Hindawi
Journal of Mathematics
Volume 2023, Article ID 3087790, 3 pages
https://doi.org/10.1155/2023/3087790

Research Article

The Relationship between E-Semigroups and R-Semigroups

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Received 6 July 2022; Revised 10 November 2022; Accepted 6 April 2023; Published 17 April 2023

Academic Editor: Xuanlong Ma

A semigroup is called an E-semigroup (R-semigroup) if the set of all idempotents (the set of all regular elements) forms a subsemigroup. In this paper, we introduce the concept of V-semigroups and establish the relationship between the three classes of semigroups.

1. Introduction and Preliminaries

Let S be a semigroup. An element $e \in S$ is called an idempotent if $e^2 = e$. An element $a \in S$ is called regular if there exists $x \in S$ such that $a =$ axa. Denote by $E(S)$ and $\mathrm{Reg}(S)$ the set of all idempotents and the set of all regular elements in S, respectively. We call S an E-semigroup if $E(S)$ forms a subsemigroup and a R-semigroup if $\mathrm{Reg}(S)$ forms a subsemigroup. S is called regular if $\mathrm{Reg}(S) = S$ and called orthodox if S is a regular E-semigroup. More details on regular semigroups and orthodox semigroups can be seen in [1]. For $a \in S$ and $e, f \in E(S)$, denote

$$\begin{aligned} V(a) &= \{x \in S \mid a = axa, x = xax\}, \\ W(a) &= \{x \in S \mid x = xax\}, \\ S(e, f) &= \{g \in V(ef) \cap E(S) \mid ge = fg = g\}. \end{aligned} \tag{1}$$

The elements of $V(a)$ (resp., $W(a)$) are called inverse elements (resp., weak inverse elements) of a. $S(e, f)$ is called the sandwich set of e and f. We now list some results about E-semigroups and R-semigroups in the following results:

Result 1 (Result 2 [2]). Let S be a semigroup and $E(S) \neq \varnothing$. Then, the following statements are equivalent.

(1) S is a R-semigroup
(2) $\langle E(S) \rangle$ is a regular subsemigroup of S
(3) $(\forall e, f \in E(S))\ ef \in \mathrm{Reg}(S)$.

Result 2 (Theorem 3.1 [3]). Let S be a semigroup and $E(S) \neq \varnothing$. Then, the following statements are equivalent.

(1) S is an E-semigroup
(2) $(\forall a, b \in S)\ V(b)V(a) \subseteq V(ab)$
(3) $(\forall e, f \in E(S))\ ef \in S(e, f)$
(4) $(\forall a, b \in S)\ W(b)W(a) \subseteq W(ab)$.

Result 3 (Proposition 3.4 [3]). Let S be an E-semigroup. Then,

(1) $(\forall e \in E(S))\ V(e) \subseteq W(e) \subseteq E(S)$
(2) $(\forall a, b \in S)\ V(a) \cap V(b) \neq \varnothing \Longrightarrow V(a) = V(b)$.

A relationship between R-semigroups and E-semigroups established by congruences can be seen in [4]. From Results 1 and 2, we know that an E-semigroup is a R-semigroup. However, the converse is not true in general (for example, a regular (not orthodox) semigroup is a R-semigroup, but it is not an E-semigroup). In this note, we introduce the concept of V-semigroups and give the conclusion that a semigroup S is an E-semigroup if and only if S is a R-semigroup and a V-semigroup.

2. Main Results

Let S be a semigroup. For $a \in S$ and $A \subseteq S$, we denote

$$V(A) = \bigcup_{a \in A} V(a),$$
$$V^2(a) = V(V(a)). \tag{2}$$

It is clear that $V(e) \subseteq V(e)V(e) \cap V^2(e)$ for any $e \in E(S)$.

Definition 1. Let S be a semigroup and $E(S) \neq \varnothing$. S is called a V-semigroup if $V(e)$ is a subsemigroup for all $e \in E(S)$, i.e., $V(e)V(e) = V(e)$.

Proposition 1. *Let S be a V-semigroup. Then, $V(e) \subseteq E(S)$ for any $e \in E(S)$.*

Proof. Let $x \in V(e)$. Then, $x^2 \in V(e)V(e) = V(e)$. Thus,

$$x = \text{xex} = x\left(\text{ex}^2 e\right)x = (\text{xex})(\text{xex}) = x^2. \tag{3}$$

Hence, $x \in E(S)$. □

Theorem 1. *Let S be a semigroup and $E(S) \neq \varnothing$. Then, S is a V-semigroup if and only if $V^2(e) = V(e)$ for all $e \in E(S)$.*

Proof. ($\Longrightarrow$) Let $e \in E(S)$ and $a \in V^2(e)$. Then, there exists $b \in V(e)$ such that $a \in V(b)$. It follows from Proposition 1 that $b \in E(S)$. Also, $a \in E(S)$. Thus, $\text{ae}, \text{ea} \in V(b)V(b) = V(b)$. Moreover, $\text{aea}, \text{eae} \in V(b)V(b) = V(b)$. Hence,

$$\text{aea} = (\text{aba})e(\text{aba}) = a(\text{baeab})a = \text{aba} = a,$$
$$\text{eae} = (\text{ebe})a(\text{ebe}) = e(\text{beaeb})e = \text{ebe} = e. \tag{4}$$

Therefore, $a \in V(e)$. From $V(e) \subseteq V^2(e)$, we obtain the result.

($\Longleftarrow$) Let $e \in E(S)$ and $x, y \in V(e)$. Then, $e \in V(\text{ex})$ and $ex \in E(S)$. Thus, $y \in V^2(\text{ex}) = V(\text{ex})$. It follows that

$$(\text{xy})e(\text{xy}) = x(\text{yexy}) = \text{xy},$$
$$e(\text{xy})e = \text{exy}(\text{exe}) = (\text{exyex})e = \text{exe} = e. \tag{5}$$

Hence, $\text{xy} \in V(e)$. Therefore, we complete this proof. □

Lemma 1. *Let S be a semigroup and $E(S) \neq \varnothing$. Then, the following two statements are equivalent.*

(1) $V(e) \subseteq E(S)$ for any $e \in E(S)$

(2) $W(e) \subseteq E(S)$ for any $e \in E(S)$

Proof. (1) $\Longrightarrow$ (2) Let $x \in W(e)$ for some $e \in E(S)$. Then, $x = \text{xex}$. It follows that

$$x(\text{exe})x = (\text{xex})ex = \text{xex} = x,$$
$$(\text{exe})x(\text{exe}) = e(\text{xex})\text{exe} = e(\text{xex})e = \text{exe}, \tag{6}$$
$$(\text{exe})(\text{exe}) = e(\text{xex})e = \text{exe}.$$

Thus, $x \in V(\text{exe})$ and $\text{exe} \in E(S)$. By condition (1), we have $x \in E(S)$.

(2) $\Longrightarrow$ (1) It follows from $V(e) \subseteq W(e)$ for any $e \in E(S)$. □

Theorem 2. *Let S be a semigroup and $E(S) \neq \varnothing$. Then, S is a V-semigroup if and only if the following two conditions hold.*

(1) $(\forall e \in E(S))V(e) \subseteq E(S)$ or (1′) $(\forall e \in E(S))W(e) \subseteq E(S)$

(2) $(\forall e, f \in E(S))V(e) \cap V(f) \neq \varnothing \Longrightarrow V(e) = V(f)$.

Proof. ($\Longrightarrow$) It follows from Proposition 1 that condition (1) holds. Next, we prove condition (2). Let $x \in V(e) \cap V(f)$. Then, we have $y \in V^2(x) \subseteq V^3(f)$ for any $y \in V(e)$. From Theorem 1, we have $V^2(f) = V(f)$. Thus, $y \in V^3(f) = V^2(f) = V(f)$. Hence, $V(e) \subseteq V(f)$. By symmetry, $V(f) \subseteq V(e)$.

($\Longleftarrow$) Let $a \in V^2(e)$. Then, there exists $x \in S$ such that $a \in V(x), e \in V(x)$. Also, $e \in V(e)$. Thus, $V(e) \cap V(x) \neq \varnothing$ and $x \in V(e) \subseteq E(S)$ by condition (1). By condition (2), we have $V(x) = V(e)$. Hence, $a \in V(e)$. Therefore, $V^2(e) = V(e)$ from $V(e) \subseteq V^2(e)$, and so, S is a V-semigroup from Theorem 1.

From Result 3 and Theorem 2, we obtain the following conclusion. □

Proposition 2. *Let S be an E-semigroup. Then, S is a V-semigroup.*

In general, the converse is not true. See the following.

Example 1

(1) Let $S = \{a, b, c, d\}$ with the following operation:

·	a	b	c	d
a	a	a	a	a
b	a	a	b	a
c	a	a	c	a
d	a	b	b	d

We can see that $E(S) = Reg(S) = \{a, c, d\}$. Moreover, $V(a) = \{a\}, V(c) = \{c\}$ and $V(d) = \{d\}$. Thus, S is a V-semigroup. However, $E(S)$ is not a subsemigroup and so S is not a E-semigroup. Indeed, $dc = b \notin E(S)$. Also, S is not a R-semigroup.

(2) Let $G = \{e\}$ be a group, $I = \{1, 2\}$ and

$$P = \left(p_{\lambda j}\right) = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix}. \tag{7}$$

Let $S = (I \times G \times I) \cup \{0\}$, and define an operation on S by

$$(i, e, \lambda)(j, e, \mu) = \begin{cases} (i, e, \mu), & \text{if } p_{\lambda j} = e, \\ 0, & \text{if } p_{\lambda j} = 0, \end{cases}$$
$$(i, e, \lambda)0 = 0(i, e, \lambda) = 00 = 0. \tag{8}$$

Then, S is a regular semigroup and so is a R-semigroup. It is easy to show that $E(S) = \{0, (1, e, 1), (2, e, 1), (2, e, 2)\}$. However, $(1, e, 1)(2, e, 2) = (1, e, 2) \notin E(S)$. Hence, S is not an E-semigroup. Moreover, we can verify that $V((2, e, 1)) = \{(1, e, 1), (1, e, 2), (2, e, 1), (2, e, 2)\}$. Also, $(2, e, 2)(1, e, 1) = 0 \notin V((2, e, 1))$. Thus, S is not a V-semigroup.

Next, we give the relationship between the three classes of semigroups.

Lemma 2. *Let S be a R-semigroup and $E(S) \neq \emptyset$. Then, $S(e, f) \neq \emptyset$ for any $e, f \in E(S)$.*

Proof. Let $e, f \in E(S)$. Since S is a R-semigroup, $V(ef) \neq \emptyset$. Let $x \in V(ef)$ and $g = \mathrm{fxe}$. Then,

$$\begin{aligned}(\mathrm{ef})g(\mathrm{ef}) &= \mathrm{ef}^2\mathrm{xe}^2 f = \mathrm{efxef} = \mathrm{ef},\\ g(\mathrm{ef})g &= \mathrm{fxe}^2 f^2\mathrm{xe} = f(\mathrm{xefx})e = \mathrm{fxe} = g.\end{aligned} \tag{9}$$

Thus, $g \in V(\mathrm{ef})$. Moreover,

$$g^2 = f(\mathrm{xefx})e = \mathrm{fxe} = g, \tag{10}$$

and so, $g \in E(S)$. Finally, it is clear that $\mathrm{ge} = g = fg$. Therefore, $g \in S(e, f)$. □

Theorem 3. *Let S be a semigroup and $E(S) \neq \emptyset$. Then, S is an E-semigroup if and only if the following two conditions hold.*

(1) S is a R-semigroup

(2) $(\forall e \in E(S))$ $V(e) \subseteq E(S)$ or $(2')$ $(\forall e \in E(S))$ $W(e) \subseteq E(S)$.

Proof. $(\Longrightarrow)$ It follows from Results 1, 2, and 3.

$(\Longleftarrow)$ Let $e, f \in E(S)$. Since S is a R-semigroup, from Lemma 2, we have that there exists $g \in E(S)$ such that $g \in V(\mathrm{ef})$. By hypothesis, $\mathrm{ef} \in V(g) \subseteq E(S)$. It follows that S is an E-semigroup.

Finally, from Theorem 2, Proposition 2, and Theorem 3, we establish the relationship between E-semigroups, R-semigroups, and V-semigroups by the following result. □

Theorem 4. *Let S be a semigroup and $E(S) \neq \emptyset$. Then, S is an E-semigroup if and only if S is a R-semigroup and a V-semigroup.*

Remark 1. Let S be a regular semigroup. Then, S is a R-semigroup. From Theorem 4, we know that S is orthodox if and only if S is a V-semigroup (see Theorem 3.2 [5]).

Data Availability

The data used to support the findings of this study are available upon request to the corresponding author.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 11701504), the Guangdong Basic and Applied Basic Research Foundation (No. 2022A1515011081), the Characteristic Innovation Project of Department of Education of Guangdong Province (No. 2020KTSCX159), the Science and Technology Innovation Guidance Project of Zhaoqing City (No. 2021040315026), the Innovative Research Team Project of Zhaoqing University, and the Scientific Research Ability Enhancement Program for Excellent Young Teachers of Zhaoqing University.

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