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Research Article

Analytical and Approximate Solutions of the Nonlinear Gas Dynamic Equation Using a Hybrid Approach

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This paper presents the study of a numerical scheme for the analytical solution of nonlinear gas dynamic equation. We use the idea of Laplace–Carson transform and associate it with the homotopy perturbation method (HPM) for obtaining the series solution of the equation. We show that this hybrid approach is excellent in agreement and converges to the exact solution very smoothly. Further, HPM combined with He's polynomial is utilized to minimize the numerical simulations in nonlinear conditions that make it easy for the implementation of Laplace–Carson transform. We also exhibit a few graphical solutions to indicate that this approach is extremely reliable and convenient for linear and nonlinear challenges.

1. Introduction

The gas dynamic equation is mathematically modeled by various physical laws such as energy, mass and momentum conservation. Gas is a collection of numerous elements in continuous chaotic motion such as molecules, atoms, ions, etc. The nonlinear gas dynamics equation is used in shock waves, centered rarified waves, contact flows, and connection discontinuities. The study of gas motion and its impact on structures using the principles of fluid dynamics and fluid mechanics is known as "gas dynamic," and it belongs to the discipline of fluid dynamics [1, 2]. Numerous researchers has studied the gas dynamic equation with different analysis [3, 4]. Srivastava and Saad [5] studied the theory of gas dynamic equation and extended it with different models. Various approaches have been introduced to solve the gas dynamics problems such as fractional reduced transform method [6], Elzaki transform homotopy perturbation approach [7], q-homotopy analysis [8], Adomian decomposition strategy [9], variational iteration method [10, 11], fractional homotopy analysis transform approach [12], homotopy perturbation method using Laplace transform [13], Homotopy analysis transform method [14] and natural decomposition method [15].

He [16–18] demonstrated the strategy HPM for the solution of nonlinear problems arising in complex models and showed that this approach has an excellent performance in obtaining the series solutions. Some scientists [19, 20] modified this study and coupled it with Laplace transform to achieve the series solution of nonlinear differential problems. Aggarwal and Kumar [21] applied Laplace–Carson to Volterra integro-differential problem of first kind. After that, Kumar and Qureshi [22] received the results of initial value problems with the Caputo derivative in the shape of series and showed the authenticity of this scheme. Thange and Gade [23] studied a few definitions of Laplace–Carson with fractional order and used the convolution theorem which was very complicated to obtain the iterations.

In this article, we study a novel scheme Laplace–Carson homotopy perturbation method ($\mathscr{L}_c$-PTM) which is constructed on the basis of Laplace–Carson and HPM. We point out that the present scheme is very connivent to use and reveals the results in the shape of a series. This approach is an independent convolution theorem that may face complications during the calculation of iterations. This article is designed as In Section 2, we present the definition of Laplace–Carson transform with basic propositions. In Section 3, we study the fundamental concept of HPM which is used to split the nonlinear elements. In Section 4, we present the numerical applications to show the ability of $\mathscr{L}_c$-PTM, and finally, we discuss the obtained results and conclusion in Sections 5 and 6 respectively.

2. Laplace and Laplace–Carson Transform

Definition 1. Consider $f(t)$ be a function with $t \geq 0$, so

$$\mathscr{L}\{f(t)\} = F(s) = \theta \int_0^{\infty} f(t)e^{-st}\mathrm{d}t, \tag{1}$$

is said to Laplace transform and s is transform function of θ.

Definition 2. Aggarwal and Kumar [21] studied a theory such that

$$\mathscr{L}_c\{g(t)\} = R(\theta) = \theta \int_0^{\infty} g(t)e^{-\theta t}\mathrm{d}t, k_1 \leq \theta \leq k_2. \tag{2}$$

Here k_1 and k_2 are arbitrary constants and $\mathscr{L}_c$ is termed as Laplace–Carson transform. Now, if $R(\theta)$ is the Laplace–Carson transform of a function $g(t)$ then $g(t)$ is the inverse of $R(\theta)$ so that,

$$\mathscr{L}_c^{-1}\{R(\theta)\} = g(t), \quad \mathscr{L}_c^{-1} \text{ is said to be inverse Laplace} - \text{Carson transform.} \tag{3}$$

Definition 3. If $g(t) = t^m$, then Laplace–Carson transform is utilized as

$$\mathscr{L}_c\{g(t)\} = R(\theta) = \frac{m!}{\theta^m}. \tag{4}$$

Properties 1. If $\mathscr{L}_c\{g(t)\} = R(\theta)$, then it has the following differential properties [21, 23].

(a) $\mathscr{L}_c\{g'(t)\} = \theta R(\theta) - \theta G(0)$,

(b) $\mathscr{L}_c\{g''(t)\} = \theta^2 R(\theta) - \theta^2 G(0) - \theta G'(0)$,

(c) $\mathscr{L}_c\{g^m(t)\} = \theta^m R(\theta) - \theta^m G(0) - \theta^{m-1} G'(0) - \cdots - \theta G^{m-1}(0)$.

3. Fundamental Concept of HPM

This segment presents the concept of HPM with the consideration of a nonlinear functional equation [24, 25]. Consider

$$T(\vartheta) - g(h) = 0, \quad h \in \Omega. \tag{5}$$

With conditions

$$S\left(\vartheta, \frac{\partial \vartheta}{\partial n}\right) = 0, \quad h \in \Gamma. \tag{6}$$

Here T and S are identified as general functional and boundary operator respectively, $g(h)$ is source term with Γ as a interval of the domain Ω. We can now split T such that T_1 is said to be a linear and T_2 be a nonlinear operator. Thus, we can write equation (5) as

$$T_1(\vartheta) + T_2(\vartheta) - g(h) = 0. \tag{7}$$

Consider $\vartheta(h, \theta)$: $\Omega \times [0, 1] \longrightarrow \mathbb{H}$ such that it is suitable for

$$H(\vartheta, \theta) = (1 - \theta)[T_1(\vartheta) - T_1(\vartheta_0)] + \theta[T_1(\vartheta) - T_2(\vartheta) - g(h)], \tag{8}$$

or

$$H(\vartheta, \theta) = T_1(\vartheta) - T_1(\vartheta_0) + qL(\vartheta_0) + \theta[T_2(\vartheta) - g(h)] = 0. \tag{9}$$

Here $\theta \in [0, 1]$ is homotopy element and ϑ_0 is an initial approximation of equation (5), which is appropriate for the boundary conditions. The study of HPM declares that θ is assumed as a minimal variable and the result of equation (5) cab be expressed in the shape of θ.

$$\vartheta = \vartheta_0 + \theta\vartheta_1 + \theta^2\vartheta_2 + \theta^3\vartheta_3 + \cdots = \sum_{i=0}^{\infty} \theta^i \vartheta_i. \tag{10}$$

Consider $\theta = 1$, we get particular of equation (10) as

$$\vartheta = \lim_{\theta \longrightarrow 1} \vartheta = \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 + \cdots = \sum_{i=0}^{\infty} \vartheta_i. \tag{11}$$

The nonlinear terms are obtained as

$$T_2\vartheta(x, t) = \sum_{n=0}^{\infty} \theta^n H_n(\vartheta), \tag{12}$$

where $H_n(\vartheta)$ is defined as

$$H_n(\vartheta_0 + \vartheta_1 + \cdots + \vartheta_n) = \frac{1}{n!} \frac{\partial^n}{\partial \theta^n} \left(T_2 \left(\sum_{i=0}^{\infty} \theta^i \vartheta_i \right) \right)_{\theta=0}. \quad n = 0, 1, 2, \cdots. \tag{13}$$

This result in equation (12) generally converges as the rate of convergence depends on the nonlinear operator T_2 .

4. Numerical Applications

In this segment, we apply the scheme of $\mathscr{L}_c$-PTM to obtain the analytical results of nonlinear gas dynamic equation. We express that this approach generates the series solution only after iterations with excellent accuracy.

4.1. Example 1. Consider the homogenous and nonlinear gas dynamic equation

$$\frac{\partial \vartheta}{\partial t} + \vartheta \frac{\partial \vartheta}{\partial x} - \vartheta(1-\vartheta) = 0. \tag{14}$$

With initial condition

$$\vartheta(x,0) = e^{-x}. \tag{15}$$

Using the Laplace–Carson transform to equation (14), we get

$$\mathscr{L}_c\left[\frac{\partial \vartheta}{\partial t} + \vartheta \frac{\partial \vartheta}{\partial x} - \vartheta(1-\vartheta)\right] = 0,$$
$$\mathscr{L}_c\left[\frac{\partial \vartheta}{\partial t}\right] = -\mathscr{L}_c\left[\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta(1-\vartheta)\right] = 0. \tag{16}$$

Employing the definition of Laplace–Carson transform, we get

$$\theta\vartheta(x,\theta) - \theta\vartheta(x,0) = -\mathscr{L}_c\left[\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta(1-\vartheta)\right]. \tag{17}$$

Which may be solved further as,

$$\vartheta(x,\theta) = \vartheta(x,0) - \frac{1}{\theta}\mathscr{L}_c\left\{\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta + \vartheta^2\right)\right\}. \tag{18}$$

Applying inverse Laplace–Carson transform, we get

$$\vartheta(x,t) = \vartheta(x,0) - \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta + \vartheta^2\right)\right\}\right]. \tag{19}$$

Utilizing HPM on equation (19), we get

$$\sum_{n=0}^{\infty} p^n \vartheta_n(x,t) = \vartheta(x,0) - p\mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\sum_{n=0}^{\infty} p^n \vartheta_n(x,t) \frac{\partial}{\partial x}\sum_{n=0}^{\infty} p^n \vartheta_n(x,t) - \sum_{n=0}^{\infty} p^n \vartheta_n(x,t) + \sum_{n=0}^{\infty} p^n \vartheta_n^2(x,t)\right\}\right]. \tag{20}$$

On comparing, the following iterations can be obtained,

$$\begin{aligned}
p^0&: \vartheta_0(x,t) = e^{-x},\\
p^1&: \vartheta_1(x,t) = \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta_0\frac{\partial \vartheta_0}{\partial x} - \vartheta_0 + \vartheta_0^2\right\}\right] = e^{-x}\frac{t^2}{2!},\\
p^2&: \vartheta_2(x,t) = \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta_0\frac{\partial \vartheta_1}{\partial x} + \vartheta_1\frac{\partial \vartheta_0}{\partial x} - \vartheta_1 + 2\vartheta_0\vartheta_1\right\}\right] = e^{-x}\frac{t^3}{3!},\\
p^3&: \vartheta_3(x,t) = \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta_0\frac{\partial \vartheta_2}{\partial x} + \vartheta_1\frac{\partial \vartheta_1}{\partial x} + \vartheta_2\frac{\partial \vartheta_0}{\partial x} - \vartheta_2 + \vartheta_1^2 + 2\vartheta_0\vartheta_2\right\}\right] = e^{-x}\frac{t^4}{4!},\\
&\vdots
\end{aligned} \tag{21}$$

Hence the solution can be expressed as

$$\begin{aligned}
\vartheta(x,t) &= \vartheta_0(x,t) + \vartheta_1(x,t) + \vartheta_2(x,t) + \vartheta_3(x,t) + \cdots,\\
\vartheta(x,t) &= e^{-x} + e^{-x}\frac{t^2}{2!} + e^{-x}\frac{t^3}{3!} + e^{-x}\frac{t^4}{4!} + \cdots,\\
\vartheta(x,t) &= e^{t-x}.
\end{aligned} \tag{22}$$

4.2. Example 2. Consider the non-homogenous and non-linear gas dynamic equation

$$\frac{\partial \vartheta}{\partial t} + \vartheta \frac{\partial \vartheta}{\partial x} - \vartheta (1 - \vartheta) = -e^{t-x}. \tag{23}$$

With initial condition

$$\vartheta (x, 0) = 1 - e^{-x}. \tag{24}$$

Using the Laplace–Carson transform to equation (23), we get

$$\begin{aligned} \mathscr{L}_c\left[\frac{\partial \vartheta}{\partial t} + \vartheta \frac{\partial \vartheta}{\partial x} - \vartheta (1 - \vartheta)\right] &= -\mathscr{L}_c\left[e^{t-x}\right], \\ \mathscr{L}_c\left[\frac{\partial \vartheta}{\partial t}\right] &= -\mathscr{L}_c\left[e^{t-x}\right] - \mathscr{L}_c\left[\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta (1 - \vartheta)\right]. \end{aligned} \tag{25}$$

Employing the definition of Laplace–Carson transform, we get

$$\theta \vartheta (x, 0) - \theta \vartheta (x, 0) = -\frac{e^{-x}}{\theta - 1} - \mathscr{L}_c\left[\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta (1 - \vartheta)\right]. \tag{26}$$

Which may be solved further as,

$$\vartheta (x, \theta) = \vartheta (x, 0) - \frac{e^{-x}}{\theta - 1} - \frac{1}{\theta}\mathscr{L}_c\left\{\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta + \vartheta^2\right\}. \tag{27}$$

Applying inverse Laplace–Carson transform, we get

$$\begin{aligned} \vartheta (x, t) &= \vartheta (x, 0) - e^{-x}\mathscr{L}_c^{-1}\left[\frac{1}{\theta - 1}\right] - \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta + \vartheta^2\right\}\right], \\ \vartheta (x, t) &= \vartheta (x, 0) - e^{t-x} + e^{-x} - \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta + \vartheta^2\right\}\right], \\ \vartheta (x, t) &= 1 - e^{-x} - e^{t-x} + e^{-x} - \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta + \vartheta^2\right\}\right], \\ \vartheta (x, t) &= 1 - e^{t-x} - \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta \frac{\partial \vartheta}{\partial x} - \vartheta + \vartheta^2\right\}\right]. \end{aligned} \tag{28}$$

Utilizing HPM on equation (28), we get

$$\sum_{n=0}^{\infty} p^n \vartheta_n (x, t) = 1 - e^{t-x} - p\mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\sum_{n=0}^{\infty} p^n \vartheta_n (x, t) \frac{\partial}{\partial x} \sum_{n=0}^{\infty} p^n \vartheta_n (x, t) - \sum_{n=0}^{\infty} p^n \vartheta_n (x, t) + \sum_{n=0}^{\infty} p^n \vartheta_n^2 (x, t)\right\}\right]. \tag{29}$$

On comparing, the following iterations can be obtained,

$$\begin{aligned} &p^0\colon\ \vartheta_0 (x, t) = 1 - e^{t-x}, \\ &p^1\colon\ \vartheta_1 (x, t) = \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta_0 \frac{\partial \vartheta_0}{\partial x} - \vartheta_0 + \vartheta_0^2\right\}\right] = 0, \\ &p^2\colon\ \vartheta_2 (x, t) = \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta_0 \frac{\partial \vartheta_1}{\partial x} + \vartheta_1 \frac{\partial \vartheta_0}{\partial x} - \vartheta_1 + 2\vartheta_0 \vartheta_1\right\}\right] = 0, \\ &p^3\colon\ \vartheta_3 (x, t) = \mathscr{L}_c^{-1}\left[\frac{1}{\theta}\mathscr{L}_c\left\{\vartheta_0 \frac{\partial \vartheta_2}{\partial x} + \vartheta_1 \frac{\partial \vartheta_1}{\partial x} + \vartheta_2 \frac{\partial \vartheta_0}{\partial x} - \vartheta_2 + \vartheta_1^2 + 2\vartheta_0 \vartheta_2\right\}\right] = 0. \\ &\quad\vdots \end{aligned} \tag{30}$$

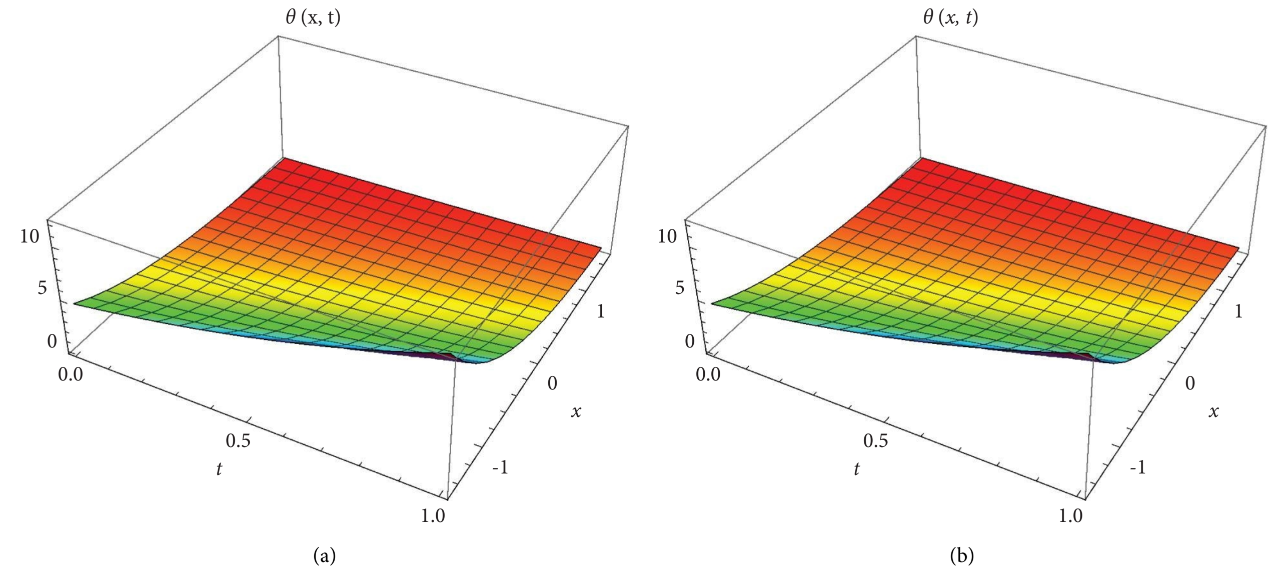

FIGURE 1: The surfaces solution of gas dynamic equation. (a) The approximate surface solution of $\vartheta(x,t)$. (b) The exact surface solution of $\vartheta(x,t)$.

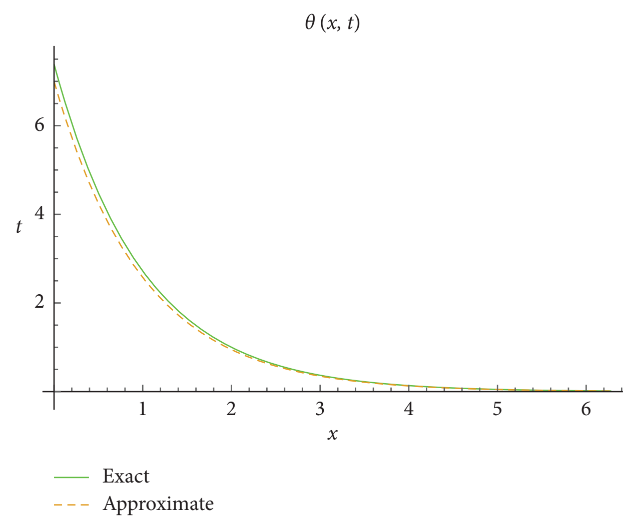

FIGURE 2: 2D plot for $\vartheta(x,t)$ with various parameter of t.

Hence the solution can be expressed as

$$\begin{aligned}\vartheta(x,t) &= \vartheta_0(x,t)+\vartheta_1(x,t)+\vartheta_2(x,t)+\vartheta_3(x,t)+\cdots,\\ \vartheta(x,t) &= 1-e^{t-x}+0+0+\cdots,\\ \vartheta(x,t) &= 1-e^{t-x}.\end{aligned} \tag{31}$$

5. Results and Discussion

In this portion, we demonstrate the graphical representation of nonlinear gas dynamic equation. Figure 1(a) represents the the approximate solution obtained by $\mathscr{L}_c$-PTM and Figure 1(b) represents the exact solution of the nonlinear gas dynamic equation. In Figure 1, we compare these graphical illustrations at $-1.5 \le x \le 1.5$ and $0 \le t \le 1$

Table 1: Absolute error among the approximate and exact solution at $t = 0.5$ and 1.

x	Approximate values at $t = 0.5$	Approximate values at $t = 1$	Exact values at $t = 1$	Absolute error at $t = 1$
0.1	1.49157	2.4506	2.4596	0.0464
0.2	1.34963	2.1274	2.22554	0.00814
0.3	1.22119	2.00638	2.01375	0.00737
0.4	1.10498	1.81545	1.82212	0.00667
0.5	0.999828	1.64269	1.64872	0.00603
0.6	0.904682	1.48636	1.49182	0.00546
0.7	0.81859	1.34492	1.34986	0.00494
0.8	0.740691	1.21693	1.2214	0.00447
0.9	0.670205	1.10113	1.10517	0.00404
1.0	0.606426	0.99634	1	0.00366

and observe that both surface solutions are in full agreement. Figure 2 represents the graphical error between the solutions obtained by $\mathscr{L}_c$-PTM and the exact solutions at $0 \leq x \leq \pi$. Table 1 presents the analysis of the absolute error at different times t and shows that the obtained values become closer to the exact solution with the increase of time. Finally, the figures and table demonstrate that our approach has high authenticity of performance and provides fast convergence results towards the exact solution.

6. Conclusion

In this article, we have successfully applied a new scheme $\mathscr{L}_c$ -PTM to determine the approximate results of gas dynamic equation. We obtained these results in the shape of series instead of discretization, linearization, or assumptions. We observe that when HPM is used with Laplace–Carson transform, we can obtain a rapid convergent series solution with less computation. We compute these iterations with the help of Mathematica Software 11.0.1. We also compare the approximate and the exact solution results and provide the absolute error to examine the efficiency of our suggested approach. 2D plot and 3D surface solutions show that we have strong agreement with the results of gas dynamic equation. Therefore, we can say that $\mathscr{L}_c$-PTM is more efficient and appropriate than other schemes. This approach is also applicable to other nonlinear problems such as fractional partial differential equations and can be expanded in a variety of scientific and engineering applications in the future.

Data Availability

This article contains all the data.

Conflicts of Interest

This article have no conflict of interest.

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