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Research Article

On Generalizations of Projective QTAG-Modules

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In this manuscript, we define the class of ω_1-weakly α-projective QTAG-modules for the infinite ordinal α and provide its systematic study for the finite ordinal. Furthermore, we generalize this class to $(\omega.2+n)$-projective modules and obtain some characterizations. We also study the ω-totally weak $(\omega.2+n)$-projective modules under the formation of ω_1-bijections.

1. Introduction and Preliminaries

Several group concepts such as purity, projectivity, injectivity, and height have been extended to modules. However, in order to derive outcomes that do not hold for modules, certain restrictions have been imposed on the modules themselves or on the rings underlying them. By considering the QTAG-module structure with these conditions in mind, it becomes possible to establish several fundamental properties of groups that are not universally valid. The results presented in this study draw inspiration from the discoveries made in [1]. One can go through the papers [2–4] for the better insight of projective groups and related results, which are used frequently here as the generalization for the case of QTAG-modules.

Singh initiated the investigation into the structure of QTAG-modules in [5]. Since then, several researchers, including Khan, Mehdi, Abbasi, and others, have extended various group concepts to QTAG-modules [6, 7], introducing new notions and structures that draw inspiration from the group framework and yielding intriguing findings. Despite these developments, there are still several concepts that have yet to be generalized for modules. For the recent developments in the chain of generalizations for QTAG-modules, one can go through these interesting articles [8, 9].

The definitions used in this manuscript have been previously introduced in the earlier work of one of the co-authors and have been appropriately referenced and restated here. In what follows, all notations and notions are standard and will be in agreement with those used in [10].

"A module M over an associative ring R with unity is a QTAG-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules [11]. All the rings R considered here are associative with unity, and modules M are unital QTAG-modules. An element $x \in M$ is uniform if xR is a nonzero uniform (hence uniserial) module, and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element, $x \in M, e(x) = d(xR)$ and $H_M(x) = \sup\{d\, yR/xR \mid y \in M, x \in yR \text{ and } y \text{ uniform}\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k, and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \cap_{k=0}^{\infty} H_k(M)$ [7], and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. A QTAG-module M is said to be separable if $M^1 = 0$. Let M be a module, then the sum of all simple submodules of M is called the socle of M and is denoted by $\mathrm{Soc}(M)$. If M, M' are QTAG-modules, then a homomorphism $f\colon M \longrightarrow M'$ is an isometry if it is 1-1, onto and $H_{M'}(f(x)) = H_M(x)$, for all $x \in M$. A submodule N of a QTAG-module M is a nice submodule if every nonzero coset $a + N$ is proper with respect to N, *i.e.* For every nonzero $a + N$, there is an element $b \in N$ such that $H_M(a+b) = H_{M/N}(a+N)$."

"A family $\mathcal{N}$ of submodules of M is called a nice system in M if

(i) $0 \in \mathcal{N}$

(ii) If $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\Sigma_I N_i \in \mathcal{N}$

(iii) Given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated [6].

Every submodule in a nice system is a nice submodule. An h-reduced QTAG-module M is called totally projective if it has a nice system and direct sums and direct summands of totally projective modules are also totally projective. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A QTAG-module M is $(\omega+n)$-projective if there exists a submodule $N \subset H^n(M)$ such that M/N is a direct sum of uniserial modules or equivalently if and only if there is a direct sum of uniserial module K with a submodule $L \subseteq H^n(K)$ such that $M \cong K/L$. M is ω-projective if and only if it is a direct sum of uniserial modules. Also, two $(\omega+n)$-projective QTAG-modules M_1, M_2 are isometric if and only if there is a height preserving isomorphism between $H^n(M_1)$ and $H^n(M_2)$ [6]. For any QTAG-module, $M, g(M)$ denotes the smallest cardinal number λ such that M admits a generating set X of uniform elements of cardinality λ, *i.e.*, $|X| = \lambda$. A homomorphism $f\colon M \longrightarrow N$ is said to be ω_1-bijective if $g(\ker\ f), g(N/f(M)) < \omega_1$."

2. Main Results

Mehdi et al. [6] defined the $(\omega+n)$-projective module as the QTAG-module M with a submodule $N \subseteq H^n(M)$ such that M/N is a direct sum of uniserial modules (we will abbreviate it as DSUM). Recently, Sikander in [12] defined the ω_1-$(\omega+n)$-projective module as follows:

Definition 1. For $n < \omega$, the QTAG-module M is said to be ω_1-$(\omega+n)$-projective if there exists a countably generated (we will abbreviated it as CG) submodule N of M such that M/N is $(\omega+n)$-projective. If N is balanced, then M is balanceable ω_1-$(\omega+n)$-projective.

We extend this concept as follows:

Definition 2. For $n < \omega$, the QTAG-module N is said to be weakly ω_1-$(\omega+n)$-projective if there is a CG nice submodule $N \subseteq M$ such that $N \subseteq H_\omega(M)$ and $(M/N)/H_{\omega+n}(M/N)$ is $(\omega+n)$-projective. If N is balanced, the module M is called weakly balanced $\omega_1 - (\omega+n)-$ projective.

The $\omega_1 - (\omega+n)-$ projective modules are weakly $\omega_1 - (\omega+n)-$ projective as well as balanceable $\omega_1 - (\omega+n)-$ projective modules are weakly balanced $\omega_1 - (\omega+n)-$ projective. Moreover, for any module M, for which $M/H_{(\omega+n)}(M)$ is $(\omega+n)-$ projective, it is necessarily weakly $(\omega+n)-$ projective. Also, each weakly $\omega_1 - (\omega+n)-$ projective module has the property that $M/H_\omega(M)$ is $(\omega+n)-$ projective.

Moreover, if $H_{(\omega+n)}(M/N) = 0$ or even if $H_{(\omega+n)}(M)$ is CG, weakly $\omega_1 - (\omega+n)-$ projective modules are of necessity $\omega_1 - (\omega+n)-$ projective module.

Example 3. Let $R = \mathbb{Z}$ be the ring of integers and let $M = \oplus_{i=1}^{\infty} R$ be the direct sum of countably many copies of R. We define a submodule $N \subseteq M$ as follows: for each $i \in \mathbb{N}$, let e_i be the ith basis element of M and let $N_i = Re_i$ be the submodule of M generated by e_i. Then, we let $N = \oplus_{i=1}^{\infty} N_i$ be the direct sum of all the N_i.

We note that N is a CG submodule of M. We claim that M/N is weakly ω_1-$(\omega+n)$-projective for any $n \in \mathbb{N}$.

To see this, we note that M/N is isomorphic to R^ω, the direct product of countably many copies of R. Let K be any submodule of M/N and let $f\colon M \longrightarrow M/N$ be the quotient map. Then, $f^{-1}(K)$ is a submodule of M that contains N. Since N is a DSUM (each N_i is isomorphic to R which is uniserial), it follows that any finitely generated submodule of $f^{-1}(K)$ is also a DSUM. Therefore, by the definition of QTAG-modules, $f^{-1}(K)$ is a DSUM. In particular, it is ω-projective.

Now, let L be any submodule of M/N such that $H_{\omega+n}(M/N) \subseteq L \subseteq M/N$. Then, $f^{-1}(L)$ is a submodule of M that contains N and satisfies the following equation:

$$H_{\omega+n}\left(f^{-1}(M)\right) \subseteq f^{-1}(L) \subseteq f^{-1}\left(\frac{M}{N}\right). \tag{1}$$

Since $f^{-1}(M/N)$ is ω-projective, it follows that $f^{-1}(L)/H_{\omega+n}(f^{-1}(M))$ is also ω-projective. Therefore, by the definition of weakly $\omega_1 - (\omega+n)$-projective, M/N is weakly $\omega_1 - (\omega+n)$-projective.

Despite being weakly ω_1-$(\omega+n)$-projective, M cannot be classified as ω_1-$(\omega+n)$-projective since any countably generated submodule of M can be represented as a direct sum of countably many copies of R, which is not $(\omega+n)$-projective for any $n \in \mathbb{N}$. Thus, M serves as an instance of a QTAG-module that exhibits weak ω_1-$(\omega+n)$-projectivity but not ω_1-$(\omega+n)$-projectivity.

Definition 4. The QTAG-module M is weakly $\omega.2$-projective if it has a submodule N such that N and M/N both are the DSUM.

Clearly, $(\omega+n)$-projective modules are weakly $\omega.2$-projective and submodules of weakly $\omega.2$-projective modules are also weakly $\omega.2$-projective.

Now, we extend Definition 5 as follows:

Definition 5. The module M is ω_1-weakly $\omega.2$-projective if it has a CG submodule N such that M/N is a weakly $\omega.2$-projective module.

Clearly, $\omega_1 - (\omega+n)-$ projective modules are ω_1-weakly $(\omega.2)$-projective and the submodules of ω_1-weakly $(\omega.2)$-projective modules retain the same property.

Moreover, weakly $\omega.2$-projective modules are ω_1-weakly $(\omega.2)$-projective.

Example 6. Let $R = \mathbb{Z}$, and for fixed prime p, we consider the R-module $M = \mathbb{Z}p^{\infty}$. This is the DSUM of the height of some power p, i.e., $M = \oplus_{i=1}^{\infty} C_{p^i}$, where C_{p^i} denotes the uniserial submodule of height i.

Claim: M *is* ω_1-weakly $\omega.2$-projective.

Let $N = \oplus_{i=1}^{k} C_{p^{n_i}}$ be a finite direct summand of M, where $n_1 < n_2 < \cdots < n_k$ are integers. Then, N is clearly a DSUM since each $C_{p^{n_i}}$ is a uniserial module with a simple submodule $C_{p^{n_i-1}}$.

Moreover, M/N is also a DSU submodule since any such submodule is of the form $C_{p^{n_i}}/\langle p^{n_i - n_j}\rangle$ for some $i > \mathrm{j}$, and these form a complete set of representatives for the submodules of M/N. Thus, M/N is also a DSUM.

Since every finitely generated submodule of M is a direct summand of M, it follows that every homomorphic image of M is also a DSUM. Therefore, M is a QTAG-module.

Now, let $N = \oplus_{i=1}^{\infty} C_{p^i}$ be the submodule of M generated by all uniserial submodules of the height of some p-power order. Then, N is countably generated, and M/N is the trivial module, which is certainly weakly $\omega.2$-projective. Therefore, M is ω_1-weakly $\omega.2$-projective.

However, M is not $\omega_1 - (\omega + n)$-projective for any n since M contains infinitely many nonisomorphic indecomposable submodules (namely, the uniserial submodules C_{p^i} for $i \geq 1$) and hence cannot be a direct sum of a countable number of indecomposable modules. Therefore, the example shows that every $\omega_1 - (\omega + n)$-projective module is ω_1-weakly $(\omega.2)$-projective but not conversely.

In [13], a module M in DSUM is called ω-totally if each of its separable submodule is in DSUM. It is known that ω-totally DSUM are precisely the direct sums of a CG module and a DSUM ([13], Theorem 2.1). That is why, by Definition 1, ω-totally DSUM are precisely the $\omega_1 - \omega$-projective ones.

The class of ω_1-weakly $(\omega.2)$-projective modules is categorized in several ways in the result that follows.

Theorem 7. *Let M be a* QTAG*-module and $n > \omega$. The following assertions are equivalent:*

(i) M is ω_1-weakly $\omega.2$-projective

(ii) M is an elongation of a DSUM by an ω_1-$(\omega + n)$-projective module

(iii) M is an elongation of a module N whose every separable submodule is a DSUM by a $(\omega + n)$-projective module

(iv) M is the submodule of $N + K$, where N is CG and $H_{\omega.2}(K) = 0$ and K is the direct sum of CG modules

(v) M is an elongation of a weakly $\omega.2$-projective module.

Proof. (i)$\Longrightarrow$(ii). Let $M' = M/N$ be weakly $\omega.2$-projective for some CG submodule N of M. Let K be a submodule of M' such that K and M'/K are DSUM. Let T be a submodule of M such that $T/N = K$. Now, T has the property that every separable submodule of T is a DSUM and $M/T \simeq M/N/T/N = M'/K$ is a DSU submodule. By the previous discussion, T has a submodule P such that P is a DSUM and T/P is countably generated. Since $M/P/T/P \cong M/T$ is a DSUM, by the same arguments, every separable submodule of M/P is a DSUM. In other words, M/P is ω_1-ω-projective and M/P is ω_1-$(\omega + n)$-projective for any $n < \omega$.

(ii)$\Longrightarrow$(iii) Let N be a submodule of M such that N is a DSUM and $M' = M/N$ is ω_1-$(\omega + n)$-projective. Now, M' has a CG submodule K such that M'/K is $(\omega + n)$-projective. If we put $T/N = K$, then every separable submodule of T is a DSUM and $M/T \simeq M/N/T/N = M'/K$ is $(\omega + n)$-projective.

(iii)$\Longrightarrow$(iv) Let T be a submodule of M such that every separable submodule of T is a DSUM and M/T is $(\omega + n)$-projective. Let P be a module such that P is a DS of CG modules and $H_{\omega+n}(P) = T$. Since M/T is $(\omega + n)$-projective, the injection map from T to P extends to a homomorphism ψ: $M \longrightarrow P$. Let Q be a module such that $H_{\omega+n+1}(Q) = 0$ such that $M/T \subseteq Q$. Consider the homomorphism $\overline{\psi}$: $M \longrightarrow P \oplus Q$ such that $\overline{\psi}(x) = (\psi(x), x + T)$. Now, $\mathrm{Ker}(\overline{\psi}) = T \cap \mathrm{Ker}(\psi)$ and if ψ is injective, therefore, $\overline{\psi}$ is also injective. As $H_{\omega.2}(P) = H_{\omega}(T)$ is countably generated, thus $P = N \oplus P'$ where N is countably generated and $H_{\omega.2}(P') = 0$. We may put $K = P' \oplus Q$ and we are done.

(iv)$\Longrightarrow$(v) If $P = M \cap K$, then $P \subseteq K$ is also weakly $\omega.2$-projective. Since M/P embeds in $K \oplus N/K \simeq N$, implying that M/P is CG.

(v)$\Longrightarrow$(i) Let P be a weakly $\omega.2$-projective module and M/P is CG. Let P be a submodule of K such that $H_{\omega.2}(K) = 0$ and $K = \underset{i\in I}{\oplus} P_i$ where each P_i is CG. Let Q be a CG submodule of M such that $M = P \oplus Q$ and $P \cap Q \subseteq \underset{j\in J \subset I}{\oplus} P_j$. If we put $= Q + (P \cap (\oplus P_i))$, then N is CG and $M/N = P + Q/N = P + N/C \simeq P/P \cap N = P/P \cap (\underset{j\in J}{\oplus} P_j)$. Since $P \cap (\underset{j\in J}{\oplus} P_j)$ is the kernel of the homomorphism ϕ: $P \longrightarrow \underset{i\in I}{\oplus} P_i \longrightarrow \underset{k\in I-J}{\oplus} P_k$, therefore, M/N is a weakly $\omega.2$-projective module and we are done. □

Example 8. We consider the following QTAG-module:

$$M = \frac{\oplus_{n=1}^{\infty} R[x_n]}{\langle x_n^2 \rangle}, \tag{2}$$

where R is any associative ring with unity. Here, $R[x_n]/\langle x_n^2\rangle$ is the quotient of the polynomial ring $R[x_n]$ by the ideal generated by x_n^2, which can be thought of as a module over R generated by the element x_n subject to the relation $x_n^2 = 0$.

We claim that M satisfies Theorem 7. First, we note that M is a DSUM (i.e., every submodule is either trivial or the module itself), so every finitely generated submodule of M is a DSUM. Thus, M is a QTAG-module.

Now, we will show that M satisfies the conditions of each part of Theorem 7.

(i) To show that M is ω_1-weakly $\omega.2$-projective, we can take N to be the submodule generated by x_1, which is countably generated. Then, M/N is isomorphic to $\oplus_{n=2}^{\infty} R[x_n]/\langle x_n^2 \rangle$, which is $(\omega+1)$-projective since every finitely generated submodule of M/N is a direct sum of cyclic modules.

(ii) We can take the DSUM to be N, as in (i), and the ω_1-$(\omega+n)$-projective module to be M/N, as shown in (i).

(iii) We can take the module N to be $\oplus_{n=1}^{\infty} Rx_n$, which is a module whose every separable submodule is a DSUM. Then, M/N is isomorphic to $\oplus_{n=1}^{\infty} R[x_n]/ < x_n^2 >$, which is $(\omega+1)$-projective.

(iv) We can take N to be the submodule generated by x_1, which is countably generated and K to be the direct sum of the modules $R[x_n]/\langle x_n^2 \rangle$ for $n > 1$. Then, $H_{\omega.2}(K) = 0$ since K is a direct sum of modules that are isomorphic to R, and M/N is isomorphic to K, which is weakly $\omega.2$-projective.

(v) Since M is a direct sum of modules that are isomorphic to $R[x_n]/\langle x_n^2 \rangle$, which are all weakly $\omega.2$-projective, M itself is also weakly $\omega.2$-projective.

Therefore, M satisfies all the conditions of Theorem 7.

Another characterization of the class of ω_1-weakly $(\omega.2)$-projective modules is given by the following. But first let us state two immediate consequences of Theorem 7.

Corollary 9. *The ω_1-weakly $\omega.2$-projective modules form the smallest class of weakly $\omega.2$-projective modules that is closed under ω_1-bijective homomorphisms.*

Proof. We have to show that the class of ω_1-weakly $\omega.2$-projective modules is closed with respect to the formation of ω_1-bijective homomorphisms. We suppose that M/N is CG, for a submodule $N \subseteq M$. It is sufficient to show M is ω_1-weakly $\omega.2$-projective if and only if N is weakly $\omega.2$-projective. We suppose that N is ω_1-weakly $\omega.2$-projective. By Theorem 7 (v), N has a weakly $\omega.2$-projective submodule K such that N/K is CG. Now, M/K is also countably generated because $M/N \simeq M/K/N/K$. Therefore, M satisfies Theorem 7 (v). Converse is trivial.

The next criterion reduces the study of ω_1-weakly $\omega.2$-projective modules to these of length $\omega.2$, and hence, it is more useful for further study. □

Corollary 10. *The* QTAG *module M is ω_1-weakly $\omega.2$-projective if and only if $H_{\omega.2}(M)$ is countably generated and $M/H_{\omega.2}(M)$ is ω_1-weakly $\omega.2$-projective.*

Proof. If M is ω_1-weakly $\omega.2$-projective such that $M = K \oplus N$ where $H_{\omega.2}(K) = 0$ and N is the DS of CG modules. Then, by Theorem 7 (iv), $H_{\omega.2}(M) \subseteq H_{\omega.2}(N)$ is countably generated and vice versa. Moreover, the natural homomorphism from M onto $M/H_{\omega.2}(M)$ is ω_1-bijective, and by Corollary 9, we are done. □

Corollary 11. *The* QTAG-*module M is ω_1-weakly $\omega.2$-projective if and only if it has a CG submodule $N \subseteq H_\omega(M)$ such that M/N is weakly $\omega.2$-projective. Moreover, every separable submodule of $H_\omega(M)$ is a DSUM, and separable ω_1-weakly $\omega.2$-projective modules are weakly $\omega.2$-projective.*

Proof. By Theorem 7 (iii), M/K is a DSUM where $K = P \oplus Q$ is the direct sum of a CG module P and a DSUM Q. Now, $H_\omega(K)$ is CG and $M/H_\omega(K)/K/H_\omega(K) \simeq M/K$ is a DSUM. Since $K/H_\omega(K)$ is also a DSUM so that $M/H_\omega(K)$ is weakly $\omega.2$-projective. We take $N = H_\omega(K)$. Since $H_\omega(M) \subseteq K$, the second part is trivial because if every separable module of a module M is a DSUM, then its submodules have the same property. For the final part, we may choose $N = 0$.

The following proposition is the immediate implication of the previous discussion: □

Proposition 12. *The* QTAG-*module M is ω_1-weakly $\omega.2$-projective if and only if there exists a submodule N such that $H_\omega(N)$ is CG and $M/H_\omega(N)$ is ω_1-weakly $\omega.2$-projective.*

Corollary 13. *Let M be a* QTAG-*module such that $H_\omega(M)$ is CG. Then, M is ω_1-weakly $\omega.2$-projective if and only if $M/H_\omega(M)$ is a weakly $\omega.2$-projective module.*

Proof. Since the homomorphism $M \longrightarrow M/H_\omega(M)$ is ω_1-bijective by Corollary 9, M is ω_1 weakly projective if and only if $M/H_\omega(M)$ is ω_1 weakly projective. Now, by Corollary 11, the result follows.

The previous corollary implies that if M is weakly $\omega.2$ projective and $H_\omega(M)$ is CG, then $M/H_\omega(M)$ is weakly $\omega.2$-projective.

Now, we investigate the relationship between ω_1-weakly $\omega.2$-projective and weakly $\omega.2$-projective module. □

Proposition 14. *Let M be a ω_1-weakly $\omega.2$-projective module of length $\leq \omega.2$. Then, there exists a submodule N of M such that N is a DSUM and M/N is weakly $\omega.2$-projective.*

Proof. By Corollary 11, there is a CG submodule $N \subseteq H_\omega(M)$ such that M/N is weakly $\omega.2$-projective. Since N is separable, it is a DSUM. □

Proposition 15. *The direct sum of ω_1-weakly $\omega.2$-projective modules is ω_1-weakly $\omega.2$-projective if and only if all but a countably many of them are weakly $\omega.2$-projective.*

Proof. We suppose that $M = \oplus_{i \in I} M_i$ is a ω_1-weakly $\omega.2$-projective module. Being the submodules of M, all the M_is are ω_1-weakly $\omega.2$-projective modules. Now, there exists a countably generated submodule N of M such that $M/N = \oplus M_i/N$ is weakly $\omega.2$-projective. If $|I| \leq \aleph_0$, then we are done; therefore, we assume that $|I| > \aleph_0$ and I contains a countable subset J such that $N \subseteq \oplus_{i \in J} M_j$. Thus, $M/N \cong (\oplus_{i \in J} M_j/N) \oplus (\oplus_{i \in I \setminus J} M_i)$. Since submodules of weakly $\omega.2$-projective modules are weakly $\omega.2$-projective, all M_is with $i \in I \setminus J$ are weakly $\omega.2$-projective.

For the converse, we suppose $M = \underset{i\in I}{\oplus} M_i$ such that there exists a countable subset $J \subseteq I$ such that all M_js, $j \in J$ are ω_1-weakly $\omega.2$-projective modules and M_is, $i \in I\setminus J$ are weakly $\omega.2$-projective modules. By definition, there exist countably generated submodules N_j's of M_js such that (M_j/N_j)s are weakly $\omega.2$-projective modules for $j \in J$. If we put $N = \underset{j\in J}{\oplus} N_j$, then N is almost countably generated and $M/N \cong (\underset{j\in J}{\oplus} M_j/N_j) \oplus (\underset{i\in I\setminus J}{\oplus} M_i)$. Since the direct sum of weakly $\omega.2$-projective modules is weakly $\omega.2$-projective, M/N is also weakly $\omega.2$-projective; thus, M is ω_1-weakly $\omega.2$-projective.

We have the following corollary as an immediate implication of the previous one: □

Corollary 16. *The countable direct sum of ω_1-weakly $\omega.2$-projective modules is a ω_1-weakly $\omega.2$-projective module.*

Now, we extend the definitions of weakly $\omega.2$-projective and ω_1-weakly $\omega.2$-projective modules as follows:

Definition 17. The $QTAG$-module M is said to be weakly $(\omega.2 + n)$-projective if there exists a $(\omega + n)$-projective submodule $N \subseteq M$ such that M/N is a DSUM.

Moreover, M is ω_1-weakly $(\omega.2 + n)$-projective if there exists a countably generated submodule K of M such that M/K is weakly $(\omega.2 + n)$-projective. Also, the submodule of ω_1-weakly $(\omega.2 + n)$-projective module is also ω_1-weakly $(\omega.2 + n)$-projective.

All of the stated and proved assertions for weakly $\omega.2$-projectives and ω_1-weakly $\omega.2$-projectives can be generalized without any difficulty to weakly $(\omega.2 + n)$-projective and ω_1-weakly $(\omega.2 + n)$-projective modules, respectively. Let us state some important ones of them.

Theorem 18. *We suppose M is a module and $n < \omega$. The following assertions are equivalent:*

(a) M is ω_1-weakly $(\omega \cdot 2 + n)$-projective

(b) M is an elongation of a $(\omega + n)$-projective module by an $\omega_1 - (\omega + n)$-projective factor

(c) M is an elongation of an $\omega_1 - (\omega + n)$-projective module (i.e., of a submodule of a direct sum of a CG module with a $(\omega + n)$-projective module) by a $(\omega + n)$-projective factor (and even by a DSUM factor)

(d) M is a submodule of a module of the form $L \oplus K$ where L is countably generated and $H_{(\omega\cdot 2+n)}(L) = 0$ and K is countably generated

(e) M is an elongation of a weakly $(\omega \cdot 2 + n)$-projective module

Proof. (a)$\Longrightarrow$(b) Let $M' = M/N$ for some $(\omega + n)$-projective submodule N. We suppose that K is a $(\omega + n)$-projective submodule of M' such that M'/K is DSUM. If T is a submodule of M such that $T/N = K$, then every submodule of T is $(\omega + n)$-projective and $M/T = (M/N)/(T/N) = M'/K$. By the same arguments, T has a submodule P which is $(\omega + n)$-projective and T/P is $\omega_1 - (\omega + n)$ projective and we are done.

(b)$\Longrightarrow$(c) Let N be a submodule of M such that N is $(\omega + n)$-projective and $M' = M/N$ is $\omega_1 - (\omega + n)$ projective. Now, M' has a CG submodule K such that M'/K is $(\omega + n)$ projective. If we put $T/N = K$, then every separable submodule of T is DSUM and $M/T \cong (M/N)/(T/N) = M'/K$ is $(\omega + n)$ projective.

(c)$\Longrightarrow$(d) Let T be a submodule of M such that every separable submodule of T is DSUM and M/T is $(\omega + n)$-projective. Let P be a submodule of M such that P is a direct sum of CG modules and $H_{\omega+n}(P) = T$. Since M/T is $(\omega + n)$-projective, the injection map i: $T \longrightarrow P$ extends to ψ: $M \longrightarrow P$. We may consider a module Q such that $H_{\omega 2+n}(Q) = 0$ and $M/T \subseteq Q$. We define $\overline{\psi} = \ker \psi \cap T = \{0\}$ and $\overline{\psi}$ is also injective. As $H_{\omega+n}(P) = T$, $H_{\omega 2+n}(P) = H_{\omega}(T)$ is also CG, thus, we have $P = L \oplus P'$, where L is CG and $H_{\omega 2+n}(P') = 0$. If we put $K = P, \oplus Q$, we are done.

(d)$\Longrightarrow$(e) Let $M \subseteq L \oplus K$, where L is CG and $H_{\omega 2+n}(L) = 0$ and K is CG. Now, K is weakly $(\omega 2 + n)$-projective; therefore, $M \cap K = P$ is also weakly $(\omega 2 + n)$-projective. Now, $M/P = M/(M \cap K) \cong (M + K)/K$ embeds in $(L + K)/K$ implying that M is an elongation of a weakly $(\omega 2 + n)$-projective module.

(e)$\Longrightarrow$(a) Let P be a weakly $(\omega 2 + n)$-projective module and M/P is CG. Let P be a submodule of K such that $H_{\omega 2+n}(K) = 0$ and $K = \underset{i\in I}{\oplus} P_i$, where each P_i is CG. If Q is a CG submodule of M such that $M = P + Q$ and $P \cap Q \subseteq \underset{j\in J\subseteq I}{\oplus} P_j$. If we put $= Q + (P \cap \underset{j\in J}{\oplus} P_j)$, then L is CG and

$$\frac{M}{L} = \frac{P+Q}{L} = \frac{P+L}{L} \cong \frac{P}{L\cap P} = \frac{P}{P\cap\left(\oplus P_j\right)}. \tag{3}$$

Now, $P \cap (\underset{i\in J}{\oplus} P_j)$ is the kernel of the composition of homomorphisms ϕ: $P \longrightarrow \underset{i\in i}{\oplus} P_i \longrightarrow \underset{k\in I\setminus J}{\oplus} P_k$. Therefore, M/L is a ω_1-weakly $(\omega 2 + n)$-projective module, which completes the proof of the theorem.

We will now study a little different class of modules: □

Definition 19. A module M is called ω-totally weak $(\omega \cdot 2 + n)$-projective if each of its separable submodule is weakly $(\omega \cdot 2 + n)$-projective.

Clearly, every submodule of a ω-totally weak $(\omega \cdot 2 + n)$-projective module retains the same property.

Proposition 20. *If M is ω_1-weakly $(\omega \cdot 2 + n)$-projective, then M is ω-totally weak $(\omega \cdot 2 + n)$-projective.*

Proof. Since as observed previously a submodule of a ω_1-weakly $(\omega \cdot 2 + n)$-projective module M is ω_1-weakly $(\omega \cdot 2 + n)$-projective as well, by what we have already shown that all separable submodules of M should be weakly $(\omega \cdot 2 + n)$-projective, as required. □

Example 21. To give an example of a QTAG-module which is ω-totally weak $(\omega \cdot 2 + n)$-projective but not ω_1-weakly $(\omega \cdot 2 + n)$-projective, let us consider the following.

Let $R = \mathbb{Z}[x_1, x_2, \ldots]$ be the polynomial ring in countably many variables and $M = \underset{n\in\mathbb{N}}{\oplus} R/(x_1, x_2, \ldots, x_n)$ be the direct sum of the quotient rings $R/(x_1, x_2, \ldots, x_n)$ with the usual module structure. Then, M is a QTAG-module.

We claim that M is ω-totally weak $(\omega \cdot 2 + n)$-projective. Indeed, every separable submodule of M is isomorphic to a direct sum of quotients of the form $R/(x_i)$ for some $i \in \mathbb{N}$. But $R/(x_i)$ is a free module over R, and hence, it is weakly $(\omega \cdot 2 + n)$-projective for any $n \in \mathbb{N}$. Thus, every separable submodule of M is weakly $(\omega \cdot 2 + n)$-projective for any $n \in \mathbb{N}$, and hence, M *is* ω-totally weak $(\omega \cdot 2 + n)$-projective.

However, M is not ω_1-weakly $(\omega \cdot 2 + n)$-projective. To see this, we suppose for contradiction that M is ω_1-weakly $(\omega \cdot 2 + n)$-projective and we let $N \subseteq M$ be a countably generated submodule such that M/N is weakly $(\omega \cdot 2 + n)$-projective. Then, M/N is a direct sum of modules of the form $R/(x_{i_1}, \ldots, x_{i_k})$, where $i_1 < \ldots < i_k$. We note that each such module is finitely generated and hence uniserial, so M/N is a DSUM. Since M/N is weakly $(\omega \cdot 2 + n)$-projective, it follows that M/N is a direct sum of modules of the form $R/(x_{i_1}, \ldots, x_{i_m})$, where $m \leq n$. But this contradicts the fact that M/N contains a direct summand isomorphic to $R/(x_{n+1})$, which is not of this form. Thus, M is not ω_1-weakly $(\omega \cdot 2 + n)$-projective.

Proposition 22. *The class of separable weakly $(\omega \cdot 2 + n)$-projectives is closed under ω_1 bijections.*

Proof. Let S be a separable module with a submodule N such that S/N is countably generated. As noted previously, if S is weakly $(\omega \cdot 2 + n)$-projective, then so is N as being a submodule.

Conversely, we assume that N is weakly $(\omega \cdot 2 + n)$-projective. We write that N/P is DSUM for some $(\omega + n)$-projective submodule P. We observe that $S/N \cong (S/P)/(N/P)$ is countably generated, and hence, one can write $S/P = (Q/P) \oplus (R/P)$, where Q/P is DSUM and R/P is countably generated for some submodules Q and R of S. Since R is separable, so R is $(\omega + n)$-projective [6]. Furthermore, $S/R \cong (S/P)/(R/P) \cong Q/R$ is DSUM, so by definition, S is weakly $(\omega.2 + n)$-projective, as desired.

We already know from Corollary 9, for any nonnegative$n$ that ω_1-weakly $(\omega \cdot 2 + n)$-projective modules are closed under ω_1-bijections and that for separable modules the ω_1-weakly $(\omega \cdot 2 + n)$-projective modules are exactly the weakly $(\omega \cdot 2 + n)$-projectives. We notice that ω-totally $(\omega + n)$-projectives are themselves ω-totally weak $(\omega \cdot 2 + n)$-projectives because $(\omega + n)$-projective modules are weakly $(\omega \cdot 2 + n)$-projective. □

Theorem 23. *The class of ω-totally weak $(\omega \cdot 2 + n)$-projective modules is closed under ω_1-bijections.*

Proof. Referring to Lemma 2.9 in [12], we suppose that $N \subseteq M$ is a submodule with M/N is CG. As indicated previously, M being ω-totally weak $(\omega \cdot 2 + n)$-projective implies the same for N.

To show the converse, let P be a separable submodule of M. Hence, $Q = P \cap N$ is a separable submodule of N, and so, by hypothesis, it is weakly $(\omega \cdot 2 + n)$-projective, but $P/Q \cong (P + N)/N \subseteq M/N$ is countably generated. Consequently, it follows from Proposition 22 that P is also weakly $(\omega \cdot 2 + n)$-projective so that M is ω-totally weak $(\omega \cdot 2 + n)$-projective, as asserted.

We continue with the following improvement of Theorem 3.6 of [12]. □

Theorem 24. *We suppose that M is a module such that $H_\omega(M)$ is CG. Then, the following assertions are equivalent:*

(i) M is ω_1-weakly $(\omega \cdot 2 + n)$-projective

(ii) M is ω-totally weak $(\omega \cdot 2 + n)$-projective

(iii) $M/H_\omega(M)$ is weakly $(\omega \cdot 2 + n)$-projective.

Proof. $(i) \Longrightarrow (ii)$ holds from Proposition 20. Moreover, the implication (ii) $\Longrightarrow$ (iii) follows from Theorem 23 because the map $M \longrightarrow M/H_\omega(M)$ is ω_1-bijective. The final implication (iii) $\Longrightarrow$ (i) follows from the corresponding generalization of Corollary 13 to (ω_1)-weakly $(\omega \cdot 2 + n)$-projectives.

Sikander et al. [14] defined the *n-layered* module M such that if for some $n < \omega$, $H^n(M) = \underset{k<\omega}{\cup} M_k$, $M_k \subseteq M_{k+1} \subseteq H^n(M)$ for all $k \geq 1$, and $M_k \cap H_k(M) = H^n(H_\omega(M))$. Moreover, an α module is α-*n*-layered, $(\alpha \leq \omega_1)$ if for every $\beta < \alpha$, each $H_\beta(M)$-high submodule of M is n-layered. □

Proposition 25. *Let N be a countably generated nice submodule of a module M with limit length such that there exists a countable ordinal $\beta <$ length(M) with $N \cap H_\beta(M) = 0$. If M is α-n-layered, then M/N is α-n-layered whenever $\beta < \alpha \leq \omega_1$.*

Proof. Let K/N be a $H_\beta(M/N)$-high submodule of M/N. Now,

$$\begin{aligned}\left(\frac{K}{N}\right) \cap \left(H_\beta\left(\frac{M}{N}\right)\right) &= \left(\frac{K}{N}\right) \cap \left(\frac{H_\beta(M) + N}{N}\right)\\ &= \frac{K \cap \left(H_\beta(M) + N\right)}{N}\\ &= \frac{\left(K \cap H_\beta(M)\right) + N}{N}\\ &= \{0\}.\end{aligned} \tag{4}$$

This implies that $K \cap H_\beta(M) \subseteq N$; therefore, $K \cap H_\beta(M) \subseteq N \cap H_\beta(M) = \{0\}$. Let $K' \supset K$ be a submodule of M such that $K' \cap H_\beta(M) = \{0\}$. Now,

$$\left(\frac{K'}{N}\right)\cap H_\beta\left(\frac{M}{N}\right)=\frac{K'\cap\left(H_\beta(M)+N\right)}{N}$$
$$=\frac{\left(K'\cap\left(H_\beta(M)\right)+N\right.}{N} \quad (5)$$
$$=\{0\},$$

with $K'/N \supset K/N$ which is not possible. Therefore, K is $H_\beta(M)$-high in M, and so, it is an isotype in M. Now, N is nice in M; therefore, it is nice in K also. Since K is *n-layered*, K/N should be n-layered when M/N is α-n-layered and we are done. □

Lemma 26. *We suppose that* $(\omega+n)\le\alpha<\omega$, *where* $n\ge1$. *Then, the* QTAG- *module M is* $(\omega+n)$*-projective and* α*-n-layered if and only if M is a DS of CG modules of length at most* $(\omega+n)$.

Proof. M is an α-projective α-n-layered module; therefore, it is n-layered. Now, by [15], M is a DSM of length at most $(\omega+n)$ as asserted. □

Theorem 27. *Let* $(\omega+n)\le\alpha\le\omega_1$ *and let M be a balanceable* ω_1-$(\omega+n)$*-projective module or an* ω-$(\omega+n)$*-projective module of the limit length for some* $n\ge1$. *Then, M is a* α*-n-layered module if and only if M is a DS of CG modules.*

Proof. We suppose that M is balanceable ω_1-$(\omega+n)$-projective. Let N be a CG nice submodule of M such that M/N is $(\omega+n)$-projective. Therefore, M/N is also α-n-layered. Now, by Lemma 26, M/N is a DS of CG modules implying that M is a DS of CG modules.

If M is a ω-$(\omega+n)$-projective module of limit length, then there is a finitely generated submodule K of M such that M/K is $(\omega+n)$-projective. Now, by Proposition 25, M/K is α-n-layered. Therefore, by Lemma 26, M/K is a direct sum of CG modules implying that M is also a direct sum of CG modules. □

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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