Research Article

Fractional Optimal Control Model of SARS-CoV-2 (COVID-19) Disease in Ghana

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Research focus on optimal control problems brought on by fractional differential equations has been extensively applied in practice. However, because they are still open ended and challenging, a number of problems with fractional mathematical modeling and problems with optimal control require additional study. Using fractional-order derivatives defined in the Atangana–Baleanu–Caputo sense, we alter the integer-order model that has been proposed in the literature. We prove the solution’s existence, uniqueness, equilibrium points, fundamental reproduction number, and local stability of the equilibrium points. The operator’s numerical approach was put into practice to obtain a numerical simulation to back up the analytical conclusions. Fractional optimum controls were incorporated into the model to identify the most efficient intervention strategies for controlling the disease. Utilizing actual data from Ghana for the months of March 2020 to March 2021, the model is validated. The simulation’s results show that the fractional operator significantly affected each compartment and that the incidence rate of the population rose when \( v \geq 0.6 \). The examination of the most effective control technique discovered that social exclusion and vaccination were both very effective methods for halting the development of the illness.

1. Introduction

The large family of viruses known as coronaviruses is responsible for a number of illnesses, including the Middle East respiratory syndrome (MERS), common cold, and severe acute respiratory syndrome (SARS-CoV-2). Never have humans been previously exposed to this novel coronavirus strain [1]. When an infected person coughs, breathes, sneezes, or talks, tiny droplets or particles, such as aerosols, are discharged into the air and transmit the disease [2, 3]. The illness’s usual symptoms include fever, coughing, exhaustion, shortness of breath, or other breathing problems, as well as loss of taste and smell [4–6]. Over 6 million people had died, and there were over 513 million confirmed infections worldwide as of May 1, 2022. Africa as a whole has recorded 11 million confirmed cases, with Ghana accounting for 161, 173 of those cases [7].

Numerous scholars have created models for the realization and control of the spread of transmissible illnesses in a population [8]. The representation of infectious diseases using fractional calculus has attracted a lot of attention lately. Examples include malaria [9], TB [10, 11], syphilis [12], chickenpox [13], and most recently COVID-19 [8, 14–23]. Fractional calculus, which is a generalization of differentiation and integration of integer orders, has been proposed to address several of the constraints of integer...
order derivatives [14]. The fractional order may be able to depict more complex dynamics and include memory effects, which are prevalent in many real-world occurrences, in comparison to the integer model [10, 15]. Using data from China, Italy, and France, Bahloul et al. [16] proposed a fractional-order SEIQRDP model to study the COVID-19 pandemic. To examine the illness transmission in Spain, the researchers in Ref. [17] proposed a new SEIRS dynamical model that uses the fractional-order derivative and adds the vaccine rate. The authors of Ref. [18] presented a Caputo fractional SIR epidemic model taking a nonlinear incidence rate into consideration. Regarding the fractal-fractional Atangana–Baleanu derivative, Khan and Atangana [19] considered a fractional model to describe the transmission of COVID-19 while accounting for the isolation and quarantine of individuals.

There are several real-world uses for research on optimum control problems brought on by fractional differential equations. However, there are a number of difficulties and problems with fractional mathematical modeling as well as challenges with optimal control that need further study. A general formulation for the optimal control problem for a family of fuzzy fractional differential systems connected to SIR and SEIR epidemic models was developed by Das and Samanta [24] using real data from Italy and South Korea. Khan et al. [8] explored a fractional COVID-19 epidemic model that included fractional optimum control and had a convex incidence rate in the sense of Atangana–Baleanu and Caputo. Baba and Bilgehan [21] devised a fractional optimum control issue that incorporates public awareness and treatment for the disease outbreak using a mathematical model with a fractional-order derivative in the Caputo sense. Nabi et al. [22] created a compartmental model combining all workable nonpharmaceutical intervention options using the classical and Caputo–Fabrizio fractional-order derivatives to study illness transmission in Bangladesh and India. The age structure and fractional-order derivatives were used to create a more accurate version of the conventional SEIR model [23]. They expanded on their methods by including follow-up controls, diagnostics, and awareness programs. In Japan, Das and Samanta [24] suggested a susceptible-asymptomatic-infectious-recovered (SAIR) compartmental model inside a fractional-order framework that included the best possible management of social distance.

We propose a fractional-order derivative defined in the Atangana–Baleanu–Caputo sense in the current study to investigate the model presented in [20]. The nonlocal characteristic of the virus dynamics is not sufficiently captured by the classical model proposed in [20]. Because the Atangana–Baleanu and Caputo derivatives have a number of desired properties, such as nonlocality and non-singularity in their kernels, and because this operator can only accurately represent the crossover behavior of the model, it was decided to utilize them to design the model. Other operators without similar qualities, such as Caputo and Caputo–Fabrizio, may or may not adequately explain the dynamics of the coronavirus [19]. Several articles using the Atangana–Baleanu–Caputo derivative are linked and can be found in [13, 25–27].

The following sections then make up the remainder of the paper. In Section 2, we create and analyze a mathematical model that makes use of the fractional-order derivative as established by Atangana, Baleanu, and Caputo. Section 3 identifies the qualitative traits of the model. We identify equilibrium locations, their stability, and the fundamental reproduction number. We incorporate time-dependent optimal control into the constructed model and analyze the optimal control model in Section 4. The numerical framework of the fractional-order model is provided in Section 5, and the numerical analysis is then presented in Section 6. In section 7, the numerical investigation of the optimal control model is presented. Finally, in Section 8, we explain and illustrate the results of our suggested model.

2. Model Formulation

In this section, utilizing the fractional-order derivatives derived in the Atangana–Baleanu in Caputo sense, we alter the model given in [20] to incorporate a compartment for quarantine individuals. The fractional-order operator is defined as \( \mathcal{D}_t^\alpha \), where \( 0 < \alpha \leq 1 \). Susceptible individuals \( S \), exposed \( E \), asymptomatic \( I_a \), symptomatic \( I_s \), vaccinated \( V \), quarantined \( Q \), and recovered \( R \) are the seven classifications that make up the model. The main premise of the model formulation is that, in contrast to [20], where only the asymptomatic transmits the virus, both symptomatic and asymptomatic persons spread the virus when they come into touch with susceptible individuals. Other presumptions in [20].

The susceptible population are recruited at the rate \( \Omega \) and die at a rate \( \mu \). These individuals get exposed to the disease when they come in contact with the asymptomatic and symptomatic at a rate \( \beta \). After being exposed to the disease, they either progress to the asymptomatic class at the rate \( (1 - \alpha) \rho \) or the symptomatic class at the rate \( \alpha \phi \). Both asymptomatic and symptomatic get quarantined at the rates \( \rho \) and \( \tau \), respectively. Those vaccinated according to this model do not get infected but may join the susceptible class at a rate \( \Gamma \). The parameters \( \mu \) and \( \delta \) are the natural and the disease-induced death rates, respectively. The parameters \( \sigma, \theta, \) and \( \gamma \) are the rate of recovery for the asymptomatic, symptomatic, and quarantine class, respectively. The schematic diagram of the model is displayed in Figure 1.

The following fractional derivatives describe the model.
We go over the definitions of the key terms used in this work and those specified in [10, 28] in this part.

2.1. Preliminaries. We go over the definitions of the key terms used in this work and those specified in [10, 28] in this part.

Definition 1. Liouville and Caputo (LC) describe the fractional derivative of the order \( \nu \) as in [10, 29] as

\[
\mathcal{L}_\nu D^\nu_t h(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-p)^{-\nu} h(p) dp, 0 < \nu \leq 1.
\]  

Definition 2. We provide the Liouville–Caputo sense definition of the Atangana–Baleanu fractional derivative [10, 28]:

\[
\mathcal{ABC}_\nu D^\nu_t h(t) = \frac{B(\nu)}{(1-\nu)} \int_0^t E^-\nu\left(\frac{t-p}{1-\nu}\right) h(p) dp,
\]  

where \( B(\nu) = 1 - \nu + \nu\Gamma(\nu) \) is the normalized function.

Definition 3. The Atangana–Baleanu–Caputo derivative’s pertinent fractional integral is given by the definition in [10, 28]

\[
\mathcal{ABC}_\nu D^\nu_t h(t) = \frac{(1-\nu)}{B(\nu)} h(t) + \frac{\nu}{B(\nu)\Gamma(\nu)} \int_0^t (t-p)^{\nu-1} h(p) dp.
\]  

They calculated both derivatives’ Laplace transforms and discovered the following:

\[
\mathcal{L}_\nu \{ \mathcal{ABC}_\nu D^\nu_t h(t) \} = \frac{B(\nu)H(q)q^\nu - q^{\nu-1} h(0)}{(1-\nu)(q^\nu + \nu - 1)}.  
\]  

Theorem 1. For a function \( h \in C[a, b] \), the following results hold [10, 30]:

\[
\| \mathcal{ABC}_\nu D^\nu_t h(t) \| < \frac{B(\nu)}{(1-\nu)} \| h(t) \|, \text{ where } \| h(t) \| = \max_{t\in[a,b]} |h(t)|.  
\]  

Additionally, the derivatives of Atangana, Baleanu, and Caputo satisfy the Lipschitz criterion [10, 30]:

\[
\| \mathcal{ABC}_\nu D^\nu_t h_1(t) - \mathcal{ABC}_\nu D^\nu_t h_2(t) \| < \alpha \| h_1(t) - h_2(t) \|.  
\]  

2.2. Existence and Uniqueness. This section establishes the existence and distinctiveness of the solutions to the system (1).

We denote a Banach space by \( D(G) \) with \( G = [0, b] \), containing real-valued continuous function with sup norm \( W = D(G) \times D(G) \times D(G) \times D(G) \times D(G) \times D(G) \) and the given norm \( \langle S, E, I_A, I_S, Q, V, R \rangle = S + E + I_A + I_S + Q + V + R \), where \( S = \sup_{t \in G} |S|, E = \sup_{t \in G} |E|, I_A = \sup_{t \in G} |I_A|, I_S = \sup_{t \in G} |I_S|, Q = \sup_{t \in G} |Q|, V = \sup_{t \in G} |V|, R = \sup_{t \in G} |R| \). Using the ABC integral operator on the system (1), we have
\[
\begin{aligned}
S(t) - S(0) &= \nu^{\text{ABC}} D_t^\nu S(t) \left\{ (1 - \eta^\nu) \Omega^\nu + \Gamma^\nu V - \beta^\nu \left( \frac{I_A(t) + I_S(t)}{N(t)} \right) S(t) - \mu^\nu S(t) \right\}, \\
E(t) - E(0) &= \nu^{\text{ABC}} D_t^\nu E(t) \left\{ \beta^\nu \left( \frac{I_A(t) + I_S(t)}{N(t)} \right) S(t) - (\phi^\nu + \mu^\nu) E(t) \right\}, \\
I_A(t) - I_A(0) &= \nu^{\text{ABC}} D_t^\nu I_A(t) \left\{ (1 - \alpha^\nu) \phi^\nu E(t) - (\delta^\nu + \rho^\nu + \mu^\nu + \sigma^\nu) I_A(t) \right\}, \\
I_S(t) - I_S(0) &= \nu^{\text{ABC}} D_t^\nu I_S(t) \left\{ \alpha^\nu E(t) - (\delta^\nu + \mu^\nu + \theta^\nu + \tau^\nu) I_S(t) \right\}, \\
Q(t) - Q(0) &= \nu^{\text{ABC}} D_t^\nu Q(t) \left\{ \rho^\nu I_A + \tau^\nu I_S - (\delta^\nu + \mu^\nu + \gamma^\nu) Q(t) \right\}, \\
V(t) - V(0) &= \nu^{\text{ABC}} D_t^\nu V(t) \left\{ \eta^\nu \Omega - (\Gamma^\nu + \mu^\nu) V(t) \right\}, \\
R(t) - R(0) &= \nu^{\text{ABC}} D_t^\nu R(t) \left\{ \gamma^\nu Q(t) + \theta^\nu I_S(t) + \sigma^\nu I_A(t) - \mu^\nu R(t) \right\}.
\end{aligned}
\]

Now, from Definition 1, we have

\[
\begin{aligned}
S(t) - S(0) &= \frac{1 - \nu}{B(v)} \Phi_1(v, t, S(t)) + \frac{\nu}{B(v) \Gamma(v)} \times \int_0^t (t - \tau)^{-1} \Phi_1(v, \tau, S(\tau)) d\tau, \\
E(t) - E(0) &= \frac{1 - \nu}{B(v)} \Phi_2(v, t, E(t)) + \frac{\nu}{B(v) \Gamma(v)} \times \int_0^t (t - \tau)^{-1} \Phi_2(v, \tau, E(\tau)) d\tau, \\
I_A(t) - I_A(0) &= \frac{1 - \nu}{B(v)} \Phi_3(v, t, I_A(t)) + \frac{\nu}{B(v) \Gamma(v)} \times \int_0^t (t - \tau)^{-1} \Phi_3(v, \tau, I_A(\tau)) d\tau, \\
I_S(t) - I_S(0) &= \frac{1 - \nu}{B(v)} \Phi_4(v, t, I_S(t)) + \frac{\nu}{B(v) \Gamma(v)} \times \int_0^t (t - \tau)^{-1} \Phi_4(v, \tau, I_S(\tau)) d\tau, \\
Q(t) - Q(0) &= \frac{1 - \nu}{B(v)} \Phi_5(v, t, Q(t)) + \frac{\nu}{B(v) \Gamma(v)} \times \int_0^t (t - \tau)^{-1} \Phi_5(v, \tau, Q(\tau)) d\tau, \\
V(t) - V(0) &= \frac{1 - \nu}{B(v)} \Phi_6(v, t, V(t)) + \frac{\nu}{B(v) \Gamma(v)} \times \int_0^t (t - \tau)^{-1} \Phi_6(v, \tau, V(\tau)) d\tau, \\
R(t) - R(0) &= \frac{1 - \nu}{B(v)} \Phi_7(v, t, R(t)) + \frac{\nu}{B(v) \Gamma(v)} \times \int_0^t (t - \tau)^{-1} \Phi_7(v, \tau, R(\tau)) d\tau,
\end{aligned}
\]

where
Furthermore, the Atangana–Baleanu–Caputo derivatives fulfill the Lipschitz condition [10, 30] only if $S(t), E(t), I_A(t), I_S(t), Q(t), V(t)$ and $R(t)$ possess an upper bound. We suppose that $S(t)$ and $S^*(t)$ are couple functions, then

$$
\left\| \Phi_1(v, t, S(t)) - \Phi_1(v, t, S^*(t)) \right\| 
= \left\| -\beta^\gamma \left( \frac{I_A(t) + I_S(t)}{N(t)} \right) - \mu^\gamma \right\| \left( S(t) - S^*(t) \right) \right\|.
$$

Considering

$$
d_1 = \left\| -\beta^\gamma \left( \frac{I_A(t) + I_S(t)}{N(t)} \right) - \mu^\gamma \right\|,
$$

equation (11) simplifies to

$$
\left\| \Phi_1(v, t, S(t)) - \Phi_1(v, t, S^*(t)) \right\| \leq d_1 \left\| (S(t) - S^*(t)) \right\|.
$$

Similarly,

$$
\left\| \Phi_2(v, t, E(t)) - \Phi_2(v, t, E^*(t)) \right\| \leq d_2 \left\| (E(t) - E^*(t)) \right\|,
$$

$$
\left\| \Phi_3(v, t, I_A(t)) - \Phi_3(v, t, I_A^*(t)) \right\| \leq d_3 \left\| (I_A(t) - I_A^*(t)) \right\|,
$$

$$
\left\| \Phi_4(v, t, I_S(t)) - \Phi_4(v, t, I_S^*(t)) \right\| \leq d_4 \left\| (I_S(t) - I_S^*(t)) \right\|,
$$

$$
\left\| \Phi_5(v, t, Q(t)) - \Phi_5(v, t, Q^*(t)) \right\| \leq d_5 \left\| (Q(t) - Q^*(t)) \right\|,
$$

$$
\left\| \Phi_6(v, t, V(t)) - \Phi_6(v, t, V^*(t)) \right\| \leq d_6 \left\| (V(t) - V^*(t)) \right\|,
$$

$$
\left\| \Phi_7(v, t, R(t)) - \Phi_7(v, t, R^*(t)) \right\| \leq d_7 \left\| (R(t) - R^*(t)) \right\|,
$$

where

$$
d_2 = \left\| -(\phi^\nu + \mu^\nu) \right\|,
$$

$$
d_3 = \left\| -(\alpha\phi^\nu + \mu^\nu + \rho^\nu + \delta^\nu + \sigma^\nu) \right\|,
$$

$$
d_4 = \left\| -(\delta^\nu + \mu^\nu + \tau^\nu + \theta^\nu) \right\|,
$$

$$
d_5 = \left\| -(\delta^\nu + \mu^\nu + \gamma^\nu) \right\|,
$$

$$
d_6 = \left\| -(\Gamma^\nu + \mu^\nu) \right\|,
$$

$$
d_7 = \left\| -\mu^\nu \right\|.
$$

Hence, the Lipschitz condition holds. Now, taking system (9) in a reiterative manner gives
\[
\begin{align*}
S_n(t) - S(0) &= \frac{1 - v}{B(v)} \Phi_1(v,t,S_{n-1}(t)) + \frac{v}{B(v) \Gamma(v)} \times \int_0^t (t - \theta)^{v-1} \Phi_1(v,\theta,S_{n-1}(\theta)) d\theta, \\
E_n(t) - E(0) &= \frac{1 - v}{B(v)} \Phi_2(v,t,E_{n-1}(t)) + \frac{v}{B(v) \Gamma(v)} \times \int_0^t (t - \theta)^{v-1} \Phi_2(v,\theta,E_{n-1}(\theta)) d\theta, \\
I_{n_1}(t) - I_{A}(0) &= \frac{1 - v}{B(v)} \Phi_3(v,t,I_{n_1}(t)) + \frac{v}{B(v) \Gamma(v)} \times \int_0^t (t - \theta)^{v-1} \Phi_3(v,\theta,I_{n_1}(\theta)) d\theta, \\
I_{n_2}(t) - I_{S}(0) &= \frac{1 - v}{B(v)} \Phi_4(v,t,I_{n_2}(t)) + \frac{v}{B(v) \Gamma(v)} \times \int_0^t (t - \theta)^{v-1} \Phi_4(v,\theta,I_{n_2}(\theta)) d\theta, \\
Q_n(t) - Q(0) &= \frac{1 - v}{B(v)} \Phi_5(v,t,Q_{n-1}(t)) + \frac{v}{B(v) \Gamma(v)} \times \int_0^t (t - \theta)^{v-1} \Phi_5(v,\theta,Q_{n-1}(\theta)) d\theta, \\
V_n(t) - V(0) &= \frac{1 - v}{B(v)} \Phi_6(v,t,V_{n-1}(t)) + \frac{v}{B(v) \Gamma(v)} \times \int_0^t (t - \theta)^{v-1} \Phi_6(v,\theta,V_{n-1}(\theta)) d\theta, \\
R_n(t) - R(0) &= \frac{1 - v}{B(v)} \Phi_7(v,t,R_{n-1}(t)) + \frac{v}{B(v) \Gamma(v)} \times \int_0^t (t - \theta)^{v-1} \Phi_7(v,\theta,R_{n-1}(\theta)) d\theta.
\end{align*}
\]

\[\text{(16)}\]

Difference of consecutive terms yields
\[
\Xi_{S_n}(t) = S_n(t) - S_{n-1}(t) = \frac{1 - v}{B(v)} \left( \Phi_1(v, t, S_{n-1}(t)) - \Phi_1(v, t, S_{n-2}(t)) \right) \\
+ \frac{v}{B(v)\Gamma(v)} \int_0^t (t-\theta)^{n-1} \left( \Phi_1(v, \theta, S_{n-1}(\theta)) - \Phi_1(v, \tau, S_{n-2}(\theta)) \right) d\theta \\
\Xi_{E_n}(t) = E_n(t) - E_{n-1}(t) = \frac{1 - v}{B(v)} \left( \Phi_2(v, t, E_{n-1}(t)) - \Phi_2(v, t, E_{n-2}(t)) \right) \\
+ \frac{v}{B(v)\Gamma(v)} \int_0^t (t-\theta)^{n-1} \left( \Phi_2(v, \theta, E_{n-1}(\theta)) - \Phi_2(v, \tau, E_{n-2}(\theta)) \right) d\theta \\
\Xi_{I_{An}}(t) = I_{An}(t) - I_{A(n-1)}(t) = \frac{1 - v}{B(v)} \left( \Phi_3(v, t, I_{A(n-1)}(t)) - \Phi_3(v, t, I_{A(n-2)}(t)) \right) \\
+ \frac{v}{B(v)\Gamma(v)} \int_0^t (t-\theta)^{n-1} \left( \Phi_3(v, \theta, I_{A(n-1)}(\theta)) - \Phi_3(v, \tau, I_{A(n-2)}(\theta)) \right) d\theta \\
\Xi_{I_{Sn}}(t) = I_{Sn}(t) - I_{S(n-1)}(t) = \frac{1 - v}{B(v)} \left( \Phi_4(v, t, I_{S(n-1)}(t)) - \Phi_4(v, t, I_{S(n-2)}(t)) \right) \\
+ \frac{v}{B(v)\Gamma(v)} \int_0^t (t-\theta)^{n-1} \left( \Phi_4(v, \theta, I_{S(n-1)}(\theta)) - \Phi_4(v, \tau, I_{S(n-2)}(\theta)) \right) d\theta \\
\Xi_{Q_n}(t) = Q_n(t) - Q_{n-1}(t) = \frac{1 - v}{B(v)} \left( \Phi_5(v, t, Q_{n-1}(t)) - \Phi_5(v, t, Q_{n-2}(t)) \right) \\
+ \frac{v}{B(v)\Gamma(v)} \int_0^t (t-\theta)^{n-1} \left( \Phi_5(v, \theta, Q_{n-1}(\theta)) - \Phi_5(v, \tau, Q_{n-2}(\theta)) \right) d\theta \\
\Xi_{V_n}(t) = V_n(t) - V_{n-1}(t) = \frac{1 - v}{B(v)} \left( \Phi_6(v, t, V_{n-1}(t)) - \Phi_6(v, t, V_{n-2}(t)) \right) \\
+ \frac{v}{B(v)\Gamma(v)} \int_0^t (t-\theta)^{n-1} \left( \Phi_6(v, \theta, V_{n-1}(\theta)) - \Phi_6(v, \tau, V_{n-2}(\theta)) \right) d\theta \\
\Xi_{R_n}(t) = R_n(t) - R_{n-1}(t) = \frac{1 - v}{B(v)} \left( \Phi_7(v, t, R_{n-1}(t)) - \Phi_7(v, t, R_{n-2}(t)) \right) \\
+ \frac{v}{B(v)\Gamma(v)} \int_0^t (t-\theta)^{n-1} \left( \Phi_7(v, \theta, R_{n-1}(\theta)) - \Phi_7(v, \tau, R_{n-2}(\theta)) \right) d\theta
\]

where \( S_n(t) = \sum_{i=0}^{n} I_{S_i}(t) \), \( E_n(t) = \sum_{i=0}^{n} I_{E_i}(t) \), \( I_{An} \) \( \Xi_{S_n}(t) = S_n(t) - S_{n-1}(t) \), \( E_n(t) = E_n(t) - E_{n-1}(t) \), \( E_{n-1}(t) = E_n(t) - E_{n-2}(t) \), \( V_n(t) = \sum_{i=0}^{n} I_{V_i}(t) \), \( R_n(t) = \sum_{i=0}^{n} R_n(t) \). Taking into consideration equations (12) – (13) and considering (17)
\[ I_{S(n-2)}(t), \Xi_{V_{n-1}}(t) = V_{n-1}(t) - V_{n-2}(t), \Xi_{R_{n-1}}(t) = R_{n-1}(t) - R_{n-2}(t), \]

\[ \begin{align*}
\|&\Xi_{S}(t)\| \leq \frac{1 - \nu}{B(v)} d_1 \|\Xi_{S_{v-1}}(t)\| \frac{\nu}{B(v)\Gamma(v)} d_4 \int_{0}^{t} (t - \Theta)^{-\nu - 1} \|\Xi_{S_{v-1}}(\Theta)\| \mathrm{d}\Theta, \\
\|&\Xi_{E}(t)\| \leq \frac{1 - \nu}{B(v)} d_3 \|\Xi_{E_{v-1}}(t)\| \frac{\nu}{B(v)\Gamma(v)} d_4 \int_{0}^{t} (t - \Theta)^{-\nu - 1} \|\Xi_{E_{v-1}}(\Theta)\| \mathrm{d}\Theta, \\
\|&\Xi_{I_A}(t)\| \leq \frac{1 - \nu}{B(v)} d_3 \|\Xi_{I_{A_{v-1}}}(t)\| \frac{\nu}{B(v)\Gamma(v)} d_4 \int_{0}^{t} (t - \Theta)^{-\nu - 1} \|\Xi_{I_{A_{v-1}}}(\Theta)\| \mathrm{d}\Theta, \\
\|&\Xi_{I_n}(t)\| \leq \frac{1 - \nu}{B(v)} d_3 \|\Xi_{I_{v_n-1}}(t)\| \frac{\nu}{B(v)\Gamma(v)} d_4 \int_{0}^{t} (t - \Theta)^{-\nu - 1} \|\Xi_{I_{v_n-1}}(\Theta)\| \mathrm{d}\Theta, \\
\|&\Xi_{V}(t)\| \leq \frac{1 - \nu}{B(v)} d_3 \|\Xi_{V_{v-1}}(t)\| \frac{\nu}{B(v)\Gamma(v)} d_4 \int_{0}^{t} (t - \Theta)^{-\nu - 1} \|\Xi_{V_{v-1}}(\Theta)\| \mathrm{d}\Theta, \\
\|&\Xi_{R}(t)\| \leq \frac{1 - \nu}{B(v)} d_3 \|\Xi_{R_{v-1}}(t)\| \frac{\nu}{B(v)\Gamma(v)} d_4 \int_{0}^{t} (t - \Theta)^{-\nu - 1} \|\Xi_{R_{v-1}}(\Theta)\| \mathrm{d}\Theta.
\end{align*} \]

(18)

**Theorem 2.** System (1) has a unique solution for \( t \in [0, b] \) subject to the condition \( 1 - \nu/B(v)d_i + \nu/B(v)\Gamma(v)b^\nu n_i < 1, i = 1, 2, 3, \ldots, 7 \) hold [27].

**Proof.** Since \( S(t), E(t), I_A(t), I_n(t), Q(t), V(t) \) and \( R(t) \) are bounded functions, equations (12) and (13) hold. In a recurring manner, (16) reaches
\[ \|S_n(t)\| \leq \|S_0(t)\| \left( \frac{1 - \nu}{B(v)} d_1 + \frac{\nu B(v)}{d_1} \right)^n, \]
\[ \|E_n(t)\| \leq \|E_0(t)\| \left( \frac{1 - \nu}{B(v)} d_2 + \frac{\nu B(v)}{d_2} \right)^n, \]
\[ \|I_{\alpha n}(t)\| \leq \|I_{\alpha 0}(t)\| \left( \frac{1 - \nu}{B(v)} d_3 + \frac{\nu B(v)}{d_3} \right)^n, \]
\[ \|I_{\beta n}(t)\| \leq \|I_{\beta 0}(t)\| \left( \frac{1 - \nu}{B(v)} d_4 + \frac{\nu B(v)}{d_4} \right)^n, \]
\[ \|S_{n+1} - S_n(t)\| \leq \sum_{i=n+1}^{n+j} T_i, \]
\[ \|E_{n+1} - E_n(t)\| \leq \sum_{i=n+1}^{n+j} T_i, \]
\[ \|I_{\alpha(n+1)}(t) - I_{\alpha n}(t)\| \leq \sum_{i=n+1}^{n+j} T_i, \]
\[ \|I_{\beta(n+1)}(t) - I_{\beta n}(t)\| \leq \sum_{i=n+1}^{n+j} T_i, \]
\[ \|Q_{n+1}(t) - Q_n(t)\| \leq \sum_{i=n+1}^{n+j} T_i, \]
\[ \|V_{n+1}(t) - V_n(t)\| \leq \sum_{i=n+1}^{n+j} T_i, \]
\[ \|R_{n+1}(t) - R_n(t)\| \leq \sum_{i=n+1}^{n+j} T_i, \]
\[ \| \Xi_{n+1}(t) \| \leq \| \Xi_n(t) \| \left( \frac{1 - \nu}{B(v)} d_1 + \frac{\nu B(v)}{d_1} \right)^n, \]
\[ \| \Xi_{E_n}(t) \| \leq \| \Xi_{E_0}(t) \| \left( \frac{1 - \nu}{B(v)} d_2 + \frac{\nu B(v)}{d_2} \right)^n, \]
\[ \| \Xi_{I_{\alpha n}}(t) \| \leq \| \Xi_{I_{\alpha 0}}(t) \| \left( \frac{1 - \nu}{B(v)} d_3 + \frac{\nu B(v)}{d_3} \right)^n, \]
\[ \| \Xi_{I_{\beta n}}(t) \| \leq \| \Xi_{I_{\beta 0}}(t) \| \left( \frac{1 - \nu}{B(v)} d_4 + \frac{\nu B(v)}{d_4} \right)^n, \]
\[ \Xi_{S_n(t)} \rightarrow 0, \Xi_{E_n(t)} \rightarrow 0, \Xi_{I_{\alpha n}(t)} \rightarrow 0, \Xi_{I_{\beta n}(t)} \rightarrow 0, \Xi_{Q_n(t)} \rightarrow 0, \Xi_{V_n(t)} \rightarrow 0, \Xi_{R_n(t)} \rightarrow 0 \]
as \( n \rightarrow \infty \). Incorporating the triangular inequality and for any \( j \), system (17) yields

\[ E_0 = (S^*, E^*, I_{\alpha}^*, I_{\beta}^*, Q^*, V^*, R^*) = \left( \frac{\Omega^* (1 - \eta^*)}{\mu^* (1 + \eta^*)}, 0, 0, 0, 0, 0, 0 \right). \]

The endemic equilibrium \((E_0)\) of system (1) is represented by \( E_0 = (S^*, E^*, I_{\alpha}^*, I_{\beta}^*, Q^*, V^*, R^*) \), where

\[ S^* = \frac{(1 - \eta^*) \Omega^* + \Gamma^*}{\beta^* (1 + I_{\alpha}^*) + \mu^*} V^*, E^* = \frac{\beta^* S^* (I_{\alpha}^* + I_{\beta}^*)}{\phi^* + \mu^*} I_{\alpha}^* = \frac{(1 - \alpha^*) \phi^* E^*}{\rho^* + \sigma^* + \mu^* + \delta^*}, \]
\[ Q^* = \frac{\mu^* (1 - \eta^*) \Omega^*}{1 + \Gamma^*} R^* = \frac{\beta^* I_{\alpha}^* + \alpha \Omega^*}{\mu^* + \delta^*} V^*. \]

We now ascertain the basic reproduction number \((R_0)\) of system (1). The total number of secondary instances that one sick person can bring about over the life of the infection in a society that is entirely susceptible is the basic reproduction number [31]. Using the next generation operator technique, by denoting \( F \) and \( V \) as matrices representing the newly produced diseases and the transition terms, we discover, respectively,
Now, the basic reproductive number is given as the spectrum radius of the matrix $FV^{-1}$.

$$R_1 = \frac{\beta^{\nu} \Omega^{\nu} (1^{\nu} + \mu^{\nu} (1 - \eta^{\nu}))}{\mu^{\nu} (\mu^{\nu} + 1^{\nu}) (\rho^{\nu} + \sigma^{\nu} + \mu^{\nu} + \delta^{\nu})},$$

and

$$R_2 = \frac{\beta^{\nu} \Omega^{\nu} (1^{\nu} + \mu^{\nu} (1 - \eta^{\nu}))}{\mu^{\nu} (\mu^{\nu} + 1^{\nu}) (\theta^{\nu} + \tau^{\nu} + \mu^{\nu} + \delta^{\nu})}.$$  (24)

The Jacobian matrix evaluated at the disease-free equilibrium point is given as

$$J = \begin{bmatrix}
-\mu^{\nu} & 0 & -\beta^{\nu} S \\
\beta^{\nu} (I_A + I_E) & -\mu^{\nu} & -\beta^{\nu} S \\
0 & (1 - \alpha) \phi^{\nu} & -\left(\rho^{\nu} + \sigma^{\nu} + \mu^{\nu} + \delta^{\nu}\right) \\
0 & \alpha \phi^{\nu} & 0 \\
0 & 0 & \rho^{\nu} \\
0 & 0 & \sigma^{\nu} \\
0 & 0 & \theta^{\nu} \\
\end{bmatrix}.$$  (25)

The characteristic equation of system (24) is

$$\begin{vmatrix}
-\mu^{\nu} & 0 & -\beta^{\nu} S \\
\beta^{\nu} S & 0 & \Gamma^{\nu} \\
0 & \beta^{\nu} S & 0 \\
0 & 0 & \theta^{\nu} \\
0 & 0 & \gamma^{\nu} \\
\end{vmatrix} = 0.$$  (26)

We need to show that all eigenvalues of system (23) are negative. The first four eigenvalues are 

$$-\mu^{\nu}, -\left(\gamma^{\nu} + \mu^{\nu} + \delta^{\nu}\right),$$

$$-\left(\Gamma^{\nu} + \mu^{\nu}\right),$$

and 

$$-\mu^{\nu}.$$ The others are obtained from the submatrix in system (24) formed by excluding the first, fifth, sixth, and seventh rows and columns of system (23). Hence, we have

$$I_{F^E} = \begin{bmatrix}
-\mu^{\nu} & 0 & -\beta^{\nu} S \\
0 & -\left(\phi^{\nu} + \mu^{\nu}\right) & \beta^{\nu} S \\
0 & (1 - \alpha) \phi^{\nu} & -\left(\rho^{\nu} + \sigma^{\nu} + \mu^{\nu} + \delta^{\nu}\right) \\
0 & \alpha \phi^{\nu} & 0 \\
0 & 0 & \rho^{\nu} \\
0 & 0 & \sigma^{\nu} \\
\end{bmatrix}.$$  (26)

The characteristic equation of system (24) is

$$\begin{vmatrix}
-\mu^{\nu} & 0 & -\beta^{\nu} S \\
\beta^{\nu} S & 0 & \Gamma^{\nu} \\
0 & \beta^{\nu} S & 0 \\
0 & 0 & \theta^{\nu} \\
0 & 0 & \gamma^{\nu} \\
\end{vmatrix} = 0.$$  (27)

The necessary conditions for the local stability of the endemic equilibrium are established in Theorem 2.

**Theorem 3.** The disease-free equilibrium is locally asymptotically stable if $R_0 < 1$ and unstable for $R_0 > 1$. 

**Proof.** The Jacobian matrix of system (1) is given as

$$I_{F} = \begin{bmatrix}
-\mu^{\nu} & 0 & -\beta^{\nu} S \\
\beta^{\nu} S & 0 & \Gamma^{\nu} \\
0 & \beta^{\nu} S & 0 \\
0 & 0 & \theta^{\nu} \\
0 & 0 & \gamma^{\nu} \\
\end{bmatrix}.$$  (26)
\[ \lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0, \]  

where

\[
\begin{aligned}
A_1 &= (\phi^\gamma + \mu^\gamma) + (\rho^\gamma + \sigma^\gamma + \mu^\gamma + \delta^\gamma) + A, \\
A_2 &= (\phi^\gamma + \mu^\gamma)[(\rho^\gamma + \sigma^\gamma + \mu^\gamma + \delta^\gamma) + A] + (\rho^\gamma + \sigma^\gamma + \mu^\gamma + \delta^\gamma)[(A - \mu^\gamma \phi^\gamma (\mu^\gamma + \Gamma^\gamma) R_0], \\
A_3 &= (\rho^\gamma + \sigma^\gamma + \mu^\gamma + \delta^\gamma)[(\phi^\gamma + \mu^\gamma) A + \alpha^\gamma \phi^\gamma \beta^\gamma S^0] + (1 - \alpha) \phi^\gamma A \beta^\gamma S^0, \\
A &= (\theta^\gamma + \tau^\gamma + \mu^\gamma + \delta^\gamma).
\end{aligned}
\]

From the Routh–Hurwitz stability criterion, if the conditions \( A_1 > 0, A_3 > 0 \) and \( A_1 A_2 - A_3 > 0 \) are satisfied, then all the roots of the characteristic equation have a negative real part which means stable equilibrium.

4. Fractional Optimal Control Problem

We add two control functions \( u_1 \) and \( u_2 \) into the system (1), where control \( u_1 \) and \( u_2 \) are social distancing and vaccination, respectively. We include the time-dependent controls into system (1), and we have

\[
\begin{aligned}
\dot{v}^{ABC} D^\gamma S &= (1 - \eta^\gamma) \Omega^\gamma + \Gamma^\gamma V - (1 - u_1) \beta S^{\gamma}(I_A + I_S) - \mu^S S - u_2 S, \\
\dot{v}^{ABC} D^\gamma E &= (1 - u_1) \beta S^{\gamma}(I_A + I_S) - (\phi^\gamma + \mu^\gamma) E, \\
\dot{v}^{ABC} D^\gamma I_A &= (1 - \alpha) \phi E - (\delta^\gamma + \tau^\gamma + \mu^\gamma + \delta^\gamma) I_A, \\
\dot{v}^{ABC} D^\gamma I_S &= \alpha \phi E - (\theta^\gamma + \tau^\gamma + \mu^\gamma + \delta^\gamma) I_S, \\
\dot{v}^{ABC} D^\gamma Q &= \rho^\gamma I_A + \tau^\gamma I_S - (\gamma^\gamma + \mu^\gamma + \delta^\gamma) Q, \\
\dot{v}^{ABC} D^\gamma V &= \eta^\gamma \Omega^\gamma - (\Gamma^\gamma + \mu^\gamma) V + u_2 S, \\
\dot{v}^{ABC} D^\gamma_0 [R(t)] &= \theta^\gamma I_S + \sigma^\gamma I_A + \gamma^\gamma Q - \mu^R R.
\end{aligned}
\]

The objective function for a fixed time \( t_f \) is given as

\[
J(u_1, u_2) = \int_0^{t_f} \left[ G_1 S(t) + G_2 E(t) + G_3 I_A(t) + G_4 I_S(t) + G_5 Q(t) + \frac{1}{2} (T_1 u_1^2 + T_2 u_2^2) \right] dt,
\]

where \( T_1 \) and \( T_2 \) are the measures of the relative cost of interventions associated with the controls \( u_1 \) and \( u_2 \). We find optimal controls \( u_1 \) and \( u_2 \) that minimize the cost function
subject to the constraint

\begin{equation}
\nu_{ABCD}^v t \Omega S(t) = \zeta_1 \nu_{ABCD}^v t \Omega E(t) = \zeta_2 \nu_{ABCD}^v t \Omega I_A(t) = \zeta_3 \nu_{ABCD}^v t \Omega I_S(t) = \zeta_4 \nu_{ABCD}^v t \Omega I_I(t) = \zeta_5
\end{equation}

where \( \zeta_i = \zeta(S, E, I_A, I_S, Q, V, R), \) \( i = 1, 2, 3, \ldots, 7, \) \( \Phi = (u_1, u_2) \) \( u_i \) is a Lebesgue measurable on \([0, 1]\) such that \( 0 \leq (u_1, u_2) \leq 1, \forall t \in [0, t_f] \), where \( t_f \) is the final time and with initial conditions \( S(0) = S_0, E(0) = E_0, I_A(0) = I_{A_0}, I_S(0) = I_{S_0}, Q(0) = Q_0, V(0) = V_0, R(0) = R_0. \)

To define the fractional optimal control, we consider the following modified cost function [10]:

\begin{equation}
J = \int_0^{t_f} \left[ H_s(S, E, I_A, I_S, Q, V, R, u_j, t) - \sum_{i=1}^{7} \lambda_i \zeta_i(S, E, I_A, I_S, Q, V, R, u_j, t) \right] dt,
\end{equation}

where \( i = 1, \ldots, 7 \) and \( j = 1, 2, 3. \) For the fractional optimal control, the Hamiltonian is

\begin{equation}
H_s(S, E, I_A, I_S, Q, V, R, u_j, t) = \nu(S, E, I_A, I_S, Q, V, R, u_j, t) + \sum_{i=1}^{7} \lambda_i \zeta_i(S, E, I_A, I_S, Q, V, R, u_j, t),
\end{equation}

where \( i = 1, \ldots, 7 \) and \( j = 1, 2, 3. \) The following are essential for the formulation of the fractional optimal control [10, 27]:

\begin{align}
\nu_{ABCD}^v t \Omega A_s &= \frac{\partial H_s}{\partial S}, \\
\nu_{ABCD}^v t \Omega A_e &= \frac{\partial H_s}{\partial E}, \\
\nu_{ABCD}^v t \Omega A_i &= \frac{\partial H_s}{\partial I_A}, \\
\nu_{ABCD}^v t \Omega A_s &= \frac{\partial H_s}{\partial I_S}, \\
\nu_{ABCD}^v t \Omega A_q &= \frac{\partial H_s}{\partial Q},
\end{align}

\begin{align}
\nu_{ABCD}^v t \Omega A_v &= \frac{\partial H_s}{\partial V}, \\
\nu_{ABCD}^v t \Omega A_r &= \frac{\partial H_s}{\partial R},
\end{align}

\begin{align}
0 &= \frac{\partial H_s}{\partial u_1},
\nu_{ABCD}^v t \Omega A_s &= \frac{\partial H_s}{\partial S}, \\
\nu_{ABCD}^v t \Omega A_e &= \frac{\partial H_s}{\partial E}, \\
\nu_{ABCD}^v t \Omega A_i &= \frac{\partial H_s}{\partial I_A}, \\
\nu_{ABCD}^v t \Omega A_s &= \frac{\partial H_s}{\partial I_S}, \\
\nu_{ABCD}^v t \Omega A_q &= \frac{\partial H_s}{\partial Q},
\end{align}

Moreover,

\begin{align}
\Lambda_s(t_f) &= \Lambda_e(t_f) = \Lambda_i(t_f) = \Lambda_s(t_f) = \Lambda_q(t_f) = \Lambda_v(t_f) = \Lambda_r(t_f) = 0,
\end{align}

are the Lagrange multipliers. Equations (31) and (32) provide the necessary conditions for the fractional optimal control in terms of the Hamiltonian for the optimal control problem defined previously. The Hamiltonian, \( H_s \), is defined by
\[ H = k_1 S^* + k_2 E^* + k_3 I_A^* + k_4 I_S^* + k_5 Q^* + \frac{1}{2} (T_1 u_1^2 + T_2 u_2^2) \]  
\[ + \nu^{ABC} D^*_1 \Lambda_S + \nu^{ABC} D^*_1 \Lambda_E + \nu^{ABC} D^*_1 \Lambda_A + \nu^{ABC} D^*_1 \Lambda_i + \nu^{ABC} D^*_1 \Lambda_Q + \nu^{ABC} D^*_1 \Lambda_V + \nu^{ABC} D^*_1 \Lambda_R. \]  

**Theorem 4.** Given an optimal control \((u_1^*, u_2^*)\) and corresponding solution \(S^*, E^*, I_A^*, I_S^*, Q^*, V^*, R^*\) of the system (26)-(27) that minimizes \(J(u)\) over \(U\), there exist adjoint variables \(\Lambda_S, \Lambda_E, \Lambda_A, \Lambda_i, \Lambda_Q, \Lambda_V, \) and \(\Lambda_R\) satisfying \([27]\)

\[
\frac{d\Lambda_i}{dt} = \frac{\partial H}{\partial u_i^*}, \tag{40}
\]

where \(i = S, E, I_A, I_S, Q, V, R\) with the transversality conditions:

\[
\begin{align*}
&\nu^{ABC} D^*_1 \Lambda_S = \beta^v (I_A - I_S)(1 - u_1)[\Lambda_S - \Lambda_E] + (\mu^v + u_2) \Lambda_S + u_2 \Lambda_V, \\
&\nu^{ABC} D^*_1 \Lambda_E = (\phi^v + \mu^v) \Lambda_E - (1 - \alpha) \phi^v \Lambda_A - \alpha \phi^v \Lambda_i, \\
&\nu^{ABC} D^*_1 \Lambda_A = (\rho^v + \sigma^v + \mu^v + \sigma^v) \Lambda_A + (1 - u_1) \beta^v S \Lambda_S - \rho^v \Lambda_Q - \sigma^v \Lambda_R, \\
&\nu^{ABC} D^*_1 \Lambda_i = (\theta^v + r^v + \mu^v + \delta^v) \Lambda_i + (1 - u_1) \beta^v S \Lambda_S - r^v \Lambda_Q - \theta^v \Lambda_R, \\
&\nu^{ABC} D^*_1 \Lambda_Q = (\gamma^v + \mu^v + \theta^v) \Lambda_Q - \gamma^v \Lambda_R, \\
&\nu^{ABC} D^*_1 \Lambda_V = -\Gamma^v \Lambda_S + (\Gamma^v + \mu^v) \Lambda_V, \\
&\nu^{ABC} D^*_1 \Lambda_R = \mu^v \Lambda_R.
\end{align*}
\]

By obtaining the solution for \(u_1^*\) and \(u_2^*\) subject to the constraints, we have

\[
0 = \frac{\partial H}{\partial u_1} = -T_1 u_1 + \beta^v S (I_A + I_S) [\Lambda_E - \Lambda_S], \tag{43}
\]

\[
0 = \frac{\partial H}{\partial u_2} = -T_2 u_2 + S [\Lambda_S - \Lambda_V].
\]

This gives

\[
\begin{align*}
u_1^* &= \min \left( 1, \max \left( 0, \frac{\beta^v S (I_A + I_S) [\Lambda_E - \Lambda_S]}{T_1} \right) \right), \\
u_2^* &= \min \left( 1, \max \left( 0, \frac{S [\Lambda_S - \Lambda_V]}{T_2} \right) \right). \tag{44}
\end{align*}
\]

**5. Numerical Scheme of the Fractional Derivative**

We apply the scheme in [10] to system (1). Let us consider the first equation of system (1).

\[
\nu^{ABC} D^*_1 [S(t)] = h(t, S(t)), S(0) = S_0. \tag{45}
\]

Applying the fundamental theorem of fractional calculus to equation (40), we obtain

\[
S(t) - S(0) = \frac{1 - v}{B(v)} h(t, S(t)) + \frac{v}{\Gamma(v) B(v)} \int_{0}^{t} g(r, S(r)) (t - r)^{-v} dr, \tag{46}
\]

where \(B(v) = 1 - v + v \Gamma(v)\) is a normalized function and at \(t_{e+1}\), we have

\[
S_{e+1} = S_0 + \frac{(1 - v) \Gamma(v)}{(1 - v) \Gamma(v) + v} h(t_e, S(t_e)) + \frac{v}{\Gamma(v) + v (1 - \Gamma(v))} \sum_{j = 0}^{e} \int_{t_j}^{t_{e+1}} h(t_e - \theta^v (t_{e+1} - \theta^v)^{-v} - \theta^v). \tag{47}
\]

Implementing two-step Lagrange's interpolation polynomial on the interval \([t_e, t_{e+1}]\) [10, 32], we have

\[
Y = \frac{h(t_e, S_0)}{g} (g - t_{e-1}) - \frac{h(t_{e-1}, S_{e-1})}{g} (g - t_e). \tag{48}
\]
Equation (43) is replaced with equation (42), and by performing the steps given in [10, 32], we obtain

\[
S(t_{e+1}) = S(t_0) + \frac{\Gamma(v)(1-v)}{\Gamma(v)(1-v) + v} h(t_e, S(t_e)) + \frac{1}{(v+1)\Gamma(v) + v} \sum_{\delta=0}^{\xi} \theta(g)^\nu h(t_{\delta}, S(t_{\delta})) (\epsilon + 1 - \delta)^\nu \\
\times (\epsilon - \partial + 2 + 2v) - (\epsilon - \partial)^\nu (\epsilon - \partial + 2 + 2v) - \theta(g)^\nu h(t_{\delta-1}, S(t_{\delta-1})) (\epsilon + 1 - \delta)^{\nu+1} (\epsilon - \partial + 2 + v) \\
- (\epsilon - \partial)^\nu (\epsilon - \partial + 1 + v).
\] (49)

To obtain high stability, we replace the step-size \( g \) in equation (44) with \( \theta(g) \) such that \( \theta(g) = g + O(g^2) \), \( 0 < \theta(g) \leq 1 \) [10, 31].

The new scheme which is called the nonstandard two-step Lagrange interpolation method (NS2LIM) is given as follows:

\[
S(t_{e+1}) = S(t_0) + \frac{\Gamma(v)(1-v)}{\Gamma(v)(1-v) + v} h(t_e, S(t_e)) + \frac{1}{(v+1)(1-v)\Gamma(v) + v} \sum_{\delta=0}^{\xi} \theta(g)^\nu h(t_{\delta}, S(t_{\delta})) (\epsilon + 1 - \delta)^\nu \\
\times (\epsilon - \partial + 2 + v) - (\epsilon - \partial)^\nu (\epsilon - \partial + 2 + 2v) - \theta(g)^\nu h(t_{\delta-1}, S(t_{\delta-1})) (\epsilon + 1 - \delta)^{\nu+1} (\epsilon - \partial + 2 + v) \\
- (\epsilon - \partial)^\nu (\epsilon - \partial + 1 + v).
\] (50)

Similarly,

\[
E(t_{e+1}) = E(t_0) + \frac{\Gamma(v)(1-v)}{\Gamma(v)(1-v) + v} h(t_e, E(t_e)) + \frac{1}{(v+1)(1-v)\Gamma(v) + v} \sum_{\delta=0}^{\xi} \theta(g)^\nu h(t_{\delta}, E(t_{\delta})) (\epsilon + 1 - \delta)^\nu \\
\times (\epsilon - \partial + 2 + v) - (\epsilon - \partial)^\nu (\epsilon - \partial + 2 + 2v) - \theta(g)^\nu h(t_{\delta-1}, E(t_{\delta-1})) (\epsilon + 1 - \delta)^{\nu+1} (\epsilon - \partial + 2 + v) \\
- (\epsilon - \partial)^\nu (\epsilon - \partial + 1 + v),
\]

\[
I_A(t_{e+1}) = I_A(t_0) + \frac{\Gamma(v)(1-v)}{\Gamma(v)(1-v) + v} h(t_e, I_A(t_e)) + \frac{1}{(v+1)(1-v)\Gamma(v) + v} \sum_{\delta=0}^{\xi} \theta(g)^\nu h(t_{\delta}, I_A(t_{\delta})) (\epsilon + 1 - \delta)^\nu \\
\times (\epsilon - \partial + 2 + v) - (\epsilon - \partial)^\nu (\epsilon - \partial + 2 + 2v) - \theta(g)^\nu h(t_{\delta-1}, I_A(t_{\delta-1})) (\epsilon + 1 - \delta)^{\nu+1} (\epsilon - \partial + 2 + v) \\
- (\epsilon - \partial)^\nu (\epsilon - \partial + 1 + v),
\]

\[
I_S(t_{e+1}) = I_S(t_0) + \frac{\Gamma(v)(1-v)}{\Gamma(v)(1-v) + v} h(t_e, I_S(t_e)) + \frac{1}{(v+1)(1-v)\Gamma(v) + v} \sum_{\delta=0}^{\xi} \theta(g)^\nu h(t_{\delta}, I_S(t_{\delta})) (\epsilon + 1 - \delta)^\nu \\
\times (\epsilon - \partial + 2 + v) - (\epsilon - \partial)^\nu (\epsilon - \partial + 2 + 2v) - \theta(g)^\nu h(t_{\delta-1}, I_S(t_{\delta-1})) (\epsilon + 1 - \delta)^{\nu+1} (\epsilon - \partial + 2 + v) \\
- (\epsilon - \partial)^\nu (\epsilon - \partial + 1 + v),
\]

\[
\frac{\Gamma(v)(1-v)}{\Gamma(v)(1-v) + v} h(t_e, S(t_e)) + \frac{1}{(v+1)\Gamma(v) + v} \sum_{\delta=0}^{\xi} \theta(g)^\nu h(t_{\delta}, S(t_{\delta})) (\epsilon + 1 - \delta)^\nu \\
\times (\epsilon - \partial + 2 + v) - (\epsilon - \partial)^\nu (\epsilon - \partial + 2 + 2v) - \theta(g)^\nu h(t_{\delta-1}, S(t_{\delta-1})) (\epsilon + 1 - \delta)^{\nu+1} (\epsilon - \partial + 2 + v) \\
- (\epsilon - \partial)^\nu (\epsilon - \partial + 1 + v),
\]
6. Numerical Simulation

In this section, we validate the model using the parameter values given in [20] and use the numerical scheme in [10]. The parameter values are given in Table 1.

Using the initial conditions given in [20], $S(0) = 30800000$, $E(0) = 0$, $I_A(0) = 2$, $I_S(0) = 0$, $Q(0) = 0$, $V(0) = 0$, $R(0) = 0$, and setting the fractional operator $\nu \in [0.6, 1.0]$ at a step-size of 0.1, the simulations performed are displayed in Figures 2–8. The figures depict the behaviour of all compartments for the first 400 days since the outbreak.

Figures 2–8 depict the behaviour of susceptible, exposed, asymptomatic, symptomatic, quarantine, vaccinated, and recovered individuals, respectively, for different values of the fractional operator $\nu$ for the period of 400 days. In Figure 2, the population of susceptible decreases as the value of the fractional operator $\nu$ reduces. The exposed, asymptomatic, symptomatic, and quarantine population is extinct when the fractional operator is 0.6 and below (Figures 3–6). Again, exposed, asymptomatic, symptomatic, and quarantine population is seen to reach an early peak when the fractional operator value is reduced from 1. The number of exposed, asymptomatic, symptomatic, and quarantine individuals decays faster at the noninteger values. However, the number of immune individuals increases as $\nu$ reduces from 1.0 to 0.6 (Figure 7). On the other hand, the recovered population is seen to become extinct at $\nu = 0.6$. This is so because the infections in the population have also become extinct at the same value of the fractional operator (Figure 8).

7. Numerical Simulation of the Fractional Optimal Control

In this section, we analyze the numerical behavior of the fractional optimal control model using the parameter values given in Table 1 and the same initial conditions $S(0) = 30800000$, $E(0) = 0$, $I_A(0) = 2$, $I_S(0) = 0$, $Q(0) = 0$, $V(0) = 0$, $R(0) = 0$. Using MATLAB OD45 Ruge–Kutta method, the results of the simulations are displayed in Figures 9–20.

Figures 9–15 depict the behaviour of the susceptible, exposed, asymptomatic, symptomatic, quarantine, immune, and recovered individuals, respectively, when the optimal control $u_1$ is fully optimized while setting $u_2 = 0$ for the entire period of 400 days. With the social distancing control, it takes a longer period of 350 days before a significant decline in the susceptible population is observed as compared to a situation without optimal control which is 250 days (Figure 9). In Figure 10, there is a drastic decline in the number of individuals that get exposed to the disease when social distancing is observed. This leads to a corresponding decline in the number of asymptomatic, symptomatic, and quarantine individuals (Figures 11–13). The social distancing measures have no effect on the immune individuals as the population remains constant with or without the optimal control (Figure 14). With a drastic fall in the number of infections in the case of an optimal control, the recovered population is also seen to decline when there is an optimal control (Figure 15). Figures 16–22 describe the dynamics of each compartment when the vaccination control is implemented.
Table 1: Parameter values and description.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Values</th>
<th>Sources</th>
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</thead>
<tbody>
<tr>
<td>$\Omega$</td>
<td>Recruitment rate</td>
<td>29.08</td>
<td>[20, 33]</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Transmission rate</td>
<td>0.9</td>
<td>[20, 27]</td>
</tr>
<tr>
<td>$\phi$</td>
<td>The rate at which exposed individuals become infectious</td>
<td>0.25</td>
<td>[20, 27, 34]</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Natural death rate</td>
<td>$0.4252912 \times 10^{-4}$</td>
<td>[20, 27]</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Disease-induced death rate</td>
<td>$1.6728 \times 10^{-5}$</td>
<td>[20, 35]</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Recovery rate of symptomatic individuals</td>
<td>1/14</td>
<td>[36]</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Rate at which vaccinated individuals lose their immunity</td>
<td>$1.52 \times 10^{-7}$</td>
<td>[20]</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Recovery rate of asymptomatic individuals</td>
<td>1/14</td>
<td>[20, 27]</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Recovery rate of quarantine individuals</td>
<td>1/14</td>
<td>[20]</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Rate at which symptomatic individuals move to the quarantine class</td>
<td>0.01</td>
<td>[36]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Rate at which asymptomatic individuals move to the quarantine class</td>
<td>$1.026 \times 10^{-7}$</td>
<td>Assumed</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Rate at which susceptible individuals are vaccinated</td>
<td>0.01624</td>
<td>[20]</td>
</tr>
</tbody>
</table>

Figure 2: Behaviour of the susceptible individuals at different values of $v$.

Figure 3: Behaviour of the exposed individuals at different values of $v$. 
Figures 16–22 show the behaviour of the susceptible, exposed, asymptomatic, symptomatic, quarantine, recovered, and immune individuals, respectively, when the vaccination control $u_2$ is fully optimized while setting $u_1 = 0$ for the entire period of 400 days. With the vaccination control, a significant decline in the number of susceptible individuals is observed for the first 300 days. However, after the 300 days, there is a slow decline in the number compared with a situation without optimal control. The vaccination reduces the number of individuals susceptible to the disease for the first 300 days (Figure 16). Vaccinating the susceptible individuals leads to a decline in the number of exposed, asymptomatic, symptomatic, and quarantine individuals as compared with a situation without the optimal control (Figures 17–20). The decline in infections leads to a decline in the recovered population (Figure 21). There are
Figure 6: Behaviour of the quarantine individuals at different values of $v$.

Figure 7: Behaviour of the vaccinated individuals at different values of $v$. 
Figure 8: Behaviour of the recovered individuals at different values of $v$.

Figure 9: Behaviour of the susceptible individuals with and without the control.
Figure 10: Behaviour of the exposed individuals with and without the optimal control.

Figure 11: Behaviour of the asymptomatic individuals with and without the control.

Figure 12: Behaviour of the symptomatic individuals with and without the control.
Figure 13: Behaviour of the quarantine individuals with and without the control.

Figure 14: Behaviour of the immune individuals with and without the control.

Figure 15: Behaviour of the recovered individuals with and without the control.
Figure 16: Behaviour of the susceptible individuals with and without the vaccination control.

Figure 17: Behaviour of the exposed individuals with and without the vaccination control.

Figure 18: Behaviour of the asymptomatic individuals with and without the vaccination control.
Figure 19: Behaviour of the symptomatic individuals with and without the control.

Figure 20: Behaviour of the quarantine individuals with and without the vaccination control.

Figure 21: Behaviour of the recovered individuals with and without the vaccination control.
a high number of individuals that develop immunity to the disease due to the vaccination of susceptible individuals (Figure 22).

8. Conclusion

In this study, the model in [20] has been modified and formulated using the fractional-order derivative defined in the Atangana–Baleanu–Caputo sense. The basic properties such as the equilibrium points, basic reproduction number, and uniqueness of the solutions have been explored. Fractional optimal controls were incorporated into the model to determine appropriate intervention strategies in curbing the spread of the disease. The model was validated using the parameter values given in [20] for a period of 400 days. The MATLAB software fourth-order Runge–Kutta method was used for the simulations. Results of the numerical simulation show that there is a significant number of individuals who become exposed, asymptomatic, symptomatic, quarantine, and recovered when the fractional operator \( v \) is above 0.6. The number of immune individuals increases with a reduction in the fractional operator value from 1 to 0.6. Contrary to the number of immune individuals, the number of susceptible individuals declines as the fractional operator value decreases. The numerical simulation of the optimal control model demonstrates that vaccination and social isolation are both very successful strategies for preventing the spread of the disease. Social isolation had no impact on the immune population, while vaccination controls produced a sizable proportion of disease-immune individuals.

Data Availability

The data supporting the formulation of the mathematical model in this paper are from the Ghana Health Service website: https://www.ghs.gov.gh/covid19/ which has been referenced in the article.

Disclosure


Conflicts of Interest

The authors declare that there are no conflicts of interest.

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References


