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Research Article

On Inequalities for q-h-Integrals via Convex Functions

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This article aims to investigate unified versions of the well-known Hermite–Hadamard inequality by considering q-h-integrals and properties of convex functions. Currently published results for q-integrals can be deduced from inequalities of this paper. Moreover, some new results are presented in terms of corollaries.

1. Introduction

Real-valued functions satisfying inequality (1) have very interesting properties. For example, they assure continuity and left and right differentiability in the interior of the domains (for more details, see [1, 2]). Especially, in the theory of mathematical inequalities, convex functions play very vital role. Several discrete and integral inequalities can be proven by analyzing properties of convex functions.

Definition 1 (see [1]). Let a real-valued function g be defined on the real line's interval I. Then g is called convex on I, if it satisfies

$$g(zx + (1 - z)y) \leq zg(x) + (1 - z)g(y), \tag{1}$$

for $z \in [0, 1]$, $x, y \in I$.

Convex functions and related inequalities are very frequently analyzed for new kinds of notions including fractional derivative and integral operators. Applications of convex functions are found in almost all fields of mathematical analysis including statistics, optimization theory, and economics. For more details, we refer the readers to [3–6]. A geometric representation of a convex function defined on real line is the well-known Hermite–Hadamard inequality stated in the following theorem.

Theorem 2. *The function satisfying (1) holds the following inequality:*

$$g\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y g(x)dx \leq \frac{g(x)+g(y)}{2}. \tag{2}$$

It states that the integral mean $\int_a^b g(x)dx/b - a$ of a convex function g over $[a, b]$ lies in between value of function g at arithmetic mean $(AM\{a, b\} \coloneqq a + b/2)$ of a, b and the arithmetic mean $(AM\{g(a), g(b)\} \coloneqq g(a) + g(b)/2)$ of values $g(a)$ and $g(b)$ of function g. The following theorem provides a generalization of the aforementioned inequality by involving a function which is symmetric about $AM\{a, b\}$.

Theorem 3 (see [7]). *The upcoming inequality is valid under the conditions of the aforementioned theorem, if p is symmetric about $AM\{a, b\}$:*

$$\int_a^b p(x)\mathrm{dx}\, g(AM\{a,b\}) \leq \int_a^b g(x)p(x)dx \leq \int_a^b p(x)\mathrm{dx}\, \mathrm{AM}\{g(a), g(b)\}. \quad (3)$$

Inequalities (2) and (3) were studied by many researchers since their appearance in the literature of mathematical inequalities. For instance, for fractional-order derivatives/integrals, one can see [8–10], while for quantum derivatives/integrals, one can see [11–15].

In this paper, we aim to give Hermite–Hadamard inequalities for q-h-integrals defined in Definition 9, see [16]. From these versions, we can get various types of q-Hermite–Hadamard inequalities. It is needful to give some important definitions and related results to elaborate the motivation behind this article. These notions and results are given in the forthcoming section.

2. Preliminaries

In this section, we give definitions of q-, q-h-derivatives, q-derivative on finite intervals, q-definite integrals, q-h-derivatives on finite intervals, and q-h-definite integrals. Also, some Hadamard type q-integral inequalities are given from some recently published articles.

Definition 4. Let $g \in C(I)$, $0 < q < 1$. The following expression:

$$D_q g(x) = \frac{g(qx) - g(x)}{(q-1)x}, \quad (4)$$

is called q-derivative of g.

Definition 5 (see [11]). Let $g \in C(I)$, $0 < q < 1$, and $h \in \mathbb{R}$. Then the q-h-derivative of g is defined by

$$C_h D_q g(x) = \frac{{}_h d_q g(x)}{{}_h d_q x} = \frac{g(q(x+h)) - g(x)}{(q-1)x + qh}. \quad (5)$$

For h = 0 in (5), we get (4), i.e.,

$$C_0 D_q g(x) = D_q g(x). \quad (6)$$

Definition 6 (see [13, 17]). Let $g \in C[a,b]$. Then the following expressions:

$$\begin{aligned} {}_a D_q g(x) &= \frac{g(x) - g(qx + (1-q)a)}{(q-1)(x-a)}, \quad x \neq a, \\ {}_b D_q g(x) &= \frac{g(x) - g(qx + (1-q)b)}{(q-1)(x-b)}, \quad x \neq b, \end{aligned} \quad (7)$$

are called q_a-derivative and q_b-derivative of g, respectively. Also, ${}_a D_q g(a) = \lim_{x \longrightarrow a} {}_a D_q g(x)$ and ${}_b D_q g(a) = \lim_{x \longrightarrow a} {}_b D_q g(x)$.

Definition 7 (see [13, 17]). Let $g \in C[a,b]$. Then the q_a- and q_b-definite integrals are given by

$$\begin{aligned} \int_a^x g(t)\,{}_a d_q t &= (x-a)(1-q)\sum_{n=0}^{\infty} q^n g((1-q^n)a + q^n x), \\ \int_x^b g(t)\,{}^b d_q t &= (b-x)(1-q)\sum_{n=0}^{\infty} q^n g(q^n x + (1-q^n)b), \end{aligned} \quad (8)$$

where $x \in [a,b]$.

One can note that $\int_a^b g(t)\,{}_a d_q t = \int_a^b g(t)\,{}^b d_q t$, when g is symmetric about the line $x = AM\{a,b\}$.

Definition 8 (see [16]). Let $f \in C[a,b]$. For $q \in (0,1)$, the q_{a-h}-derivative of f at $x \in [a,b]$ is defined by the expression

$$C_h D_q^a f(x) = \frac{f(x) - f(qx + (1-q)a + qh)}{(x-a)(1-q) - qh}, x \neq \frac{a(1-q) + qh}{1-q} := x_\circ. \quad (9)$$

Analogously, the q_{b-h}-derivative of f at $x \in [a,b]$ is given by

$$C_h D_q^b f(x) = \frac{f(x) - f(qx + (1-q)b + qh)}{(1-q)(x-b) - qh}, x \neq \frac{b(1-q) + qh}{1-q} := y_\circ. \quad (10)$$

Also, ${}_{C_h}D_q^a f(x_\circ) = \lim_{x \longrightarrow x_\circ} {}_{C_h}D_q^a f(x)$ and ${}_{C_h}D_q^b f(x_\circ) = \lim_{x \longrightarrow y_\circ} {}_{C_h}D_q^b f(x)$.

Definition 9 (see [16]). Let $f \in C[a,b]$, $0 < q < 1$, and $h \in \mathbb{R}$. Then the q_{a-h}- and q_{b-h}-integrals denoted by $I_{q-h}^a f$ and $I_{q-h}^b f$, respectively, are given by

$$\begin{aligned} I_{q-h}^a f(x) &:= \int_a^x f(t)_h d_q^a t \\ &= ((x-a)(1-q)+qh)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a + nq^n h), \quad x > a, \\ I_{q-h}^b f(x) &:= \int_x^b f(t)_h d_q^b t \\ &= ((b-x)(1-q)+qh)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b + nq^n h), \quad x < b. \end{aligned} \tag{11}$$

Recent research [11] employing q-definite integrals has demonstrated the following q-H-H inequality for convex functions.

$$f\left(\frac{b+aq}{1+q}\right) \leq \frac{\int_a^b f(x) d_q^a x}{b-a} \leq \frac{qf(a)+f(b)}{1+q}. \tag{12}$$

Theorem 10. *A differentiable convex function f: $[a,b] \longrightarrow \mathbb{R}$ must satisfy the upcoming inequality for q_a-integrals:*

Theorem 11. *The upcoming inequality holds for q_a-integrals under the conditions of the aforementioned theorem:*

$$f\left(\frac{a+bq}{1+q}\right) + f'\left(\frac{a+bq}{1+q}\right)\frac{(b-a)(1-q)}{1+q} \leq \frac{\int_a^b f(x) d_q^a x}{b-a} \leq \frac{qf(a)+f(b)}{1+q}. \tag{13}$$

Theorem 12. *A differentiable convex function f on $[a,b]$ must satisfy the following q_a-integral inequality:*

$$f(AM\{a,b\}) + f'(AM\{a,b\})\frac{(b-a)(1-q)}{2(1+q)} \leq \frac{\int_a^b f(x) d_q^a x}{b-a} \leq \frac{qf(a)+f(b)}{1+q}. \tag{14}$$

In [13], the following q-H-H inequality for convex functions was proved.

Theorem 13. *A differentiable convex function f on $[a,b]$ satisfies the upcoming inequality:*

$$f\left(\frac{a+bq}{1+q}\right) \leq \frac{\int_a^b f(x) d_q^b x}{b-a} \leq \frac{f(a)+qf(b)}{1+q}. \tag{15}$$

The following example is needful for upcoming results.

Example 1. If $f(x) = x$, $x \in [a,b]$ and $0 < q < 1$. Then $I_{q-h}^a(f(x))$, $I_{q-h}^b(f(x))$ are given by

$$\begin{aligned} I_{q-h}^a(f(x)) = \int_a^x f(x) d_q x &= ((x-a)(1-q)+qh)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a + nq^n h) \\ &= ((x-a)(1-q)+qh)\left(\frac{x+aq}{1-q^2} + h\sum_{n=0}^{\infty} nq^{2n}\right) \end{aligned}$$

$$
\begin{aligned}
&= ((x-a)(1-q)+qh)\left(\frac{x+aq}{1-q^2}+\frac{hq^2}{\left(1-q^2\right)^2}\right),\\
I_{q-h}^{b}(f(x)) &= \int_x^b f(x)d_qx = ((b-x)(1-q)+qh)\sum_{n=0}^{\infty} q^n f\left(q^n x+\left(1-q^n\right)b+nq^n h\right)\\
&= ((b-x)(1-q)+qh)\left(\frac{x+bq}{1-q^2}+h\sum_{n=0}^{\infty} nq^{2n}\right)\\
&= ((b-x)(1-q)+qh)\left(\frac{x+bq}{1-q^2}++\frac{hq^2}{\left(1-q^2\right)^2}\right).
\end{aligned}
\tag{16}
$$

3. Generalized q-h-H-H Inequalities

We establish generalized q-h-H-H type inequalities for convex functions. In particular cases, a number of q-H-H inequality versions are deducible. In the whole paper, we consider S to be the sum of the series $\sum_{n=0}^{\infty} n q^{2n}$, that is, $S = hq^2/(1-q^2)^2$.

Theorem 14. *A convex function $\mathscr{H}$: $[a,b] \longrightarrow \mathbb{R}$ satisfies the upcoming q_{a-h}-integral inequality:*

$$
\begin{aligned}
&\mathscr{H}\left(\frac{b+aq}{1+q}\right)\frac{(x-a)(1-q)+qh}{1-q}+m((x-a)(1-q)+qh)\left(\frac{x-b}{1-q^2}+Sh\right)\\
&\le \int_a^x \mathscr{H}(x)_h d_q^a x \le ((x-a)(1-q)+qh)\left(\frac{\mathscr{H}(a)}{1-q}+\frac{\mathscr{H}(b)-\mathscr{H}(a)}{b-a}\left(\frac{x-a}{1-q^2}+hS\right)\right).
\end{aligned}
\tag{17}
$$

Proof. It is given that $\mathscr{H}$ is convex function; therefore, a tangent to $\mathscr{H}$ at any point must be line of support for $\mathscr{H}$ at that point. Let f denotes the function which describes the line of support to the function H at $b+aq/1+q$. Then in equation form, we have

$$
f(x) = \mathscr{H}\left(\frac{b+aq}{1+q}\right)+m\left(x-\frac{b+aq}{1+q}\right), m \in \left[\mathscr{H}_-'\left(\frac{b+aq}{1+q}\right), \mathscr{H}_+'\left(\frac{b+aq}{1+q}\right)\right].
\tag{18}
$$

Furthermore, f and $\mathscr{H}$ must satisfy the inequality $f(x) \le \mathscr{H}(x)$, since g is a convex function. Hence, the upcoming inequality holds.

$$
\mathscr{H}\left(\frac{b+aq}{1+q}\right)+m\left(x-\frac{b+aq}{1+q}\right) \le \mathscr{H}(x), x \in (a,b).
\tag{19}
$$

Taking q_{a-h}-integral on both sides, we get the upcoming inequality:

$$
\int_a^x\left(\mathscr{H}\left(\frac{b+aq}{1+q}\right)+m\left(x-\frac{b+aq}{1+q}\right)\right)_h d_q^a x \le \int_a^x \mathscr{H}(x)_h d_q^a x.
\tag{20}
$$

From Example 1, one can see that

$$
\int_a^x x_h d_q^a x = ((x-a)(1-q)+qh)\left(\frac{x+aq}{1-q^2}+hS\right),
\tag{21}
$$

and

$$
\int_a^x h d_q^a x = \frac{(x-a)(1-q)+qh}{1-q}.
\tag{22}
$$

The first inequality of (17) can be obtained by using (21) and (22) in (20).

For getting the other inequality of (17), we proceed as follows. The line passing through the points $(a, \mathscr{H}(a))$ and $(b, \mathscr{H}(b))$ is defined by the function L:

$$
L(x) = \mathscr{H}(a)+\frac{\mathscr{H}(b)-\mathscr{H}(a)}{b-a}(x-a),
\tag{23}
$$

and it satisfies the inequality $\mathscr{H}(x) \leq L(x)$ for all $x \in [a,b]$ because g is convex on $[a,b]$. Therefore, we have

$$\mathscr{H}(x) \leq \mathscr{H}(a) + \frac{\mathscr{H}(b) - \mathscr{H}(a)}{b-a}(x-a). \quad (24)$$

The following q_{a-h}-integral inequality is obtained:

$$\int_a^x \mathscr{H}(x)_h d_q^a x \leq \int_a^x \left(\mathscr{H}(a) + \frac{\mathscr{H}(b) - \mathscr{H}(a)}{b-a}(x-a)\right) {}_h d_q^a x. \quad (25)$$

The second inequality of (17) can be found after some computation.

Next, we provide a few implications of the aforementioned theorem. □

Corollary 15. *If, in addition, $\mathscr{H}$ is differentiable in Theorem 14, then we have the following inequality:*

$$\begin{aligned}&\mathscr{H}\left(\frac{b+aq}{1+q}\right)\frac{(x-a)(1-q)+qh}{1-q} + \mathscr{H}'\left(\frac{b+aq}{1+q}\right)((x-a)(1-q)+qh)\left(\frac{x-b}{1-q^2} + Sh\right)\\ &\leq \int_a^x \mathscr{H}(x)_h d_q^a x \leq ((x-a)(1-q)+qh)\left(\frac{\mathscr{H}(a)}{1-q} + \frac{\mathscr{H}(b) - \mathscr{H}(a)}{b-a}\left(\frac{x-a}{1-q^2} + hS\right)\right).\end{aligned} \quad (26)$$

Proof. Since $\mathscr{H}$ is differentiable function, $m = \mathscr{H}'(b + aq/1 + q)$. From inequality (17), one can obtain (26). □

Corollary 16. *If $h = 0$ in Theorem 14, we get the upcoming inequality:*

$$\begin{aligned}&\mathscr{H}\left(\frac{b+aq}{1+q}\right) + \mathscr{H}'\left(\frac{b+aq}{1+q}\right)\frac{x-b}{1+q} \leq \frac{\int_a^x \mathscr{H}(x)_0 d_q^a x}{x-a}\\ &\leq \left(\mathscr{H}(a) + \frac{(\mathscr{H}(b) - \mathscr{H}(a))(x-a)}{(b-a)(1+q)}\right).\end{aligned} \quad (27)$$

Remark 17. Under the assumption of aforementioned theorem, one can get the following results:

(1) If $x = b$ in (27), we get the inequality (12) stated in Theorem 10. Further, if $\mathscr{H}$ is symmetric about $AM\{a,b\}$, inequality (27) holds for q_b-integrals.

(2) If $h = 0, x = b$ and $q \longrightarrow 1$ in Theorem 14, one can have inequality (2).

A generalization of the above theorem is given in the following result.

Theorem 18. *Additionally, if p is a nonnegative function, the upcoming inequality for q_{a-h}-integrals also holds under the conditions of the aforementioned theorem:*

$$\begin{aligned}&\left(\mathscr{H}\left(\frac{b+aq}{1+q}\right) - \frac{m(b+aq)}{1+q}\right)\int_a^x p(x) d_q^a x + m\int_a^x x p(x) d_q^a x\\ &\leq \int_a^x \mathscr{H}(x) p(x)_h d_q^a x \leq \left(\frac{b\mathscr{H}(a) - a\mathscr{H}(b)}{b-a}\right)\int_a^x p(x) d_q^a x + \frac{\mathscr{H}(b) - \mathscr{H}(a)}{b-a}\int_a^x x p(x) d_q^a x.\end{aligned} \quad (28)$$

Proof. By multiplying inequality (19) with $p(x)$ and then taking q_{a-h}-integral on both sides, one can get the first inequality of (28) after some simplifications. The second inequality of (28) can be obtained in a similar way by using inequality (24) instead of (19). □

Theorem 19. *Inequality (29) holds for q_{a-h}-integrals under the same conditions of the above theorem:*

$$\begin{aligned}&\mathscr{H}\left(\frac{a+bq}{1+q}\right)\frac{(x-a)(1-q)+qh}{1-q}+\mathscr{H}'\left(\frac{a+bq}{1+q}\right)((x-a)(1-q)+qh)\\&\cdot\left(\frac{(x-a)-q(b-a)}{1-q^2}+Sh\right)\leq\int_a^x\mathscr{H}(x)_h d_q x\leq((x-a)(1-q)+qh)\\&\times\left(\frac{\mathscr{H}(a)}{1-q}+\frac{\mathscr{H}(b)-\mathscr{H}(a)}{b-a}\left(\frac{x-a}{1-q^2}+hS\right)\right),\end{aligned}\tag{29}$$

provided $\mathscr{H}$ is differentiable.

Proof. The tangent line of function $\mathscr{H}$ at point $a+bq/1+q\in(a,b)$ is defined by function T which satisfies the inequality $T(x)\leq\mathscr{H}(x)$. Therefore, we have

$$\mathscr{H}\left(\frac{a+bq}{1+q}\right)+\mathscr{H}'\left(\frac{a+bq}{1+q}\right)\left(x-\frac{a+bq}{1+q}\right)\leq\mathscr{H}(x),\tag{30}$$

for all $x\in[a,b]$. Taking q_a-h-integral on both sides, we get the upcoming inequality:

$$\int_a^x\left(\mathscr{H}\left(\frac{a+bq}{1+q}\right)+\mathscr{H}'\left(\frac{a+bq}{1+q}\right)\left(x-\frac{a+bq}{1+q}\right)\right){}_h d_q x\leq\int_a^x\mathscr{H}(x)_h d_q x.\tag{31}$$

By using (21) and (22), we get

$$\begin{aligned}&\mathscr{H}\left(\frac{a+bq}{1+q}\right)\frac{(x-a)(1-q)+qh}{1-q}+\mathscr{H}'\left(\frac{a+bq}{1+q}\right)\left(((x-a)(1-q)+qh)\left(\frac{x+aq}{1-q^2}+hS\right)\right.\\&\left.-\frac{((x-a)(1-q)+qh)}{1-q}\frac{a+bq}{1+q}\right)\leq\int_a^x\mathscr{H}(x)_h d_q x.\end{aligned}\tag{32}$$

The first inequality of (29) is proved. The second inequality can be proved on the same lines as in proof of Theorem 14.

A few implications of the aforementioned theorem are given as follows. □

Corollary 20. *If $h=0$ in Theorem 19, we get the upcoming inequality:*

$$\begin{aligned}&\mathscr{H}\left(\frac{a+bq}{1+q}\right)+\mathscr{H}'\left(\frac{a+bq}{1+q}\right)\frac{x-a-q(b-a)}{1+q}\leq\frac{\int_a^x\mathscr{H}(x)_0 d_q^a x}{x-a}\\&\leq\mathscr{H}(a)+\frac{(\mathscr{H}(b)-\mathscr{H}(a))(x-a)}{(b-a)(1+q)}.\end{aligned}\tag{33}$$

Remark 21

(1) If $x = b$ in (27), we get the inequality (13) stated in Theorem 11. Further, if f is symmetric about $AM\{a,b\}$, inequality (33) holds for q_b-integrals.

(2) If $h = 0, x = b$ and $q \longrightarrow 1$ in Theorem 14, one can have inequality (2).

Theorem 22. *Additionally, if p is a nonnegative function, the upcoming inequality for q_{a-h}-integrals also holds under the conditions of Theorem 14:*

$$\begin{aligned}&\left(\mathscr{H}\left(\frac{a+bq}{1+q}\right)-\frac{a+bq}{1+q}\mathscr{H}'\left(\frac{a+bq}{1+q}\right)\right)\int_a^x p(x)d_q^a x+\mathscr{H}'\left(\frac{a+bq}{1+q}\right)\int_a^x xp(x)d_q^a x\\&\leq\int_a^x \mathscr{H}(x)_h p(x)d_q^a x\leq\left(\frac{b\mathscr{H}(a)-a\mathscr{H}(b)}{b-a}\right)\int_a^x p(x)d_q^a x+\frac{\mathscr{H}(b)-\mathscr{H}(a)}{b-a}\int_a^x xp(x)d_q^a x,\end{aligned} \tag{34}$$

provided $\mathscr{H}$ is differentiable.

Proof. By multiplying inequality (30) with $p(x)$ and then taking q_{a-h}-integral on both sides, one can get the first inequality of (34) after some simplifications. The second inequality of (34) can be obtained in a similar way by using inequality (24) instead of (30). □

Remark 23. It can be noted that for $p(x) = 1$, Theorem 22 provides Theorem 19.

Theorem 24. *The upcoming inequality holds under the conditions of Theorem 14:*

$$\begin{aligned}&\mathscr{H}(AM\{a,b\})\frac{(x-a)(1-q)+qh}{1-q}+\mathscr{H}'(AM\{a,b\})((x-a)(1-q)+qh)\\&\times\left(\frac{2x-(b-a)q-(b+a)}{2(1-q^2)}+Sh\right)\leq\int_a^x \mathscr{H}(x)_h d_q^a x\leq((x-a)(1-q)+qh)\\&\times\left(\frac{\mathscr{H}(a)}{1-q}+\frac{\mathscr{H}(b)-\mathscr{H}(a)}{b-a}\left(\frac{x-a}{1-q^2}+hS\right)\right),\end{aligned} \tag{35}$$

provided $\mathscr{H}$ is differentiable.

Proof. The tangent line of the function $\mathscr{H}$ at point $AM\{a,b\}$ of (a,b) is defined by the function T_2 as follows:

$$T_2(x) = \mathscr{H}(AM\{a,b\}) + \mathscr{H}'(AM\{a,b\})(x - AM\{a,b\}). \tag{36}$$

It is given that $\mathscr{H}$ is convex, and we have $T_2(x) \leq \mathscr{H}(x)$, i.e.,

$$\mathscr{H}(AM\{a,b\}) + \mathscr{H}'(AM\{a,b\})(x - AM\{a,b\}) \leq \mathscr{H}(x), \tag{37}$$

for all $x \in [a,b]$ from which the following $q_a - h$-integral inequality holds:

$$\int_a^x\left(\mathscr{H}(AM\{a,b\})+\mathscr{H}'(AM\{a,b\})(x-AM\{a,b\})\right)_h d_q^a x\leq\int_a^b \mathscr{H}(x)_h d_q x. \tag{38}$$

By using (21) and (22), we get the upcoming inequality:

$$\mathscr{H}(AM\{a,b\})\frac{(x-a)(1-q)+qh}{1-q}+\mathscr{H}'(AM\{a,b\})\times\left(((x-a)(1-q)+qh)\left(\frac{x+qa}{1-q^2}+hS\right)-((x-a)(1-q)+qh)\frac{a+b}{2(1-q)}\right)\leq\int_a^b\mathscr{H}(x)_h d_q x. \quad (39)$$

The first inequity of (35) is proved. The second inequality can be proved on the same lines as in proof of Theorem 14.

In the following, we provide a few implications of the aforementioned theorem. □

Corollary 25. *If $h=0$ in Theorem 24, we get the upcoming inequality:*

$$\mathscr{H}(AM\{a,b\})+\mathscr{H}'(AM\{a,b\})\frac{2x-q(b-a)-(b+a)}{2(1+q)}\leq\frac{\int_a^x\mathscr{H}(x)_0 d_q^a x}{x-a}\leq\mathscr{H}(a)+\frac{(\mathscr{H}(b)-\mathscr{H}(a))(x-a)}{(b-a)(1+q)}. \quad (40)$$

Remark 26

(1) By setting $x=b$ in (40), we get (14) as stated in Theorem 12. Further, if f is symmetric about $AM\{a,b\}$, inequality (40) holds for q_b-integrals.

(2) If $h=0, x=b$ and $q\longrightarrow 1$ in Theorem 14, we get the H-H inequality.

(3) It can be noted that for $p(x)=1$, Theorem 18 provides Theorem 14.

4. Concluding Remarks

Hermite–Hadamard inequalities for convex functions using q-h-integrals were proved. Inequalities for q-integrals proved in [11, 13] were mentioned in particular cases. By using definitions and properties of different classes related to convex functions, new inequalities can be established for q-h-integrals. We are interested to explore further q-h-integral versions of classical inequalities including Opial inequality, Ostrowski inequality, and Grüss inequality.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study.

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