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Research Article

Research on the Reliability of a Two-Robot Security System with Early Warning Function

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In this paper, the mathematical model of a kind of two-robot security system with an early warning function is studied. By using strongly continuous operator semigroup theory and Volterra integral equation theory, the properties of the semigroup of the system operator, the existence and uniqueness of nonnegative solution, and the well-posedness of solution are discussed, respectively. Under the assumption that the failure rate and repair rate of the system are constants, the equations of the robot system are transformed into an ordinary differential equation group, and then the instantaneous reliability and stable-state reliability of the system are obtained. The reliability and zero-state controllability of the system are proved. Finally, the numerical solution of the system model is obtained by using MATLAB mathematical software, and the corresponding numerical simulation diagram is given. The results show that the conclusions of numerical calculation and numerical simulation are in accordance with the results of reliability theory, thereby the reliability of the robot safety system is verified.

1. Introduction

With the progress and development of the times, robots are becoming more and more widely used. Robots are mechanical devices that automatically perform work, replacing or assisting human work, such as manufacturing, construction, or hazardous work. Therefore, the stability and reliability of the robot security system has become a hot issue. Early warning function refers to the ability to provide early warning and timely issuance of alarm signals before a system or component malfunctions, allowing sufficient time to predict various impending disasters and reduce economic losses caused to humans by disasters. The reliability [1] of a system means the ability or property of the system to accomplish the specified functions under the specified time and the specified using conditions. It is also one of the important characteristics of a repairable system. The robot security system with early warning function can accurately detect the abnormal condition of the parts of the robot working system according to the early warning signal and carry out advanced control, continue to work, or carry out maintenance, so as to reduce the economic losses. In order to ensure the security and reliability of the system and avoid the occurrence of unexpected accidents, the early warning function is introduced into the repairable robot security system in this paper. Based on the abstract Cauchy problem theory, the system reliability is analyzed.

The repairable system is an important system in the research of reliability mathematics, and reliability is one of the important contents in the research of the repairable system. Many scholars and researchers at home and abroad have done a lot of research on this kind of system, and achieved fruitful results. Some of the literatures adopt the method of qualitative analysis; for example, Wang et al. used linear operator semigroup theory in literatures [2–9] to study the semigroup properties of the main operator of repairable systems composed of faulty components and repairmen and discussed the well-posedness of the system model solution by using functional analysis methods. In literatures [10–15], Kamranfar et al. studied the stability of repairable systems. On the basis of discussing the asymptotic stability of the solutions of a repairable system with two different components connected in parallel, Wang et al. studied the controllability of the system in the zero state and the criterion of using the controllability of the system in literatures [16–24], and obtained the corresponding optimal control element of the system by using the minimization sequence, thus solving the optimal control problem of the system solution. In other literatures, the reliability index of the system is obtained by quantitative analysis. For example, Marco et al. obtained the reliability index of the system in the literatures [25–32] by using the Laplace transformation inversion method, MATLAB mathematical software calculation, linear differential equation with constant coefficients satisfying initial conditions, and so on. In addition, some literatures combine the quantitative analysis method with computer technology to perform the numerical calculation and simulation of the repairable system. For example, Gao et al. studied a repairable system with early warning function in literature [33], utilizing that when the risk coefficient $\alpha \longrightarrow \infty$, the warning system approximates a new model with weak solution—the model of a nonearly warning system; the corresponding conclusions were drawn and numerical simulation examples were given by computer, which greatly enriches the theory and practice of the repairable system with an early warning function; Zhou et al. studied the reliability of the system in literatures [34–37] and simulated the graphs of transient and steady-state reliability of the system by using Maple software. In summary, in the existing literatures and materials both domestically and internationally, the method of combining qualitative analysis with quantitative analysis is less effective than that of using qualitative or quantitative analysis alone. In this article, a kind of two-robot security system with an early warning function is taken as the research object, and attempts are made to combine qualitative analysis with quantitative analysis to study the reliability of the system.

The main contents of this article are the modeling of the robot safety system, the well-posedness and controllability of the system solution, the reliability index of the system, and the numerical experiment of the reliability theory results. However, with the development and application of system reliability, when the system models become more complex, new methods and theories are needed to guide them, which is an important development problem of system reliability research in the future.

The reliability of repairable system has achieved certain achievements in both theory and application. Nowadays, in the 21st century, the research on reliability has been elevated both domestically and internationally to the high level of understanding of saving resources and energy. Products (or equipment) are developing in the direction of gradually improving reliability and gradually reducing maintenance time, maintenance personnel, and maintenance costs. At the same time, reliability technology is the result of the joint development of multiple fields and technologies. It belongs to a comprehensive basic industry, and the current development trend is moving towards comprehensiveness, usability, automation, informatization, virtualization, and intelligence. Thus, higher economic benefits and stronger competitiveness can be achieved.

2. Mathematical Model

A robot is a complex system including mechanical, electronic, electrical, hydraulic, pneumatic, computer, and other types of components and control software. It is relatively complicated to study its reliability and safety [15, 16]. In order to consider this issue more clearly, this article assumes that the robot security system consists of two robots、security devices and a repairman [17]. Assuming that the entire system does not consider the repair and replacement process and starts to run at the time $t = 0$. At the time $t = 0$, the two robots and the safety device are brand new, the system starts to operate normally, and the maintenance personnel goes on holiday. If the system fails, the repairman immediately terminates the vacation and repairs the faulty system immediately. If $N(t)$ represents the state at the moment t, the system has the following situations:

State 0. One robot and safety device work, and one robot is in hot standby state

State 1. One robot and safety device are working, and one robot is in a malfunction state

State 2. The state in which both robots are malfunctioning

State 3. The state of the system when the safety device fails

State c. The state of the normal faulty system

The state-transfer chart of the system is shown in Figure 1.

In order to facilitate the modeling and model analysis, the following general assumptions can be made according to the state transition diagram of the repairable system:

(1) Various faults are independent of each other in the statistical sense
(2) Only when both robots fail (or the safety device fails due to conventional reasons), the whole system is in a fault state
(3) The failure rate of the system is constant, and the repair rate after system failure is nonconstant
(4) The normal working time of the robot security system follows the exponential distribution function $F = 1 - e^{-\lambda t}$, $t \geq 0$, and $\lambda > 0$
(5) The repair time of faulty parts of robot system follows the general distribution function $G = 1 - e^{-\mu_i(x)t}$, $t \geq 0$, $\mu_i(x) > 0$, and $i = 1, 2$
(6) The two robots are exactly the same and repaired as new

Since the time distribution of transitions between states of the system does not completely obey the negative exponential distribution, it can be known from the above hypothesis that $N(t)$ represents the state at the moment t, so $\{N(t), t \geqslant 0\}$ is not a Markov process. But we can make it a high-dimensional Markov process by using the method of supplementary variable, assuming that the fault system is in

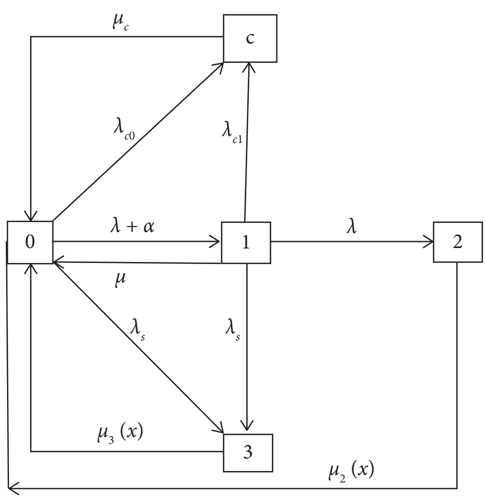

Figure 1: State-transfer chart of the system.

a state of maintenance, the supplementary variable $X_i(t)\,(i = 2, 3)$ represents the maintenance time from the beginning of maintenance to the present, and let

$$Z(t) = \begin{cases} N(t), & N(t) = 0, 1, c, \\ (N(t), X_2(t)), & N(t) = 2, \\ (N(t), X_3(t)), & N(t) = 3. \end{cases} \quad (1)$$

Then, $\{Z(t), t \geqslant 0\}$ constitutes a Markov process; that is, $\{N(t), X_i(t) \mid t \geqslant 0\}\,(i = 2, 3)$ is a continuous-time two-dimensional Markov process; that is, t at any time, given the concrete values of $N(t)$ and $X_i(t)$, then the probability law of the process $\{N(t), X_i(t) \mid t \geqslant 0\}\,(i = 2, 3)$ after time t has nothing to do with the history of the process before time t. Let $P_i(t)\,(i = 0, 1, c)$ denote the probability that the system is in state i at time t, $P_i(t, x)\,(i = 2, 3)$ represents the probability density of the time t that the system is in state i and the faulty part has been repaired x, i.e.,

$$\begin{aligned} P_i(t) &= P\{N(t) = i\},\ (i = 0, 1, c), \\ P_i(t, x)dx &= P\{x < X_i(t) \leqslant x + dx, N(t) = i\},\ (i = 2, 3). \end{aligned} \quad (2)$$

Here, it should be noted that although $P_i(t, x)\,(i = 2, 3)$ is only defined as $0 \leqslant x < t$, but for the sake of discussion, $P_i(t, x)\,(i = 2, 3)$ can be defined according to the actual physical background of the system and $P_i(t, x)\,(i = 2, 3)$ is extended on $x > t$, that is, supplementary definition $P_i(t, x) = 0, x > t, i = 2, 3$. At the same time, the system is out of state $i\,(i = 2, 3)$ of the risk function; that is, the fix quotiety of the faulty component in the system in the state $i\,(i = 2, 3)$ can be defined as follows by the conditional probability [21, 22]:

$$\begin{aligned} \mu_i(x)\Delta t &= P\big(x < X_i(t) \leq x + \Delta t \,\big|\, X_i(t) > x\big), \\ &= \frac{P(x < X_i(t) \leq x + dx)}{P(X_i(t) > x)}, \\ &= \frac{G_i(x + \Delta t) - G_i(x)}{1 - G_i(x)}, \\ &= \frac{dG_i(x)/dx}{1 - G_i(x)}\Delta t, i = 2, 3. \end{aligned} \quad (3)$$

And from the actual physical meaning of $\mu_i(x)$, the following reasonable assumptions can be made:

$$\begin{aligned} &0 \leqslant \mu_i(x) < \infty, M = \sup_{x \in [0,\infty)} \mu_i(x), \int_0^\infty \mu_i(t)\mathrm{d}t = \infty, \\ &0 < \lim_{x \longrightarrow \infty} \frac{1}{x}\int_0^x \mu_i(t)\mathrm{d}t = \mu_i < \infty, (i = 2, 3). \end{aligned} \quad (4)$$

The following is a discussion of the system state transition after Δt time. Therefore, let λ represents the damage rate of the running system robot caused by its own reasons, λ_{ci} indicates the normal failure rate of the system in the state $i\,(i = 0, 1)$, λ_s indicates the human fault quotiety of the operating system, and α represents the damage quotiety of hot standby robot, μ indicates the constant fix quotiety of the running robot, μ_c indicates the constant fix quotiety of the running system, and $P_i(t)$ indicates the probability that the system is in state i at the moment $t\,(i = 0, 1, c)$, $P_i(t, x)$ represents the probability that the system is in the state i and the repaired time x at the moment t, (t, x) in $[0, \infty] \times [0, \infty]$. $\mu_i(x)$ represents the fix quotiety when the system is in state i and the repaired time x. Then, it is deduced from the formula of total probability and the properties of Markov process (for convenience, it is assumed that Δx is the same as Δt):

$P_0(t + \Delta t)$ = P (at t when the system is in state 0, Δt the system remains in state 0) + P (at t when the system is in state 1, Δt the system is fixed to state 0) + P (at t when the system is in state c, Δt the system is fixed to state 0) + P (at t when the system is in state 2, Δt the system is fixed to state 0) + P (at t when the system is in state 3, Δt the system is fixed to state 0):

$$P_0(t + \Delta t) = P_0(t)\left[1 - (\lambda + \alpha + \lambda_{c0} + \lambda_s)\Delta t\right] + \mu P_1(t)\Delta t + \mu_c P_c(t)\Delta t + \sum_{i=2}^{3}\int_0^\infty \mu_i(x)P_i(t, x)\Delta t\mathrm{d}x + o(\Delta t). \quad (5)$$

From formula (5) and the definition of derivative,

$$\begin{aligned}
P_0(t+\Delta t)-P_0(t) &= -(\lambda+\alpha+\lambda_{c0}+\lambda_s)P_0(t)\Delta t+\mu P_1(t)\Delta t \\
&\quad +\mu_c P_c(t)\Delta t+\sum_{i=2}^{3}\int_0^{\infty}\mu_i(x)P_i(t,x)\mathrm{d}x\Delta t+o(\Delta t), \Rightarrow \\
\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} &= \frac{-(\lambda+\alpha+\lambda_{c0}+\lambda_s)P_0(t)\Delta t}{\Delta t}+\frac{\mu P_1(t)\Delta t}{\Delta t} \\
&\quad +\frac{\mu_c P_c(t)\Delta t}{\Delta t}+\frac{\sum_{i=2}^{3}\int_0^{\infty}\mu_i(x)P_i(t,x)\mathrm{d}x\Delta t}{\Delta t}+\frac{o(\Delta t)}{\Delta t}, \Rightarrow \\
\lim_{\Delta t\longrightarrow 0}\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} &= -\lim_{\Delta t\longrightarrow 0}(\lambda+\alpha+\lambda_{c0}+\lambda_s)P_0(t)+\lim_{\Delta t\longrightarrow 0}\mu P_1(t) \\
&\quad +\lim_{\Delta t\longrightarrow 0}\mu_c P_c(t)+\lim_{\Delta t\longrightarrow 0}\sum_{i=2}^{3}\int_0^{\infty}\mu_i(x)P_i(t,x)\mathrm{d}x+\lim_{\Delta t\longrightarrow 0}\frac{o(\Delta t)}{\Delta t}, \Rightarrow \\
\frac{\mathrm{d}P_0(t)}{\mathrm{d}t} &= -(\lambda+\alpha+\lambda_{c0}+\lambda_s)P_0(t)+\mu P_1(t)+\mu_c P_c(t)+\sum_{i=2}^{3}\int_0^{\infty}P_i(t,x)\mu_i(x)\mathrm{d}x.
\end{aligned} \tag{6}$$

$P_1(t+\Delta t)$ = P (at t the system is in state 1, the faulty robot in Δt has not been repaired, and the other robot has not failed, and the system in Δt has not left state 1) + P (at t the system is in state 0, and one robot in Δt has failed):

$$\begin{aligned}
P_1(t+\Delta t) &= P_1(t)\left[1-(\mu+\lambda+\lambda_{c1}+\lambda_s)\Delta t\right] \\
&\quad +(\lambda+\alpha)P_0(t)\Delta t+o(\Delta t).
\end{aligned} \tag{7}$$

From formula (7) and the definition of derivative,

$$\begin{aligned}
P_1(t+\Delta t)-P_1(t) &= -(\mu+\lambda+\lambda_{c1}+\lambda_s)P_1(t)\Delta t+(\lambda+\alpha)P_0(t)\Delta t+o(\Delta t), \Rightarrow \\
\frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} &= \frac{-(\mu+\lambda+\lambda_{c1}+\lambda_s)P_1(t)\Delta t}{\Delta t}+\frac{(\lambda+\alpha)P_0(t)\Delta t}{\Delta t}+\frac{o(\Delta t)}{\Delta t}, \Rightarrow \\
\lim_{\Delta t\longrightarrow 0}\frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} &= -\lim_{\Delta t\longrightarrow 0}(\mu+\lambda+\lambda_{c1}+\lambda_s)P_1(t)+\lim_{\Delta t\longrightarrow 0}(\lambda+\alpha)P_0(t)+\lim_{\Delta t\longrightarrow 0}\frac{o(\Delta t)}{\Delta t}, \Rightarrow \\
\frac{\mathrm{d}P_1(t)}{\mathrm{d}t} &= -(\mu+\lambda+\lambda_{c1}+\lambda_s)P_1(t)+(\lambda+\alpha)P_0(t).
\end{aligned} \tag{8}$$

$P_c(t+\Delta t)$ = P (at t when the system is in state c, Δt the system routine fault has not been repaired, and the system has not left the state c) + P (at t when the system is in state 0, Δt the system routine fault) + P (at t when the system is in state 1, Δt the system routine fault):

$$\begin{aligned}
P_c(t+\Delta t) &= P_c(t)(1-\mu_c\Delta t)+\lambda_{c0}P_0(t)\Delta t \\
&\quad +\lambda_{c1}P_1(t)\Delta t+o(\Delta t).
\end{aligned} \tag{9}$$

From formula (9) and the definition of derivative,

$$\begin{aligned}
P_c(t+\Delta t)-P_c(t) &= -\mu_c P_c(t)\Delta t+\lambda_{c0}P_0(t)\Delta t+\lambda_{c1}P_1(t)\Delta t+o(\Delta t), \Rightarrow \\
\frac{P_c(t+\Delta t)-P_c(t)}{\Delta t} &= \frac{-\mu_c P_c(t)\Delta t}{\Delta t}+\frac{\lambda_{c0}P_0(t)\Delta t}{\Delta t}+\frac{\lambda_{c1}P_1(t)\Delta t}{\Delta t}+\frac{o(\Delta t)}{\Delta t}, \Rightarrow \\
\lim_{\Delta t\longrightarrow 0}\frac{P_c(t+\Delta t)-P_c(t)}{\Delta t} &= -\lim_{\Delta t\longrightarrow 0}\mu_c P_c(t)+\lim_{\Delta t\longrightarrow 0}\lambda_{c0}P_0(t)+\lim_{\Delta t\longrightarrow 0}\lambda_{c1}P_1(t)+\lim_{\Delta t\longrightarrow 0}\frac{o(\Delta t)}{\Delta t}, \Rightarrow \\
\frac{\mathrm{d}P_c(t)}{\mathrm{d}t} &= \lambda_{c0}P_0(t)+\lambda_{c1}P_1(t)-\mu_c P_c(t).
\end{aligned} \tag{10}$$

Similarly, there are

$$\begin{aligned} P_i(t+\Delta t, x+\Delta t) &= P\ (\text{at } t \text{ the system is in state i, and the repair time of the faulty components is } x, \\ &\quad \Delta t \text{ the system has not left state i}), \\ &= P\ (\text{at } t \text{ the system is in state i, and the repair time of the faulty components is } x) \\ &\quad \times P\ (\Delta t \text{ the failure components are not repaired yet}), \\ &= P_i(t,x)\left[1-\mu_i(x)\Delta t\right] + o(\Delta t), i = 2, 3. \end{aligned} \tag{11}$$

Derived from formula (11) and the definition of partial derivative,

$$\begin{aligned} &P_i(t+\Delta t, x+\Delta t) - P_i(t,x) = -P_i(t,x)\mu_i(x)\Delta t + o(\Delta t), \Rightarrow \\ &P_i(t+\Delta t, x+\Delta t) - P_i(t, x+\Delta t) + P_i(t, x+\Delta t) - P_i(t,x) = -P_i(t,x)\mu_i(x)\Delta t + o(\Delta t), \Rightarrow \\ &\frac{P_i(t+\Delta t, x+\Delta t) - P_i(t, x+\Delta t)}{\Delta t} + \frac{P_i(t, x+\Delta t) - P_i(t,x)}{\Delta t} = \frac{-P_i(t,x)\mu_i(x)\Delta t}{\Delta t} + \frac{o(\Delta t)}{\Delta t}, \Rightarrow \\ &\lim_{\Delta t \longrightarrow 0} \frac{P_i(t+\Delta t, x+\Delta t) - P_i(t, x+\Delta t)}{\Delta t} \\ &+ \lim_{\Delta t \longrightarrow 0} \frac{P_i(t, x+\Delta t) - P_i(t,x)}{\Delta t} \\ &= -\lim_{\Delta t \longrightarrow 0} P_i(t,x)\mu_i(x) + \lim_{\Delta t \longrightarrow 0} \frac{o(\Delta t)}{\Delta t}, \Rightarrow \\ &\frac{\partial P_i(t,x)}{\partial t} + \frac{\partial P_i(t,x)}{\partial x} = -\mu_i(x)P_i(t,x). \end{aligned} \tag{12}$$

The boundary conditions and initial conditions of the system are discussed below [20].

Because $P_2(t,0)$ represents the probability that the system just enters the state 2 at t, that is, the probability that the system just leaves the state 1 at t, i.e.,

$$P_2(t+\Delta t, 0)\Delta t = \lambda P_1(t)\Delta t + o(\Delta t). \tag{13}$$

From formula (13) and the definition of derivative,

$$\begin{aligned} &\frac{P_2(t+\Delta t, 0)\Delta t}{\Delta t} = \frac{\lambda P_1(t)\Delta t}{\Delta t} + \frac{o(\Delta t)}{\Delta t}, \Rightarrow \\ &\lim_{\Delta t \longrightarrow 0} P_2(t+\Delta t, 0) = \lim_{\Delta t \longrightarrow 0} \lambda P_1(t) + \lim_{\Delta t \longrightarrow 0} \frac{o(\Delta t)}{\Delta t}, \Rightarrow \\ &P_2(t,0) = \lambda P_1(t). \end{aligned} \tag{14}$$

In addition, $P_3(t,0)$ represents the probability that the system just enters state 3 at t, that is, the probability that the system just leaves state 0 or state 1 at t, i.e.,

$$P_3(t+\Delta t, 0)\Delta t = \lambda_s\left[P_0(t) + P_1(t)\right]\Delta t + o(\Delta t). \tag{15}$$

From formula (15) and the definition of derivative

$$\frac{P_3(t+\Delta t, 0)\Delta t}{\Delta t} = \frac{\lambda_s\left[P_0(t) + P_1(t)\right]\Delta t}{\Delta t} + \frac{o(\Delta t)}{\Delta t}, \Rightarrow$$

$$\lim_{\Delta t\longrightarrow 0} P_3(t+\Delta t,0) = \lim_{\Delta t\longrightarrow 0} \lambda_s\left[P_0(t)+P_1(t)\right] + \lim_{\Delta t\longrightarrow 0} \frac{o(\Delta t)}{\Delta t}, \Rightarrow$$
$$P_3(t,0) = \lambda_s \sum_{i=0}^{1} P_i(t). \tag{16}$$

Assuming that time $t = 0$, both parts are good, that is, the initial condition is

$$P_0(0) = 1, P_1(0) = P_c(0) = P_2(0,x) = P_3(0,x) = 0, x \geq 0. \tag{17}$$

At this time, from probability analysis formulas (5) to (17) [18, 19], the integro-differential equations of the mathematical model of the two-robot security system with early warning function can be described in the Figure 1 are

$$\frac{\mathrm{d}P_0(t)}{\mathrm{d}t} = -(\lambda+\alpha+\lambda_{c0}+\lambda_s)P_0(t) + \mu P_1(t) + \mu_c P_c(t) + \sum_{i=2}^{3}\int_0^\infty P_i(t,x)\mu_i(x)\mathrm{d}x, \tag{18}$$

$$\frac{\mathrm{d}P_1(t)}{\mathrm{d}t} = -(\mu+\lambda+\lambda_{c1}+\lambda_s)P_1(t) + (\lambda+\alpha)P_0(t), \tag{19}$$

$$\frac{\mathrm{d}P_c(t)}{\mathrm{d}t} = \lambda_{c0}P_0(t) + \lambda_{c1}P_1(t) - \mu_c P_c(t), \tag{20}$$

$$\frac{\partial P_i(t,x)}{\partial x} + \frac{\partial P_i(t,x)}{\partial t} = -\mu_i(x)P_i(t,x), i = 2,3, \tag{21}$$

$$P_2(t,0) = \lambda P_1(t), P_3(t,0) = \lambda_s \sum_{i=0}^{1} P_i(t), \tag{22}$$

$$P_0(0) = 1, P_1(0) = P_c(0) = P_2(0,x) = P_3(0,x) = 0. \tag{23}$$

3. Well-Posedness of the System Solution

Since the repairable robot system with early warning function contains both integral and differential, it is difficult to directly deal with it. Therefore, it is necessary to perform the necessary conversion before the reliability analysis [23]. In order to facilitate the subsequent discussion of the adaptability of the system solution, the system equations (18)–(23) are translated into an abstract Cauchy problem in Banach space [24, 25]. For this definition, make $b_0 = \lambda + \alpha + \lambda_{c0} + \lambda_s, b_1 = \mu + \lambda + \lambda_{c1} + \lambda_s$.

Take state space

$$X := \left\{ P \in \mathbb{R}^3 \times \left(L^1[0,+\infty)\right)^2 \middle| \|P\| = \sum_{i=0}^{1}|P_i| + |P_c| + \sum_{i=2}^{3}\|P_i(x)\|_{L^1[0,+\infty]} < \infty \right\}, \tag{24}$$

among which $P = (P_0, P_1, P_c, P_2(x), P_3(x))$. At this point, it can be proved $(X, \|\cdot\|)$ is a Banach space. The following operators A and B and their domains $D(A), D(B)$ are defined as follows:

$$D(A)=\left\{P\in X\left|\begin{array}{l}\left(\frac{d}{dx}+\mu_i(x)\right)P_i(x)\in L^1[0,+\infty),P_i(x)(i=2,3)\\ \text{is a strictly succession function and meets}\\ P(0)=\left(P_0,P_1,P_c,P_2(0),P_3(0)\right)=\left(P_0,P_1,P_c,\lambda P_1,\lambda_s\sum_{i=0}^{1}P_i\right)\end{array}\right.\right\}. \tag{25}$$

For $P\in D(A)$, the range $R(A)=\{AP\mid P\in D(A)\}$ of the operator A is defined as follows:

$$AP=\operatorname{diag}\left(-b_0,-b_1,-\mu_c,-\left(\frac{d}{dx}+\mu_2(x)\right),-\left(\frac{d}{dx}+\mu_3(x)\right)\right)P. \tag{26}$$

And for $P\in X=D(B)$, the range $R(B)=\{BP\mid P\in D(B)\}$ of the operator B is defined as follows:

$$BP=\begin{pmatrix}0&\mu&\mu_c&\int_0^\infty\mu_2(x)dx&\int_0^\infty\mu_3(x)dx\\ \lambda+\alpha&0&0&0&0\\ \lambda_{c0}&\lambda_{c1}&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{pmatrix}\begin{pmatrix}P_0\\ P_1\\ P_c\\ P_2(x)\\ P_3(x)\end{pmatrix},D(B)=X, \tag{27}$$

Therefore, the system equations (18)–(23) can be rewritten as an abstract Cauchy problem in Banach space:

$$\begin{cases}\frac{dP(t,x)}{dt}=(A+B)P(t,x),t\geq 0,\\ P(0,x)\triangleq P^0=(1,0,0,0,0)^T,\\ P(t,x)\triangleq\left(P_0(t),P_1(t),P_c(t),P_2(t,x),P_3(t,x)\right)^T.\end{cases} \tag{28}$$

Several lemmas are given below, which play a significant part in describing the semigroup characteristics of system operators.

Lemma 1 (see [26]). *When* $\gamma>0,\gamma\in\rho(A)$, *have* $\|(\gamma I-A)^{-1}\|<1/\gamma$.

Proof. For any $y=(y_0,y_1,y_c,y_2(x),y_3(x))\in X$, discuss the equation $(\gamma I-A)P=y$, which is equivalent to

$$\begin{cases}(\gamma+b_i)P_i=y_i,(i=0,1),\\ (\gamma+b_2)P_c=y_c,\\ \frac{dP_i(x)}{dx}=-(\gamma+\mu_i(x))P_i(x)+y_i(x),(i=2,3),\\ P_2(0)=\lambda P_1,P_3(0)=\lambda_sP_0+\lambda_sP_1.\end{cases} \tag{29}$$

Therefore,

$$\begin{cases} P_i = \dfrac{y_i}{\gamma + b_i}, \ (i = 0, 1), \\ P_c = \dfrac{y_c}{\gamma + b_2}, \\ P_i(x) = P_i(0) e^{-\gamma x - \int_0^x \mu_i(\xi) \mathrm{d}\xi} + e^{-\gamma x - \int_0^x \mu_i(\xi)\mathrm{d}\xi} \displaystyle\int_0^x e^{\gamma\tau + \int_0^\tau \mu_i(\xi)\mathrm{d}\xi} y_i(\tau)\mathrm{d}\tau, \ (i = 2, 3). \end{cases} \tag{30}$$

So, from equations (29) and (30) and Fubini theorem, we have

$$\begin{aligned}
\|P\| &= \sum_{i=0}^{1} |P_i| + |P_c| + \sum_{i=2}^{3} \|P_i(x)\|_{L^1[0,\infty]} \\
&\leq \sum_{i=0}^{1} |P_i| + |P_c| + \sum_{i=2}^{3} \left(\int_0^\infty |P_i(x)| \mathrm{d}x \right) \\
&= \sum_{i=0}^{1} |P_i| + |P_c| + \sum_{i=2}^{3} \left(P_i(0) \int_0^\infty e^{-\gamma x - \int_0^x \mu_i(\xi)\mathrm{d}\xi} \mathrm{d}x \right. \\
&\quad \left. + \int_0^\infty e^{-\gamma x - \int_0^x \mu_i(\xi)\mathrm{d}\xi} \mathrm{d}x \int_0^x e^{\gamma\tau + \int_0^\tau \mu_i(\xi)\mathrm{d}\xi} |y_i(\tau)| \mathrm{d}\tau \right) \\
&= \sum_{i=0}^{1} |P_i| + |P_c| + \sum_{i=2}^{3} \left(P_i(0) \int_0^\infty e^{-\gamma x - \int_0^x \mu_i(\xi)\mathrm{d}\xi} \mathrm{d}x \right. \\
&\quad \left. + \int_0^\infty |y_i(\tau)| e^{\gamma\tau + \int_0^\tau \mu_i(\xi)\mathrm{d}\xi} \mathrm{d}\tau \int_\tau^\infty e^{-\gamma x - \int_0^x \mu_i(\xi)\mathrm{d}\xi} \mathrm{d}x \right) \\
&\leq \sum_{i=0}^{1} |P_i| + |P_c| + \sum_{i=2}^{3} \left(P_i(0) \int_0^\infty e^{-\gamma x} \mathrm{d}x \right. \\
&\quad \left. + \int_0^\infty |y_i(\tau)| e^{\gamma\tau} \mathrm{d}\tau \int_\tau^\infty e^{-\gamma x} \mathrm{d}x \right) \\
&= \sum_{i=0}^{1} |P_i| + |P_c| + \sum_{i=2}^{3} \left(\frac{1}{\gamma} P_i(0) + \frac{1}{\gamma} \int_0^\infty |y_i(\tau)| \mathrm{d}\tau \right) \\
&= \frac{1}{\gamma + b_0} |y_0| + \frac{1}{\gamma + b_1} |y_1| + \frac{1}{\gamma + \mu_c} |y_c| + \frac{\lambda}{\gamma} \frac{1}{\gamma + b_1} |y_1| \\
&\quad + \frac{\lambda_s}{\gamma} \left(\frac{1}{\gamma + b_0} |y_0| + \frac{1}{\gamma + b_1} |y_1| \right) + \frac{1}{\gamma} \int_0^\infty |y_2(\tau)| \mathrm{d}\tau + \frac{1}{\gamma} \int_0^\infty |y_3(\tau)| \mathrm{d}\tau \\
&< \frac{1}{\gamma} \left(|y_0| + |y_1| + |y_c| + \|y_2\|_{L^1[0,\infty} + \|y_3\|_{L^1[0,\infty} \right) = \\
&= \frac{1}{\gamma} \|y\|.
\end{aligned} \tag{31}$$

Among them, $\int_0^\infty e^{-\gamma x}\mathrm{d}x = 1/\gamma, e^{-\gamma\tau} \le 1, \gamma > 0, \tau \in [0,\infty)$ and since $\gamma > 0$, $(\gamma I - A)^{-1}: X \longrightarrow X$ is a bounded linear operator. $\gamma \in \rho(A)$ and $\|(\gamma I - A)^{-1}\| < 1/\gamma$. □

Lemma 2 (see [27]). *$D(A)$ Dense in X.*

Proof. Suppose $P_i(t) \in C_0^\infty[0,\infty)$ and there is a constant C_i, so that for any $t \in [0, C_i]$, there are $P_i(t) = 0$, $(i = 2, 3)$. Construct a set

$$L = \left\{ \begin{array}{c} P(t) = (P_0, P_1, P_c, P_2(t), P_3(t)) \mid P_i(t) \in C_0^\infty[0,\infty), \text{and there is a constant } C_i > 0, \\ \text{makes for arbitrary } t \in [0, C_i], \text{always have } P_i(t) = 0, i = 2, 3. \end{array} \right\}. \tag{32}$$

At this time, using the knowledge of functional analysis, it is not difficult to verify that L is dense in Banach space X. Therefore, to verify that $D(A)$ is dense in X, just prove $D(A)$ is dense in L. For this, take $P = (P_0, P_1, P_c, P_2(t), P_3(t)) \in L$; then, for any $i (i = 2, 3)$, there is a constant $C_i > 0$; when $x \in [0, C_i]$, there is always$P_i(t) = 0$, so when $t \in [0, 2s]$, $0 < 2s < \min\{C_1, C_2\}$, there is always $P_i(t) = 0$, $(i = 2, 3)$. Let

$$\begin{aligned} f^s(0) &= (P_0, P_1, P_c, f_2^s(0), f_3^s(0)) = (P_0, P_1, P_c, \lambda P_1, \lambda_s P_0 + \lambda_s P_1), \\ f^s(t) &= (P_0, P_1, P_c, f_2^s(t), f_3^s(t)) = (P_0, P_1, P_c, P_2(t), P_3(t)). \end{aligned} \tag{33}$$

Among them

$$f_i^s(t) = \begin{cases} f_i^s(0)\left[1 - \dfrac{t}{s}\right]^2, & t \in [0, s), \\ -\mu_i (t-s)^2 (t-2s)^2, & t \in [s, 2s), \\ P_i(t), & t \in [2s, \infty). \end{cases} \quad i = 2, 3.$$

$$\mu_i = \frac{f_i^s(0)\int_0^{2s}\mu_i(t)[1-t/s]^2\mathrm{d}x}{\int_0^{2s}\mu_i(t)(t-s)^2(t-2s)^2\mathrm{d}t}, \ i = 2, 3. \tag{34}$$

At this point, it is easy to verify $f^s(t) \in D(A)$, and

$$\begin{aligned} \|P - f^s(t)\| &= \sum_{i=1}^{2}\int_0^\infty |P_i(t) - f_i^s(t)|\mathrm{d}t \\ &= \sum_{i=2}^{3}\int_0^{2s} |P_i(t) - f_i^s(t)|\mathrm{d}t \\ &= \sum_{i=2}^{3}\left(\int_0^s |f_i^s(0)|\left(1-\frac{t}{s}\right)^2\mathrm{d}t + \int_s^{2s}|\mu_i|(t-s)^2(t-2s)^2\mathrm{d}t\right) \\ &= \sum_{i=2}^{3}\left(-s\int_0^s |f_i^s(0)|\left(1-\frac{t}{s}\right)^2\mathrm{d}\left(1-\frac{t}{s}\right) + \int_s^{2s}|\mu_i|(t-s)^2\mathrm{d}\frac{(t-2s)^3}{3}\right) \\ &= \sum_{i=2}^{3}\left(-\frac{s}{3}|f_i^s(0)|\left(1-\frac{t}{s}\right)^3\Big|_0^s - \int_s^{2s}|\mu_i|2(t-s)\mathrm{d}\frac{(t-2s)^4}{12}\right) \\ &= \sum_{i=2}^{3}\left(\frac{s}{3}|f_i^s(0)| - 2|\mu_i|\frac{(t-s)(t-2s)^4}{12}\Bigg|_s^{2s} + \int_s^{2s}|\mu_i|\frac{(t-2s)^4}{6}\mathrm{d}(t-s)\right) \\ &= \sum_{i=2}^{3}\left(|f_i^s(0)|\frac{s}{3} + |\mu_i|\frac{s^5}{30}\right) \longrightarrow 0, \ (s \longrightarrow 0). \end{aligned} \tag{35}$$

The above mentioned formula indicates that $D(A)$ is dense in L, so $D(A)$ is dense in X. □

Lemma 3 (see [28]). *Operator $A+B$ creates positive C_0 compressed hemigroup $T(t)$.*

Proof. We split the proof this theorem into three steps.

The first step is to verify that $A+B$ creates C_0 hemigroup $T(t)$.

It can be verified by the demi-closed operator theorem that A is a closed linear operator; from the definition of B operator, B is a bounded linear operator, and

$$\|B\| \le \max\{\lambda+\alpha+\lambda_{c0}, \lambda_{c1}+\mu, \mu_c, M\}. \tag{36}$$

From the above mentioned proof and the Hille–Yosida generation theorem, we know that the operator A creates a C_0 hemigroup, and then from the perturbation theorem of the semigroup, the operator $A+B$ creates a C_0 hemigroup, denoted as $T(t)$.

The second step is to verify that $A+B$ creates a hemigroup $T(t)$ that is a positive C_0 semigroup.

In fact, when $y_i\,(i=0,1,c,2,3)$ is a nonnegative vector, P is a nonnegative vector, so $(\gamma I-A)^{-1}$ is a positive operator. At the same time, it is known from the meaning of B that B is also a positive operator. Hence,

$$(\gamma I-A-B)^{-1} = \left[I-(\gamma I-A)^{-1}B\right]^{-1}(\gamma I-A)^{-1}. \tag{37}$$

So, when $\gamma > \max\{\lambda+\alpha+\lambda_{c0}, \lambda_{c1}+\mu, \mu_c, M\}$, there is$\|(\gamma I-A)^{-1}\|<1$, which $[I-(\gamma I-A)^{-1}B]^{-1}$ exists and is bounded because

$$\left[I-(\gamma I-A)^{-1}B\right]^{-1} = \sum_{k=0}^{\infty}\left[(\gamma I-A)^{-1}B\right]^{k}. \tag{38}$$

Therefore, $[I-(\gamma I-A)^{-1}B]^{-1}$ is also a positive operator, so when $\gamma > \max\{\lambda+\alpha+\lambda_{c0}, \lambda_{c1}+\mu, \mu_c, M\}$, $(\gamma I-A-B)^{-1}$ is a positive operator, so the semigroup $T(t)$ created by $A+B$ is a positive C_0 semigroup.

The third step is to prove that $A+B$ creates positive C_0 compressed hemigroup $T(t)$.

In fact, whatever $P=(P_0,P_1,P_c,P_2(t),P_3(t))\in D(A)$, let

$$Q_P = \left[\frac{[P_0]^+}{P_0}, \frac{[P_1]^+}{P_1}, \frac{[P_c]^+}{P_c}, \frac{[P_2(x)]^+}{P_2(x)}, \frac{[P_3(x)]^+}{P_3(x)}\right]. \tag{39}$$

Among them $[P_i]^+=\max\{P_i,0\}$, $(i=0,1,c)$, $[P_i(x)]^+=\max\{P_i(x),0\}$, $(i=2,3)$.

Therefore,

$$\begin{aligned}
\langle (A+B)P, Q_P\rangle &= \left(-b_0P_0+\mu P_1+\mu_c P_c+\sum_{i=2}^{3}\int_0^\infty P_i(x)\mu_i(x)\mathrm{d}x\right)\frac{[P_0]^+}{P_0}\\
&\quad [(\lambda+\alpha)P_0-b_1P_1]\frac{[P_1]^+}{P_1}+(\lambda_{c0}P_0+\lambda_{c1}P_1-\mu_cP_c)\frac{[P_c]^+}{P_c}\\
&\quad -\sum_{i=2}^{3}\int_0^\infty\left(\frac{\mathrm{d}P_i(x)}{\mathrm{d}x}+\mu_i(x)P_i(x)\right)\frac{[P_i(x)]^+}{P_i(x)}\mathrm{d}x\\
&\le -b_0[P_0]^+ +\mu[P_1]^+ +\mu_c[P_c]^+ +\sum_{i=2}^{3}\int_0^\infty \mu_i(x)[P_i(x)]^+\mathrm{d}x\\
&\quad +(\lambda+\alpha)[P_0]^+ - b_1[P_1]^+ +\lambda_{c0}[P_0]^+ +\lambda_{c1}[P_1]^+ -\mu_c[P_c]^+\\
&\quad +\sum_{i=2}^{3}[P_i(0)]^+ - \sum_{i=2}^{3}\int_0^\infty \mu_i(x)[P_i(x)]^+\mathrm{d}x\\
&= -(\lambda+\alpha+\lambda_{c0}+\lambda_s)[P_0]^+ +\mu[P_1]^+ +\mu_c[P_c]^+ +(\lambda+\alpha)[P_0]^+\\
&\quad -(\mu+\lambda+\lambda_{c1}+\lambda_s)[P_1]^+ +\lambda_{c0}[P_0]^+ +\lambda_{c1}[P_1]^+ -\mu_c[P_c]^+\\
&\quad +\lambda[P_1]^+ +\lambda_s[P_0]^+ +\lambda_s[P_1]^+ = 0.
\end{aligned} \tag{40}$$

Then, $A+B$ is a diffusion operator. From the above mentioned proof and Pillips theorem, $A+B$ creates a positive contraction C_0 hemigroup $T(t)$. Therefore, by the uniqueness theorem of a C_0 hemigroup, $T(t)$ is a positive contraction C_0 hemigroup.

The well-posedness of the system solution is discussed below; because the system equations (18)–(23) are complex equations composed of differential, partial differential and integral, it is difficult to solve them directly. Therefore, in order to discuss the existence and uniqueness of the nonnegative solution of the system, system (18)–(23) is converted into the form of convolution Volterra integral equation. To this end, notation is introduced:

$$\begin{aligned} &b_0 = \lambda + \alpha + \lambda_{c0} + \lambda_s, \\ &b_1 = \mu + \lambda + \lambda_{c1} + \lambda_s \\ &\widetilde{P}_i(t,x) = \begin{cases} P_i(t,x), & 0 \le x \le t, \ (i = 2,3) \\ 0, & x > t. \end{cases} \\ &\widetilde{P}_i(t) = \begin{cases} P_i(t), & t \ge 0, \\ 0, & t < 0. \end{cases} (i = 0,1,c) \\ &\widetilde{\mu}_i(x) = \begin{cases} \mu_i(x), & x \ge 0, \\ 0, & x < 0. \end{cases} (i = 2,3) \end{aligned} \tag{41}$$

For convenience, $\widetilde{P}_i(t,x), \widetilde{P}_i(t), \widetilde{\mu}_i(x)$ is still represented by $P_i(t,x), P_i(t), \mu_i(x)$ in the following system equations. $P_i(t,x)$ can be obtained from the partial differential equations (21):

$$P_i(t,x) = P_i(t-x,0)e^{-\int_0^x \mu_i(\xi)\mathrm{d}\xi}, \ (i = 2,3). \tag{42}$$

First, substituting (42) into (18), then make $t - x = \tau$ as a variable to obtain

$$\begin{aligned} \frac{\mathrm{d}P_0(t)}{\mathrm{d}t} &= -b_0P_0(t) + \mu P_1(t) + \mu_c P_c(t) + \sum_{i=2}^{3} \int_0^{\infty} P_i(t,x)\mu_i(x)\mathrm{d}x \\ &= -b_0P_0(t) + \mu P_1(t) + \mu_c P_c(t) + \sum_{i=2}^{3} \int_0^{t} P_i(t-x,0)e^{-\int_0^x \mu_i(\xi)\mathrm{d}\xi}\mu_i(x)\mathrm{d}x \\ &= -b_0P_0(t) + \mu P_1(t) + \mu_c P_c(t) + \sum_{i=2}^{3} \int_0^{t} P_i(\tau,0)e^{-\int_0^{t-\tau} \mu_i(\xi)\mathrm{d}\xi}\mu_i(t-\tau)\mathrm{d}\tau. \end{aligned} \tag{43}$$

Since the initial condition $P_0(0) = 1$, $s - \tau = v$, $P_0(t)$ can be obtained as follows:

$$\begin{aligned} P_0(t) &= P_0(0)e^{-\int_0^t b_0\mathrm{d}x} + e^{-\int_0^t b_0\mathrm{d}x}\int_0^t \Big[\mu P_1(s) \\ &\quad + \mu_c P_c(s) + \sum_{i=2}^{3}\int_0^s P_i(\tau,0)e^{-\int_0^{s-\tau}\mu_i(\xi)\mathrm{d}\xi}\mu_i(s-\tau)\mathrm{d}\tau\Big]e^{\int_0^s b_0\mathrm{d}x}\mathrm{d}s \\ &= e^{-b_0t} + \int_0^t\Big[\mu P_1(s) + \mu_c P_c(s) + \sum_{i=2}^{3}\int_0^s P_i(\tau,0)e^{-\int_0^{s-\tau}\mu_i(\xi)\mathrm{d}\xi}\mu_i(s-\tau)\mathrm{d}\tau\Big]e^{-b_0(t-s)}\mathrm{d}s \\ &= e^{-b_0t} + \mu\int_0^t P_1(s)e^{-b_0(t-s)}\mathrm{d}s + \mu_c\int_0^t P_c(s)e^{-b_0(t-s)}\mathrm{d}s \\ &\quad + \sum_{i=2}^{3}\int_0^t\Big[\int_0^s P_i(\tau,0)e^{-\int_0^{s-\tau}\mu_i(\xi)\mathrm{d}\xi}\mu_i(s-\tau)\mathrm{d}\tau\Big]e^{-b_0(t-s)}\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= e^{-b_0 t} + \int_0^t \mu P_1(s) e^{-b_0(t-s)} \mathrm{d}s + \int_0^t \mu_c P_c(s) e^{-b_0(t-s)} \mathrm{d}s \\
&\quad + \sum_{i=2}^{3} \int_0^t P_i(\tau,0)\mathrm{d}\tau \int_0^{t-\tau} e^{-b_0(t-\tau)} e^{b_0 \nu - \int_0^{\nu} \mu_i(\xi)\mathrm{d}\xi} \mu_i(\nu)\mathrm{d}\nu \\
&= e^{-b_0 t} + \int_0^t \mu P_1(\tau) e^{-b_0(t-\tau)} \mathrm{d}\tau + \int_0^t \mu_c P_c(\tau) e^{-b_0(t-\tau)} \mathrm{d}\tau \\
&\quad + \sum_{i=2}^{3} \int_0^t P_i(\tau,0)\mathrm{d}\tau \int_0^{t-\tau} e^{-b_0(t-\tau)} e^{b_0 \nu - \int_0^{\nu} \mu_i(\xi)\mathrm{d}\xi} \mu_i(\nu)\mathrm{d}\nu \\
&= e^{-b_0 t} + \int_0^t k_0 \mu P_1(\tau)\mathrm{d}\tau + \int_0^t k_0 \mu_c P_c(\tau)\mathrm{d}\tau + \sum_{i=2}^{3} \int_0^t k_i(t-\tau) P_i(\tau,0)\mathrm{d}\tau.
\end{aligned} \tag{44}
$$

Among them,

$$
\begin{aligned}
k_0 &= e^{-b_0(t-\tau)}, \\
k_i(t-\tau) &= \int_0^{t-\tau} k_0 e^{b_0 \nu - \int_0^{\nu} \mu_i(\xi)\mathrm{d}\xi} \mu_i(\nu)\mathrm{d}\nu, i = 2, 3.
\end{aligned} \tag{45}
$$

Similarly, $P_1(t)$ and $P_c(t)$ can be solved by (19) and (20), respectively:

$$
\begin{aligned}
P_1(t) &= P_1(0) e^{-\int_0^t b_1 \mathrm{d}x} + e^{-\int_0^t b_1 \mathrm{d}x} \int_0^t (\lambda+\alpha) P_0(\tau) e^{\int_0^{\tau} b_1 \mathrm{d}x} \mathrm{d}\tau \\
&= \int_0^t e^{-b_1(t-\tau)} (\lambda+\alpha) P_0(\tau)\mathrm{d}\tau \\
&= \int_0^t (\lambda+\alpha) k_1 P_0(\tau)\mathrm{d}\tau.
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
P_c(t) &= P_c(0) e^{-\int_0^t \mu_c \mathrm{d}x} + e^{-\int_0^t \mu_c \mathrm{d}x} \int_0^t \left[\lambda_{c0} P_0(\tau) + \lambda_{c1} P_1(\tau)\right] e^{\int_0^{\tau} \mu_c \mathrm{d}x} \mathrm{d}\tau \\
&= \int_0^t e^{-\mu_c(t-\tau)} \left[\lambda_{c0} P_0(\tau) + \lambda_{c1} P_1(\tau)\right]\mathrm{d}\tau \\
&= \int_0^t \lambda_{c0} k_c P_0(\tau)\mathrm{d}\tau + \int_0^t \lambda_{c1} k_c P_1(\tau)\mathrm{d}\tau.
\end{aligned} \tag{47}
$$

Among them,

$$
k_1 = e^{-b_1(t-\tau)}, k_c = e^{-\mu_c(t-\tau)}. \tag{48}
$$

Substituting formulas (44) and (46) into formula (22), respectively:

$$
P_2(t,0) = \lambda P_1(t) = \int_0^t (\lambda+\alpha)\lambda k_1 P_0(\tau)\mathrm{d}\tau, \tag{49}
$$

$$
\begin{aligned}
P_3(t,0) &= \lambda_s P_0(t) + \lambda_s P_1(t) = \lambda_s e^{-b_0 t} + \int_0^t (\lambda+\alpha)\lambda_s k_1 P_0(\tau)\mathrm{d}\tau + \int_0^t k_0 \mu \lambda_s P_1(\tau)\mathrm{d}\tau + \int_0^t k_0 \mu_c \lambda_s P_c(\tau)\mathrm{d}\tau \\
&\quad + \sum_{i=2}^{3} \int_0^t \lambda_s k_i(t-\tau) P_i(\tau,0)\mathrm{d}\tau,
\end{aligned} \tag{50}
$$

Put expressions (44), (46), (47), (49), and (50) together to form the convolution Volterra integral equations:

$$\begin{cases} P_0(t) = e^{-b_0 t} + \int_0^t k_0\mu P_1(\tau)\mathrm{d}\tau + \int_0^t k_0\mu_c P_c(\tau)\mathrm{d}\tau + \sum_{i=2}^{3}\int_0^t k_i(t-\tau)P_i(\tau,0)\mathrm{d}\tau, \\ P_1(t) = \int_0^t (\lambda+\alpha)k_1 P_0(\tau)\mathrm{d}\tau, \\ P_c(t) = \int_0^t \lambda_{c0}k_c P_0(\tau)\mathrm{d}\tau + \int_0^t \lambda_{c1}k_c P_1(\tau)\mathrm{d}\tau, \\ P_2(t,0) = \lambda P_1(t) = \int_0^t (\lambda+\alpha)\lambda k_1 P_0(\tau)\mathrm{d}\tau, \\ P_3(t,0) = \lambda_s e^{-b_0 t} + \int_0^t (\lambda+\alpha)\lambda_s k_1 P_0(\tau)\mathrm{d}\tau + \int_0^t k_0\mu\lambda_s P_1(\tau)\mathrm{d}\tau \\ + \int_0^t k_0\mu_c\lambda_s P_c(\tau)\mathrm{d}\tau + \sum_{i=2}^{3}\int_0^t \lambda_s k_i(t-\tau)P_i(\tau,0)\mathrm{d}\tau. \end{cases} \tag{51}$$

And write the Volterra integral equation in vector form

$$P(t) = f(t) + \int_0^t k(t-\tau)P(\tau)\mathrm{d}\tau. \tag{52}$$

Among them,

$$\begin{aligned} P(t) &= \left(P_0(t), P_1(t), P_c(t), P_2(t,0), P_3(t,0)\right)^T, \\ f(t) &= \left(f_0(t), f_1(t), f_c(t), f_2(t,0), f_3(t,0)\right)^T \\ &= \left(e^{-b_0 t}, 0, 0, 0, \lambda_s e^{-b_0 t}\right), \\ k(t-\tau) &= \begin{pmatrix} 0 & \mu k_0 & \mu_c k_c & k_2(t-\tau) & k_3(t-\tau) \\ (\lambda+\alpha)k_1 & 0 & 0 & 0 & 0 \\ \lambda_{c0}k_c & \lambda_{c1}k_c & 0 & 0 & 0 \\ (\lambda+\alpha)\lambda k_1 & 0 & 0 & 0 & 0 \\ (\lambda+\alpha)\lambda_s k_1 & \lambda_s\mu k_0 & \lambda_s\mu_c k_0 & \lambda_s k_2(t-\tau) & \lambda_s k_3(t-\tau) \end{pmatrix}. \end{aligned} \tag{53}$$

According to the above mentioned equation, the following theorem holds. □

Theorem 4. *The existence and uniqueness of nonnegative solutions for the system (18)–(23) are a necessary and sufficient condition for the existence and uniqueness of nonnegative solutions for the Volterra integral equation (52).*

For any $T > 0$, from the expressions of $f(t)$ and $k(t-\tau)$, each component $f_i(t)(i = 0, 1, c, 2, 3)$ of $f(t)$ and each $k_i(t-\tau)(i = 2, 3)$ of $k(t-\tau)$ are nonnegative bounded functions, according to documents [36, 37] the following theorem holds.

Theorem 5. *The Volterra integral equation (52) has a unique nonnegative solution on $C[0, T]$.*

According to Theorem 5 and the specific expression of the solution of $P_i(t, x)(i = 2, 3)$, we know that $P_i(t, x)(i = 2, 3)$ has a unique nonnegative solution on $C[0, T]$. So, the Volterra integral (52) has a unique nonnegative solution on $C[0, T]$, that is, $P(t) = (P_0(t), P_1(t), P_c(t), P_2(t, 0), P_3(t, 0))^T$ exists and is unique on $C[0, T]$.

To sum up, the following theorem holds.

Theorem 6. *The system (18)–(23) has a unique nonnegative solution on $C[0, T]$.*

The main results of this paper and the well-posedness conclusions of the system solution are given below.

Theorem 7 (see [29]). *Abstract Cauchy problem has a sole nonnegative solution $P(t, x)$ which satisfies*

$$\|P(t,\cdot)\| = 1, T \geq 0. \tag{54}$$

Proof. From the above Lemmas 1–3, it can be seen that the abstract Cauchy problem has a sole nonnegative solution $P(t,x)$ which satisfies $P(t,x) = T(t)p(0) = T(t)(1,0,0,0,0)$, also $\|T(t)\| \leq 1$; therefore,

$$\|P(t,\cdot)\| = \|T(t)(1,0,0,0,0)\| \leq \|(1,0,0,0,0)\| = 1, t \geq 0. \quad (55)$$

At the same time, since $P_i(t,x)$, $(i = 2,3)$ satisfies the system model, if $\|P(t,\cdot)\|$ is regarded as time t function, there are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|P(t,\cdot)\| = \sum_{i=0}^{1}\frac{\mathrm{d}}{\mathrm{d}t}|P_i(t)| + \frac{\mathrm{d}}{\mathrm{d}t}|P_c(t)| \\ + \sum_{i=2}^{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty P_i(t,x)\mathrm{d}x = 0.\end{aligned} \quad (56)$$

Therefore, $\|P(t,\cdot)\| = \|P(0,\cdot)\| = 1$. □

Theorem 8 (see [30]). *0 is the simple eigenvalue of the operator $A + B$.*

Proof. Discuss the equation $(A + B)P = 0$, that is,

$$\begin{cases} -b_0P_0 + \mu P_1 + \mu_c P_c + \sum_{i=2}^{3}\int_0^\infty P_i(x)\mu_i(x)\mathrm{d}x = 0, \\ (\lambda + \alpha)P_0 - b_1P_1 = 0 \\ \lambda_{c0}P_0 + \lambda_{c1}P_1 - \mu_c P_c = 0 \\ \frac{\mathrm{d}P_i(x)}{\mathrm{d}x} + \mu_i(x)P_i(x) = 0, i = 2,3, \\ P_2(0) = \lambda P_1, P_3(0) = \lambda_s P_0 + \lambda_s P_1. \end{cases} \quad (57)$$

Solve the above equations to get

$$\begin{cases} P_i(x) = P_i(0)e^{-\int_0^x \mu_i(\tau)\mathrm{d}\tau}, i = 2,3, \\ -b_0P_0 + \mu P_1 + \mu_c P_c + \sum_{i=2}^{3}\int_0^\infty \mu_i(x)P_i(0)e^{-\int_0^x \mu_i(\tau)\mathrm{d}\tau}\mathrm{d}x = 0. \end{cases} \quad (58)$$

Assume $P_0 > 0$, from the system of equations (57) and (58),

$$\begin{cases} P_1 = \left(\frac{\lambda + \alpha}{b_1}\right)P_0, \\ P_c = \left(\frac{\lambda_{c0}b_1 + \lambda\lambda_{c1} + \alpha\lambda_{c1}}{\mu_c b_1}\right)P_0, \\ P_2(x) = \lambda\left(\frac{\lambda + \alpha}{b_1}\right)e^{-\int_0^x \mu_2(\tau)\mathrm{d}\tau}P_0, \\ P_3(x) = \lambda_s\left(\frac{\lambda + \alpha + b_1}{b_1}\right)e^{-\int_0^x \mu_3(\tau)\mathrm{d}\tau}P_0. \end{cases} \quad (59)$$

It can be seen that, $P_i(x) \in L^1[0,\infty), i = 2,3,$ so $P = (P_0, P_1, P_c, P_2(x), P_3(x))$ is the eigenvector of 0 eigenvalue corresponding to the operator $A + B$.

Take $Q = (1,1,1,1,1)$, then we have $\langle P, Q\rangle = \sum_{i=0}^{1}P_i + P_c + \sum_{i=2}^{3}\int_0^\infty P_i(x)\mathrm{d}x > 0$. For arbitrary, $P \in D(A + B)$ have $\langle (A + B)P, Q\rangle = 0$; thus, 0 is the simple eigenvalue of $A + B$. □

4. Reliability of the System Solution

The reliability of the system is one of the significant contents of the repairable system model. In the above system (18)–(23), because the size of $P_0(t)$ indicates the probability that both robots are in good condition and can work normally at $t = 0$, the larger the $P_0(t)$ is, the closer the system is to working properly; therefore, the size of $P_0(t)$ determines the probability that the system will work properly and thus gets one of the important factors affecting the reliability of the system.

So as to discuss the reliability of the system, several definitions are given at first.

Definition 9. $P_0(t)$ is called the transient reliability of system (18)–(23).

Definition 10. If $\lim_{t\rightarrow 0}P_0(t) = P_0^*$ exists, then P_0^* is called the stable-state reliability of the system (18)–(23).

Definition 11. If $P_0(t) \geq P_0^*$, the system is said to be reliable [31].

Theorem 12. *The system (18)–(23) has a time-dependent asymptotically steady stable-state solution $P^*(x)$, namely,*

$$\lim_{x\longrightarrow\infty} P(t,\cdot) = P^*(x) = (P_0, P_1, P_c, P_2(x), P_3(x)), \tag{60}$$

where $P^*(x) = (P_0, P_1, P_c, P_2(x), P_3(x))$ *is called the stable-state solution of the system (18)–(23).*

Theorem 13. *When the fault quotiety and fix quotiety are constants, the system (18)–(23) has reliability.*

Proof. Let the fault quotiety and fix quotiety be constants, i.e.,

$$\begin{aligned}\lambda &= \lambda_{c0} = \lambda_{c1} = \lambda_s = \alpha = \kappa,\\ \mu &= \mu_c = \mu_2(x) = \mu_3(x) = \beta.\end{aligned} \tag{61}$$

Simultaneously let

$$\begin{aligned}\frac{d}{dt}\int_0^\infty P_i(t,x)dx &= \int_0^\infty \frac{\partial P_i(t,x)}{\partial t}dx, i = 2,3,\\ P_i(t) &= \int_0^\infty P_i(t,x)dx, i = 2,3.\end{aligned} \tag{62}$$

Then, the actual physical background of the system (18)–(23) is $\sum_{i=0}^{3} P_i(t) + P_c(t) = 1$, which can be transformed into formula (18):

$$\frac{dP_0(t)}{dt} = -4\kappa P_0(t) + \beta P_1(t) + \beta P_c(t) + \beta P_2(t) + \beta P_3(t). \tag{63}$$

A system of ordinary differential equations can be converted from the upper system (18)–(23):

$$\begin{cases}\dfrac{dP_0(t)}{dt} = -(4\kappa+\beta)P_0(t) + \beta,\\ \dfrac{dP_1(t)}{dt} = -(3\kappa+\beta)P_1(t) + 2\kappa P_0(t),\\ \dfrac{dP_c(t)}{dt} = \kappa P_0(t) + \kappa P_1(t) - \beta P_c(t),\\ \dfrac{dP_2(t)}{dt} = \kappa P_1(t) - \beta P_2(t),\\ \dfrac{dP_3(t)}{dt} = \kappa P_0(t) + \kappa P_1(t) - \beta P_3(t),\\ P_0(0) = 1, P_1(0) = P_c(0) = P_2(0) = P_3(0) = 0.\end{cases} \tag{64}$$

At this point, if

$$\begin{aligned}P(t) &= \begin{pmatrix} P_0(t) & P_1(t) & P_c(t)\end{pmatrix}^T,\\ \mathbf{b} &= \begin{pmatrix}\beta & 0 & 0\end{pmatrix}^T,\\ A &= \begin{pmatrix} -(4\kappa+\beta) & 0 & 0\\ 2\kappa & -(3\kappa+\beta) & 0\\ \kappa & \kappa & -\beta\end{pmatrix}.\end{aligned} \tag{65}$$

Then, the first three equations of system (18)–(23) can be converted into an abstract Cauchy problem:

$$\begin{cases}\dfrac{dP(t)}{dt} = AP(t) + \mathbf{b},\\ P(0) = \begin{pmatrix}1 & 0 & 0\end{pmatrix}^T.\end{cases} \tag{66}$$

The solution of the system (66) is obtained by using ordinary differential equation theory and advanced algebra knowledge, including the following four steps:

Step 1: Find all eigenvalues of matrix A

$$\det(rE - A) = \begin{vmatrix} r + (4\kappa+\beta) & 0 & 0\\ -2\kappa & r + (3\kappa+\beta) & 0\\ -\kappa & -\kappa & r+\beta\end{vmatrix} = 0. \tag{67}$$

That is, $(r + 4\kappa + \beta)(r + 3\kappa + \beta)(r + \beta) = 0$, then the eigenvalues of the matrix A are $r_1 = -4\kappa - \beta$, $r_2 = -3\kappa - \beta, r_3 = -\beta$.

Step 2: Seek e^{At}, let

$$e^{At} = q_1(t)Q_0 + q_2(t)Q_1 + q_3(t)Q_2. \tag{68}$$

Among them,

$$Q_0 = E, Q_1 = -(r_1E - A), Q_2 = -(r_2E - A)Q_1, \tag{69}$$

$$\begin{aligned}q_1(t) &= e^{r_1 t},\\ q_2(t) &= \int_0^t e^{r_2(t-s)}q_1(s)ds,\\ q_3(t) &= \int_0^t e^{r_3(t-s)}q_2(s)ds.\end{aligned} \tag{70}$$

Substitute the formulas (69) and (70) into the formula (68) and sort them out

$$e^{At} = \begin{pmatrix} e^{-(4\kappa+\beta)t} & 0 & 0\\ e^{2\kappa t} & e^{-(3\kappa+\beta)t} & 0\\ e^{\kappa t} & e^{\kappa t} & e^{-\beta t}\end{pmatrix}. \tag{71}$$

Step 3: Find A^{-1}. From the knowledge of linear algebra, it is easy to find the inverse matrix A^{-1} of A, that is,

$$A^{-1} = \begin{pmatrix} -\dfrac{1}{4\kappa+\beta} & 0 & 0\\ -\dfrac{2\kappa}{(4\kappa+\beta)(3\kappa+\beta)} & -\dfrac{1}{3\kappa+\beta} & 0\\ -\dfrac{\kappa(5\kappa+\beta)}{(4\kappa+\beta)(3\kappa+\beta)\beta} & -\dfrac{\kappa}{\beta(3\kappa+\beta)} & -\dfrac{1}{\beta}\end{pmatrix}. \tag{72}$$

Step 4: Finding the solution of the system (66), we have

$$\begin{aligned}P(t) &= e^{\int_0^t A\mathrm{d}x}\left[\int_0^t e^{-\int_0^\tau A\mathrm{d}\xi}\mathbf{b}\mathrm{d}\tau + P(0)\right]\\ &= e^{At}\left[-\frac{1}{A}\int_0^t e^{-A\tau}\mathbf{b}\mathrm{d}(-A\tau) + P(0)\right]\\ &= e^{At}P(0) - A^{-1}\left(E - e^{At}\right)\mathbf{b}.\end{aligned} \tag{73}$$

Substitute the above formulas e^{At} and A^{-1} into the formula (73) and sort them out

$$P(t) = \begin{pmatrix} \dfrac{\beta}{4\kappa+\beta} + \dfrac{4\kappa}{4\kappa+\beta}e^{-(4\kappa+\beta)t} \\ \dfrac{2\kappa\beta}{(3\kappa+\beta)(4\kappa+\beta)} + \dfrac{6\kappa}{3\kappa+\beta}e^{-(3\kappa+\beta)t} - \dfrac{8\kappa}{4\kappa+\beta}e^{-(4\kappa+\beta)t} \\ \dfrac{5\kappa^2+\kappa\beta}{(3\kappa+\beta)(4\kappa+\beta)} - \dfrac{2\kappa}{3\kappa+\beta}e^{-(3\kappa+\beta)t} + \dfrac{\kappa}{4\kappa+\beta}e^{-(4\kappa+\beta)t} \end{pmatrix}. \tag{74}$$

So, the transient reliability of the system (18)–(23) is

$$P_0(t) = \frac{\beta}{4\kappa+\beta} + \frac{4\kappa}{4\kappa+\beta}e^{-(4\kappa+\beta)t}, \tag{75}$$

Thus, by definition of 10, the stable-state reliability of system (18)–(23) is

$$P_0^* = \lim_{t\longrightarrow\infty} P_0(t) = \frac{\beta}{4\kappa+\beta}, \tag{76}$$

From the above mentioned discussion, it is obvious that you can get $P_0(t) \geq P_0^*$, Therefore, according to the definition of 11, the system (18)–(23) is reliable. □

5. Controllability of System Zero State

Using the method of functional analysis to find a control element $\mu^* \in U$, to study the controllability of $P_0(t)$, $P_0(t)$ in the system model can be transferred to the specified state in a finite time $T\,(T > 0)$, and the allowable control set is selected as follows:

$$U = \left\{ \mu(x) \,\middle|\, \begin{array}{l} \mu(x) = (\mu_2(x), \mu_3(x)) \in L^\infty[0,\infty) \times L^\infty[0,\infty), 0 \leq \mu_i(x) < \infty \\ M = \sup\limits_{x\in[0,\infty} \mu_i(x) < \infty, \displaystyle\int_0^\infty \mu_i(x)\mathrm{d}x = \infty, i = 2, 3 \end{array} \right\}. \tag{77}$$

Theorem 14. *Let the failure rate $\lambda = \lambda_{c0} = \lambda_{c1} = \lambda_s = \alpha = \kappa$, the repair rate $\mu = \mu_c = \mu_2(x) = \mu_3(x) = \beta$, and let η be the probability that the system reaches the desired state at finite time $T\,(T > 0)$ and fits $e^{-4\kappa T} < \eta < 1$, then there is $\mu^* \in U$ such that $P_0(T) = \eta$.*

Proof. By the formula of (75),

$$P_0(T) = \frac{\beta}{4\kappa+\beta} + \frac{4\kappa}{4\kappa+\beta}e^{-(4\kappa+\beta)T}. \tag{78}$$

Considering $P_0(T)$ as a function of the variable β, we derive

$$\begin{aligned}\frac{\mathrm{d}P_0(T)}{\mathrm{d}\beta} &= \frac{4\kappa}{(4\kappa+\beta)^2} - \frac{4\kappa}{(4\kappa+\beta)^2}\\ &\quad\times e^{-(4\kappa+\beta)T}[1 + (4\kappa+\beta)T],\end{aligned} \tag{79}$$

since

$$e^x > x + 1, x > 0, \tag{80}$$

and therefore

$$e^{(4\kappa+\beta)T} > 1 + (4\kappa+\beta)T. \tag{81}$$

Thus, we have

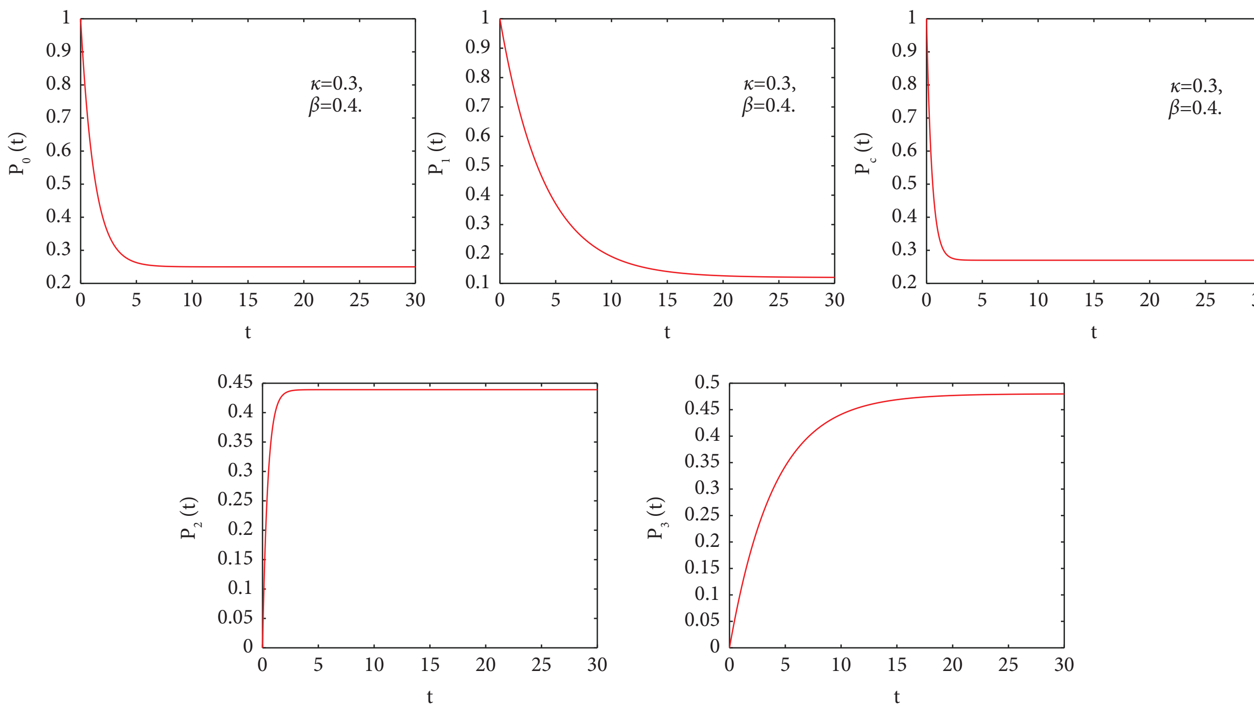

Figure 2: Numerical solution of the system (18)–(23) ($\mu_2(x) = \mu_3(x) = \text{constant}$).

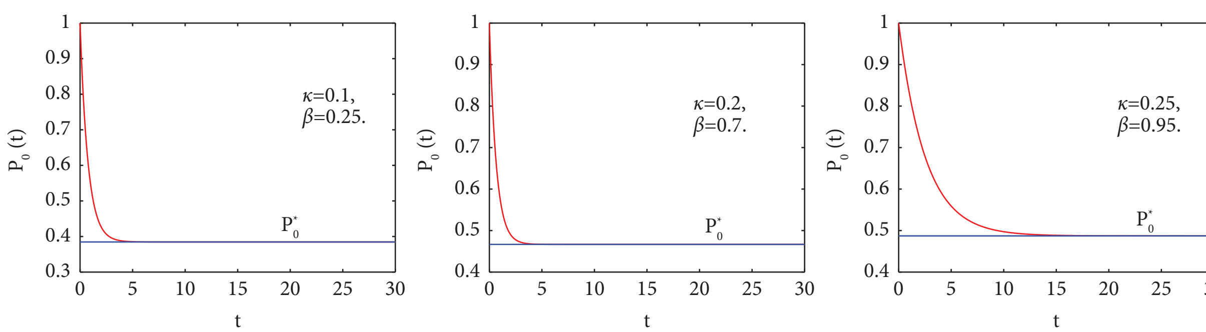

Figure 3: Transient reliability and stable-state reliability of the system (18)–(23).

$$\frac{dP_0(T)}{d\beta} \geq \frac{4\kappa}{(4\kappa+\beta)^2} - \frac{4\kappa}{(4\kappa+\beta)^2} \times e^{-(4\kappa+\beta)T} e^{(4\kappa+\beta)T} = 0. \quad (82)$$

This formula shows that $P_0(T)$ is a monotonically increasing function about the variable β, Note that $\lim_{\beta\to 0} P_0(T) = e^{-4\kappa T}$, $\lim_{\beta\to\infty} P_0(T) = 1$. Therefore, for any η: $e^{-4\kappa T} < \eta < 1$, according to the intermediate value theorem, there is $\beta^* \in U$, such that

$$P_0(\mathrm{T}) = \eta. \quad (83)$$

The above mentioned proves that the zero state $P_0(T)$ of the system is controllable.

Theorem 14 indicates that the zero state $P_0(t)$ of the system is controllable, in other words, the initial state $P_0(t)$ of the system is controllable, but it does not mean that other states of the system are controllable, that is, the conclusion of controllability of the whole system is not necessarily derived from zero-state controllability, Similarly, to prove that the system is completely controllable, it is also necessary to prove other states of the system, such as $P_1(t), P_c(t), P_2(t,x), P_3(t,x)$, etc. are controllable. □

6. Numerical Simulation

According to Theorem 7, there is a unique nonnegative time-dependent solution for the model of a repairable robot system with early warning function. In addition, the reliability of the two-robot security system with the function of early warning is discussed by using the theory and method of ordinary differential equation, On this basis, the controllability of the zero state of the system is proved by using the method of functional analysis. The following is a numerical simulation of the above results by using the numerical calculation method, which verifies the correctness of the results of the system reliability theory [32]. Assuming that the system failure rate and repair rate are constants κ, β, respectively, i.e.,

$$\begin{aligned} &\lambda = \lambda_{c0} = \lambda_{c1} = \lambda_s = \alpha = \kappa, \\ &\mu = \mu_c = \mu_2(x) = \mu_3(x) = \beta. \end{aligned} \tag{84}$$

To do this, as shown by the above proof of reliability Theorem 13, the system is converted into an ordinary differential equation system (64), solve the linear equations, we have

$$\begin{cases} P_0(t) = \dfrac{\beta}{4\kappa+\beta} + \dfrac{4\kappa}{4\kappa+\beta} e^{-(4\kappa+\beta)t}, \\ P_1(t) = \dfrac{2\kappa\beta}{(3\kappa+\beta)(4\kappa+\beta)} + \dfrac{6\kappa}{3\kappa+\beta} e^{-(3\kappa+\beta)t} - \dfrac{8\kappa}{4\kappa+\beta} e^{-(4\kappa+\beta)t}, \\ P_c(t) = \dfrac{5\kappa^2+\kappa\beta}{(3\kappa+\beta)(4\kappa+\beta)} - \dfrac{2\kappa}{3\kappa+\beta} e^{-(3\kappa+\beta)t} + \dfrac{\kappa}{4\kappa+\beta} e^{-(4\kappa+\beta)t}. \end{cases} \tag{85}$$

At this time, if $\kappa = 0.3, \beta = 0.4$, the numerical solution of system (18)–(23) can be obtained by using Matlab mathematical software, and the result is shown in Figure 2.

It can be seen from Figure 2 that the system equations (18)–(23) has a time-dependent asymptotically steady stable-state solution $P^*(x)$, which is consistent with the conclusion of the Theorem 12, at the same time, Theorem 12 indicates that the system equations (18)–(23) is steady. It should be pointed out here that the importance of the Theorem 12 lies in that it not only proves the asymptotic stability [33] of the solution of the system equations (18)–(23), but also proves the existence of the stable-state solution $P^*(x)$ of the system related to the stable-state reliability P_0^* of the two-robot safety system with early warning function.

In addition, by selecting $\kappa = 0.1, 0.2, 0.25, \beta = 0.25, 0.7, 0.95$, respectively, the 3 groups of instantaneous reliability $P_0(t)$ and stable-state reliability P_0^* of the system equations (18)–(23) can be obtained by using MATLAB software. For more intuitive comparison, put the charts of $P_0(t)$ and P_0^* together. The result is shown in Figure 3:

It can be seen from the Figure 3 that the transient reliability curves of the system equations (18)–(23) are all above the stable-state reliability curves [34], when $t \longrightarrow \infty$, $P_0(t) \longrightarrow P_0^*$ can be obtained by mathematical limit analysis, which is consistent with the reliability of the conclusion system equations (18)–(23) of the Theorem 13. To some extent, the above results show that the numerical method can reflect and depict that each state of the system tends to be stable with the change of time, and the results are in line with the actual situation of the robot security system.

7. Concluding Remarks

In this paper, the mathematical model of a two-robot safety system with an early warning function is studied, the semigroup characteristics of system operators are discussed by using linear operator semigroup theory, and the well-posedness of the solution of the system is proved. Under the assumption that the fault quotiety and fix quotiety of the system are constants, the early warning model equations are converted into ordinary differential equations, the transient reliability and stable-state reliability of the system are obtained, and the reliability and zero-state controllability of the system are proved. Finally, the numerical solution of the ordinary differential equation set of (64) is obtained by using Matlab mathematical software, and the graphs of the transient reliability and stable-state reliability of the system of equations (18)–(23) are simulated, which shows that the results obtained by numerical calculation and numerical simulation are consistent with the proof of the above theory [35].

The research on the repairable system model with an early warning function is mainly carried out by qualitative analysis, quantitative analysis, and the combination of qualitative analysis and quantitative analysis. Most of the existing literatures adopt qualitative analysis methods, but few use quantitative analysis methods. And the application of the qualitative analysis combined with the quantitative analysis method in the repairable system model is rarely reported in the repairable system model. First, the innovation of this paper is to combine the qualitative analysis with the quantitative analysis method in order to perfect and enrich the theory and method of repairable systems. Second, in the early warning system, when the warning prompt is invalid, the robot system model with an early warning function approaches to a repairable system model without an early warning function; the relationship between the early warning system and the nonearly warning system should be further studied, and the relationship between their steady-state solutions should be discussed. Finally, when studying the important indexes of system reliability, transient reliability, and steady-state reliability because of the relative complexity of the solution of the system (66) model, this paper uses the method of solving linear differential equations with constant coefficients that satisfy initial conditions, which is computationally complex. Whether the solution process can be simplified by the methods of Laplace transformation inversion, MATLAB mathematical software coding and compiling, and so on remains to be further studied.

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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