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Research Article

Characterizations of Lie n-Centralizers on Certain Trivial Extension Algebras

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In this paper, we describe the structure of Lie n-centralizers of a trivial extension algebra. We then present some conditions under which a Lie n-centralizer on a trivial extension algebra is proper. As an application, we consider Lie n-centralizers on a triangular algebra.

1. Introduction

Let R be a unital commutative ring, $\mathscr{U}$ be a unital algebra over R, and $Z(\mathscr{U})$ be the center of $\mathscr{U}$. Let $[x, y] = xy - yx$ denote the Lie product of elements $x, y \in \mathscr{U}$. An R-linear map $\phi: \mathscr{U} \longrightarrow \mathscr{U}$ is called a left (right) centralizer if $\phi(xy) = \phi(x)y$ ($\phi(xy) = x\phi(y)$) holds for all $x, y \in \mathscr{U}$. Furthermore, δ is called a centralizer if it is both a left centralizer and a right centralizer. Centralizers on rings as well as algebras have been extensively investigated (see [1–5]). An R-linear map $\delta: \mathscr{U} \longrightarrow \mathscr{U}$ is called a Lie centralizer if $\delta([x, y]) = [\delta(x), y]$ for all $x, y \in \mathscr{U}$. It is easy to check that δ is a Lie centralizer on $\mathscr{U}$ if and only if $\delta([x, y]) = [x, \delta(y)]$ for all $x, y \in \mathscr{U}$. If a Lie centralizer $\delta: \mathscr{U} \longrightarrow \mathscr{U}$ can be expressed as $\delta(x) = \lambda x + \tau(x)$, where $\lambda \in Z(\mathscr{U})$ and $\tau: \mathscr{U} \longrightarrow Z(\mathscr{U})$ is a linear map vanishing at commutators $[x, y]$ for all $x, y \in \mathscr{U}$. Then, the R-linear map $\delta: \mathscr{U} \longrightarrow \mathscr{U}$ is called a proper Lie centralizer. Recently, the structure of Lie centralizers is studied by many mathematicians (see [6–10]).

Let $\mathscr{U}$ be a unital algebra over R and $\mathscr{M}$ be a $\mathscr{U}$-bimodule. Then, the direct product $\mathscr{U} \times \mathscr{M}$ equipped with the pairwise addition, scalar product, and the algebra product is given by

$$(u, m)(u', m') = (uu', um' + mu'), \quad \text{for all } u, u' \in \mathscr{U}, m, m' \in \mathscr{M}, \tag{1}$$

which forms a unital algebra, which is called a trivial extension algebra of $\mathscr{U}$ by $\mathscr{M}$ and will be denoted by $\mathscr{U} \propto \mathscr{M}$. The center of $\mathscr{U} \propto \mathscr{M}$ is given by

$$Z(\mathscr{U} \propto \mathscr{M}) = \{(u, m) | u \in Z(\mathscr{U}), [u', m] = 0 = [u, m'], \text{for all } u' \in \mathscr{U}, m' \in \mathscr{M}\}. \tag{2}$$

Trivial extension algebras have been extensively studied in algebra and analysis (see [11–16]). In this paper, we will study the structure of Lie n-centralizers on a trivial extension algebra.

2. Preliminaries

In this paper, we mainly discuss the trivial extension algebra $\mathscr{U} \propto \mathscr{M}$ for which $\mathscr{U}$ has a nontrivial idempotent α satisfying

$$\alpha m\beta = m, \quad \text{for all } m \in M, \tag{3}$$

where $\beta = I - \alpha$. A triangular algebra is an important example of a trivial extension algebra that satisfies (3). Let A and B be unital algebras over a unital commutative ring R, and let M be a unital (A, B)-bimodule. Then, the set

$$\mathrm{Tri}(A, M, B) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mid a \in A, m \in M, b \in B \right\}, \tag{4}$$

forms an algebra under the usual matrix addition and formal matrix multiplication. Such an algebra is called a triangular algebra (see [17]). It is easy to prove that $\mathrm{Tri}(A, M, B)$ is isomorphic to the trivial extension algebra $(A \oplus B) \propto \mathscr{M}$, where the algebra $A \oplus B$ is equipped with the usual pairwise operations and M is regarded as an $(A \oplus B)$-bimodule equipped with the module operations $(a, b)m = \mathrm{am}$ and $m(a, b) = \mathrm{mb}$ for all $(a, b) \in A \oplus B$ and $m \in M$. Assuming $\alpha = (I_A, 0)$ and $\beta = (0, I_B)$, it is easy to show that α is a nontrivial idempotent of $A \oplus B$ and $\alpha m\beta = m$ for all $m \in M$. In addition, we can get

$$\begin{aligned} &\alpha(A \oplus B)\alpha \cong A, \alpha(A \oplus B)\beta \cong \{0\}, \beta(A \oplus B)\alpha \cong \{0\}, \\ &\beta(A \oplus B)\beta \cong B. \end{aligned} \tag{5}$$

If a trivial extension algebra $\mathscr{U} \propto \mathscr{M}$ satisfies (3), then the center of $\mathscr{U} \propto \mathscr{M}$ coincides with

$$Z(\mathscr{U} \propto \mathscr{M}) = \{(u, 0) | u \in Z(\mathscr{U}), [u, m] = 0, \text{for all } m \in M\}. \tag{6}$$

Next, we give the definition of Lie n-centralizers. Let us define the following sequence of polynomials:

$$\begin{aligned} p_1(x_1) &= x_1, \\ p_2(x_1, x_2) &= [p_1(x_1), x_2] = [x_1, x_2], \\ p_3(x_1, x_2, x_3) &= [p_2(x_1, x_2), x_3] = [[x_1, x_2], x_3], \\ &\cdots\cdots \\ p_n(x_1, x_2, \ldots, x_n) &= [p_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n]. \end{aligned} \tag{7}$$

The polynomial $p_n(x_1, x_2, \ldots, x_n)$ is said to be an $(n-1)$-th commutator $(n \geq 2)$. A Lie n-centralizer is an R-linear map $f\colon \mathscr{U} \longrightarrow \mathscr{U}$ which satisfies the rule

$$f(p_n(x_1, x_2, \ldots, x_n)) = p_n(f(x_1), x_2, \ldots, x_n), \tag{8}$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{U}$. If there exists an element $\lambda \in Z(\mathscr{U})$ and an R-linear map $\tau\colon \mathscr{U} \longrightarrow Z(\mathscr{U})$ vanishing on each $(n-1)$-th commutator $p_n(x_1, x_2, \ldots, x_n)$ such that $f(x) = \lambda x + \tau(x)$ for all $x \in \mathscr{U}$, then the Lie n-centralizer f is called a proper Lie n-centralizer.

Now, we state some lemmas which are very important for proving the main results.

Lemma 1 (see [13] Proposition 2.5.) *Let $\mathscr{U}$ be a unital algebra and $\mathscr{M}$ be a $\mathscr{U}$-bimodule. Suppose that $\mathscr{U}$ has a nontrivial idempotent α and denote $\beta = I - \alpha$. Then, the following statements are equivalent:*

(i) For all $m \in \mathscr{M}$, $\alpha m\beta = m$.

(ii) For all $m \in \mathscr{M}$, $\beta m = 0 = m\alpha$.

(iii) For all $m \in \mathscr{M}$, $\alpha m = m = m\beta$.

(iv) For all $m \in \mathscr{M}$ and $u \in \mathscr{U}$, $um = \alpha u\alpha m$ and $mu = m\beta u\beta$.

Lemma 2. *Let $\mathscr{U}$ be a unital algebra containing a nontrivial idempotent α and $\mathscr{M}$ be a $\mathscr{U}$-bimodule satisfying $\alpha m\beta = m$ for all $m \in M$, where $\beta = I - \alpha$. Then, every Lie n-centralizer $f\colon \mathscr{U} \longrightarrow \mathscr{U}$ satisfies $[f(\alpha u\beta), m] + [f(\beta u\alpha), m] = 0$ for all $u \in \mathscr{U}, m \in \mathscr{M}$.*

Proof. Since f is a Lie n-centralizer, it follows that

$$\begin{aligned} f(\alpha u\beta) &= f(p_n(\alpha, \alpha u\beta, \beta, \ldots, \beta)) \\ &= p_n(f(\alpha), \alpha u\beta, \beta, \ldots, \beta), \end{aligned} \tag{9}$$

for all $u \in \mathscr{U}$. According to Lemma 1, we obtain $[f(\alpha u\beta), m] = 0$ for all $u \in \mathscr{U}$ and $m \in \mathscr{M}$. In a similar way, we get $[f(\beta u\alpha), m] = 0$ for all $u \in \mathscr{U}$ and $m \in \mathscr{M}$. Therefore, f satisfies $[f(\alpha u\beta), m] + [f(\beta u\alpha), m] = 0$ for all $u \in \mathscr{U}, m \in \mathscr{M}$. □

In particular, based on the fact that every centralizer is a Lie n-centralizer, if ϕ is a centralizer, then we have $[\phi(\alpha u\beta), m] + [\phi(\beta u\alpha), m] = 0$ for all $u \in \mathscr{U}, m \in \mathscr{M}$.

Lemma 3 (see [16] Lemma 2.2). *Let $f\colon U \propto M \longrightarrow U \propto M$ be an R-linear map and f have the following form $f((u, m)) = (f_{\mathscr{U}}(u) + h_{\mathscr{U}}(m), f_{\mathscr{M}}(u) + h_{\mathscr{M}}(m))$, then f is a centralizer if and only if the following conditions hold:*

(1) $f_{\mathscr{U}}\colon \mathscr{U} \longrightarrow \mathscr{U}$ is a centralizer;

(2) $f_{\mathscr{M}}\colon \mathscr{U} \longrightarrow \mathscr{M}$ is a centralizer;

(3) $\mathrm{uh}_{\mathscr{U}}(m) = h_{\mathscr{U}}(um) = 0 = h_{\mathscr{U}}(mu) = h_{\mathscr{U}}(m)u$ for all $u \in \mathscr{U}$ and $m \in \mathscr{M}$;

(4) $\mathrm{uh}_{\mathscr{M}}(m) = h_{\mathscr{M}}(um) = f_{\mathscr{U}}(u)m$ and $mf_{\mathscr{U}}(u) = h_{\mathscr{M}}(mu) = h_{\mathscr{M}}(m)u$ for all $u \in \mathscr{U}$ and $m \in \mathscr{M}$;

(5) $h_{\mathscr{U}}(m)n = 0 = \mathrm{mh}_{\mathscr{U}}(n)$ for all $m, n \in \mathscr{M}$.

Lemma 4 (see [12], Lemma 3.11). *Assume that $\mathscr{U} \propto \mathscr{M}$ is a trivial extension algebra satisfying (3). Then, the following statements hold:*

(1) The center of $\mathscr{U} \propto \mathscr{M}$ is given by

$$Z(\mathcal{U} \propto \mathcal{M}) = \{(u,0); u \in \mathcal{U}, \alpha u\alpha \in Z(\alpha\mathcal{U}\alpha), \beta u\beta \in Z(\beta\mathcal{U}\beta), \alpha u\alpha u_{12} = u_{12}\beta u\beta \\ u_{21}\alpha u\alpha = \beta u\beta u_{21}, [u,m] = 0, \text{for all } u_{12} \in \alpha\mathcal{U}\beta, u_{21} \in \beta\mathcal{U}\alpha, m \in \mathcal{M}\}. \tag{10}$$

(2) $[Z(\mathcal{U}), \mathcal{M}] = 0$, *if one of the following conditions holds:*

(i) $Z(\alpha\mathcal{U}\alpha) = \pi_{\alpha\mathcal{U}\alpha}(Z(\mathcal{U} \propto \mathcal{M}))$ *and* $\alpha\mathcal{U}\beta$ *is faithful as a right* $\beta\mathcal{U}\beta$*-module.*

(ii) $Z(\beta\mathcal{U}\beta) = \pi_{\beta\mathcal{U}\beta}(Z(\mathcal{U} \propto \mathcal{M}))$ *and* $\alpha\mathcal{U}\beta$ *is faithful as a left* $\alpha\mathcal{U}\alpha$*-module.*

Lemma 5. *An R-linear map* $\phi: \mathcal{U} \longrightarrow \mathcal{U}$ *is a centralizer if and only if there exists an element* $\lambda \in Z(\mathcal{U})$ *such that* $\phi(x) = \lambda x$ *for all* $x \in \mathcal{U}$.

Proof. Suppose that $\phi: \mathcal{U} \longrightarrow \mathcal{U}$ is a centralizer, then we have

$$\phi(x) = \phi(Ix) = \phi(I)x = \phi(xI) = x\phi(I), \tag{11}$$

for all $x \in \mathcal{U}$. Set $\phi(I) = \lambda$, then we get $\lambda \in Z(\mathcal{U})$ and $\phi(x) = \lambda x$ for all $x \in \mathcal{U}$.

Conversely, it is clear. □

3. Lie n-Centralizers on $\mathcal{U} \propto \mathcal{M}$

The following result gives the structure of a Lie n-centralizer on a trivial extension algebra.

Theorem 6. *Let* $f: \mathcal{U} \propto \mathcal{M} \longrightarrow \mathcal{U} \propto \mathcal{M}$ *be an R-linear map and* f *have the following form:*

$$f((u,m)) = (f_{\mathcal{U}}(u) + h_{\mathcal{U}}(m), f_{\mathcal{M}}(u) + h_{\mathcal{M}}(m)), \tag{12}$$

where $f_{\mathcal{U}}: \mathcal{U} \longrightarrow \mathcal{U}; h_{\mathcal{U}}: \mathcal{M} \longrightarrow \mathcal{U}; f_{\mathcal{M}}: \mathcal{U} \longrightarrow \mathcal{M}; h_{\mathcal{M}}: \mathcal{M} \longrightarrow \mathcal{M}$ *are R-linear maps. Then,* f *is a Lie n-centralizer if and only if the following conditions are satisfied:*

(i) $f_{\mathcal{U}}: \mathcal{U} \longrightarrow \mathcal{U}$ *is a Lie n-centralizer;*

(ii) $f_{\mathcal{M}}: \mathcal{U} \longrightarrow \mathcal{M}$ *is a Lie n-centralizer;*

(iii) $h_{\mathcal{U}}(p_n(m_1, u_2, \ldots, u_n)) = p_n(h_{\mathcal{U}}(m_1), u_2, \ldots, u_n) = 0$ *and* $h_{\mathcal{U}}(p_n(u_1, \ldots, m_i, \ldots, u_n)) = 0$ *for all* $u_1, u_2, \ldots, u_n \in \mathcal{U}$, $m_i \in \mathcal{M}$, *and* $i \in \{2, \ldots, n\}$;

(iv) $h_{\mathcal{M}}(p_n(m_1, u_2, \ldots, u_n)) = p_n(h_{\mathcal{M}}(m_1), u_2, \ldots, u_n)$ *and* $h_{\mathcal{M}}(p_n(u_1, \ldots, m_i, \ldots, u_n)) = p_n(f_{\mathcal{U}}(u_1), \ldots, m_i, \ldots, u_n)$ *for all* $u_1, \ldots, u_n \in \mathcal{U}$, $m_i \in \mathcal{M}$, *and* $i \in \{2, \ldots, n\}$;

(v) $p_n(h_{\mathcal{U}}(m_1), u_2, \ldots, m_i, \ldots, u_n) = 0$ *for all* $u_2, \ldots, u_n \in \mathcal{U}$, $m_1, m_i \in \mathcal{M}$, *and* $i \in \{2, \ldots, n\}$.

Proof. Since f is a Lie n-centralizer on $\mathcal{U} \propto \mathcal{M}$, it follows that

$$f(p_n(x_1, x_2, \ldots, x_n)) = p_n(f(x_1), x_2, \ldots, x_n), \tag{13}$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{U} \propto \mathcal{M}$.

Let us choose $x_1 = (u_1, 0), x_2 = (u_2, 0), \ldots, x_n = (u_n, 0)$ in (13). Then, we obtain

$$\begin{aligned} f(p_n(x_1, x_2, \ldots, x_n)) &= (f_{\mathcal{U}}(p_n(u_1, u_2, \ldots, u_n)), f_{\mathcal{M}}(p_n(u_1, u_2, \ldots, u_n))), \\ f(p_n(x_1, x_2, \ldots, x_n)) &= p_n(f(x_1), x_2, \ldots, x_n) \\ &= p_n((f_{\mathcal{U}}(u_1), f_{\mathcal{M}}(u_1)), (u_2, 0), \ldots, (u_n, 0)) \\ &= (p_n(f_{\mathcal{U}}(u_1), u_2, \ldots, u_n), p_n(f_{\mathcal{M}}(u_1), u_2, \ldots, u_n)), \end{aligned} \tag{14}$$

for all $u_1, u_2, \ldots, u_n \in \mathcal{U}$. Comparing the above equations, we have that $f_{\mathcal{U}}: \mathcal{U} \longrightarrow \mathcal{U}$ and $f_{\mathcal{M}}: \mathcal{U} \longrightarrow \mathcal{M}$ are Lie n-centralizers.

Let us consider $x_1 = (u_1, 0), x_2 = (u_2, 0), \ldots, x_n = (0, m_n)$ in (13). Then, we deduce

$$\begin{aligned} f(p_n(x_1, x_2, \ldots, x_n)) &= (h_{\mathcal{U}}(p_n(u_1, u_2, \ldots, m_n)), h_{\mathcal{M}}(p_n(u_1, u_2, \ldots, m_n))), \\ f(p_n(x_1, x_2, \ldots, x_n)) &= p_n(f(x_1), x_2, \ldots, x_n) = (0, p_n(f_{\mathcal{U}}(u_1), u_2, \ldots, m_n)), \end{aligned} \tag{15}$$

for all $u_1, u_2, \ldots, u_{n-1} \in \mathcal{U}, m_n \in \mathcal{M}$. Comparing the abovementioned relations, we conclude

$$\begin{aligned} &h_{\mathscr{U}}(p_n(u_1,u_2,\ldots,m_n))=0,\\ &h_{\mathscr{M}}(p_n(u_1,u_2,\ldots,m_n))=p_n(f_{\mathscr{U}}(u_1),u_2,\ldots,m_n). \end{aligned} \tag{16}$$

Similarly, considering $x_1=(u_1,0), x_2=(u_2,0),\ldots,x_i=(0,m_i),\ldots,x_n=(u_n,0)$ in (13), $i\in\{2,\ldots,n-1\}$, we find

$$\begin{aligned} &h_{\mathscr{U}}(p_n(u_1,\ldots,m_i,\ldots,u_n))=0,\\ &h_{\mathscr{M}}(p_n(u_1,\ldots,m_i,\ldots,u_n))=p_n(f_{\mathscr{U}}(u_1),\ldots,m_i,\ldots,u_n), \end{aligned} \tag{17}$$

for all $u_1,\ldots,u_n\in\mathscr{U}, m_i\in\mathscr{M}$, where $i\in\{2,\ldots,n\}$. Setting $x_1=(0,m_1), x_2=(u_2,0),\ldots,x_n=(u_n,0)$ in (13), we get

$$\begin{aligned} &h_{\mathscr{U}}(p_n(m_1,u_2,\ldots,u_n))=p_n(h_{\mathscr{U}}(m_1),u_2,\ldots,u_n),\\ &h_{\mathscr{M}}(p_n(m_1,u_2,\ldots,u_n))=p_n(h_{\mathscr{M}}(m_1),u_2,\ldots,u_n). \end{aligned} \tag{18}$$

Since $h_{\mathscr{U}}(p_n(m_1,u_2,\ldots,u_n))=-h_{\mathscr{U}}(p_n(u_2,m_1,\ldots,u_n))$, it follows from (17) and (18) that

$$h_{\mathscr{U}}(p_n(m_1,u_2,\ldots,u_n))=p_n(h_{\mathscr{U}}(m_1),u_2,\ldots,u_n)=0, \tag{19}$$

for all $m_1\in\mathscr{M}, u_2,\ldots,u_n\in\mathscr{U}$.

If we take $x_1=(0,m_1), x_2=(0,m_2), x_3=(u_3,0),\ldots,x_n=(u_n,0)$ in (13), then we arrive at

$$\begin{aligned} &f(p_n(x_1,x_2,\ldots,x_n))=0,\\ &f(p_n(x_1,x_2,\ldots,x_n))=p_n(f(x_1),x_2,\ldots,x_n)=(0,p_n(h_{\mathscr{U}}(m_1),m_2,u_3,\ldots,u_n)). \end{aligned} \tag{20}$$

Hence, $p_n(h_{\mathscr{U}}(m_1),m_2,u_3,\ldots,u_n)=0$ for all $m_1,m_2\in\mathscr{M}, u_3,\ldots,u_n\in\mathscr{U}$. In an analogous way, we obtain $p_n(h_{\mathscr{U}}(m_1),u_2,\ldots,m_i,\ldots,u_n)=0$ for all $m_1,m_i\in\mathscr{M}, u_2,\ldots,u_n\in\mathscr{U}$, where $i\in\{2,\ldots,n\}$.

Conversely, taking $x_1=(u_1,m_1), x_2=(u_2,m_2),\ldots,x_n=(u_n,m_n)$, we get from (i)–(v) that

$$\begin{aligned} p_n(f(x_1),x_2,\ldots,x_n)&=p_n((f_{\mathscr{U}}(u_1)+h_{\mathscr{U}}(m_1),f_{\mathscr{M}}(u_1)+h_{\mathscr{M}}(m_1)),(u_2,m_2),\ldots,(u_n,m_n))\\ &=(p_n(f_{\mathscr{U}}(u_1),u_2,\ldots,u_n),p_n(f_{\mathscr{U}}(u_1)+h_{\mathscr{U}}(m_1),u_2,\ldots,u_{n-1},m_n)\\ &\quad+p_n(f_{\mathscr{U}}(u_1)+h_{\mathscr{U}}(m_1),u_2,\ldots,m_{n-1},u_n)\\ &\quad+\cdots+p_n(f_{\mathscr{M}}(u_1)+h_{\mathscr{M}}(m_1),u_2,\ldots,u_n))\\ &=(f_{\mathscr{U}}(p_n(u_1,u_2,\ldots,u_n))+h_{\mathscr{U}}(p_n(u_1,\ldots,u_{n-1},m_n))\\ &\quad+\cdots+h_{\mathscr{U}}(p_n(m_1,u_2,\ldots,u_n)),f_{\mathscr{M}}(p_n(u_1,u_2,\ldots,u_n))\\ &\quad+h_{\mathscr{M}}(p_n(u_1,\ldots,u_{n-1},m_n))+\cdots+h_{\mathscr{M}}(p_n(m_1,u_2,\ldots,u_n)))\\ &=f(p_n((u_1,m_1),(u_2,m_2),\ldots,(u_n,m_n)))\\ &=f(p_n(x_1,x_2,\ldots,x_n)). \end{aligned} \tag{21}$$

Hence, f is a Lie n-centralizer on $\mathscr{U}\propto\mathscr{M}$. □

Now, we can present the first main result of this paper, which provides the necessary and sufficient conditions for a Lie n-centralizer on a trivial extension algebra $\mathscr{U}\propto\mathscr{M}$ satisfying (3) to be proper.

Theorem 7. *Let $\mathscr{U}\propto\mathscr{M}$ be a trivial extension algebra satisfying (3). Suppose that $f\colon\mathscr{U}\propto\mathscr{M}\longrightarrow\mathscr{U}\propto\mathscr{M}$ is a Lie n-centralizer and has the form*

$$f((u,m))=(f_{\mathscr{U}}(u)+h_{\mathscr{U}}(m),f_{\mathscr{M}}(u)+h_{\mathscr{M}}(m)), \tag{22}$$

then f is proper if and only if the following conditions are satisfied:

(1) There exists an R-linear map $\tau_{\mathscr{U}}\colon\mathscr{U}\longrightarrow Z(\mathscr{U})$ such that

(i) $f_{\mathscr{U}}-\tau_{\mathscr{U}}$ is a centralizer on $\mathscr{U}$ and

(ii) $[\tau_{\mathscr{U}}(\alpha u\alpha),m]=0=[\tau_{\mathscr{U}}(\beta u\beta),m]$ for all $u\in\mathscr{U}, m\in\mathscr{M}$.

(2) $f_{\mathscr{M}}(\beta u\alpha)=0$ for all $u\in\mathscr{U}$.

Proof. Since f is a Lie n-centralizer on $\mathscr{U}\propto\mathscr{M}$, it follows that f satisfies Theorem 6. Assume that the assumptions (1) and (2) hold, we define two maps $\phi,\tau\colon\mathscr{U}\propto\mathscr{M}\longrightarrow\mathscr{U}\propto\mathscr{M}$ satisfying $\phi((u,m))=((f_{\mathscr{U}}-\tau_{\mathscr{U}})(u)+h_{\mathscr{U}}(m),f_{\mathscr{M}}(u)+h_{\mathscr{M}}(m))$ and $\tau((u,m))=(\tau_{\mathscr{U}}(u),0)$. Clearly, $f=\phi+\tau$. We claim that $\tau(\mathscr{U}\propto\mathscr{M})\subseteq Z(\mathscr{U}\propto\mathscr{M})$. Indeed, according to (6), it suffices to prove that $[\tau_{\mathscr{U}}(u),m]=0$ for all $u\in\mathscr{U}, m\in\mathscr{M}$. Since $f_{\mathscr{U}}-\tau_{\mathscr{U}}$ is a centralizer and $f_{\mathscr{U}}$ is a Lie n-centralizer, it follows from the assumption (1)(ii) and Lemma 2 that

$$\begin{aligned}[\tau_{\mathcal{U}}(u), m] &= [\tau_{\mathcal{U}}(\alpha u\beta) + \tau_{\mathcal{U}}(\beta u\alpha), m] \\ &= [(f_{\mathcal{U}} - (f_{\mathcal{U}} - \tau_{\mathcal{U}}))(\alpha u\beta) + (f_{\mathcal{U}} - (f_{\mathcal{U}} - \tau_{\mathcal{U}}))(\beta u\alpha), m] \\ &= [f_{\mathcal{U}}(\alpha u\beta) + f_{\mathcal{U}}(\beta u\alpha), m] - [(f_{\mathcal{U}} - \tau_{\mathcal{U}})(\alpha u\beta) + (f_{\mathcal{U}} - \tau_{\mathcal{U}})(\beta u\alpha), m] \\ &= 0,\end{aligned} \tag{23}$$

for all $u \in \mathcal{U}, m \in \mathcal{M}$. Therefore, $\tau(\mathcal{U} \propto \mathcal{M}) \subseteq Z(\mathcal{U} \propto \mathcal{M})$. By Lemma 5, it remains to show that ϕ is a centralizer on $\mathcal{U} \propto \mathcal{M}$.

According to Lemma 3 and the assumption (1)(i), it suffices to prove that ϕ satisfies the following conditions: f_M is a centralizer; $uh_{\mathcal{M}}(m) = h_{\mathcal{M}}(\mathrm{um}) = (f_{\mathcal{U}} - \tau_{\mathcal{U}})(u)m$, $m(f_{\mathcal{U}} - \tau_{\mathcal{U}})(u) = h_{\mathcal{M}}(\mathrm{mu}) = h_{\mathcal{M}}(m)u$, $\mathrm{uh}_{\mathcal{U}}(m) = h_{\mathcal{U}}(\mathrm{um}) = 0 = h_{\mathcal{U}}(\mathrm{mu}) = h_{\mathcal{U}}(m)u$, and $h_{\mathcal{U}}(m)n = 0 = \mathrm{mh}_{\mathcal{U}}(n)$ for all $u \in \mathcal{U}, m, n \in \mathcal{M}$.

Since $f_{\mathcal{M}}$ satisfies $f_{\mathcal{M}}(p_n(u_1, u_2, \ldots, u_n)) = p_n(f_{\mathcal{M}}(u_1), u_2, \ldots, u_n)$, it follows from

$$f_{\mathcal{M}}(p_n(\alpha u\alpha, \beta, \ldots, \beta)) = 0, \tag{24}$$

that $f_{\mathcal{M}}(\alpha u\alpha)\beta = 0$. Similarly, $\alpha f_{\mathcal{M}}(\beta u\beta) = 0$ for all $u \in \mathcal{U}$. Therefore,

$$f_{\mathcal{M}}(\alpha u\alpha) = f_{\mathcal{M}}(\beta u\beta) = 0. \tag{25}$$

Next, we define an R-linear map δ: $\mathcal{U} \longrightarrow \mathcal{M}$ by $\delta(u) = f_{\mathcal{M}}(\alpha u\beta)$ for all $u \in \mathcal{U}$. For each $u_1, u_2 \in \mathcal{U}$, we get

$$\begin{aligned}\delta(u_1u_2) &= f_{\mathcal{M}}(\alpha u_1u_2\beta) \\ &= f_{\mathcal{M}}(\alpha u_1\alpha u_2\beta) + f_{\mathcal{M}}(\alpha u_1\beta u_2\beta) \\ &= f_{\mathcal{M}}(p_n(\alpha u_1\alpha, \alpha u_2\beta, \beta, \ldots, \beta)) + f_{\mathcal{M}}(p_n(\alpha u_1\beta, \beta u_2\beta, \beta, \ldots, \beta)) \\ &= p_n(f_{\mathcal{M}}(\alpha u_1\alpha), \alpha u_2\beta, \beta, \ldots, \beta) + p_n(f_{\mathcal{M}}(\alpha u_1\beta), \beta u_2\beta, \beta, \ldots, \beta) \\ &= f_{\mathcal{M}}(\alpha u_1\beta)\beta u_2\beta \\ &= f_{\mathcal{M}}(\alpha u_1\beta)u_2 \\ &= \delta(u_1)u_2.\end{aligned} \tag{26}$$

On the other hand, we have

$$\begin{aligned}\delta(u_1u_2) &= f_{\mathcal{M}}(\alpha u_1u_2\beta) \\ &= p_n(\alpha u_1\alpha, f_{\mathcal{M}}(\alpha u_2\beta), \beta, \ldots, \beta) + p_n(\alpha u_1\beta, f_{\mathcal{M}}(\beta u_2\beta), \beta, \ldots, \beta) \\ &= \alpha u_1\alpha f_{\mathcal{M}}(\alpha u_2\beta) \\ &= u_1 f_{\mathcal{M}}(\alpha u_2\beta) \\ &= u_1\delta(u_2).\end{aligned} \tag{27}$$

According to the assumption (2) and (25), we obtain $f_{\mathscr{M}}(u) = \delta(u)$. Therefore, $f_{\mathscr{M}}$ satisfies $f_{\mathscr{M}}(u_1u_2) = f_{\mathscr{M}}(u_1)u_2 = u_1 f_{\mathscr{M}}(u_2)$. That is, $f_{\mathscr{M}}$ is a centralizer.

Using $[\tau_{\mathscr{U}}(\alpha u\alpha), m] = 0$ and the fact that $f_{\mathscr{U}} - \tau_{\mathscr{U}}$ is a centralizer, we arrive at

$$\begin{aligned} h_{\mathscr{M}}(\mathrm{um}) &= h_{\mathscr{M}}(p_n(\alpha u\alpha, m, \beta, \ldots, \beta)) \\ &= p_n(f_{\mathscr{U}}(\alpha u\alpha), m, \beta, \ldots, \beta) \\ &= [f_{\mathscr{U}}(\alpha u\alpha), m] \\ &= [(f_{\mathscr{U}} - \tau_{\mathscr{U}})(\alpha u\alpha), m] + [\tau_{\mathscr{U}}(\alpha u\alpha), m] \\ &= [\alpha(f_{\mathscr{U}} - \tau_{\mathscr{U}})(u)\alpha, m] \\ &= (f_{\mathscr{U}} - \tau_{\mathscr{U}})(u)m, \\ h_{\mathscr{M}}(\mathrm{um}) &= -h_{\mathscr{M}}(p_n(m, \alpha u\alpha, \beta, \ldots, \beta)) \\ &= -p_n(h_{\mathscr{M}}(m), \alpha u\alpha, \beta, \ldots, \beta) \\ &= [\alpha u\alpha, h_M(m)] \\ &= \mathrm{uh}_{\mathscr{M}}(m), \end{aligned} \tag{28}$$

for all $u \in \mathscr{U}, m \in \mathscr{M}$. Similarly, we get $h_{\mathscr{M}}(\mathrm{mu}) = h_{\mathscr{M}}(m)u = m(f_{\mathscr{U}} - \tau_{\mathscr{U}})(u)$ for all $u \in \mathscr{U}, m \in \mathscr{M}$.

Applying Theorem 6 yields that $h_{\mathscr{U}}(m) = h_{\mathscr{U}}(p_n(\alpha, m, \beta, \ldots, \beta)) = 0$ for all $m \in \mathscr{M}$. Hence, $\mathrm{uh}_{\mathscr{U}}(m) = h_{\mathscr{U}}(\mathrm{um}) = 0 = h_{\mathscr{U}}(\mathrm{mu}) = h_{\mathscr{U}}(m)u$ and $h_{\mathscr{U}}(m)n = 0 = \mathrm{mh}_{\mathscr{U}}(n)$ for all $u \in \mathscr{U}, m, n \in \mathscr{M}$. Therefore, ϕ is a centralizer. Finally,

$$\begin{aligned} \tau(p_n(x_1, x_2, \ldots, x_n)) &= f(p_n(x_1, x_2, \ldots, x_n)) - \phi(p_n(x_1, x_2, \ldots, x_n)) \\ &= p_n(f(x_1), x_2, \ldots, x_n) - p_n(\phi(x_1), x_2, \ldots, x_n) \\ &= p_n(f(x_1) - \phi(x_1), x_2, \ldots, x_n) \\ &= p_n(\tau(x_1), x_2, \ldots, x_n) \\ &= 0, \end{aligned} \tag{29}$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{U} \propto \mathscr{M}$.

Conversely, suppose that f is proper, then there exists a centralizer $\phi\colon \mathscr{U} \propto \mathscr{M} \longrightarrow \mathscr{U} \propto \mathscr{M}$ and an R-linear map $\tau\colon \mathscr{U} \propto \mathscr{M} \longrightarrow Z(\mathscr{U} \propto \mathscr{M})$ such that $f = \phi + \tau$. In view of (6), we get $\tau((u, m)) = (\tau_{\mathscr{U}}(u), 0)$, where $\tau_{\mathscr{U}}\colon \mathscr{U} \longrightarrow Z(\mathscr{U})$ is an R-linear map satisfying $[\tau_{\mathscr{U}}(u), m] = 0$ for all $u \in \mathscr{U}, m \in \mathscr{M}$. On the other hand, $f - \tau = \phi$ is a centralizer on $\mathscr{U} \propto \mathscr{M}$ and by Lemma 3, $f_{\mathscr{U}} - \tau_{\mathscr{U}}, f_{\mathscr{M}}$ are centralizers. According to Lemma 1, we get $f_{\mathscr{M}}(\beta u\alpha) = f_{\mathscr{M}}(\beta)u\alpha = 0$ for all $u \in \mathscr{U}$. □

Using Theorem 7 and Lemma 4, we can give the next main result, which provides the sufficient conditions for any Lie n-centralizer on a trivial extension algebra to be proper.

Corollary 8. *Assume that $\mathscr{U} \propto \mathscr{M}$ is a trivial extension algebra satisfying (3) and f is a Lie n-centralizer on $\mathscr{U} \propto \mathscr{M}$ with the form*

$$f((u, m)) = (f_{\mathscr{U}}(u) + h_{\mathscr{U}}(m), f_{\mathscr{M}}(u) + h_{\mathscr{M}}(m)). \tag{30}$$

Then, f is proper if the following conditions are satisfied:

(1) Every Lie n-centralizer on $\mathscr{U}$ is proper;

(2) $f_{\mathscr{M}}(\beta u\alpha) = 0$ for all $u \in \mathscr{U}$;

(3) One of the following two conditions holds:

(i) $Z(\alpha\mathscr{U}\alpha) = \pi_{\alpha\mathscr{U}\alpha}(Z(\mathscr{U} \propto \mathscr{M}))$ and $\alpha\mathscr{U}\beta$ is faithful as a right $\beta\mathscr{U}\beta$-module;

(ii) $Z(\beta\mathscr{U}\beta) = \pi_{\beta\mathscr{U}\beta}(Z(\mathscr{U} \propto \mathscr{M}))$ and $\alpha\mathscr{U}\beta$ is faithful as a left $\alpha\mathscr{U}\alpha$-module.

Proof. Since f is a Lie n-centralizer on $\mathscr{U} \propto \mathscr{M}$, it follows from Theorem 6 that $f_{\mathscr{U}}$ is a Lie n-centralizer on $\mathscr{U}$. According to the assumption (1), there exists an R-linear map $\tau_{\mathscr{U}}\colon \mathscr{U} \longrightarrow Z(\mathscr{U})$ such that $f_{\mathscr{U}} - \tau_{\mathscr{U}}$ is a centralizer and $\tau_{\mathscr{U}}$ vanishes on all $(n-1)$-th commutators of $\mathscr{U}$. By Theorem 7, it is sufficient to show that $\tau_{\mathscr{U}}$ satisfies $[\tau_{\mathscr{U}}(\alpha u\alpha), m] = 0 = [\tau_{\mathscr{U}}(\beta u\beta), m]$ for all $u \in \mathscr{U}, m \in \mathscr{M}$. Using Lemma 4, if the assumption (3)(i) or (3)(ii) holds, then we have $[Z(\mathscr{U}), \mathscr{M}] = 0$, which implies $[\tau_{\mathscr{U}}(\alpha u\alpha), m] = 0 = [\tau_{\mathscr{U}}(\beta u\beta), m]$ for all $u \in \mathscr{U}, m \in \mathscr{M}$. □

Applying Theorem 7 to triangular algebras, we can obtain the following result.

Corollary 9. *Let f be a Lie n-centralizer on a triangular algebra $Tri(A, M, B)$, then f has the form*

$$f(((a, b), m)) = (f_{A\oplus B}((a, b)), f_M((a, b)) + h_M(m)), \tag{31}$$

where $(a,b) \in A\oplus B, m \in M$, *and* f *is proper if and only if there exists a linear map* $\tau_{A\oplus B}$: $A\oplus B \longrightarrow Z(A\oplus B)$, *satisfying the following conditions:*

(1) $f_{A\oplus B} - \tau_{A\oplus B}$ *is a centralizer on* $A\oplus B$;

(2) $[\tau_{A\oplus B}((a,b)), m] = 0$ *for all* $(a,b) \in A\oplus B$ *and* $m \in M$.

Proof. In view of Theorem 6, we have

$$h_{A\oplus B}(m) = h_{A\oplus B}(p_n(\alpha, m, \beta, \ldots, \beta)) = 0, \tag{32}$$

for all $m \in M$. That is,

$$f(((a,b),m)) = (f_{A\oplus B}((a,b)), f_M((a,b)) + h_M(m)), \tag{33}$$

where $(a,b) \in A\oplus B, m \in M$. According to Theorem 7, the remaining part is true. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed to the study conception and design.

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