10). The proof completed. □

Remark 10.

(1) If $n = 1$ and $h(\alpha) = 1 - \alpha$ then inequality (10) reduced to the Hermite-Hadamard inequality for classical convex functions [10].

(2) If $n = 1$, then inequality (10) reduced to the Hermite-Hadamard inequality for modified h-convex functions [11].

The following result can be easily obtain by elementary analysis.

Corollary 11. *Let $\sum_{i=1}^{m}\psi_i : J \longrightarrow \mathbb{R}$ be the sum of (n,h)-polynomial convex functions on J. Then*

$$\sum_{i=1}^{m}\psi_i\left(\frac{a_1+b_1}{2}\right) \le \frac{1}{b_1-a_1}\sum_{i=1}^{m}\int_{a_1}^{b_1}\psi_i(x)dx \le \sum_{i=1}^{m}\psi_i(a_1) + \left[\sum_{i=1}^{m}\psi_i(b_1) - \sum_{i=1}^{m}\psi_i(a_1)\right]\frac{1}{n}\sum_{s=1}^{n}\int_0^1[h(\alpha)]^s d\alpha. \quad (17)$$

Proof. If we replace ψ by $\sum_{i=1}^{m}\psi_i$ in Theorem 9, we get the required result. □

In the next theorem we will derive Hermite-Hadamard type inequality for product of two (n,h)-polynomial convex functions.

Theorem 12. *Consider ψ, ϕ are (n,h)-polynomial convex on J, such that ψ and ϕ are similarly ordered functions. If*

$$\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\right]^2 + \left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\right]^2 \neq 0, \quad (18)$$

then

$$\frac{1}{\left[1/n\sum_{s=1}^{n}[1-[h(1/2)]^s]\right]^2 + \left[1/n\sum_{s=1}^{n}[h(1/2)]^s\right]^2} \cdot \left[\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right) - \left(\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\right) \cdot \left(\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\right)M(a_1,b_1)\right] \le \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)\phi(x)dx \le \psi(a_1)\phi(a_1) + [\psi(b_1)\phi(b_1) - \psi(a_1)\phi(a_1)]\frac{1}{n}\sum_{s=1}^{n}\int_0^1[h(\alpha)]^s d\alpha, \quad (19)$$

where

$$M(a_1,b_1) = \psi(a_1)\phi(a_1) + \psi(b_1)\phi(b_1). \quad (20)$$

Proof. As ψ, ϕ are (n,h)-polynomial convex on J, so

$$\begin{aligned}
&\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right) = \psi\left(\frac{((1-\alpha)a_1+\alpha b_1) + (\alpha a_1 + (1-\alpha)b_1)}{2}\right) \\
&\times\phi\left(\frac{((1-\alpha)a_1+\alpha b_1) + (\alpha a_1 + (1-\alpha)b_1)}{2}\right) \\
&\le \left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\psi((1-\alpha)a_1+\alpha b_1) + \frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\psi(\alpha a_1 + (1-\alpha)b_1)\right] \\
&\times\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\phi((1-\alpha)a_1+\alpha b_1) + \frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\phi(\alpha a_1 + (1-\alpha)b_1)\right] \\
&\le \left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\right]^2\psi((1-\alpha)a_1+\alpha b_1)\phi((1-\alpha)a_1+\alpha b_1) \\
&+ \left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\right]^2\psi(\alpha a_1 + (1-\alpha)b_1)\phi(\alpha a_1 + (1-\alpha)b_1) \\
&+ \left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\right]\cdot\left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\right]\times[\psi((1-\alpha)a_1+\alpha b_1)\phi(\alpha a_1 + (1-\alpha)b_1) \\
&+ \psi(\alpha a_1 + (1-\alpha)b_1)\phi((1-\alpha)a_1+\alpha b_1)] \\
&\le \left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\right]^2\psi((1-\alpha)a_1+\alpha b_1)\phi((1-\alpha)a_1+\alpha b_1) \\
&+ \left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\right]^2\psi(\alpha a_1 + (1-\alpha)b_1)\phi(\alpha a_1 + (1-\alpha)b_1) \\
&+ \left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\right]\cdot\left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\right] \\
&\times\left[\left(\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\psi(a_1) + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi(b_1)\right)\right. \\
&\times\left(\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\phi(a_1) + \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\phi(b_1)\right) \\
&+ \left(\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi(a_1) + \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\psi(b_1)\right) \\
&\left.\cdot\left(\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\phi(a_1) + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\phi(b_1)\right)\right] \\
&\le \left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\right]^2\psi((1-\alpha)a_1+\alpha b_1)\phi((1-\alpha)a_1+\alpha b_1) \\
&+ \left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\right]^2\psi(\alpha a_1 + (1-\alpha)b_1)\phi(\alpha a_1 + (1-\alpha)b_1)
\end{aligned}$$

$$+\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\right]\cdot\left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\right]$$
$$\cdot\left[\left(\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right](\psi(a_1)\phi(a_1)+\psi(b_1)\psi(b_1))\right.\right.$$
$$+\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right]^{2}\psi(b_1)\phi(a_1)+\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]^{2}\psi(a_1)\phi(b_1))$$
$$+\left(\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right](\psi(a_1)\phi(a_1)+\psi(b_1)\psi(b_1))\right.$$
$$\left.\left.+\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right]^{2}\psi(a_1)\phi(b_1)+\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]^{2}\psi(b_1)\phi(a_1))\right]\right.$$
$$\leq\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\right]^{2}\psi((1-\alpha)a_1+\alpha b_1)\phi((1-\alpha)a_1+\alpha b_1)$$
$$+\left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\right]^{2}\psi(\alpha a_1+(1-\alpha)b_1)\phi(\alpha a_1+(1-\alpha)b_1)$$
$$+\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\right]\cdot\left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\left[\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right]\right.$$
$$\cdot(2\psi(a_1)\phi(a_1)+2\psi(b_1)\psi(b_1))+\left(\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]^{2}+\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right]^{2}\right)$$
$$\left.\times(\psi(a_1)\phi(b_1)+\psi(b_1)\phi(a_1))\right].\tag{21}$$

Since, ψ and ϕ are similarly ordered, so

$$\leq\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\right]^{2}\psi((1-\alpha)a_1+\alpha b_1)\phi((1-\alpha)a_1+\alpha b_1)$$
$$+\left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\right]^{2}\psi(\alpha a_1+(1-\alpha)b_1)\phi(\alpha a_1$$
$$+(1-\alpha)b_1)+\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\right]\cdot\left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\right]$$
$$\cdot\left[\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right](2\psi(a_1)\phi(a_1)\right.$$
$$+2\psi(b_1)\psi(b_1))+\left(\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\right]^{2}+\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\right]^{2}\right)$$
$$\left.\times(\psi(a_1)\phi(a_1)+\psi(b_1)\phi(b_1))\right]\leq\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\right]^{2}$$
$$\cdot\psi((1-\alpha)a_1+\alpha b_1)\phi((1-\alpha)a_1+\alpha b_1)+\left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\right]^{2}$$
$$\cdot\psi(\alpha a_1+(1-\alpha)b_1)\phi(\alpha a_1+(1-\alpha)b_1)$$
$$+\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\right]\cdot\left[\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\right]M(a_1,b_1).\tag{22}$$

Integrating (22) w. r. t "α" from 0 to 1, we obtain

$$\frac{1}{[1/n\sum_{s=1}^{n}[1-[h(1/2)]^{s}]]^{2}+[1/n\sum_{s=1}^{n}[h(1/2)]^{s}]^{2}}\cdot\left[\psi\left(\frac{a_1+b_1}{2}\right)\phi\left(\frac{a_1+b_1}{2}\right)-\left(\frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^{s}\right)\right.$$
$$\left.\cdot\left(\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^{s}\right]\right)M(a_1,b_1)\right]\leq\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)\phi(x)dx.\tag{23}$$

The LHS of (19) can be obtained easily.

To prove right hand side of (19), we will use the following inequality

$$\psi(\alpha a_1+(1-\alpha)b_1)\phi(\alpha a_1+(1-\alpha)b_1)$$
$$\leq\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(a_1)+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(b_1)\right]$$
$$\cdot\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\phi(a_1)+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\phi(b_1)\right].\tag{24}$$

Since, ψ and ϕ are similarly ordered, so

$$\leq\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(a_1)\phi(a_1)+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(b_1)\phi(b_1).\tag{25}$$

Integrating (25) w. r. t "α' from 0 to 1

$$\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)\phi(x)dx\leq\psi(a_1)\phi(a_1)$$
$$+[\psi(b_1)\phi(b_1)-\psi(a_1)\phi(a_1)]\frac{1}{n}\sum_{s=1}^{n}\int_{0}^{1}[h(\alpha)]^{s}d\alpha,\tag{26}$$

which is the required right hand side of (19). This completes the proof. □

Remark 13. Inserting $n=1$ in Theorem 12 we obtain [11], Theorem 4.

The following lemma, shows that (n,h)-polynomial convex functions have the same property of convex functions.

Lemma 14. *Let ψ be (n,h)-polynomial convex functions, then*

$$\psi(a_1+b_1-x)\leq\psi(a_1)+\psi(b_1)-\psi(x),\forall x\in[a_1,b_1],\tag{27}$$

where $x=\alpha(a_1)+(1-\alpha)b_1$, $\alpha\in[0,1]$.

Proof. Let ψ be (n,h)-polynomial convex functions and $x=\alpha(a_1)+(1-\alpha)b_1$, we have

$$\begin{aligned}\psi(a_1+b_1-x)&=\psi(a_1+b_1-\alpha a_1-(1-\alpha)b_1)\\&\leq\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(a_1)+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(b_1)\\&\leq\psi(a_1)+\psi(b_1)\\&\quad-\left[\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^{s}\psi(a_1)+\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^{s}]\psi(b_1)\right]\\&\leq\psi(a_1)+\psi(b_1)-\psi(\alpha a_1+(1-\alpha)b_1)\\&\leq\psi(a_1)+\psi(b_1)-\psi(x).\end{aligned}\tag{28}$$

Which completes the proof. □

4. Hermite-Hadamard Type Inequalities in Riemann-Liouville Integral Operator

Now, we will derive Hermite-Hadamard inequality for (n,h)-polynomial convex functions via Riemann-Liouville fractional integral.

Definition 15 (see [12]). Let $\psi \in L[a, b]$. The right-hand side and left-hand side Riemann-Liouville integral operators of order $\sigma > 0$ with $b > a > 0$, are defined by

$$J_{a+}^{\sigma}\psi(x) = \frac{1}{\Gamma(\sigma)}\int_a^x (x-k)^{\sigma-1}\psi(k)dk, x > a$$
$$J_{b-}^{\sigma}\psi(x) = \frac{1}{\Gamma(\sigma)}\int_x^b (k-x)^{\sigma-1}\psi(k)dk, x < b \tag{29}$$

respectively, where $\Gamma(\sigma)$ is the Gamma function defined as $\Gamma(\sigma) = \int_0^\infty e^{-k}x^{\sigma-1}dk$.

It is to be noted that $J_{a^+}^0\psi(x) = J_{b^-}^0\psi(x) = \psi(x)$.

Theorem 16. *Let $\psi : J \longrightarrow \mathbb{R}$ is (n,h)-polynomial convex and $\psi \in L[a_1, b_1]$. Then the following inequalities hold*

$$\frac{1}{\sigma}\psi\left(\frac{a_1+b_1}{2}\right) \leq \frac{\Gamma(\sigma)}{(b_1-a_1)^{\sigma}}\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]J_{a_1^+}^{\sigma}\psi(b_1) + \frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s J_{b_1^-}^{\sigma}\psi(a_1)\right] \tag{30}$$

$$\frac{\Gamma(\sigma)}{(b_1-a_1)^{\sigma}}\left[J_{a_1^+}^{\sigma}\psi(b_1) + J_{b_1^-}^{\sigma}\psi(a_1)\right] \leq \frac{\psi(a_1)+\psi(b_1)}{\sigma}. \tag{31}$$

Proof. Inserting $\alpha = 1/2$ in Definition 3, we have

$$\psi\left(\frac{x+y}{2}\right) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\psi(x) + \frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\psi(y). \tag{32}$$

Taking $x = \alpha a_1 + (1-\alpha)b_1$ and $y = \alpha b_1 + (1-\alpha)a_1$, we have

$$\psi\left(\frac{a_1+b_1}{2}\right) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-\left[\left(\frac{1}{2}\right)\right]^s\right]\psi(\alpha a_1 + (1-\alpha)b_1) + \frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\psi(\alpha b_1 + (1-\alpha)a_1). \tag{33}$$

Multiplying (33) with $\alpha^{\sigma-1}$ and integrating w. r. t. α, we get

$$\int_0^1\psi\left(\frac{a_1+b_1}{2}\right)\alpha^{\sigma-1}d\alpha \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right] \cdot \int_0^1\alpha^{\sigma-1}\psi(\alpha a_1+(1-\alpha)b_1) \cdot d\alpha + \frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\int_0^1\alpha^{\sigma-1}\psi(\alpha b_1+(1-\alpha)a_1)d\alpha, \tag{34}$$

implies that

$$\frac{1}{\sigma}\psi\left(\frac{a_1+b_1}{2}\right) \leq \frac{1}{n(b_1-a_1)}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]\int_{a_1}^{b_1}\left(\frac{b_1-x}{b_1-a_1}\right)^{\sigma-1} \cdot \psi(x)dx + \frac{1}{n(b_1-a_1)}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s\int_{a_1}^{b_1}\left(\frac{x-a_1}{b_1-a_1}\right)^{\sigma-1} \cdot \psi(x)dx \leq \frac{\Gamma(\sigma)}{(b_1-a_1)^{\sigma}}\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-\left[h\left(\frac{1}{2}\right)\right]^s\right]J_{a_1^+}^{\sigma}\psi(b_1) + \frac{1}{n}\sum_{s=1}^{n}\left[h\left(\frac{1}{2}\right)\right]^s J_{b_1^-}^{\sigma}\psi(a_1)\right], \tag{35}$$

which is (30).

Since ψ is (n,h)-polynomial convex functions, so

$$\psi((1-\alpha)a_1+\alpha b_1) + \psi(\alpha a_1+(1-\alpha)b_1) \leq \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi(a_1) + \frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s] + \psi(b_1)\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]\psi(a_1) + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\psi(b_1) = \psi(a_1)+\psi(b_1). \tag{36}$$

Multiplying (33) with $\alpha^{\sigma-1}$ and integrating w. r. t. α, we get

$$\frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\left(\frac{x-a_1}{b_1-a_1}\right)^{\sigma-1}\psi(x)dx + \frac{1}{(b_1-a_1)}\int_{a_1}^{b_1}\left(\frac{b_1-x}{b_1-a_1}\right)^{\sigma-1} \cdot \psi(x)dx \leq \frac{\psi(a_1+\psi(b_1))}{\sigma}\frac{\Gamma(\sigma)}{(b_1-a_1)^{\sigma}} \cdot \left[J_{a_1^+}^{\sigma}\psi(b_1) + J_{b_1^-}^{\sigma}\psi(a_1)\right] \leq \frac{\psi(a_1)+\psi(b_1)}{\sigma}. \tag{37}$$

Which is (4.4). This completes the proof. □

Remark 17. Inserting $n = 1$ in Theorem 16 we obtain [11], Theorem 6.

5. New Inequalities for (n,h)-Polynomial Convex Functions

Now we will establish some new inequalities for (n,h)-polynomial convex functions. To proceed we begin with the following lemma which will be needed to obtain results of our desired type.

Lemma 18 (see [13], Lemma 2.1]). *Let $\psi : J \longrightarrow \mathbb{R}$ be a differentiable map on J and $\psi' \in L[a_1, b_1]$, then we have*

$$\begin{aligned}&\frac{\psi(a_1)+\psi(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx \\ &= \frac{b_1-a_1}{2}\int_0^1 (1-2\alpha)\psi'(\alpha a_1 - (1-\alpha)b_1)d\alpha.\end{aligned} \tag{38}$$

Theorem 19. *Let $\psi : J \longrightarrow \mathbb{R}$ be a differentiable map on J and $\psi' \in L[a_1, b_1]$. If the function $|\psi'|$ (n,h)-polynomial convex on J, then for $\alpha \in [0,1]$, we have*

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right| \\ &\le \frac{b_1-a_1}{2}\left[\frac{|\psi'(a_1)|}{2} + \frac{|\psi'(b_1)|-|\psi'(a_1)|}{n}\sum_{s=1}^{n}\int_0^1 |1-2\alpha|[h(\alpha)]^s d\alpha\right].\end{aligned} \tag{39}$$

Proof. Using Lemma 18 and (n,h)-polynomial convexity of $|\psi'|$, we get

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right| \\ &\le \frac{b_1-a_1}{2}\int_0^1 |1-2\alpha||\psi'(\alpha a_1 - (1-\alpha)b_1)| \\ &\cdot d\alpha \le \frac{b_1-a_1}{2}\int_0^1 |1-2\alpha|\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]|\psi'(a_1)| \right. \\ &\left. + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s|\psi'(b_1)|\right]d\alpha \le \frac{b_1-a_1}{2} \\ &\cdot\left[\frac{|\psi'(a_1)|}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|[1-[h(\alpha)]^s]d\alpha + \frac{|\psi'(b_1)|}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|[h(\alpha)]^s\right. \\ &\cdot d\alpha \le \frac{b_1-a_1}{2}\left[\frac{|\psi'(a_1)|}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|d\alpha - \frac{|\psi'(a_1)|}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|[h(\alpha)]^s\right. \\ &\left. + \frac{|\psi'(b_1)|}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|[h(\alpha)]^s d\alpha\right] \\ &\le \frac{b_1-a_1}{2}\left[\frac{|\psi'(a_1)|}{2} + \frac{|\psi'(b_1)|-|\psi'(a_1)|}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|[h(\alpha)]^s d\alpha\right].\end{aligned} \tag{40}$$

□

Remark 20. If we set $n = 1$, and $h(\alpha) = 1-\alpha$ in the Theorem 19 we can immediately obtain [13], Theorem 2.2.

Theorem 21. *Let $\psi : J \longrightarrow \mathbb{R}$ be a differentiable map on J and $\psi' \in L[a_1, b_1]$ with $q > 1$, $(1/p) + (1/q) = 1$. If the $|\psi'|^q$ is (n,h)-polynomial convex on J, then for $\alpha \in [0,1]$, we have*

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right| \le \frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{1/q} \\ &\times\left[|\psi'(a_1)|^q + \frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1[h(\alpha)]^s d\alpha\right]^{1/q}.\end{aligned} \tag{41}$$

Proof. Unliving Lemma 18, the Hölder's inequality and the fact that $|\psi'|^q$ is (n,h)-polynomial convex, we get

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right| \\ &\le \frac{b_1-a_1}{2}\int_0^1|1-2\alpha||\psi'(\alpha a_1-(1-\alpha)b_1)|d\alpha \\ &\le \frac{b_1-a_1}{2}\times\left[\left(\int_0^1|1-2\alpha|^p d\alpha\right)^{1/p}\left(\int_0^1|\psi'(\alpha a_1-(1-\alpha)b_1)|^q d\alpha\right)^{1/q}\right] \\ &\le \frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{1/p}\times\left(\int_0^1|\psi'(\alpha a_1-(1-\alpha)b_1)|^q d\alpha\right)^{1/q} \\ &\le \frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{1/p}\times\left(\int_0^1\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]|\psi'(a_1)|^q + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s|\psi'(b_1)|^q\right]d\alpha\right)^{1/q} \\ &\le \frac{b_1-a_1}{2}\left(\frac{1}{p+1}\right)^{1/p}\times\left[|\psi'(a_1)|^q + \frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1[h(\alpha)]^s d\alpha\right]^{1/q}.\end{aligned} \tag{42}$$

□

Remark 22. By setting $n = 1$ and $h(\alpha) = 1-\alpha$ in the Theorem 21, one obtain [13], Theorem 2.3.

Theorem 23. *Let $\psi : J \longrightarrow \mathbb{R}$ be a differentiable map on J and $\psi' \in L[a_1, b_1]$ with $q \ge 1$. If $|\psi'|^q$ is (n,h)-polynomial convex on $[a_1, b_1]$, then for $\alpha \in [0,1]$, we have*

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right| \\ &\le \frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-1/q} \\ &\times\left[\frac{|\psi'(a_1)|^q}{2} + \frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|[h(\alpha)]^s\right]^{1/q}.\end{aligned} \tag{43}$$

Proof. Assume that $q > 1$. Using Lemma 18, power mean inequality and convexity of $|\psi'|^q$, we achieve

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right| \le \frac{b_1-a_1}{2}\int_0^1|1-2\alpha||\psi'(\alpha a_1-(1-\alpha)b_1)|d\alpha \\ &\le \frac{b_1-a_1}{2}\times\left[\left(\int_0^1|1-2\alpha|d\alpha\right)^{1-1/q}\left(\int_0^1|1-2\alpha||\psi'(\alpha a_1-(1-\alpha)b_1)|^q d\alpha\right)^{1/q}\right] \le \frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-1/q} \\ &\times\left(\int_0^1|1-2\alpha||\psi'(\alpha a_1-(1-\alpha)b_1)|^q d\alpha\right)^{1/q} \le \frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-1/q} \\ &\times\left(\int_0^1|1-2\alpha|\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]|\psi'(a_1)|^q + \frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s|\psi'(b_1)|^q\right]d\alpha\right)^{1/q} \\ &\le \frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-1/q}\times\left(\frac{|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|[1-[h(\alpha)]^s]d\alpha + \frac{|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|[h(\alpha)]^s d\alpha\right)^{1/q} \\ &\le \frac{b_1-a_1}{2}\left(\frac{1}{2}\right)^{1-1/q}\times\left[\frac{|\psi'(a_1)|^q}{2} + \frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1|1-2\alpha|[h(\alpha)]^s\right]^{1/q}\end{aligned} \tag{44}$$

For $q = 1$, the result can be proved in a similar fashion as of Theorem 19. □

Next we need the following result to refine the power-mean inequality;

Lemma 24 (see [14]). *Let $p > 1$ and $(1/p) + (1/q) = 1$. If ψ and ϕ are real functions defined on interval J and if $|\psi|^q$, $|\phi|^q$ are integrable functions on J, then*

$$\int_{a_1}^{b_1}|\psi(x)\phi(x)|dx \leq \frac{1}{b_1-a_1}\left[\left(\int_{a_1}^{b_1}(b_1-x)|\psi(x)|^p dx\right)^{1/p}\cdot\left(\int_{a_1}^{b_1}(b_1-x)|\phi(x)|^q dx\right)^{1/q}+\left(\int_{a_1}^{b_1}(x-a_1)|\psi(x)|^p dx\right)^{1/p}\cdot\left(\int_{a_1}^{b_1}(x-a_1)|\phi(x)|^q dx\right)^{1/q}\right]. \tag{45}$$

The refined version of integral version of power-mean inequality is as follow:

Theorem 25 (Improved power-mean integral inequality [15]). *Let $q \geq 1$. If ψ and ϕ are real functions defined on interval J and if $|\psi|$, $|\psi||g|^q$ are integrable functions on J, then*

$$\int_{a_1}^{b_1}|\psi(x)\phi(x)|dx \leq \frac{1}{b_1-a_1}\left[\left(\int_{a_1}^{b_1}(b_1-x)|\psi(x)|dx\right)^{1-1/q}\cdot\left(\int_{a_1}^{b_1}(b_1-x)|\psi(x)||\phi(x)|^q dx\right)^{1/q}+\left(\int_{a_1}^{b_1}(x-a_1)|\psi(x)|dx\right)^{1-1/q}\cdot\left(\int_{a_1}^{b_1}(x-a_1)|\psi(x)||\phi(x)|^q dx\right)^{1/q}\right]. \tag{46}$$

Theorem 26. *Let $\psi : J \longrightarrow \mathbb{R}$ be a differentiable map on J and $\psi' \in L[a_1, b_1]$ with $q > 1, (1/p) + (1/q) = 1$. If $|\psi'|^q$ is (n,h)-polynomial convex on J, then for $\alpha \in [0, 1]$, we have*

$$\begin{aligned}\left|\frac{\psi(a_1)+\psi(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right| &\leq \frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{1/p}\\ &\times\left[\frac{|\psi'(a_1)|^q}{2}+\frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\alpha)[h(\alpha)]^s d\alpha\right]^{1/q}\\ &+\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{1/p}\\ &\times\left[\frac{|\psi'(a_1)|^q}{2}+\frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1\alpha[h(\alpha)]^s d\alpha\right]^{1/q}.\end{aligned} \tag{47}$$

Proof. Utilizing convexity of $|\psi'|^q$, integral version of Holder-Iscan inequality and the Lemma 18, we arrive at

$$\begin{aligned}&\left|\frac{\psi(a_1)+\psi(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right|\\ &\leq \frac{b_1-a_1}{2}\int_0^1|1-2\alpha||\psi'(\alpha a_1-(1-\alpha)b_1)|d\alpha\\ &\leq \frac{b_1-a_1}{2}\times\left[\left(\int_0^1(1-\alpha)|1-2\alpha|^p d\alpha\right)^{1/p}\times\left(\int_0^1(1-\alpha)|\psi'(\alpha a_1-(1-\alpha)b_1)|^q d\alpha\right)^{1/q}\right]+\frac{b_1-a_1}{2}\\ &\times\left[\left(\int_0^1\alpha|1-2\alpha|^p d\alpha\right)^{1/p}\left(\int_0^1\alpha|\psi'(\alpha a_1-(1-\alpha)b_1)|^q d\alpha\right)^{1/q}\right]\\ &\leq \frac{b_1-a_1}{2}\times\left[\left(\frac{1}{2(p+1)}\right)^{1/p}\left(\int_0^1(1-\alpha)|\psi'(\alpha a_1-(1-\alpha)b_1)|^q d\alpha\right)^{1/q}\right]\\ &+\frac{b_1-a_1}{2}\times\left[\left(\frac{1}{2(p+1)}\right)^{1/p}\left(\int_0^1\alpha|\psi'(\alpha a_1-(1-\alpha)b_1)|^q d\alpha\right)^{1/q}\right]\\ &\leq \frac{b_1-a_1}{2}\times\left[\left(\frac{1}{2(p+1)}\right)^{1/p}\left(\int_0^1(1-\alpha)\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]|\psi'(a_1)|^q+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s|\psi'(b_1)|^q\right]d\alpha\right)^{1/q}\right]+\frac{b_1-a_1}{2}\\ &\times\left[\left(\frac{1}{2(p+1)}\right)^{1/p}\left(\int_0^1\alpha\left[\frac{1}{n}\sum_{s=1}^{n}[1-[h(\alpha)]^s]|\psi'(a_1)|^q+\frac{1}{n}\sum_{s=1}^{n}[h(\alpha)]^s\right]d\alpha\right)^{1/q}\right]\\ &\leq \frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{1/p}\times\left(\frac{|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\alpha)[1-[h(\alpha)]^s]d\alpha\right.\\ &\left.+\frac{|\psi'(b_1)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\alpha)[h(\alpha)]^s d\alpha\right)^{1/q}+\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{1/p}\\ &\times\left(\frac{|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1\alpha[1-[h(\alpha)]^s]d\alpha+\frac{|\psi'(b_1)|^q}{n}\sum_{s=1}^{n}\int_0^1\alpha[h(\alpha)]^s d\alpha\right)^{1/q}\\ &\leq \frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{1/p}\times\left[\frac{|\psi'(a_1)|^q}{2}+\frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\right.\\ &\left.\cdot\sum_{s=1}^{n}\int_0^1(1-\alpha)[h(\alpha)]^s d\alpha\right]^{1/q}+\frac{b_1-a_1}{2}\left(\frac{1}{2(p+1)}\right)^{1/p}\\ &\times\left[\frac{|\psi'(a_1)|^q}{2}+\frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1\alpha[h(\alpha)]^s d\alpha\right]^{1/q}.\end{aligned} \tag{48}$$

□

Remark 27. Taking $n = 1$ and $h(\alpha) = 1 - \alpha$ in the Theorem 26 give [14], Theorem 3.2.

Theorem 28. *Let $\psi : J \longrightarrow \mathbb{R}$ be a differentiable map on J and $\psi' \in L[a_1, b_1]$ with $q \geq 1$. If $|\psi'|^q$ is (n,h)-polynomial convex on $[a_1, b_1]$, then for $\alpha \in [0, 1]$, we have*

$$\begin{aligned}\left|\frac{\psi(a_1)+\psi(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right| &\leq \frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\\ &\times\left[\frac{|\psi'(a_1)|^q}{4}+\frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\right.\\ &\left.\cdot\sum_{s=1}^{n}\int_0^1(1-\alpha)|1-2\alpha|[h(\alpha)]^s d\alpha\right]^{1/q}+\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\\ &\times\left[\frac{|\psi'(a_1)|^q}{4}+\frac{|\psi'(b_1)|^q-|\psi'(a_1)|^q}{n}\sum_{s=1}^{n}\int_0^1\alpha|1-2\alpha|[h(\alpha)]^s d\alpha\right]^{1/q}.\end{aligned} \tag{49}$$

Proof. Assuming $q > 1$, utilizing Lemma 18, integral version of improved power-mean inequality and convexity of the $|\psi'|^q$, we get

$$\begin{aligned}
&\left|\frac{\psi(a_1)+\psi(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\psi(x)dx\right| \\
&\le \frac{b_1-a_1}{2}\int_0^1 |1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|d\alpha \le \frac{b_1-a_1}{2} \\
&\quad\times\left[\left(\int_0^1 (1-\alpha)|1-2\alpha|d\alpha\right)^{1-(1/q)}\right. \\
&\quad\left.\cdot\left(\int_0^1 (1-\alpha)|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q d\alpha\right)^{1/q}\right] \\
&\quad+\frac{b_1-a_1}{2}\times\left[\left(\int_0^1 \alpha|1-2\alpha|d\alpha\right)^{1-(1/q)}\right. \\
&\quad\left.\cdot\left(\int_0^1 \alpha|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q d\alpha\right)^{1/q}\right] \\
&\le \frac{b_1-a_1}{2}\times\left[\left(\frac{1}{4}\right)^{1-(1/q)}\left(\int_0^1 (1-\alpha)|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q d\alpha\right)^{1/q}\right] \\
&\quad+\frac{b_1-a_1}{2}\times\left[\left(\frac{1}{4}\right)^{1-(1/q)}\left(\int_0^1 \alpha|1-2\alpha|\left|\psi'(\alpha a_1-(1-\alpha)b_1)\right|^q d\alpha\right)^{1/q}\right] \\
&\le \frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left(\int_0^1 (1-\alpha)|1-2\alpha|\left[\frac{1}{n}\sum_{s=1}^n\left[1-[h(\alpha)]^s\right]\left|\psi'(a_1)\right|^q\right.\right. \\
&\quad\left.\left.+\frac{1}{n}\sum_{s=1}^n [h(\alpha)]^s\left|\psi'(b_1)\right|^q\right]d\alpha\right)^{1/q}+\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)} \\
&\quad\times\left(\int_0^1 \alpha|1-2\alpha|\left[\frac{1}{n}\sum_{s=1}^n\left[1-[h(\alpha)]^s\right]\left|\psi'(a_1)\right|^q+\frac{1}{n}\sum_{s=1}^n [h(\alpha)]^s\left|\psi'(b_1)\right|^q\right]d\alpha\right)^{1/q} \\
&\le \frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)}\times\left[\frac{\left|\psi'(a_1)\right|^q}{4}+\frac{\left|\psi'(b_1)\right|^q-\left|\psi'(a_1)\right|^q}{n}\right. \\
&\quad\left.\cdot\sum_{s=1}^n\int_0^1 (1-\alpha)|1-2\alpha|[h(\alpha)]^s d\alpha\right]^{1/q}+\frac{b_1-a_1}{2}\left(\frac{1}{4}\right)^{1-(1/q)} \\
&\quad\times\left[\frac{\left|\psi'(a_1)\right|^q}{4}+\frac{\left|\psi'(b_1)\right|^q-\left|\psi'(a_1)\right|^q}{n}\sum_{s=1}^n\int_0^1 \alpha|1-2\alpha|[h(\alpha)]^s d\alpha\right]^{1/q}.
\end{aligned} \tag{50}$$

For $q = 1$, we use the estimates of Theorem 19 which also follows step by step the above estimates. This completes the proof of theorem. □

6. Application to the Means

Consider $a_1, b_1 > 0$ are two numbers. The arithmetic, geometric, logarithmic, and p-logarithmic means for a_1 and b_1 are defined by,

$$\begin{aligned}
&A(a_1,b_1)=\frac{a_1+b_1}{2},\ G(a_1,b_1)=\sqrt{a_1b_1}, \\
&L := L(a_1,b_1)=\begin{cases}\dfrac{b_1-a_1}{\ln b_1-\ln a_1}, & a_1\ne b_1 \\ a_1, & a_1=b_1\end{cases} \\
&L_p=L_p(a_1,b_1)=\begin{cases}\left[\dfrac{b_1^{p+1}-a_1^{p+1}}{(p+1)(b_1-a_1)}\right]^{1/p}, & a_1\ne b_1,\ p\in\mathbb{R}\setminus\{-1,0\} \\ a_1, & a_1=b_2,\end{cases}
\end{aligned} \tag{51}$$

Proposition 29. *Let $a_1, b_1 \in (0,\infty)$ with $a_1 < b_1$, then the following inequalities hold:*

$$A^n(a_1,b_1) \le Ł_n^n(a_1,b_1) \le a_1^n + (b_1^n - a_1^n)\frac{1}{n}\sum_{s=1}^n\int_0^1 [h(\alpha)]^s d\alpha. \tag{52}$$

Proof. Taking $\psi(x) = x^n$, $x \in [0,\infty)$ in Theorem 9 we obtain the required result. □

Proposition 30. *Let $a_1, b_1 \in (0,\infty)$ with $a_1 < b_1$, then the following inequalities hold:*

$$A^{-1}(a_1,b_1) \le L^{-1}(a_1,b_1) \le a_1^{-1} + \frac{(b_1-a_1)}{a_1b_1}\frac{1}{n}\sum_{s=1}^n\int_0^1 [h(\alpha)]^s d\alpha. \tag{53}$$

Proof. Taking $\psi(x) = x^{-1} \in (0,\infty)$ in Theorem 9 we obtain the required result. □

7. Conclusion

In this paper we introduced a new more generalized class of (n,h)-polynomial convex functions and gave some of its basic interesting properties. We also established Hermite-Hadamard type inequalities for (n,h)-polynomial convex functions. Some applications of the results to the special means are also given. The remarks presented in the paper justify that our results are extension and generalization of many existing results. It will be interesting to establish Hermite-Hadamard, Fejér, and Jensen type inequalities for the different fractional integral operators.

Data Availability

All data is included within this paper.

Conflicts of Interest

The authors have no competing interests.

Authors' Contributions

All authors contributed equally in this paper.

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