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Research Article

Multi-Criteria Decision-Making with Novel Pythagorean Fuzzy Aggregation Operators

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Unpredictability and fuzziness coexist in decision-making analysis due to the complexity of the decision-making environment. “Pythagorean fuzzy numbers” (PFNs) outperform “intuitionistic fuzzy numbers” (IFNs) when dealing with unclear data. The “Pythagorean fuzzy set” (PFS) is a useful tool because it removes the restriction that the sum of membership degrees be less than or equal to one by substituting the square sum for the sum of membership degrees. This study proposes two aggregating operators (AOs). The recommended operators outperform the already specified PFN operators. The proposed operator is utilised in the multicriteria decision-making process to identify the best candidate for instruction (MCDM).

1. Introduction

Today’s decision-making mechanism is becoming increasingly complicated, rendering it even more challenging for decision-makers to make sound judgments. This is mostly due to the fact that the intelligence acquired has a huge number of discrepancies. The data are given as discrete or interval numbers acquired from the reflecting journals or corresponding centers. Nonetheless, with the rising complexity of everyday activities, it is difficult to pinpoint actual information. In other sense, the major disadvantage of crisp sets is their failure to handle uncertainty. To conclude, if we analyse the recorded data as they are, the computed findings may lead to the conflicting selection. To deal with ambiguities in data, Zadeh [1] proposed a fuzzy set (FS) theory, in which every element is defined by its membership degree (MSD), which ranges from 0 to 1. Later, Atanassov [2] extended the FSs to intuitionistic FSs (IFSs) by combining nonmembership degrees (NMSDs) and MDs in such a way that their sum does not exceed one. The PFS was created by Yager [3–5], in which the squared sum of the MSDs and NMSDs is less than one. The main advantages of such prolonged FSs are that they use MSD and NMSD to express ambiguous information.

Data analysis is critical for making decision in the sectors of organizational, societal, clinical, scientific, cognitive, and machine intelligence. Generally, understanding of the alternate has been viewed as a crisp number or linguist number. Unfortunately, due to its unpredictability, the data cannot simply be pooled. In reality, AOs are crucial in the context of MCDM difficulties, as their primary goal is to agglomerate a bunch of inputs into a single value. The famous “Maclaurin symmetric mean” (MSM) AOs linked to IFSs were introduced by Liu and Qin [6]. Gul [7] pioneered the concept of Fermatean fuzzy SAW, VIKOR, and ARAS, which he applied to the COVID-19 testing laboratory prediction phase. MCDM technique based on fuzzy rough sets was introduced by Ye et al. [8]. Mu et al. [9] constructed power MSM AOs using PFS extension as interval-valued.

Pythagorean probabilistic hesitant fuzzy AOs were proposed by Batool et al. [10]. Peng and Yuan [11] introduced Pythagorean fuzzy averaging AOs, while Rehman et al. [12] proposed geometric AOs. Deli and Çagman [13] proposed intuitionistic fuzzy parameterized soft set.

Wang and Garg [14] proposed the idea of "Archimedean based Pythagorean fuzzy interactive" based operations and AOs with application to MCDM. Wang et al. [15] gave the "PF-interactive Hamacher power" AOs with applications to the assessment of express service quality. Huang et al. [16] initiated the idea of PF-MULTIMOORA approach with applications. Lin et al. [17] and Lin et al. [18] proposed some measures for PFSs. Meng et al. [19] proposed the idea of knowledge diffusion trajectories PFSs. Lin et al. [20] gave the "bibliometric analysis" for the PFSs. Chen et al. [21] proposed the framework of MCDM for the "sustainable building material selection." Using comparable linguistic ELECTRE III, he [22] also established expert knowledge bid assessment for building project selection. Chen et al. [23] presented the novel idea of using online-review analysis to determine passenger requirements and assess the level of customer satisfaction. Wei and Lu [24] gave the idea of "PF-power AOs," Wu and Wei [25] presented the idea of "PF-Hamacher AOs" and Garg [26] proposed "confidence levels based PF AOs" with application.

The remainder of this article will be organised in the following manner. Section 2 discusses several important PFS concepts. Section 3 considers a number of hybrid AOs for PFSs. Section 4 describes a technique for solving MCDM issues with new AOs. Section 5 is a call for details on proposed AOs. Section 6 concludes with some final remarks and future recommendations.

2. Preliminaries

In this part, we will go through the fundamentals of FSs, IFs, and PFS-sets.

Definition 1. [1] Let $\mathfrak{X}^{\Theta}$ be the reference set. A fuzzy set (FS) $\mathfrak{H}$ is

$$\mathfrak{H} = \left\{ < \wp^{\Lambda}, \mu^{\gimel}_{\mathfrak{H}}\left(\wp^{\Lambda}\right) > : \wp^{\Lambda} \in \mathfrak{X}^{\Theta} \right\}, \tag{1}$$

where $\mu^{\gimel}_{\mathfrak{H}}: \mathfrak{X}^{\Theta} \longrightarrow [0,1]$ is the MSD of $\mathfrak{H}$, which assigns a single real value to each alternative in the unit closed interval $[0,1]$.

Definition 2. [2] An intuitionistic fuzzy sets (IFSs) is

$$T = \left\{ < \wp^{\Lambda}, \mu^{\gimel}_{T}\left(\wp^{\Lambda}\right), \nu^{\gimel}_{T}\left(\wp^{\Lambda}\right) > : \wp^{\Lambda} \in \mathfrak{X}^{\Theta} \right\}, \tag{2}$$

which is represented by MSD $\mu^{\gimel}_{T}(\wp^{\Lambda}): \mathfrak{X}^{\Theta} \longrightarrow [0,1]$ and N-MSD $\nu^{\gimel}_{T}(\wp^{\Lambda}): \mathfrak{X}^{\Theta} \longrightarrow [0,1]$ with the constraint $0 \leq \mu^{\gimel}_{T}(\wp^{\Lambda}) + \nu^{\gimel}_{T}(\wp^{\Lambda}) \leq 1, \forall \wp^{\Lambda} \in \mathfrak{X}^{\Theta}$.

Definition 3. [4] A Pythagorean fuzzy set (PFS) in a universe $\mathfrak{X}^{\Theta}$ is

$$\vartheta^{\aleph} = \left\{ < \wp^{\Lambda}, \mu^{\gimel}_{\vartheta^{\aleph}}\left(\wp^{\Lambda}\right), \nu^{\gimel}_{\vartheta^{\aleph}}\left(\wp^{\Lambda}\right) > : \wp^{\Lambda} \in \mathfrak{X}^{\Theta} \right\}, \tag{3}$$

where $\mu^{\gimel}_{\vartheta^{\aleph}}(\wp^{\Lambda}): \mathfrak{X}^{\Theta} \longrightarrow [0,1]$ shows the MSD and $\nu^{\gimel}_{\vartheta^{\aleph}}(\wp^{\Lambda}): \mathfrak{X}^{\Theta} \longrightarrow [0,1]$ shows the N-MSD of the element $\wp^{\Lambda} \in \mathfrak{X}^{\Theta}$ to the set $\vartheta^{\aleph}$, respectively, with the condition that $0 \leq \mu^{\gimel}_{\vartheta^{\aleph}}(\wp^{\Lambda})^2 + \nu^{\gimel}_{\vartheta^{\aleph}}(\wp^{\Lambda})^2 \leq 1$.

A basic element of the form $\left\langle \mu^{\gimel}_{\vartheta^{\aleph}}(\wp^{\Lambda}), \nu^{\gimel}_{\vartheta^{\aleph}}(\wp^{\Lambda}) \right\rangle$ in a PFS $\vartheta^{\aleph}$ is called "Pythagorean fuzzy number" (PFN). It is denoted by $\vartheta^{\aleph} = \left\langle \mu^{\gimel}_{\vartheta^{\aleph}}, \nu^{\gimel}_{\vartheta^{\aleph}} \right\rangle$.

2.1. Operational Laws for PFSs

Definition 4. [4] Let $\vartheta_1^{\aleph} = \left\langle \mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}), \nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}) \right\rangle$ and $\vartheta_2^{\aleph} = \left\langle \mu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda}), \nu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda}) \right\rangle$ be PFSs on a $\mathfrak{X}^{\Theta}$. Then,

(1) $\overline{\vartheta_1^{\aleph}} = \left\langle \nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}), \mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}) \right\rangle$.

(2) $\vartheta_1^{\aleph} \subseteq \vartheta_2^{\aleph}$ iff $\mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}) \leq \mu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda})$ and $\nu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda}) \leq \nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda})$.

(3) $\vartheta_1^{\aleph} = \vartheta_2^{\aleph}$ iff $\vartheta_1^{\aleph} \subseteq \vartheta_2^{\aleph}$ and $\vartheta_2^{\aleph} \subseteq \vartheta_1^{\aleph}$.

(4) $\vartheta_1^{\aleph} \cup \vartheta_2^{\aleph} = \{\langle \wp^{\Lambda}, ma\wp^{\Lambda} \left\{ \mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}), \mu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda}) \right\}, \min\left\{ \nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}), \nu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda}) \right\} \rangle | \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\}$.

(5) $\vartheta_1^{\aleph} \cap \vartheta_2^{\aleph} = \left\{ \left\langle \wp^{\Lambda}, \min\left\{ \mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}), \mu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda}) \right\}, ma\wp^{\Lambda} \left\{ \nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}), \nu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda}) \right\} \right\rangle | \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\right\}$.

(6) $\vartheta_1^{\aleph} + \vartheta_2^{\aleph} = \left\{ \left\langle \wp^{\Lambda}, \sqrt{(\mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda})^2 + \mu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda})^2 - \mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda})^2 \mu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda})^2)}, \nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}) \nu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda}) \right\rangle | \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\right\}$.

(7) $\vartheta_1^{\aleph} \cdot \vartheta_2^{\aleph} = \left\{ \left\langle \wp^{\Lambda}, (\mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda}) \mu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda}), \sqrt{\nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda})^2 + \nu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda})^2 - \nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda})^2 \nu^{\gimel}_{\vartheta_2^{\aleph}}(\wp^{\Lambda})^2} \right\rangle | \wp^{\Lambda} \in \mathfrak{X}^{\Theta}\right\}$.

(8) $\sqsupset \vartheta_1^{\aleph} = \left\{ \left\langle \wp^{\Lambda}, \sqrt{(1 - (1 - \mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda})^2)^{\sqsupset}))}, \nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda})^{\sqsupset} \right\rangle \right\}$.

(9) $\vartheta_1^{\aleph \sqsupset} = \left\{ \left\langle \wp^{\Lambda}, \mu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda})^{\sqsupset}, \sqrt{(1 - (1 - \nu^{\gimel}_{\vartheta_1^{\aleph}}(\wp^{\Lambda})^2)^{\sqsupset}))}, \right\rangle \right\}$.

Theorem 1. *[4] Let* $\mathrm{P}^{\gimel}$, $\mathrm{B}^{\gimel}$, *and* $\mathrm{C}^{\gimel}$ *be any PFSs over the reference set* $\mathfrak{X}^{\Theta}$. *Let* $\tilde{U}$ *be absolute PFS and* $\tilde{\varnothing}$ *be the null PFS. Then,*

(i) $\mathrm{P}^{\gimel} \cup \mathrm{P}^{\gimel} = \mathrm{P}^{\gimel}$.

(ii) $\mathrm{P}^{\gimel} \cap \mathrm{P}^{\gimel} = \mathrm{P}^{\gimel}$.

(iii) $(\mathrm{P}^{\gimel} \cup \mathrm{B}^{\gimel}) \cup \mathrm{C}^{\gimel} = \mathrm{P}^{\gimel} \cup (\mathrm{B}^{\gimel} \cup \mathrm{C}^{\gimel})$.

(iv) $(\mathrm{P}^{\gimel} \cap \mathrm{B}^{\gimel}) \cap \mathrm{C}^{\gimel} = \mathrm{P}^{\gimel} \cap (\mathrm{B}^{\gimel} \cap \mathrm{C}^{\gimel})$.

(v) $\mathrm{P}^{\gimel} \cup (\mathrm{B}^{\gimel} \cap \mathrm{C}^{\gimel}) = (\mathrm{P}^{\gimel} \cup \mathrm{B}^{\gimel}) \cap (\mathrm{P}^{\gimel} \cup \mathrm{C}^{\gimel})$.

(vi) $\mathrm{P}^{\gimel} \cap (\mathrm{B}^{\gimel} \cup \mathrm{C}^{\gimel}) = (\mathrm{P}^{\gimel} \cap \mathrm{B}^{\gimel}) \cup (\mathrm{P}^{\gimel} \cap \mathrm{C}^{\gimel})$.

(vii) $\mathrm{P}^{\gimel} \cup \tilde{\varnothing} = \mathrm{P}^{\gimel}$ *and* $\mathrm{P}^{\gimel} \cap \tilde{\varnothing} = \tilde{\varnothing}$

(viii) $\mathrm{P}^{\gimel} \cup \tilde{U} = \tilde{U}$ *and* $\mathrm{P}^{\gimel} \cap \tilde{U} = \mathrm{P}^{\gimel}$

(ix) $(\mathrm{P}^{\gimel c})^{c} = \mathrm{P}^{\gimel}$.

(x) $\tilde{U}^{c} = \tilde{\varnothing}$ ***<i>and*** $\tilde{\varnothing}^{c} = \tilde{U}$.

Theorem 2. *Let* $\mathrm{P}^{\gimel}$ *and* $\mathrm{B}^{\gimel}$ *be two PFSs over the reference set* $\mathfrak{X}^{\Theta}$. *Then,*

(a) $(\mathrm{P}^{\gimel} \cup \mathrm{B}^{\gimel})^{c} = \mathrm{P}^{\gimel c} \cap \mathrm{B}^{\gimel c}$ *and*

(b) $(\mathrm{P}^{\gimel} \cap \mathrm{B}^{\gimel})^{c} = \mathrm{P}^{\gimel c} \cup \mathrm{B}^{\gimel c}$.

2.2. Operational Laws for PFNs

Definition 5. [4] Suppose $\Upsilon_1^{\zeta} = \langle \mu_1^{\gimel}, \nu_1^{\gimel} \rangle$ and $\Upsilon_2^{\zeta} = \langle \mu_2^{\gimel}, \nu_2^{\gimel} \rangle$ are the two PFNs. Then,

(1) $\overline{\Upsilon_1^{\zeta}} = \langle \nu_1^{\gimel}, \mu_1^{\gimel} \rangle$

(2) $\Upsilon_1^{\zeta} \vee \Upsilon_2^{\zeta} = \langle \max\{\mu_1^{\gimel}, \nu_1^{\gimel}\}, \min\{\mu_2^{\gimel}, \nu_2^{\gimel}\} \rangle$

(3) $\Upsilon_1^{\zeta} \wedge \Upsilon_2^{\zeta} = \langle \min\{\mu_1^{\gimel}, \nu_1^{\gimel}\}, \max\{\mu_2^{\gimel}, \nu_2^{\gimel}\} \rangle$

(4) $\Upsilon_1^{\zeta} \oplus \Upsilon_2^{\zeta} = \left\langle \sqrt{\mu_1^{\gimel 2} + \mu_2^{\gimel 2} - \mu_1^{\gimel 2}\mu_2^{\gimel 2}}, \nu_1^{\gimel}\nu_2^{\gimel} \right\rangle$

(5) $\Upsilon_1^{\zeta} \otimes \Upsilon_2^{\zeta} = \left\langle \mu_1^{\gimel}\mu_2^{\gimel}, \sqrt{\nu_1^{\gimel 2} + \nu_2^{\gimel 2} - \nu_1^{\gimel 2}\nu_2^{\gimel 2}} \right\rangle$

(6) $\sqsupset \Upsilon_1^{\zeta} = \left\langle \sqrt{(1 - (1 - \mu_1^{\gimel 2})^{\sqsupset})}, \nu_1^{\gimel \sqsupset} \right\rangle$

(7) $\Upsilon_1^{\zeta \Upsilon_1^{\zeta}} = \left\langle \mu_1^{\gimel \sqsupset}, \sqrt{(1 - (1 - \nu_1^{\gimel 2})^{\sqsupset})} \right\rangle$

Theorem 3. *[4] Suppose $\Upsilon_1^{\zeta} = \langle \mu_1^{\gimel}, \nu_1^{\gimel} \rangle$ and $\Upsilon_2^{\zeta} = \langle \mu_2^{\gimel}, \nu_2^{\gimel} \rangle$ are any PFNs on a $\mathfrak{X}^{\Theta}$, and $n_1, n_2 > 0$, then*

(1) $\Upsilon_1^{\zeta} \oplus \Upsilon_2^{\zeta} = \Upsilon_2^{\zeta} \oplus \Upsilon_1^{\zeta}$

(2) $\Upsilon_1^{\zeta} \otimes \Upsilon_2^{\zeta} = \Upsilon_2^{\zeta} \otimes \Upsilon_1^{\zeta}$

(3) $n(\Upsilon_1^{\zeta} \oplus \Upsilon_2^{\zeta}) = n\Upsilon_1^{\zeta} \oplus n\Upsilon_2^{\zeta}$

(4) $n_1\Upsilon_1^{\zeta} \oplus n_2\Upsilon_2^{\zeta} = (n_1 + n_2)\Upsilon_1^{\zeta}$

(5) $\Upsilon_1^{\zeta n_1} \otimes \Upsilon_1^{\zeta n_2} = \Upsilon_1^{\zeta n_1 + n_2}$

(6) $\Upsilon_1^{\zeta n} \otimes \Upsilon_1^{\zeta n} = (\Upsilon_1^{\zeta} \otimes \Upsilon_2^{\zeta})^n$

Definition 6. [4] Let $\vartheta^{\aleph} = \langle \mu^{\gimel}, \nu^{\gimel} \rangle$ be the PFN, then a "score function" (SF) $\tilde{\daleth}$ of $\vartheta^{\aleph}$ is given as

$$\tilde{\daleth}(\vartheta^{\aleph}) = \mu^{\gimel 2} - \nu^{\gimel 2}. \quad (4)$$

$\tilde{\daleth}(\vartheta^{\aleph}) \in [-1, 1]$. If the SF is high, the PFN is also significant. Unfortunately, in the many situations of PFN, the SF is ineffective. For example: let $\Upsilon_1^{\zeta} = \langle 0.6138, 0.2534 \rangle$ and $\Upsilon_2^{\zeta} = \langle 0.7147, 0.4453 \rangle$ be two PFNs. Then, SF of Υ_1^{ζ} is $\tilde{\daleth}(\Upsilon_1^{\zeta}) = 0.3125$ and the SF of Υ_2^{ζ} is $\tilde{\daleth}(\Upsilon_2^{\zeta}) = 0.3125$. This demonstrates that the SF is insufficient for comparing the PFNs. We employ another function called the "accuracy function" (AF) to tackle this difficulty.

Definition 7. [4] Let $\vartheta^{\aleph} = \langle \mu^{\gimel}, \nu^{\gimel} \rangle$ be the PFN. Then, the AF $\coprod$ of $\vartheta^{\aleph}$ is defined as

$$\coprod(\vartheta^{\aleph}) = \mu^{\gimel 2} + \nu^{\gimel 2}, \quad (5)$$

where $\coprod(\vartheta^{\aleph}) \in [0, 1]$. If the AF is high, the PFN is also significant.

For example, $\Upsilon_1^{\zeta} = \langle 0.6138, 0.2534 \rangle$ and $\Upsilon_2^{\zeta} = \langle 0.7147, 0.4453 \rangle$. The AFs are $\coprod(\Upsilon_1^{\zeta}) = 0.4410$ and $\coprod(\Upsilon_2^{\zeta}) = 0.7091$. Thus we can write $\Upsilon_1^{\zeta} < \Upsilon_2^{\zeta}$.

Definition 8. [4] Let $s = \langle \mu_s^{\gimel}, \nu_s^{\gimel} \rangle$ and $t = \langle \mu_t^{\gimel}, \nu_t^{\gimel} \rangle$ be any two PFNs. Let $\tilde{\daleth}(s), \tilde{\daleth}(t)$ be the SFs of s and t and $\coprod(s), \coprod(t)$ be the AFs of s and t, respectively. Then,

(1) If $\tilde{\daleth}(s), \tilde{\daleth}(t)$, then $s > t$

(2) If $\tilde{\daleth}(s), \tilde{\daleth}(t)$, then

If $\coprod(s) > \coprod(t)$, then $s > t$,
If $\coprod(s) > \coprod(t)$, then $s = t$.

2.3. Some Basic AOs Related to PFNs

Definition 9. [11] Let $\breve{\Upsilon}_k^{\zeta} = \langle \mu_k^{\gimel}, \nu_k^{\gimel} \rangle$ be the conglomeration of PFNs. Define (PFWA): $\mathfrak{X}^{\Theta n} \longrightarrow \mathfrak{X}^{\Theta}$ given by

$$\begin{aligned}(\text{PFWG})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) &= \sum_{k=1}^{n} \mathfrak{G}_k^{\gimel} \breve{\Upsilon}_k^{\zeta} \\ &= \mathfrak{G}_1^{\gimel} \breve{\Upsilon}_1^{\zeta} \otimes \mathfrak{G}_2^{\gimel} \breve{\Upsilon}_2^{\zeta} \otimes, \ldots, \mathfrak{G}_n^{\gimel} \breve{\Upsilon}_n^{\zeta},\end{aligned} \quad (6)$$

where T^n is the set of all PFNs and $\mathfrak{G}^{\gimel} = (\mathfrak{G}_1^{\gimel}, \mathfrak{G}_2^{\gimel}, \ldots, \mathfrak{G}_n^{\gimel})^T$ is the "weight vector" (WV) of $(\breve{\Upsilon}_1, \breve{\Upsilon}_2, \ldots, \breve{\Upsilon}_n)$, s.t. $0 \leq \mathfrak{G}_k^{\gimel} \leq 1$, and $\sum_{k=1}^{n} \mathfrak{G}_k^{\gimel} = 1$. Then, the PFWA is the "Pythagorean fuzzy weighted averaging (PFWA) operator."

We can evaluate PFWA using the operating laws of PFNs, as shown by the preceding theorem.

Theorem 4. *[11] Let $\breve{\Upsilon}_k^{\zeta} = \langle \mu_k^{\gimel}, \nu_k^{\gimel} \rangle$ $(k = 1, 2, \ldots, n)$ be the conglomeration of PFNs, we also evaluate the PFWA by*

$$(\text{PFWA})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) = \left\langle \sqrt{\left(1 - \prod_{k=1}^{n} \left(1 - \mu_k^{\gimel}\right)^{\mathfrak{G}_k^{\gimel}}\right)}, \prod_{k=1}^{n} \nu_k^{\gimel \mathfrak{G}_k^{\gimel}} \right\rangle. \quad (7)$$

Example 1. Let $\breve{\Upsilon}_1^{\zeta} = (0.70, 0.50), \breve{\Upsilon}_2^{\zeta} = (0.30, 0.50)$, and $\breve{\Upsilon}_3 = (0.60, 0.70)$ be the three PFNs and $w = (0.30, 0.30, 0.40)$ be the WV of $(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \breve{\Upsilon}_3^{\zeta})$. We use PFWA operator to aggregate the three PFNs by using equation (1):

$$\begin{aligned}(\text{PFWA})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \breve{\Upsilon}_3^{\zeta}\right) &= \left\langle \sqrt{\left(1 - \prod_{k=1}^{3} \left(1 - \mu_k^{\gimel}\right)^{\mathfrak{G}_k^{\gimel}}\right)}, \prod_{k=1}^{3} \nu_k^{\gimel \mathfrak{G}_k^{\gimel}} \right\rangle \\ &= (0.591, 0.572).\end{aligned} \quad (8)$$

Definition 10. [12] Let $\breve{\Upsilon}_k^{\zeta} = \langle \mu_k^{\beth}, \nu_k^{\beth} \rangle$ be the conglomeration of PFN, and (PFWG): $\mathfrak{X}^{\Theta n} \longrightarrow \mathfrak{X}^{\Theta}$, if

$$(\mathrm{PFWG})\left(\breve{\Upsilon}\zeta_1 t, n\breve{\Upsilon}q\zeta_2 h,_{\ldots} x, 7\breve{\Upsilon}C\zeta_n\right) = \sum_{k=1}^{n} \breve{\Upsilon}\zeta_k^{\mathfrak{G}_k^{\beth}} = \breve{\Upsilon}\zeta_1^{\mathfrak{G}_1^{\beth}} \otimes \breve{\Upsilon}\zeta_2^{\mathfrak{G}_2^{\beth}} \otimes, \ldots, \breve{\Upsilon}\zeta_n^{\mathfrak{G}_n^{\beth}}, \tag{9}$$

where $\mathfrak{G}^{\beth} = (\mathfrak{G}_1^{\beth}, \mathfrak{G}_2^{\beth}, \ldots, \mathfrak{G}_n^{\beth})^T$ is WV of $(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta})$, s.t. $0 \leq \mathfrak{G}_k^{\beth} \leq 1$, and $\sum_{k=1}^{n} \mathfrak{G}_k^{\beth} = 1$. Then, the PFWG is the "Pythagorean fuzzy weighted geometric (PFWG) operator."

We can evaluate PFWG using the operating laws of PFNs, as shown by the preceding theorem.

Theorem 5. *[12] Let $\breve{\Upsilon}_k^{\zeta} = \langle \mu_k^{\beth}, \nu_k^{\beth} \rangle$ be the conglomeration of PFNs, we can find PFWG by*

$$(\mathrm{PFWG})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) = \left\langle \prod_{k=1}^{n} \mu_k^{\beth \mathfrak{G}_k^{\beth}}, \left(1 - \prod_{k=1}^{n} \left(1 - \nu_k^{\beth}\right)^{\mathfrak{G}_k^{\beth}}\right)^{1/2} \right\rangle. \tag{10}$$

Example 2. Let $\breve{\Upsilon}_1^{\zeta} = (0.70, 0.50), \breve{\Upsilon}_2^{\zeta} = (0.30, 0.50)$, and $\breve{\Upsilon}_3^{\zeta} = (0.60, 0.70)$ be the three PFNs and $\mathfrak{G}^{\beth} = (0.30, 0.30, 0.40)$ be the WV of $(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \breve{\Upsilon}_3^{\zeta})$. We use PFWG operator to aggregate the three PFNs by using equation (2):

$$\begin{aligned}(\mathrm{PFWG})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \breve{\Upsilon}_3^{\zeta}\right) &= \left\langle \prod_{k=1}^{3} \mu_k^{\beth \mathfrak{G}_k^{\beth}}, \left(1 - \prod_{k=1}^{3} \left(1 - \nu_k^{\beth q}\right)^{\mathfrak{G}_k^{\beth}}\right)^{1/2} \right\rangle \\ &= (0.510, 0.603).\end{aligned} \tag{11}$$

2.4. Some Deficiencies of PFWA and PFWG Operators. As we all know, PFWA and PFWG operators are utilised to accumulate knowledge in different MCDM issues. Therefore, whenever some values go toward the upper justifications or highest weights, their summed values may imply some absurd results. In this section, we will look at two scenarios.

Case 1. Take two PFNs s.t. $\breve{\Upsilon}_1^{\zeta} = (0.001, 0), \breve{\Upsilon}_2^{\zeta} = (1, 0)$ with weights $\mathfrak{G}_1^{\beth} = 0.9$ and $\mathfrak{G}_2^{\beth} = 0.1$. By equations (7) and (10) we get

$$\begin{aligned}\mathrm{PFWA}\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}\right) &= (1, 0), \\ \mathrm{PFWG}\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}\right) &= (0.002, 0).\end{aligned} \tag{12}$$

Case 2. Take two PFNs s.t. $\breve{\Upsilon}_1^{\zeta} = (0.001, 0), \breve{\Upsilon}_2^{\zeta} = (1, 0)$ with weights $\mathfrak{G}_1^{\beth} = 0.1$ and $\mathfrak{G}_2^{\beth} = 0.9$. By equations (7) and (10) we get

$$\begin{aligned}\mathrm{PFWA}\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}\right) &= (1, 0), \\ \mathrm{PFWG}\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}\right) &= (0.501, 0).\end{aligned} \tag{13}$$

We can see from these data that the PFWA and PFWG operators managed to provide reasonable outcomes in these two circumstances. As a result, in order to address these inadequacies or limitations, we must strengthen the AOs.

3. Some Hybrid Aggregation Operators of PFNs

In this part, we suggest a novel hybrid AOs to fill the gaps left by the PFWA and PFGA operators.

3.1. PFHWAGA Operator. Assume that $\breve{\Upsilon}_k^{\zeta} = \langle \mu_k^{\beth}, \nu_k^{\beth} \rangle$ $(k = 1, 2, \ldots, n)$ is a conglomeration of PFN and (PFHWAGA): $\mathfrak{X}^{\Theta n} \longrightarrow \mathfrak{X}^{\Theta}$, if

$$(\mathrm{PFHWAGA})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) = \left(\sum_{k=1}^{n} \mathfrak{G}_k^{\beth} \breve{\Upsilon}_2^{\zeta}\right)^{\sqsupset} \left(\sum_{k=1}^{n} \breve{\Upsilon}_2^{\zeta^{\mathfrak{G}_k^{\beth}}}\right)^{1-\sqsupset}, \tag{14}$$

where, $\sqsupset$ is any real number in $[0, 1]$ and $\mathfrak{G}^{\beth} = (\mathfrak{G}_1^{\beth}, \mathfrak{G}_2^{\beth}, \ldots, \mathfrak{G}_n^{\beth})^T$ is WV of $(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta})$, s.t. $0 \leq \mathfrak{G}_k^{\beth} \leq 1$ and $\sum_{k=1}^{n} \mathfrak{G}_k^{\beth} = 1$. Then, the PFHWAGA is called the PFHWAGA operator. The preceding theorem can be used to find PFHWAGA based on the operating principles of PFNs.

Theorem 6. *Let $\breve{\Upsilon}_k^{\zeta} = \left\langle \mu_k^{\gimel}, \nu_k^{\gimel} \right\rangle$ be the conglomeration of PFN, we can find PFHWAGA by*

$$\begin{aligned}
&(\text{PFHWAGA})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) \\
&= \left(\sum_{k=1}^{n} \mathfrak{G}_k^{\gimel} \breve{\Upsilon}_k^{\zeta}\right)^{\sqsupset} \left(\sum_{k=1}^{n} \breve{\Upsilon}_k^{\zeta^{\mathfrak{G}_k^{\gimel}}}\right)^{1-\sqsupset} \\
&= \left\langle \left(1 - \prod_{k=1}^{n}\left(1 - \left(\mu_k^{\gimel}\right)^2\right)^{w_k}\right)^{\sqsupset/2} \left(\prod_{k=1}^{n} \mu_k^{\gimel w_k}\right)^{1-\sqsupset}, \sqrt{1 - \left(1 - \left(\prod_{k=1}^{n}\left(\nu_k^{\gimel}\right)^{w_k}\right)^2\right)^{\sqsupset} \left(\prod_{k=1}^{n}\left(1 - \left(\nu_k^{\gimel}\right)^2\right)^{w_k}\right)^{1-\sqsupset}} \right\rangle,
\end{aligned} \tag{15}$$

where $\sqsupset$ is any number from $[0, 1]$. $\mathfrak{G}^{\gimel} = (\mathfrak{G}_1^{\gimel}, \mathfrak{G}_2^{\gimel}, \ldots, \mathfrak{G}_n^{\gimel})^T$ *is the WV of* $(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta})$, *s.t.* $0 \leq \mathfrak{G}_k^{\gimel} \leq 1$, *and* $\sum_{k=1}^{n} \mathfrak{G}_k^{\gimel} = 1$.

Proof. Based on PFWA and PFGA operators and the operational laws of PFSs.

$$\begin{aligned}
&(\text{PFHWAGA})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) \\
&= \left(\sum_{k=1}^{n} \mathfrak{G}_k^{\gimel} \breve{\Upsilon}_k^{\zeta}\right)^{\sqsupset} \left(\sum_{k=1}^{n} \breve{\Upsilon}_k^{\zeta^{\mathfrak{G}_k^{\gimel}}}\right)^{1-\sqsupset} \\
&= \left\langle \left(\left(1 - \prod_{k=1}^{n}\left(1 - \mu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{1/2}, \prod_{k=1}^{n} \nu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)\right\rangle^{\sqsupset} \left\langle\left(\prod_{k=1}^{n} \mu_k^{\gimel \mathfrak{G}_k^{\gimel}}, \left(1 - \prod_{k=1}^{n}\left(1 - \nu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{1/2}\right)\right\rangle^{1-\sqsupset} \\
&= \left\langle\left(\left(1 - \prod_{k=1}^{n}\left(1 - \mu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{\sqsupset/2}, \left(1 - \left(1 - \left(\prod_{k=1}^{n} \nu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)^2\right)^{\sqsupset}\right)^{1/2}\right)\left(\left(\prod_{k=1}^{n} \mu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset}, \left(1 - \left(\prod_{k=1}^{n}\left(1 - \nu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset}\right)^{1/2}\right)\right\rangle \\
&= \left\langle \left(1 - \prod_{k=1}^{n}\left(1 - \mu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{\sqsupset/2} \cdot \left(\prod_{k=1}^{n} \mu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset}, \right. \\
&\left.\sqrt{\left(1 - \left(1 - \left(\prod_{k=1}^{n} \nu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)^2\right)^{\sqsupset}\right)^{2/2} + \left(1 - \left(\prod_{k=1}^{n}\left(1 - \nu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset}\right)^{2/2} - \left(1 - \left(1 - \left(\prod_{k=1}^{n} \nu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)^2\right)^{\sqsupset}\right)\left(1 - \left(\prod_{k=1}^{n}\left(1 - \nu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset}\right)} \right\rangle \\
&= \left\langle \left(1 - \prod_{k=1}^{n}\left(1 - \left(\mu_k^{\gimel}\right)^2\right)^{\mathfrak{G}_k^{\gimel}}\right)^{\sqsupset/2} \left(\prod_{k=1}^{n} \mu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset}, \right. \\
&\left.\sqrt{\left(1 - \left(1 - \left(\prod_{k=1}^{n} \nu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)^2\right)^{\sqsupset}\right) + \left(1 - \left(\prod_{k=1}^{n}\left(1 - \nu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset}\right)^{2/2} - 1 + \left(\prod_{k=1}^{n}\left(1 - \nu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset} + \left(1 - \left(\prod_{k=1}^{n} \nu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)^2\right)^{\sqsupset} - \left(1 - \left(\prod_{k=1}^{n}\left(1 - \nu_k^{\gimel 2}\right)^{\mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset}\right)\left(1 - \left(\prod_{k=1}^{n} \nu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right)^2\right)^{\sqsupset}} \right\rangle \\
&= \left\langle \left(1 - \prod_{k=1}^{n}\left(1 - \left(\mu_k^{\gimel}\right)^2\right)^{\mathfrak{G}_k^{\gimel}}\right)^{\sqsupset/2} \left(\prod_{k=1}^{n} \mu_k^{\gimel \mathfrak{G}_k^{\gimel}}\right), \sqrt{1 - \left(1 - \left(\prod_{k=1}^{n}\left(\nu_k^{\gimel}\right)^{\mathfrak{G}_k^{\gimel}}\right)^2\right)^{\sqsupset} \left(\prod_{k=1}^{n}\left(1 - \left(\nu_k^{\gimel}\right)^2\right)^{\mathfrak{G}_k^{\gimel}}\right)^{1-\sqsupset}} \right\rangle.
\end{aligned} \tag{16}$$

Therefore, this complete the proof of equation (15).
ə1 □

Remark 1. It is feasible to explore the many families of the PFHWAGA operator independently for different values of $\sqsupset \in [0, 1]$. When we consider a particular situation, such as $\sqsupset = 1$, the PFHWAGA operator is converted to the PFWA operator. The PFHWAGA operator is simplified to the PFWG operator if $\sqsupset = 0$. The PFHWAGA operator is the mean of the PFWA and PFWG operators if $\sqsupset = 0.5$.

Example 3. Let $\breve{\Upsilon}_1^{\zeta} = (0.71, 0.52)$, $\breve{\Upsilon}_2^{\zeta} = (0.34, 0.56)$, and $\breve{\Upsilon}_3^{\zeta} = (0.57, 0.68)$ be the three PFNs, $\mathfrak{G}^{\gimel} = (0.5, 0.3, 0.2)$ be the WV of $(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \breve{\Upsilon}_3^{\zeta})$, and $\sqsupset = 0.5$. We use PFHWAGA operator to aggregate the three PFNs by using equation (15).

$$
\begin{aligned}
(\text{PFHWAGA})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \breve{\Upsilon}_3^{\zeta}\right) &= \left(1 - \prod_{k=1}^{3}\left(1-\left(\mu_k^{\daleth}\right)^2\right)^{w_k}\right)^{0.5/2}\left(\prod_{k=1}^{3}\mu_k^{\daleth w_k}\right)^{1-0.5}, \\
&\cdot \sqrt{1-\left(1-\left(\prod_{k=1}^{3}\left(\nu_k^{\daleth}\right)^{w_k}\right)^2\right)^{0.5}\left(\prod_{k=1}^{3}\left(1-\left(\nu_k^{\daleth}\right)^2\right)^{w_k}\right)^{1-0.5}} \\
&= (0.582, 0.567).
\end{aligned} \tag{17}
$$

It is clear from the characteristics of the PFWA and PFWG operators that the PFHWAGA operator has idempotency, boundedness, and monotonicity as well.

Theorem 7. *Let* $\breve{\Upsilon}_k^{\zeta} = \left\langle \mu_k^{\daleth}, \nu_k^{\daleth}\right\rangle (k = 1, 2, \ldots, n)$*is a conglomeration of PFNs. Then,*

(1) (Idempotency) if$\breve{\Upsilon}_k^{\zeta} = \breve{\Upsilon}^{\zeta} = \langle \mu^{\daleth}, \nu^{\daleth}\rangle$*for allk, then*

$$
\text{PFHWAGA}\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) = \breve{\Upsilon}^{\zeta} \tag{18}
$$

(2) (Boundedness) if$\breve{\Upsilon}^{\zeta^-} = (\min(\mu_k^{\daleth}), \max\ (\nu_k^{\daleth}))$*and*$\breve{\Upsilon}^{\zeta^+} = (\max(\mu_k^{\daleth}), \min(\nu_k^{\daleth}))$*, then we obtain*

$$
\breve{\Upsilon}^{\zeta^-} \le \text{PFHWAGA}\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) \le \breve{\Upsilon}^{\zeta^+}. \tag{19}
$$

(3) (Monotonicity) if $\breve{\Upsilon}_k^{\zeta} = \left\langle \mu_k^{\daleth}, \nu_k^{\daleth}\right\rangle$ *and* $\breve{\Upsilon}_k^{\zeta^*} = \left\langle \mu_k^{\daleth *}, \nu_k^{\daleth *}\right\rangle$ *are two sets of PFNs.If*$\mu_k^{\daleth} \ge \mu_k^{\daleth *}, \nu_k^{\daleth} \le \nu_k^{\daleth *}$*for allkthen,*

$$
\text{PFHWAGA}\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) \ge \text{PFHWAGA}\left(\breve{\Upsilon}_1^{\zeta^*}, \breve{\Upsilon}_2^{\zeta^*}, \ldots, \breve{\Upsilon}_n^{\zeta^*}\right). \tag{20}
$$

3.2. PFHOWAGA Operator. Assume that $\breve{\Upsilon}_k^{\zeta} = \left\langle \mu_k^{\daleth}, \nu_k^{\daleth}\right\rangle$ is a conglomeration of PFNs, and (PFHOWAGA): $\mathfrak{X}^{\Theta n} \longrightarrow \mathfrak{X}^{\Theta}$, if

$$
(\text{PFHWAGA})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) = \left(\sum_{k=1}^{n} \mathfrak{G}_k^{\daleth}\breve{\Upsilon}_{\daleth(k)}^{\zeta}\right)^{\daleth}\left(\sum_{k=1}^{n}\breve{\Upsilon}_{\daleth(k)}^{\zeta^{\mathfrak{G}_k^{\daleth}}}\right)^{1-\daleth}, \tag{21}
$$

where T^n is the set of all PFNs, $(\daleth(1), \daleth(2), \ldots, \daleth(k))$ is a permutation of $(1, 2, \ldots, n)$ s.t. $\breve{\Upsilon}_{\daleth(j-1)}^{\zeta} \ge \breve{\Upsilon}_{\daleth(j)}^{\zeta}$ for any k, $\daleth$ is any real number in the interval $[0, 1]$, and $\mathfrak{G}^{\daleth} = (\mathfrak{G}_1^{\daleth}, \mathfrak{G}_2^{\daleth}, \ldots, \mathfrak{G}_n^{\daleth})^T$ is WV of $(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta})$, s.t. $0 \le \mathfrak{G}_k^{\daleth} \le 1$ and $\sum_{k=1}^{n} \mathfrak{G}_k^{\daleth} = 1$. Then, the PFHOWAGA is called the PFHOWAGA operator.

We can also find PFHOWAGA operator by the following theorem.

Theorem 8. *Let* $\breve{\Upsilon}_k^{\zeta} = \left\langle \mu_k^{\daleth}, \nu_k^{\daleth}\right\rangle$ *be a conglomeration of PFNs. We can find PFHOWAGA by*

$$
\begin{aligned}
(\text{PFHWAGA})\left(\breve{\Upsilon}_1^{\zeta}, \breve{\Upsilon}_2^{\zeta}, \ldots, \breve{\Upsilon}_n^{\zeta}\right) &= \left(\sum_{k=1}^{n} \mathfrak{G}_k^{\daleth}\breve{\Upsilon}_{\daleth(k)}^{\zeta}\right)^{\daleth}\left(\sum_{k=1}^{n}\breve{\Upsilon}_{\daleth(k)}^{\zeta^{\mathfrak{G}_k^{\daleth}}}\right)^{1-\daleth} \\
&= \left(1 - \prod_{k=1}^{n}\left(1-\left(\mu_{\daleth(k)}^{\daleth}\right)^2\right)^{\mathfrak{G}_k^{\daleth}}\right)^{\daleth/2}\left(\prod_{k=1}^{n}\mu_{\daleth(k)}^{\daleth^{\mathfrak{G}_k^{\daleth}}}\right)^{1-\daleth} \\
&\cdot \sqrt{1-\left(1-\left(\prod_{k=1}^{n}\left(\nu_{\daleth(k)}^{\daleth}\right)^{\mathfrak{G}_k^{\daleth}}\right)^2\right)^{\daleth}\left(\prod_{k=1}^{n}\left(1-\left(\nu_{\daleth(k)}^{\daleth}\right)^2\right)^{\mathfrak{G}_k^{\daleth}}\right)^{1-\daleth}},
\end{aligned} \tag{22}
$$

where $\beth$ *is any real number in the interval* $[0,1]$.

Proof. The proof can be made by similar way to proof of Theorem 6, so we omit the proof. ∎

Example 4. Let $\ddot{a}_1=(0.71,0.52), \ddot{a}_2=(0.34,0.56)$, and $\ddot{a}_3=(0.57,0.68)$ be the three PFNs, $\mathfrak{G}^{\beth}=(0.5,0.3,0.2)$ be the WV of $(\ddot{a}_1,\ddot{a}_2,\ddot{a}_3)$, and $\beth=0.5$.By SF, we rank these PFNs:

$$\begin{aligned}\tilde{\daleth}(\ddot{a}_1)&=0.181,\\ \tilde{\daleth}(\ddot{a}_2)&=-0.085,\\ \tilde{\daleth}(\ddot{a}_3)&=-0.108.\end{aligned} \tag{23}$$

Now, $\breve{\Upsilon}_1^{\zeta}=\ddot{a}_1, \breve{\Upsilon}_2^{\zeta}=\ddot{a}_2, \breve{\Upsilon}_3^{\zeta}=\ddot{a}_3$. We use PFHOWAGA operator to aggregate by using equation (24).

$$\begin{aligned}(\text{PFHOWAGA})\left(\breve{\Upsilon}_1^{\zeta},\breve{\Upsilon}_2^{\zeta},\breve{\Upsilon}_3^{\zeta}\right)&=\left(1-\prod_{k=1}^{3}\left(1-\left(\mu_{\beth(k)}^{\beth}\right)^2\right)^{w_k}\right)^{0.5/2}\left(\prod_{k=1}^{3}\mu_{\beth(k)}^{\beth w_k}\right)^{1-0.5}\\ &\cdot\sqrt{1-\left(1-\left(\prod_{k=1}^{3}\left(\nu_{\beth(k)}^{\beth}\right)^{w_k}\right)^2\right)^{0.5}\left(\prod_{k=1}^{3}\left(1-\left(\nu_{\beth(k)}^{\beth}\right)^2\right)^{w_k}\right)^{1-0.5}}\\ &=(0.582,0.567).\end{aligned} \tag{24}$$

Theorem 9. *Let* $\breve{\Upsilon}_k^{\zeta}=\left\langle\mu_k^{\beth},\nu_k^{\beth}\right\rangle$ *is a conglomeration of PFNs. Then,*

(1) *(Idempotency) If* $\breve{\Upsilon}_k^{\zeta}=\breve{\Upsilon}^{\zeta}=\langle\mu^{\beth},\nu^{\beth}\rangle$ *for allk, then*

$$\text{PFHOWAGA}\left(\breve{\Upsilon}_1^{\zeta},\breve{\Upsilon}_2^{\zeta},\ldots,\breve{\Upsilon}_n^{\zeta}\right)=\breve{\Upsilon}^{\zeta} \tag{25}$$

(2) *(Boundedness) if* $\breve{\Upsilon}^{\zeta^-}=(\min(\mu_k^{\beth}),\max(\nu_k))$ *and* $\breve{\Upsilon}^{\zeta^+}=(\max(\mu_k^{\beth}),\min(\nu_k^{\beth}))$, *then we have*

$$\breve{\Upsilon}^{\zeta^-}\leq\text{PFHOWAGA}\left(\breve{\Upsilon}_1^{\zeta},\breve{\Upsilon}_2^{\zeta},\ldots,\breve{\Upsilon}_n^{\zeta}\right)\leq\breve{\Upsilon}^{\zeta^+}. \tag{26}$$

(3) *(Monotonicity) if* $\breve{\Upsilon}_k^{\zeta}=\left\langle\mu_k^{\beth},\nu_k^{\beth}\right\rangle$ *and* $\breve{\Upsilon}_k^{\zeta^*}=\left\langle\mu_k^{\beth*},\nu_k^{\beth*}\right\rangle$ *are two sets of PFNs. If* $\mu_k^{\beth}\geq\mu_k^{\beth*}, \nu_k^{\beth}\leq\nu_k^{\beth*}$ *for allk, then*

$$\text{PFHOWAGA}\left(\breve{\Upsilon}_1^{\zeta},\breve{\Upsilon}_2^{\zeta},\ldots,\breve{\Upsilon}_n^{\zeta}\right)\geq\text{PFHOWAGA}\left(\breve{\Upsilon}_1^{\zeta^*},\breve{\Upsilon}_2^{\zeta^*},\ldots,\breve{\Upsilon}_n^{\zeta^*}\right). \tag{27}$$

3.3. Numerical Example. To demonstrate the correctness of the aggregated values of the PFHWAGA and PFHOWAGA operations, we consider the first scenario in Section 2.4. If $\beth=0.5$, we will utilize the PFHWAGA and PFHOWAGA operations.

For Case 1, by equation (15), there is $\text{PFHWAGA}(\breve{\Upsilon}_1^{\zeta},\breve{\Upsilon}_2^{\zeta})=(0.045,0)$, which is between $\text{PFWA}(\breve{\Upsilon}_1^{\zeta},\breve{\Upsilon}_2^{\zeta})=(1,0)$ and $\text{PFWG}(\breve{\Upsilon}_1^{\zeta},\breve{\Upsilon}_2^{\zeta})=(0.002,0)$.

The moderate values are indicated in the above case by new advanced operators. These operators are clearly capable of overcoming the shortcomings of PFWA and PFWG operators. As a result, the PFHWAGA and PFHOWAGA operators are more efficient and acceptable in aggregating data.

4. Multi-Criteria Decision-Making Method

Suppose that $\breve{\Upsilon}^{\zeta}=\left\{(\breve{\Upsilon}_1^{\zeta},\breve{\Upsilon}_2^{\zeta},\ldots,\breve{\Upsilon}_p^{\zeta})\right\}$ and $\mathscr{V}^{\varpi}=\left\{\mathscr{V}_1^{\varpi},\mathscr{V}_2^{\varpi},\ldots,\mathscr{V}_q^{\varpi}\right\}$ are the assemblage of alternatives and attributes. Consider $\mathfrak{G}^{\beth}$ be the WV of all criterion, s.t. $\mathfrak{G}_j^{\beth}\in[0,1], \sum_{j=1}^{n}\mathfrak{G}_j^{\beth}=1$, and $\mathfrak{G}_j^{\beth}$ represent the weight of $\mathscr{V}_j^{\varpi}$. The DM assesses alternatives based on parameters and the evaluation parameters are in PFNs. Consider $(\breve{\coprod}_{ij})_{p\times q}=\left\langle\mu_{ij}^{\beth},\nu_{ij}^{\beth}\right\rangle$ is the decision matrix given by the DM, $(\breve{\coprod}_{ij})$ represents a PFNs for alternative $\breve{\Upsilon}_i^{\zeta}$ associated with the criterions $\mathscr{V}_j^{\varpi}$. With this Algorithm 1, some constraints are included s.t

(1) $\mu_{ij}^{\beth}$ *and* $\nu_{ij}^{\beth}\in[0,1]$

(2) $0\leq\mu_A^{\beth}(\mathfrak{G}^{\beth})^2+\nu_A^{\beth}(\mathfrak{G}^{\beth})^2\leq1, (q\geq1)$.

We now design Algorithm 1 to tackle the specified issue. The flow chart of Algorithm 1 is given by Figure 1.

5. MCDM Problem Related to Selection of Appropriate Candidate

Example 5. Consider the choosing of a university professor as a fairly straightforward decision-making dilemma. For the choosing, there are four teachers accessible,

Phase i. Obtain the decision matrix from DMs. $(\breve{\coprod}_{ij})_{p\times q} = \langle \mu_{ij}^{\beth}, \nu_{ij}^{\beth} \rangle$.
Phase ii. The decision matrix should be normalised. When we have various kinds of criteria or attributes, such as cost and benefit, we normalise the decision matrix by taking the complement of the cost criteria.
Phase iii. Find $\beta_i^{\gamma} = \text{PFHWAGA}\,(\beta_{i1}^{\gamma}, \beta_{i2}^{\gamma}, \ldots, \beta_{in}^{\gamma})$ or $\beta_i^{\gamma} = \text{PFHOWAGA}\,(\beta_{i1}^{\gamma}, \beta_{i2}^{\gamma}, \ldots, \beta_{in}^{\gamma})$ for each $i = 1, 2, \ldots, q$.
Phase iv. Evaluate the SFs for all β_i^{γ} for the collective overall PFNs.
Phase v. Rank all the $\beta_i^{\gamma}\,(i = 1, 2, \ldots, p)$ according to the score values.

Algorithm 1: Decision-making algorithm.

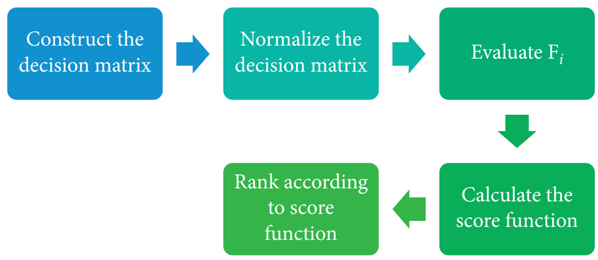

Figure 1: Flow chart of Algorithm 1.

	$\mathcal{V}_1^{\varpi}$	$\mathcal{V}_2^{\varpi}$	$\mathcal{V}_3^{\varpi}$	$\mathcal{V}_4^{\varpi}$
$\widetilde{\breve{\coprod}}_1$	(0.320, 0.610)	(0.410, 0.450)	(0.330, 0.560)	(0.660, 0.530)
$\widetilde{\breve{\coprod}}_2$	(0.610, 0.520)	(0.340, 0.560)	(0.570, 0.680)	(0.610, 0.490)
$\widetilde{\breve{\coprod}}_3$	(0.720, 0.320)	(0.360, 0.490)	(0.600, 0.420)	(0.700, 0.210)
$\widetilde{\breve{\coprod}}_4$	(0.760, 0.120)	(0.820, 0.320)	(0.910, 0.120)	(0.600, 0.130)

$D = \left\{ \widetilde{\breve{\coprod}}_i, i = 1, 2, 3, 4 \right\}$. To evaluate the professor, we consider given criterion $P = \left\{ \mathcal{V}_i^{\varpi}, i = 1, 2, 3, 4 \right\}$ given as

$$\begin{aligned} \mathcal{V}_1^{\varpi} &= \text{experience}, \\ \mathcal{V}_2^{\varpi} &= \text{research background}, \\ \mathcal{V}_3^{\varpi} &= \text{teaching methodology}, \\ \mathcal{V}_4^{\varpi} &= \text{personality}. \end{aligned} \tag{28}$$

PFNs are used in this challenge to evaluate the four feasible alternatives $\widetilde{\breve{\coprod}}_i\,(i = 1, 2, 3, 4)$ based on the four criteria listed above. Let $q = 3$, a WV $\mathfrak{G}^{\beth}$ is $(0.20, 0.30, 0.10, 0.40)^T$, and the controlling index is $\sqsupset = 0.5$.

Now, we will solve the MCDM issue using Algorithm 1. The following sections detail the procedure phases:

Phase i. Evaluating the choice matrix provided by the individual based on PF information.

Phase ii. The decision matrix is already in normalised form.

Phase iii. Compute $\beta_i^{\gamma} = \text{PFHWAGA}\,(\beta_{i1}^{\gamma}, \beta_{i2}^{\gamma}, \ldots, \beta_{in}^{\gamma})$ for each i. Thus we find aggregated PFNs by using equation (15).

$$\begin{aligned} \beta_1^{\gamma} &= (0.4980, 0.5270), \\ \beta_2^{\gamma} &= (0.5310, 0.5390), \\ \beta_3^{\gamma} &= (0.6020, 0.3510), \\ \beta_4^{\gamma} &= (0.7420, 0.1990). \end{aligned} \tag{29}$$

Phase iv. Evaluate the SFs for all β_i^{γ} for the collective overall PFNs.

$$\begin{aligned} \tilde{\daleth}(\beta_1^{\gamma}) &= -0.0230, \\ \tilde{\daleth}(\beta_2^{\gamma}) &= -0.0070, \\ \tilde{\daleth}(\beta_3^{\gamma}) &= 0.1750, \\ \tilde{\daleth}(\beta_4^{\gamma}) &= 0.4010. \end{aligned} \tag{30}$$

Phase v. Rank all the $\beta_i^{\gamma}\,(i = 1, 2, 3, 4)$ according to the score values.

$$\beta_4^{\gamma} \succ \beta_3^{\gamma} \succ \beta_2^{\gamma} \succ \beta_1^{\gamma}, \tag{31}$$

and thus β_4^γ is the most desirable alternative.

6. Conclusion

AOs, such as the PFWA and PFWG operators, are significant mathematical tools for integrating PF data. We designed two operators, namely the "Pythagorean fuzzy hybrid weighted arithmetic geometric aggregation (PFHWAGA) operator" and the "Pythagorean fuzzy hybrid ordered weighted arithmetic geometric aggregation (PFHOWAGA) operator" to address some of the shortcomings of the PFWA and PFWG operators in various real-world problems. Several characteristics of the PFHWAGA and PFHOWAGA operators were discovered. The recommended operators outperform the existing PFN-defined operators. With the use of examples, we enlarged on the proposed operators. Underneath the PF environment, designed operators are more robust and efficient than existing operators. Based on the PFHWAGA and PFHOWAGA operators, we devised an MCDM technique for selecting the best candidate for the position of teacher. We will use this concept in future to develope better other AOs, namely Einstein AOs, Hamacher AOs, and Dombi AOs in future.

Data Availability

The paper includes the information used to verify the study's findings.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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