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Research Article

Sums of One Prime Power and Four Prime Cubes in Short Intervals

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Let $k \geqslant 1$ be an integer. In this study, we derive an asymptotic formula for the average number of representations of integers $n = p_1^k + p_2^3 + p_3^3 + p_4^3 + p_5^3$ in short intervals, where p_1, p_2, p_3, p_4, p_5 are prime numbers.

1. Introduction

Let $N, k_1, k_2, \dots, k_r$ be integers with $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_r$. The Waring–Goldbach problem for unlike powers of primes concerns the representation of N as the form

$$n = p_1^{k_1} + p_2^{k_2} + \cdots + p_r^{k_r}, \tag{1}$$

is classical. These topics have attracted mathematicians' attention.

In 1953, Prachar [1] considered the representation of N as the form

$$n = p_1^2 + p_2^3 + p_3^5 + p_4^k, \tag{2}$$

and he obtained the exceptional set is $O(N(\log N)^{-(30/47)+\epsilon})$. This result has been improved by Bauer [2], Bauer [3], and Zhao [4], and the latest result is $O(N^{1-(1/16)+\epsilon})$. For general $k \geqslant 5$, the best result was given by Hoffman and Yu [5] which is $O(N^{1-(47/420\cdot 2^s)+\epsilon})$ where $s = [k + 1/2]$.

In 1961, Schwarz et al. [6] also considered the representation of N as the form

$$n = p_1^2 + p_2^3 + p_3^6 + p_4^k, \tag{3}$$

and they obtained the exceptional set is $O(N(\log N)^{-A})$ for any fixed $A > 0$. Later, Brüdern [7] improved it to $O(N^{1-(1/8k^2)+\epsilon})$.

Recently, Feng and Ma [8] considered a special case,

$$n = p_1^k + p_2^3 + p_3^3 + p_4^3 + p_5^3, \tag{4}$$

with $k \geqslant 4$. Let $E(k, N)$ be the number of positive even integers n up to N which cannot be written in the form (4). They established that $E(k, N) \ll N^{1-\theta(k)+\epsilon}$, here

$$\theta(k) = \begin{cases} \dfrac{1}{24}, & k = 4, \\ \dfrac{1}{54}, & k = 5, \\ \dfrac{1}{9x}, & k \geqslant 6, \end{cases} \tag{5}$$

where

$$x = \begin{cases} \left[\dfrac{14k}{3} - 20\right], & k = 6, 7, \\ \left[\left(\dfrac{2k}{3} - \dfrac{1}{2}\left[\dfrac{2k}{3}\right]\right)\left(\left[\dfrac{2k}{3}\right] + 1\right)\right], & k \geqslant 8. \end{cases} \tag{6}$$

Let

$$r_k(n) = \sum_{n = p_1^k + p_2^3 + p_3^3 + p_4^3 + p_5^3} \log p_1 \log p_2 \log p_3 \log p_4 \log p_5. \tag{7}$$

In this study, we want to reconsider the result of Feng and Ma by studying the average behaviour of $r_k(n)$ over short intervals $[N, N+P]$ and $P = o(N)$.

Theorem 1. *Let $N \geqslant 2, 1 \leqslant P \leqslant N, k \geqslant 4$ be integers. Then, for every $\epsilon > 0$, there exists $C = C(\epsilon) \geqslant 0$, such that*

$$\sum_{n=N+1}^{N+P} r_k(n) = \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)} PN^{1/3+1/k} + O_k\left(PN^{1/3+1/k}\exp\left(-C\left(\frac{L}{\log L}\right)^{1/3}\right)\right), \tag{8}$$

as $N \longrightarrow \infty$, uniformly for $N^{1-5/(6k)+\epsilon} \leqslant P \leqslant N^{1-\epsilon}$ and Γ is Euler's function.

Comparing the result in Feng and Ma, from Theorem 1 we can say that, for N sufficiently large, every interval of size large than $N^{1-5/(6k)+\epsilon}$ contains the expected amount of integers which are a sum of one prime power and four prime cubes.

Assuming that the Riemann Hypothesis (RH) holds, we can further improve the size of P.

Theorem 2. *Let $\epsilon > 0, N \geqslant 2, 1 \leqslant P \leqslant N, k \geqslant 4$ be integers and assume the Riemann Hypothesis holds. Then, there exists a suitable positive constant B such that*

$$\sum_{n=N+1}^{N+P} r_k(n) = \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)} PN^{1/3+1/k} + O_k\left(P^2N^{1/k-2/3} + P^{1/2}N^{5/6+1/(2k)}L^B\right), \tag{9}$$

as $N \longrightarrow \infty$, uniformly for $\infty(N^{1-1/k}L^{2B}) \leqslant P \leqslant o(N)$. Here, $f = \infty(g)$ means $g = o(f)$ and Γ is Euler's function.

The proofs of Theorems 1 and 2 use the original Hardy–Littlewood circle method and the strategies adopted in the works of Languasco and Zaccagnini [9–11].

2. Preliminaries

In this study, we assume that N is a sufficiently large integer. Let $e(\alpha) = e^{2\pi i\alpha}, z = 1/N - 2\pi i\alpha, L = \log N$ and $l \geqslant 1$ be an integer,

$$\widetilde{S}_l(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^l/N} e\left(n^l\alpha\right),$$
$$\widetilde{V}_l(\alpha) = \sum_{p=2}^{\infty} (\log p) e^{-p^l/N} e\left(p^l\alpha\right). \tag{10}$$

We have

$$|z|^{-1} \ll \min\left(N, |\alpha|^{-1}\right). \tag{11}$$

We also set

$$U(\alpha, P) = \sum_{m=1}^{P} e(m\alpha). \tag{12}$$

From Montgomery [12], p. 39, we have

$$|U(\alpha, P)| \leqslant \min\left(P; |\alpha|^{-1}\right). \tag{13}$$

Now, we need some lemmas as follows.

Lemma 3 ([11], Lemma 3]). *Let $l \geqslant 1$ be an integer. Then, we have*

$$\left|\widetilde{S}_l(\alpha) - \widetilde{V}_l(\alpha)\right| \ll_l N^{1/(2l)}. \tag{14}$$

Lemma 4 ([10], Lemma 2]). *Let $N \geqslant 2$ be a sufficiently large integer and $l \geqslant 1$ be an integer. Then,*

$$\widetilde{S}_l(\alpha) = \frac{\Gamma(1/l)}{lz^{1/l}} - \frac{1}{l}\sum_{\varrho} z^{-\varrho/l}\Gamma(\varrho/l) + O_l(1), \tag{15}$$

where $\varrho = \beta + i\gamma$ runs over the nontrivial zeros of the Riemann-zeta function $\zeta(s)$.

Lemma 5 ([7], Lemma 4]). *Let $\mu > 0$. Then,*

$$\int_{-1/2}^{1/2} z^{-\mu} e(-n\alpha)\mathrm{d}\alpha = e^{-n/N}\frac{n^{\mu-1}}{\Gamma(\mu)} + O_\mu\left(\frac{1}{n}\right), \tag{16}$$

uniformly for $n \geqslant 1$.

Lemma 6 ([11], Lemma 1]). *Let $l \geqslant 1$ be an integer and ϵ be an arbitrarily small positive constant. Then, there exists a positive constant $c_1 = c_1(\epsilon)$, which does not depend on l, such that*

$$\int_{-\xi}^{\xi} \left|\widetilde{S}_l(\alpha) - \frac{\Gamma(1/l)}{lz^{1/l}}\right|^2 \mathrm{d}\alpha \ll_l N^{2/l-1}\exp\left(-c_1\left(\frac{L}{\log L}\right)^{1/3}\right), \tag{17}$$

uniformly for $0 \leqslant \xi \leqslant N^{-1+5/(6l)-\epsilon}$. Assuming RH holds we obtain

$$\int_{-\xi}^{\xi} \left|\widetilde{S}_l(\alpha) - \frac{\Gamma(1/l)}{lz^{1/l}}\right|^2 \mathrm{d}\alpha \ll_l N^{1/l}\xi L^2, \tag{18}$$

uniformly for $0 \leqslant \xi \leqslant 1/2$.

Lemma 7 ([9], Lemma 5]). *Let l, k be integers with $l \geqslant 1, 1 \leqslant k \leqslant l$. There exists a suitable positive constant $A = A(k, l)$, such that*

$$\int_{-1/2}^{1/2} \left|\widetilde{S}_l(\alpha)\right|^{2^k} \mathrm{d}\alpha \ll_{k,l} N^{(2^k-k)/l}L^A,$$
$$\int_{-1/2}^{1/2} \left|\widetilde{V}_l(\alpha)\right|^{2^k} \mathrm{d}\alpha \ll_{k,l} N^{(2^k-k)/l}L^A. \tag{19}$$

Lemma 8 ([9], Lemma 6]). *Let $l > 1$ and $\tau > 0$. Then, we have*

$$\int_{-\tau}^{\tau} \left|\widetilde{S}_l(\alpha)\right|^4 \mathrm{d}\alpha \ll_l \left(\tau N^{2/l} + N^{4/l-1}\right)N^{\epsilon},$$
$$\int_{-\tau}^{\tau} \left|\widetilde{V}_l(\alpha)\right|^4 \mathrm{d}\alpha \ll_l \left(\tau N^{2/l} + N^{4/l-1}\right)N^{\epsilon}. \tag{20}$$

Lemma 9 ([9], Lemma 7]). *Let* $l>2, c\geqslant 1$ *and* $N^{-c}\leqslant \omega \leqslant N^{2/l-1}$. *We also let* $I(\omega) := [-1/2,-\omega]\cup[\omega,1/2]$. *Then, we have*

$$\int_{I(w)} \left|\widetilde{S}_l(\alpha)\right|^4 \frac{d\alpha}{|\alpha|} \ll_l \frac{N^{4/l-1+\epsilon}}{\omega},$$
$$\int_{I(w)} \left|\widetilde{V}_l(\alpha)\right|^4 \frac{d\alpha}{|\alpha|} \ll_l \frac{N^{4/l-1+\epsilon}}{\omega}. \tag{21}$$

3. Proof of Theorem 1

Let $P>2B$ with

$$B = \exp\left(d\left(\frac{\log N}{\log\log N}\right)^{1/3}\right), \tag{22}$$

where $d=d(\epsilon)>0$ will be chosen later. Recalling (7), we can write

$$\sum_{n=N+1}^{N+P} e^{-n/N} r_k(n) = \int_{-1/2}^{1/2} \widetilde{V}_k(\alpha)\widetilde{V}_3(\alpha)^4 U(-\alpha,P)e(-N\alpha)\mathrm{d}\alpha. \tag{23}$$

We find it also convenient to set

$$\widetilde{E}_l(\alpha) = \widetilde{S}_l(\alpha) - \frac{\Gamma(1/l)}{lz^{1/l}}. \tag{24}$$

Let $I(B,P) = [-1/2,-B/P]\cup[B/P,1/2]$, we can obtain

$$\begin{aligned}
&\sum_{n=N+1}^{N+P} e^{-n/N} r_k(n)\\
&= \int_{-B/P}^{B/P} \frac{\Gamma(1/k)\Gamma(4/3)^4}{kz^{4/3+1/k}} U(-\alpha,P)e(-N\alpha)\mathrm{d}\alpha\\
&+ \int_{-B/P}^{B/P} \frac{\Gamma(1/k)}{kz^{1/k}}\left(S_3(\alpha)^4 - \frac{\Gamma(4/3)^4}{z^{4/3}}\right) U(-\alpha,P)e(-N\alpha)\mathrm{d}\alpha\\
&+ \int_{-B/P}^{B/P} \widetilde{E}_k(\alpha)\widetilde{S}_3(\alpha)^4 U(-\alpha,P)e(-N\alpha)\mathrm{d}\alpha\\
&+ \int_{-1/2}^{1/2} \widetilde{V}_k(\alpha)\left(\widetilde{V}_3(\alpha)^4 - \widetilde{S}_3(\alpha)^4\right) U(-\alpha,P)e(-N\alpha)\mathrm{d}\alpha\\
&+ \int_{-1/2}^{1/2} \widetilde{S}_3(\alpha)^4\left(\widetilde{V}_k(\alpha) - \widetilde{S}_k(\alpha)\right) U(-\alpha,P)e(-N\alpha)\mathrm{d}\alpha\\
&+ \int_{I(B,P)} \widetilde{S}_k(\alpha)\widetilde{S}_3(\alpha)^4 U(-\alpha,P)e(-N\alpha)\mathrm{d}\alpha\\
&:= \mathcal{F}_1+\mathcal{F}_2+\mathcal{F}_3+\mathcal{F}_4+\mathcal{F}_5+\mathcal{F}_6.
\end{aligned} \tag{25}$$

Now, we need to estimate these terms.

3.1. Estimate of $\mathcal{F}_1$. From the approximation $e^{-n/N} = e^{-1} + O(PN^{-1})$, Lemma 5, and (11) we can obtain

$$\begin{aligned}
\mathcal{F}_1 &= \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)} \sum_{n=N+1}^{N+P} n^{1/3+1/k} e^{-n/N} + O_k\left(\frac{P}{N}\right)\\
&+ O_k\left(\int_{B/P}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^{7/3+1/k}}\right)\\
&= \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)e} PN^{1/3+1/k}\\
&+ O_k\left(P^2N^{1/k-2/3} + N^{1/k+1/3} + \left(\frac{P}{B}\right)^{4/3+1/k}\right).
\end{aligned} \tag{26}$$

3.2. Estimate of $\mathcal{F}_2$. From the identity $f^2-g^2 = 2g(f-g)+(f-g)^2$, (24), and $\widetilde{S}_3(\alpha)\ll N^{1/3}$, we obtain

$$\begin{aligned}
&\frac{\Gamma(1/k)}{kz^{1/k}}\left(\widetilde{S}_3(\alpha)^4 - \frac{\Gamma(4/3)^4}{z^{4/3}}\right)\\
&= \frac{\Gamma(1/k)}{kz^{1/k}}\left(\widetilde{S}_3(\alpha)^2 + \frac{\Gamma(4/3)^2}{z^{2/3}}\right)\left(\frac{2\Gamma(4/3)}{z^{1/3}}\widetilde{E}_3(\alpha) + \widetilde{E}_3(\alpha)^2\right)\\
&= \frac{\Gamma(1/k)}{kz^{1/k}}\left(\widetilde{S}_3(\alpha)^2\frac{2\Gamma(4/3)}{z^{1/3}}\widetilde{E}_3(\alpha) + \widetilde{S}_3(\alpha)^2\widetilde{E}_3(\alpha)^2\right.\\
&\left. + \frac{2\Gamma(4/3)^3}{z}\widetilde{E}_3(\alpha) + \frac{\Gamma(4/3)^2}{z^{2/3}}\widetilde{E}_3(\alpha)\right)\\
&\ll \left|\widetilde{S}_3(\alpha)\right|^2 \frac{\left|\widetilde{E}_3(\alpha)\right|}{|z|^{1/3+1/k}} + \frac{\left|\widetilde{E}_3(\alpha)\right|}{|z|^{1+1/k}} + N^{2/3+1/k}\left|\widetilde{E}_3(\alpha)\right|^2.
\end{aligned} \tag{27}$$

Using (11) and (27), we obtain

$$\begin{aligned}
\mathcal{F}_2 &\ll P\int_{-B/P}^{B/P} \left|\widetilde{S}_3(\alpha)\right|^2 \frac{\left|\widetilde{E}_3(\alpha)\right|}{|z|^{1/3+1/k}}\mathrm{d}\alpha + P\int_{-B/P}^{B/P} \frac{\left|\widetilde{E}_3(\alpha)\right|}{|z|^{1+1/k}}\mathrm{d}\alpha\\
&+ PN^{2/3+1/k}\int_{-B/P}^{B/P} \left|\widetilde{E}_3(\alpha)\right|^2\mathrm{d}\alpha\\
&:= P\left(\mathcal{E}_1 + \mathcal{E}_2 + N^{2/3+1/k}\mathcal{E}_3\right).
\end{aligned} \tag{28}$$

By Lemma 6 we can obtain, for every $\epsilon>0$, there exists $c_1 = c_\epsilon > 0$ such that

$$\mathcal{E}_3 \ll_k N^{-1/3}\exp\left(-c_1\left(\frac{L}{\log L}\right)^{1/3}\right). \tag{29}$$

provided that $B/P\leqslant N^{-13/18-\epsilon}$, i.e., $P\geqslant BN^{13/18+\epsilon}$. By (11) and (29) and the Cauchy–Schwarz inequality we have, for every $\epsilon>0$, there exists $c_1 = c_\epsilon>0$, such that

$$\begin{aligned}\mathscr{E}_2 &\ll_k \left(\int_{-B/P}^{B/P}\left|\widetilde{E}_3(\alpha)\right|^2\mathrm{d}\alpha\right)^{1/2}\left(\int_{-1/N}^{1/N}N^{2+2/k}\mathrm{d}\alpha+2\int_{1/N}^{B/P}\frac{\mathrm{d}\alpha}{|\alpha|^{2+2/k}}\right)^{1/2}\\&\ll_k N^{1/3+1/k}\exp\left(-\frac{c_1}{2}\left(\frac{L}{\log L}\right)^{1/3}\right),\end{aligned}\tag{30}$$

provided that $P\geqslant BN^{13/18+\epsilon}$. By Lemma 7, (11), (29), and the Cauchy–Schwarz inequality we also have, for every $\epsilon>0$, there exists $c_1=c_\epsilon>0$ such that

$$\begin{aligned}\mathscr{E}_1 &\ll_k \left(\int_{-B/P}^{B/P}\left|\widetilde{E}_3(\alpha)\right|^2\mathrm{d}\alpha\right)^{1/2}\left(\int_{-B/P}^{B/P}\frac{\left|\widetilde{S}_3(\alpha)\right|^4}{|z|^{2/3+2/k}}\mathrm{d}\alpha\right)^{1/2}\\&\ll_k \left(\int_{-B/P}^{B/P}\left|\widetilde{E}_3(\alpha)\right|^2\mathrm{d}\alpha\right)^{1/2}\left(\int_{-1/2}^{1/2}\left|\widetilde{S}_3(\alpha)\right|^8\mathrm{d}\alpha\right)^{1/4}\left(\int_{-B/P}^{B/P}\frac{\mathrm{d}\alpha}{|z|^{4/3+4/k}}\right)^{1/4}\\&\ll_k \mathscr{E}_3^{1/2}N^{5/12}L^{A/4}\left(\int_{-B/P}^{B/P}\frac{\mathrm{d}\alpha}{|z|^{4/3+4/k}}\right)^{1/4}\\&\ll_k \mathscr{E}_3^{1/2}N^{5/12}L^{A/4}\left(\int_{-1/N}^{1/N}N^{4/3+4/k}\mathrm{d}\alpha+2\int_{1/N}^{B/P}\frac{\mathrm{d}\alpha}{\alpha^{4/3+4/k}}\right)^{1/4}\\&\ll_k \mathscr{E}_3^{1/2}N^{1/2+1/k}L^{A/4}\ll_k N^{1/3+1/k}\exp\left(-\frac{c_1}{4}\left(\frac{L}{\log L}\right)^{1/3}\right),\end{aligned}\tag{31}$$

provided that $P\geqslant BN^{13/18+\epsilon}$. Hence, by (28)–(31), we finally obtain that for every $\epsilon>0$, there exists $c_1=c_\epsilon>0$ such that

$$\mathscr{F}_2\ll_k PN^{1/3+1/k}\exp\left(-\frac{c_1}{4}\left(\frac{L}{\log L}\right)^{1/3}\right),\tag{32}$$

provided that $P\geqslant BN^{13/18+\epsilon}$.

3.3. Estimate of $\mathscr{F}_3$. Now, we estimate $\mathscr{F}_3$. By (13), Lemmas 6 and 7, and the Cauchy–Schwarz inequality, we have, for every $\epsilon>0$, there exists $c_1=c_\epsilon>0$ such that

$$\begin{aligned}\mathscr{F}_3 &\ll_k \left(\int_{-1/2}^{1/2}\left|\widetilde{S}_3(\alpha)\right|^8\mathrm{d}\alpha\right)^{1/2}\left(\int_{-B/P}^{B/P}\left|\widetilde{E}_k(\alpha)\right|^2|U(\alpha,P)|^2\mathrm{d}\alpha\right)^{1/2}\\&\ll_k PN^{5/6}L^{A/2}\left(\int_{-B/P}^{B/P}\left|\widetilde{E}_k(\alpha)\right|^2\mathrm{d}\alpha\right)^{1/2}\\&\ll_k PN^{1/3+1/k}\exp\left(-\frac{c_1}{2}\left(\frac{L}{\log L}\right)^{1/3}\right),\end{aligned}\tag{33}$$

provided that $P \geqslant BN^{1-5/(6k)+\epsilon}$.

3.4. Estimate of $\mathcal{F}_4$. From Lemma 3 and $\widetilde{V}_k(\alpha) \ll_k N^{1/k}$, we have

$$\begin{aligned}\widetilde{V}_k(\alpha)\left(\widetilde{V}_3(\alpha)^4 - \widetilde{S}_3(\alpha)^4\right) &\ll_k \left|\widetilde{V}_k(\alpha)\right|\left|\widetilde{V}_3(\alpha) - \widetilde{S}_3(\alpha)\right|\left(\left|\widetilde{V}_3(\alpha)\right| + \left|\widetilde{S}_3(\alpha)\right|\right)^3 \\ &\ll_k N^{1/6+1/k}\max\left(\left|\widetilde{V}_3(\alpha)\right|^3, \left|\widetilde{S}_3(\alpha)\right|^3\right).\end{aligned} \tag{34}$$

Then, we have

$$\begin{aligned}\mathcal{F}_4 &\ll_k N^{1/6+1/k}\int_{-1/2}^{1/2}\left(\left|\widetilde{V}_3(\alpha)\right|^3 + \left|\widetilde{S}_3(\alpha)\right|^3\right)|U(-\alpha, P)|\mathrm{d}\alpha \\ &\coloneqq N^{1/6+1/k}(J_1 + J_2).\end{aligned} \tag{35}$$

By (13), Lemmas 8, and 9, we have

$$\begin{aligned}J_2 &\ll P\int_{-1/P}^{1/P}\left|\widetilde{S}_3(\alpha)\right|^3\mathrm{d}\alpha + \int_{I(1,P)}\left|\widetilde{S}_3(\alpha)\right|^3\frac{\mathrm{d}\alpha}{|\alpha|} \\ &\ll P^{3/4} \leqslant \left(\int_{-1/P}^{1/P}\left|\widetilde{S}_3(\alpha)\right|^4\mathrm{d}\alpha\right)^{3/4} \\ &\quad + \leqslant\left(\int_{I(1,P)}\left|\widetilde{S}_3(\alpha)\right|^4\frac{\mathrm{d}\alpha}{|\alpha|}\right)^{3/4} \leqslant \left(\int_{I(1,P)}\frac{\mathrm{d}\alpha}{|\alpha|}\right)^{1/4} \\ &\ll P^{3/4}N^{1/4+\epsilon} + \left(HN^{1/3+\epsilon}\right)^{3/4}L^{1/4} \\ &\ll P^{3/4}N^{1/4+\epsilon},\end{aligned} \tag{36}$$

provided that $P \gg N^{1/3}$. Similarly, we have

$$J_1 \ll P^{3/4}N^{1/4+\epsilon}. \tag{37}$$

By (35)–(37), we have

$$\mathcal{F}_4 \ll_k P^{3/4}N^{5/12+1/k+\epsilon}, \tag{38}$$

provided that $P \gg N^{1/3}$.

3.5. Estimate of $\mathcal{F}_5$. By Lemmas 3, 8, 9, (13), and a partial integration, we have

$$\begin{aligned}\mathcal{F}_5 &\ll_k PN^{1/(2k)}\int_{-1/P}^{1/P}\left|\widetilde{S}_3(\alpha)\right|^4\mathrm{d}\alpha + N^{1/(2k)}\int_{I(1/P)}\frac{\left|\widetilde{S}_3(\alpha)\right|^4}{|\alpha|}\mathrm{d}\alpha \\ &\ll_k PN^{1/(2k)}\left(\frac{N^{2/3+\epsilon}}{P} + N^{1/3+\epsilon}\right) + PN^{1/(2k)}N^{1/3+\epsilon} \\ &\ll_k N^{1/(2k)}\left(N^{2/3+\epsilon} + PN^{1/3+\epsilon}\right),\end{aligned} \tag{39}$$

provided that $P \gg N^{1/3}$.

3.6. Estimate of $\mathcal{F}_6$. Clearly, by Lemma 9 and (13), we have

$$\begin{aligned}\mathcal{F}_6 &\ll_k N^{1/k}\int_{I(B,P)}\frac{\left|\widetilde{S}_3(\alpha)\right|^4}{|\alpha|}\mathrm{d}\alpha \\ &\ll_k \frac{P}{B}N^{1/3+1/k+\epsilon},\end{aligned} \tag{40}$$

provided that $P \gg BN^{1/3}$.

3.7. Completion of the Proof. Let $k \geqslant 4$. By (26)–(40) we can obtain, for every $\epsilon > 0$, there exists $c_1 = c_1(\epsilon) > 0$, such that

$$\begin{aligned}&\sum_{n=N+1}^{N+P} e^{-n/N}r_k(n) \\ &= \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)e}PN^{1/3+1/k} \\ &\quad + O_k\left(PN^{1/3+1/k}\exp\left(-\frac{c_1}{4}\left(\frac{L}{\log L}\right)^{1/3}\right) + \frac{P}{B}N^{1/3+1/k+\epsilon}\right),\end{aligned} \tag{41}$$

provided that $P \geqslant BN^{1-5/(6k)+\epsilon}$. We choose $d = c_1$ in (22). Then, for every $k \geqslant 4$, we have for every $\epsilon > 0$, there exists $C = C(\epsilon) > 0$, such that

$$\begin{aligned}\sum_{n=N+1}^{N+P} e^{-n/N}r_k(n) &= \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)e}PN^{1/3+1/k} \\ &\quad + O_k\left(PN^{1/3+1/k}\exp\left(-C\left(\frac{L}{\log L}\right)^{1/3}\right)\right),\end{aligned} \tag{42}$$

provided that $P \geqslant N^{1-5/(6k)+\epsilon}$. We note that $e^{-n/N} = e^{-1} + O(P/N)$ for $n \in [N+1, N+P]$, $1 \leqslant P \leqslant N$. Then, we have for every $\epsilon > 0$, there exists $C = C(\epsilon) > 0$, such that

$$\begin{aligned}\sum_{n=N+1}^{N+P} r_k(n) &= \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)}PN^{1/3+1/k} \\ &\quad + O_k\left(PN^{1/3+1/k}\exp\left(-C\left(\frac{L}{\log L}\right)^{1/3}\right)\right) \\ &\quad + O_k\left(\frac{P}{N}\sum_{n=N+1}^{N+P} r_k(n)\right),\end{aligned} \tag{43}$$

provided that $P \leqslant N$ and $P \geqslant N^{1-5/(6k)+\epsilon}$ for $k \geqslant 4$. Using $e^{n/N} \leqslant e^2$ and (41), the last error term is $\ll_k P^2 N^{1/k-2/3}$. Thus,

$$\sum_{n=N+1}^{N+P} r_k(n) = \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)} P N^{1/3+1/k} + O_k\left(P N^{1/3+1/k} \exp\left(-C\left(\frac{L}{\log L}\right)^{1/3}\right)\right), \tag{44}$$

uniformly for $N^{1-5/(6k)+\epsilon} \leqslant P \leqslant N^{1-\epsilon}$. Now, Theorem 1 follows.

4. Proof of Theorem 2

Let $k \geqslant 4$, $P \geqslant 2$, and $P = o(N)$ be an integer. We recall that we set $L = \log N$ for brevity. From now on we assume that RH holds, we may write

$$\sum_{n=N+1}^{N+P} e^{-n/N} r_k(n) = \int_{-1/2}^{1/2} \widetilde{V}_k(\alpha)\widetilde{V}_3(\alpha)^4 U(-\alpha, P) e(-N\alpha)\mathrm{d}\alpha. \tag{45}$$

In this conditional case, we can simplify the proof. Recalling Lemma 4 and (24), we have

$$\begin{aligned}\sum_{n=N+1}^{N+P} e^{-n/N} r_k(n) &= \int_{-1/2}^{1/2} \frac{\Gamma(1/k)\Gamma(4/3)^4}{k z^{4/3+1/k}} U(-\alpha, P) e(-N\alpha)\mathrm{d}\alpha \\ &+ \int_{-1/2}^{1/2} \frac{\Gamma(1/k)}{k z^{1/k}} \left(S_3(\alpha)^4 - \frac{\Gamma(4/3)^4}{z^{4/3}}\right) U(-\alpha, P) e(-N\alpha)\mathrm{d}\alpha \\ &+ \int_{-1/2}^{1/2} \widetilde{E}_k(\alpha)\widetilde{S}_3(\alpha)^4 U(-\alpha, P) e(-N\alpha)\mathrm{d}\alpha \\ &+ \int_{-1/2}^{1/2} \widetilde{V}_k(\alpha)\left(\widetilde{V}_3(\alpha)^4 - \widetilde{S}_3(\alpha)^4\right) U(-\alpha, P) e(-N\alpha)\mathrm{d}\alpha \\ &+ \int_{-1/2}^{1/2} \widetilde{S}_3(\alpha)^4\left(\widetilde{V}_k(\alpha) - \widetilde{S}_k(\alpha)\right) U(-\alpha, P) e(-N\alpha)\mathrm{d}\alpha \\ &\coloneqq \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5.\end{aligned} \tag{46}$$

Now, we can evaluate these terms.

4.1. Evaluation of $\mathcal{I}_1$. Using Lemma 5, a direct calculation gives

$$\begin{aligned}\mathcal{I}_1 &= \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)} \sum_{n=N+1}^{N+P} n^{1/3+1/k} e^{-n/N} + O_k\left(\frac{P}{N}\right) \\ &= \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)e} P N^{1/3+1/k} + O_k\left(P^2 N^{1/k-2/3} + N^{1/3+1/k}\right).\end{aligned} \tag{47}$$

4.2. Evaluation of $\mathcal{I}_2$. By (27), we have

$$\begin{aligned}\mathcal{I}_2 &\ll \int_{-1/2}^{1/2} \frac{\left|\widetilde{S}_3(\alpha)\right|^2}{|z|^{1/3+1/k}} \left|\widetilde{E}_3(\alpha)\right|\left|U(-\alpha, P)\right|\mathrm{d}\alpha + \int_{-1/2}^{1/2} \frac{\widetilde{E}_3(\alpha)}{|z|^{1+1/k}} |U(\alpha, P)|\mathrm{d}\alpha \\ &+ \int_{-1/2}^{1/2} N^{2/3} \frac{\left|\widetilde{E}_3(\alpha)\right|^2}{|z|^{1/k}} |U(\alpha, P)|\mathrm{d}\alpha \\ &\coloneqq \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3.\end{aligned} \tag{48}$$

Let

$$\psi = \int_{-1/2}^{1/2} \frac{\left|\widetilde{E}_3(\alpha)\right|^2}{|z|^{1/k}} |U(-\alpha, P)|\mathrm{d}\alpha. \tag{49}$$

Using Lemma 6, (13), and integration by parts, we have

$$\begin{aligned}\psi &\ll P N^{1/k} \int_{-1/N}^{1/N} \left|\widetilde{E}_3(\alpha)\right|^2 \mathrm{d}\alpha + 2P \int_{1/N}^{1/P} \frac{\left|\widetilde{E}_3(\alpha)\right|^2}{\alpha^{1/k}} \mathrm{d}\alpha + 2\int_{1/P}^{1/2} \frac{\left|\widetilde{E}_3(\alpha)\right|^2}{\alpha^{1+1/k}} \mathrm{d}\alpha \\ &\ll_k P N^{1/k-2/3} L^2 + P^{1/k} N^{1/3} L^2 \ll_k P^{1/k} N^{1/3} L^2.\end{aligned} \tag{50}$$

Then, we have

$$\mathscr{K}_3 \ll_k P^{1/k} N L^2. \tag{51}$$

By the Cauchy–Schwarz inequality, (11), (13), and (50), we have

$$\begin{aligned}\mathscr{K}_2 &\ll \psi^{1/2}\left(\int_{-1/2}^{1/2} \frac{|U(-\alpha, P)|}{|z|^{2+1/k}} \mathrm{d}\alpha\right)^{1/2} \\ &\ll \psi^{1/2}\left(PN^{2+1/k}\int_{-1/N}^{1/N} \mathrm{d}\alpha + 2P\int_{1/N}^{1/P} \frac{\mathrm{d}\alpha}{\alpha^{2+1/k}} + 2\int_{1/P}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^{3+1/k}}\right)^{1/2} \\ &\ll_k P^{1/2+1/(2k)} N^{2/3+1/(2k)} L.\end{aligned} \tag{52}$$

By the Cauchy–Schwarz inequality, (11), (13), and (50), we have

$$\begin{aligned}\mathscr{K}_1 &\ll \psi^{-1/2}\left(\int_{-1/2}^{1/2} \left|\widetilde{S}_3(\alpha)\right|^8 \mathrm{d}\alpha\right)^{1/4}\left(\int_{-1/2}^{1/2} \frac{|U(-\alpha, P)|^2}{|z|^{4/3+2/k}} \mathrm{d}\alpha\right)^{1/4} \\ &\ll P^{1/(2k)} N^{1/6} L N^{5/12} L^{A/4}\left(P^2 N^{4/3+2/k}\int_{-1/N}^{1/N} \mathrm{d}\alpha\right) \\ &\quad + 2P^2\int_{1/N}^{1/P} \frac{\mathrm{d}\alpha}{\alpha^{4/3+2/k}} + 2\int_{1/P}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^{10/3+2/k}} \\ &\ll_k P^{1/2+1/(2k)} N^{2/3+1/(2k)} L^{1+A/4}.\end{aligned} \tag{53}$$

Summing up by (48)–(54), we have

$$\mathscr{I}_2 \ll_k P^{1/2+1/(2k)} N^{2/3+1/(2k)} L^{1+A/4}, \tag{54}$$

for every $k \geqslant 4$.

4.3. Evaluation of $\mathscr{I}_3$. Using the Cauchy–Schwarz inequality, (13), and Lemma 6, we can obtain

$$\begin{aligned}\mathscr{I}_3 &\ll_k \left(\int_{-1/2}^{1/2} \left|\widetilde{S}_3(\alpha)\right|^8 \mathrm{d}\alpha\right)^{1/2}\left(\int_{-1/2}^{1/2} \left|\widetilde{E}_k(\alpha)\right|^2 |U(\alpha, P)|^2 \mathrm{d}\alpha\right)^{1/2} \\ &\ll_k N^{5/6} L^{A/2}\left(P^2\int_{-1/P}^{1/P} \left|\widetilde{E}_k(\alpha)\right|^2 \mathrm{d}\alpha + 2\int_{1/P}^{1/2} \left|\widetilde{E}_k(\alpha)\right|^2 \frac{\mathrm{d}\alpha}{\alpha^2}\right)^{1/2} \\ &\ll_k P^{1/2} N^{5/6+1/(2k)} L^{1+A/2}.\end{aligned} \tag{55}$$

4.4. Evaluation of $\mathscr{I}_4$. Clearly, $\mathscr{I}_4 = \mathscr{F}_4$ of Section 3.4. So, we can obtain

$$\mathscr{I}_4 \ll_k P^{3/4} N^{5/12+1/k+\epsilon}, \tag{56}$$

provided that $P \gg N^{1/3}$.

4.5. Evaluation of $\mathscr{I}_5$. Clearly, $\mathscr{I}_5 = \mathscr{F}_5$ of Section 3.5. So, we can obtain

$$\mathscr{I}_5 \ll_k N^{1/(2k)}\left(N^{2/3+\epsilon} + PN^{1/3+\epsilon}\right), \tag{57}$$

provided that $P \gg N^{1/3}$.

4.6. Completion of the Proof. By (46) and (47) and (54)–(57), there exist $B = B(A)$ such that for $P \geqslant N^{1/3}$,

$$\sum_{n=N+1}^{N+P} e^{-n/N} r_k(n) = \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)e} PN^{1/3+1/k} + O_k\left(P^2 N^{1/k-2/3} + P^{1/2} N^{5/6+1/(2k)} L^B\right), \tag{58}$$

which is an asymptotic formula for $\infty(N^{1-1/k}L^{2B}) \leqslant P \leqslant o(N)$. From $e^{-n/N} = e^{-1} + O(P/N)$ for $n \in [N+1, N+P]$ and $1 \leqslant P \leqslant N$, we can obtain the following:

$$\sum_{n=N+1}^{N+P} r_k(n) = \frac{\Gamma(1/k)\Gamma(4/3)^4}{k\Gamma(4/3+1/k)} PN^{1/3+1/k} + O_k\left(P^2 N^{1/k-2/3} + P^{1/2} N^{5/6+1/(2k)} L^B\right) + O_k \leqslant \left(\frac{P}{N}\sum_{n=N+1}^{N+P} r_k(n)\right). \tag{59}$$

Using $e^{n/N} \leqslant e^2$ and (58), the last error term is dominated by all of the previous ones. Thus, Theorem 2 follows.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Gen Li wrote Section 2 and Section 4, Xianjiu Huang wrote Section 1 and some English expressions in Sections 2–4, Xiaoming Pan wrote Sections 3.1–3.3, and Li Yang wrote Sections 3.4–3.7, abstract, and references.

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