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Research Article

Application of Asymptotic Analysis of a High-Dimensional HJB Equation to Portfolio Optimization

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In this paper, we consider a portfolio optimization problem where the wealth consists of investing into a risky asset with a slow mean-reverting volatility and receiving an uncontrollable stochastic cash flow under the exponential utility. The Hamilton–Jacobi–Bellman equation formulated from the optimal investment problem is a high-dimensional nonlinear partial differential equation and difficult to find its analytical or numerical solutions. The paper provides a tractable asymptotic approach which treats the initial problem as a perturbation around the constant volatility problem. In this paper, we present a formal derivation of asymptotic approximation and prove the accuracy of the value function. Moreover, an illustrative example is provided to assess our approximate strategy and value function.

1. Introduction

The problem of portfolio optimization in continuous time has a long history dating from Merton's seminal paper [1] which provides explicit solutions for how best to allocate wealth between a single risky asset modelled by Geometric Brownian Motion and a riskless asset. Since then, this optimal portfolio choice problem has attracted numerous studies [2–6] not only in academia but also in the financial industry. Browne [2] considered that the portfolio optimization problem is affected by an uncontrollable stochastic cash flow or random risk process. For instance, companies such as insurers or defined benefit pension managers undertake to accept random unavoidable risks that they must match with returns [7, 8].

The main goal of this paper is to study Browne's investment model with a slow mean-reverting volatility. That is, the coefficients of the risky asset are affected by a slow random factor which is able to capture some of the well-known features of volatility, such as the volatility smile and skew in [9, 10]. The asymptotic approach, in this paper, has been developed in [11], where this method is used for linear option pricing. Here, we provide some new results for the nonlinear portfolio optimization problem with a random risk process.

We assume the risky asset is defined as P_t and its volatility σ and growth rate μ are the functions of a slow factor Z. The price process of P_t and its volatility-driving factor Z_t are given by

$$\begin{aligned} \mathrm{dP}_t &= \mu(Z_t)P_t\mathrm{dt} + \sigma(Z_t)P_t\mathrm{dW}_t^{(1)}, \\ \mathrm{dZ}_t &= \delta c(Z_t)\mathrm{dt} + \sqrt{\delta}g(Z_t)\mathrm{dW}_t^{(\delta)}, \end{aligned} \tag{1}$$

where the correlation of the standard Brownian motions $(W_t^{(1)}, W_t^{(\delta)})$ satisfies

$$d\langle W^{(1)}, W^{(\delta)}\rangle_t = \rho_1\mathrm{dt}, |\rho_1| < 1, \tag{2}$$

and Z_t is described as slow mean-reverting factor or slow factor for short when δ goes to 0. The more details of the model and asymptotic calculation are presented in the following sections.

The novelties of this paper are described as follows. First, the new portfolio optimization problem takes both an uncontrollable stochastic cash flow and a slow factor into account. Second, compared with the results in Fouque et al.'s paper [11], the asymptotic analysis is extended to a class of nonlinear equation and the accuracy of the approximate solution is proved. Third, we analyze the feasibility of the approximate strategy.

The rest of this paper is organized as follows. In Section 2, we introduce the investment model with a random risk process in a slowly varying stochastic environment. In Section 3, the first order approximate solutions of the value function and the optimal strategy are provided. In Section 4, we prove the theoretical accuracy of the first order approximation to the value function. Numerical experiments are conducted in Section 5 to illustrate the tractability of the approximate solutions. Section 6 concludes and suggests directions of extension.

2. Reformulation of an Investment Model: Slow Factor-Random Risk Process Situation

In this section, we formulate the investment model over a finite time horizon $[0, T]$ and introduce the background in the real world. The dynamic processes of the risky asset P_t, the slow factor Z_t, and the random cash Y_t are given by

$$\begin{aligned} \mathrm{dP}_t &= \mu(Z_t)P_t\mathrm{dt} + \sigma(Z_t)P_t\mathrm{dW}_t^{(1)},\\ \mathrm{dZ}_t &= \delta c(Z_t)\mathrm{dt} + \sqrt{\delta}g(Z_t)\mathrm{dW}_t^{(\delta)},\\ \mathrm{dY}_t &= \alpha\mathrm{dt} + \beta\mathrm{dW}_t^{(2)}, \end{aligned} \tag{3}$$

where μ, σ, c, and g are the functions of the slow factor Z_t and α and β are the constants. The correlations of the standard Brownian motions $(W_t^{(1)}, W_t^{(\delta)}, W_t^{(2)})$ on $(\Omega, \mathscr{F}, (\mathscr{F})_t, \mathbb{P})$ satisfy

$$\begin{aligned} d\langle W^{(1)}, W^{(\delta)}\rangle_t &= \rho_1\mathrm{dt}, \quad |\rho_1| < 1,\\ d\langle W^{(1)}, W^{(2)}\rangle_t &= \rho_2\mathrm{dt}, \quad |\rho_2| < 1,\\ d\langle W^{(2)}, W^{(\delta)}\rangle_t &= 0. \end{aligned} \tag{4}$$

The uninteresting case $\rho_2^2 = 1$ is not considered where the model we study is reduced to the classical Merton model [12].

The total amount of money which the firm invest in the risky asset is defined by f_t. Assume f_t is a suitable admissible adapted control process, that is, f_t is a non-anticipative function that satisfies $\mathbb{E}\int_0^T f_t^2\,\mathrm{dt} < \infty$. The wealth of the company is denoted by X_t^f which is given by

$$\mathrm{dX}_t^f = \frac{f_t\mathrm{dP}_t}{P_t} + \mathrm{dY}_t, \quad X_0 = x. \tag{5}$$

Inserting (3) into (5), the wealth process is given by

$$\begin{aligned} \mathrm{dX}_t &= [f_t\mu(z) + \alpha]\mathrm{dt} + f_t\sigma(z)\mathrm{dW}_t^{(1)} + \beta\mathrm{dW}_t^{(2)},\\ \mathrm{dZ}_t &= \delta c(Z_t)\mathrm{dt} + \sqrt{\delta}g(Z_t)\mathrm{dW}_t^{(\delta)}, \end{aligned} \tag{6}$$

where the initial values are given by $X_0 = x$ and $Z_0 = z$.

The firm's objective is to find a trading strategy that maximizes expected utility conditioned on current value of x and z: $\mathbb{E}\{U(X_T) \mid X_t = x, Z_t = z\}$. Thus, the value function is defined as

$$V(t, x, z) = \sup_f \mathbb{E}\{U(X_T) \mid X_t = x, Z_t = z\}. \tag{7}$$

Exponential utility is a concept in economics and finance that is used to model the preferences of individuals or investors when making decisions involving uncertain outcomes or risks. It is particularly important in the field of decision theory, especially in situations where individuals are making choices that involve uncertain future payoffs, such as investment decisions or decisions related to insurance. Exponential utility function $U(x)$ is given by

$$U(x) = d - \frac{l}{r}e^{-\mathrm{rx}}, \tag{8}$$

where d, l, and r are the positive constants and parameter r represents constant absolute risk aversion. Exponential utility function plays an important role in insurance mathematics and actuarial practice due to the principle of zero utility under which a fair premium is independent of the level of reserves of an insurance company.

On the basis of the dynamic programming principle, the HJB equation of value function V is formulated as

$$\begin{cases} V_t + \delta\mathscr{M}V + \frac{1}{2}\beta^2 V_{\mathrm{xx}} + \alpha V_x + \mathrm{NL} = 0,\\ V(T, x, z) = U(x), t \in [0, T], x \in \mathbb{R}, z \in \mathbb{R}, \end{cases} \tag{9}$$

where the infinitesimal generator $\mathscr{M}$ is given as

$$\mathscr{M} = \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2} + c(z)\frac{\partial}{\partial z}, \tag{10}$$

and NL is given by

$$\begin{aligned} \mathrm{NL} = \max_f \Big(&\frac{1}{2}\sigma^2(z)f_t^2 V_{\mathrm{xx}}\\ &+ f_t\big(\rho_2\sigma(z)\beta V_{\mathrm{xx}} + \mu(z)V_x + \sqrt{\delta}\rho_1 g(z)\sigma(z)V_{xz}\big)\Big). \end{aligned} \tag{11}$$

3. Approximate Solutions of Value Function and Optimal Strategy

Due to the difficulty of finding an explicit solution in (11), we express the value function in terms of the solution of a linear parabolic equation and perform an asymptotic analysis for this linear parabolic equation.

3.1. Expansion of Value Function

Assumption 1. Before that, we make some assumptions as follows:

(1) $Z_t = Z_{\delta t}^{(1)}$ in distribution, where $Z^{(1)}$ has unique invariant distribution ϕ which is independent of δ and δ goes to 0

(2) The value function $V(t, x, z)$ satisfies the following conditions: smooth on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$, strictly increasing, and strictly concave in x.

(3) The coefficients $\mu(z)$ and $\sigma(z)$ to be differentiable

(4) The value function V is the unique solution of the HJB PDE (9)

It is obvious that the "max" part of the HJB equation is a quadratic function of f, and hence, the maximizer f^* is given by

$$f_t^* = -\frac{\mu(z)}{\sigma^2(z)}\frac{V_x}{V_{xx}} - \frac{\rho_2\beta}{\sigma(z)} - \frac{\rho_1 g(z)\sqrt{\delta}}{\sigma(z)}\frac{V_{xz}}{V_{xx}}. \tag{12}$$

Inserting f^* into (9), we obtain the following PDE.

$$\begin{cases} V_t + \delta\mathscr{M}V + \frac{1}{2}\beta^2 V_{xx} + \alpha V_x - \frac{1}{2}\sigma^2(z)V_{xx}\left(\frac{\mu(z)}{\sigma^2(z)}\frac{V_x}{V_{xx}} + \frac{\rho_2\beta}{\sigma(z)} + \frac{\sqrt{\delta}\rho_1 g(z)}{\sigma(z)}\frac{V_{xz}}{V_{xx}}\right)^2 = 0, \\ V(T,x,z) = U(x), t \in [0,T], x \in \mathbb{R}, z \in \mathbb{R}. \end{cases} \tag{13}$$

Equation (13) is a nonlinear PDE where variables x, y, and z are coupled, meaning changes in one variable affect others. Thus, PDE (13) is not easily solved, either analytically or numerically. When δ goes to zero, it is a regular perturbation and we construct an asymptotic approximation of the solution which is originally introduced in [11].

Lemma 2. *Under the assumptions listed above, the value function V in (13) is given by*

$$V(t,x,z) = d - \frac{l}{r}e^{-\mathrm{rx}}\varphi^q(t,z), \tag{14}$$

where d,l, and r are positive, $q = 1/1-\rho_1^2$, and φ: $[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}_+$ solves the linear parabolic equation:

$$\varphi_t + \delta\mathscr{M}\varphi + \frac{1}{2}\beta^2 r^2\frac{\varphi}{q} - \alpha r\frac{\varphi}{q} - \frac{1}{2}r^2\theta^2\frac{\varphi}{q} + \sqrt{\delta}\rho_1 \mathrm{gr}\theta\varphi_z = 0, \tag{15}$$

where $\theta(z) = \rho_2\beta - (\mu(z)/\sigma(z)r)$, $\mathscr{M}$ is defined in (10), and the terminal condition $\varphi(T,z) = 1$.

Proof. Suggest that the value function is of the form

$$V(t,x,z) = d - \frac{l}{r}e^{-\mathrm{rx}}\varphi^q(t,z), \tag{16}$$

where $q = 1/1-\rho_1^2$. Direct substitution in the HJB equation (13) yields that φ solves

$$\varphi_t + \delta\mathscr{M}\varphi + \frac{1}{2}\beta^2 r^2\frac{\varphi}{q} - \alpha r\frac{\varphi}{q} - \frac{1}{2}r^2\theta^2\frac{\varphi}{q} + \sqrt{\delta}\rho_1 \mathrm{gr}\theta\varphi_z = 0, \tag{17}$$

with terminal condition $\varphi(T,z) = 1$. □

Theorem 3. *The value function $V(t,x,z)$ in (13) can be approximated by*

$$V(t,x,z) \approx \widetilde{V}(t,x,z) := V^{(0)}(t,x,z) + \sqrt{\delta}V^{(1)}(t,x,z), \tag{18}$$

where

$$V^{(0)}(t,x,z) = d - \frac{l}{r}e^{-\mathrm{rx}}\cdot\left(\varphi^{(0)}\right)^q, \tag{19}$$

$$V^{(1)}(t,x,z) = \frac{l}{2r}e^{-\mathrm{rx}}(T-t)^2\rho_1 r^3 g(z)\theta(z)\theta'(z)\cdot\left(\varphi^{(0)}\right)^q, \tag{20}$$

$$\varphi^{(0)}(t,z) = \exp\left\{\frac{1}{q}(T-t)\left(\frac{1}{2}\beta^2 r^2 - \alpha r - \frac{1}{2}r^2\theta^2(z)\right)\right\}, \tag{21}$$

$$\theta(z) = \rho_2\beta - \frac{\mu(z)}{\sigma(z)r}, \quad q = \frac{1}{1-\rho_1^2}. \tag{22}$$

Proof. To obtain the approximate solution of the value function $V(t,x,z)$, we just need to analyze asymptotic approximation of the new function φ.

First, we look for an expansion for the function φ in power of $\sqrt{\delta}$:

$$\varphi(t,z) = \varphi^{(0)}(t,z) + \sqrt{\delta}\varphi^{(1)}(t,z) + \delta\varphi^{(2)}(t,z)\cdots. \tag{23}$$

Inserting this expansion into (15) and collecting the order 1 terms lead to

$$\varphi_t^{(0)} + \left(\frac{1}{2}\beta^2 r^2 - \alpha r - \frac{1}{2}r^2\theta^2\right)\frac{\varphi^{(0)}}{q} = 0, \tag{24}$$

with the terminal condition $\varphi^{(0)}(T,z) = 1$. It is obvious that from (24), the explicit solution of $\varphi^{(0)}(T,z)$ is given by

$$\varphi^{(0)}(t,z) = \exp\left\{\frac{1}{q}(T-t)\left(\frac{1}{2}\beta^2 r^2 - \alpha r - \frac{1}{2}r^2\theta^2(z)\right)\right\}. \tag{25}$$

Taking the order $\sqrt{\delta}$ terms after inserting the expansion (23) into PDE (15) leads to

$$\varphi_t^{(1)} + \left(\frac{1}{2}\beta^2 r^2 - \alpha r - \frac{1}{2}r^2\theta^2\right)\frac{\varphi^{(1)}}{q} = -\rho_1 \mathrm{rg}(z)\theta(z)\varphi_z^{(0)}, \tag{26}$$

with zero terminal condition $\varphi^{(1)}(T,z)=0$. Through (25), $\varphi_z^{(0)}$ is given by

$$\varphi_z^{(0)}(t,z)=-(T-t)\frac{r^2\theta(z)\theta'(z)}{q}\varphi^{(0)}(t,z). \tag{27}$$

Inserting equation above into (26), we have

$$\begin{aligned}&\varphi_t^{(1)}+\left(\frac{1}{2}\beta^2r^2-\alpha r-\frac{1}{2}r^2\theta^2\right)\frac{\varphi^{(1)}}{q}\\&=(T-t)\frac{\rho_1r^3g(z)\theta(z)\theta'(z)}{q}\varphi^{(0)},\quad \varphi^{(1)}(T,z)=0.\end{aligned} \tag{28}$$

From the linear PDE (28), we have the solution of $\varphi^{(1)}(t,z)$:

$$\varphi^{(1)}(t,z)=-\frac{1}{2}(T-t)^2\frac{\rho_1r^3g(z)\theta(z)\theta'(z)}{q}\varphi^{(0)}(t,z). \tag{29}$$

Thus, the approximate solution of φ is given by

$$\varphi(t,z)\approx\widetilde{\varphi}(t,z)\coloneqq\varphi^{(0)}(t,z)+\sqrt{\delta}\varphi^{(1)}(t,z), \tag{30}$$

where $\varphi^{(0)}$ and $\varphi^{(1)}$ are given in (25) and (29).

Second, through the approximate solution of φ in (30) and value function (14), we have

$$\begin{aligned}V(t,x,z)&\approx d-\frac{l}{r}e^{-rx}\left(\varphi^{(0)}+\sqrt{\delta}\,\varphi^{(1)}\right)^q\\&\approx d-\frac{l}{r}e^{-rx}\left(\left(\varphi^{(0)}\right)^q+q\sqrt{\delta}\,\varphi^{(1)}\left(\varphi^{(0)}\right)^{q-1}\right)\\&\approx d-\frac{l}{r}e^{-rx}\left(1-\sqrt{\delta}\ \frac{1}{2}(T-t)^2\rho_1r^3g(z)\theta(z)\theta'(z)\right)\cdot\left(\varphi^{(0)}\right)^q\\&\coloneqq V^{(0)}(t,x,z)+\sqrt{\delta}V^{(1)}(t,x,z)\\&\coloneqq\widetilde{V}(t,x,z),\end{aligned} \tag{31}$$

where $V^{(0)}(t,x,z)$ and $V^{(1)}(t,x,z)$ are given by

$$\begin{aligned}V^{(0)}(t,x,z)&=d-\frac{l}{r}e^{-rx}\cdot\left(\varphi^{(0)}\right)^q,\\V^{(1)}(t,x,z)&=\frac{l}{2r}e^{-rx}(T-t)^2\rho_1r^3g(z)\theta(z)\theta'(z)\cdot\left(\varphi^{(0)}\right)^q.\end{aligned} \tag{32}$$

The function $\theta(z)$ and parameter ρ are defined by (22). □

3.2. Expansion of Optimal Portfolio

Theorem 4. *The optimal strategy f^* in (12) can be approximated by*

$$f^*\approx\widetilde{f}\coloneqq f^{(0)}+\sqrt{\delta}f^{(1)}, \tag{33}$$

where $f^{(0)}$ and $f^{(1)}$ are given by

$$\begin{aligned}f^{(0)}&=\frac{\mu(z)}{\sigma^2(z)r}-\frac{\rho_2\beta}{\sigma(z)},\\f^{(1)}&=-\frac{\rho_1g(z)}{\sigma(z)}(T-t)r\theta(z)\theta'(z).\end{aligned} \tag{34}$$

The function $\theta(z)$ is defined by (22).

Proof. Inserting approximate solution of value function (18) into (12) gives an approximation for f^* which leads to an approximate feedback policy of the form

$$f^*=f^{(0)}+\sqrt{\delta}f^{(1)}+\cdots, \tag{35}$$

where $f^{(0)}$ and $f^{(1)}$ satisfy

$$\begin{aligned}f^{(0)}&=-\frac{\mu(z)}{\sigma^2(z)}\frac{V_x^{(0)}}{V_{xx}^{(0)}}-\frac{\rho_2\beta}{\sigma(z)},\\f^{(1)}&=-\frac{\mu(z)}{\sigma^2(z)}\left(\frac{V_x^{(1)}}{V_{xx}^{(0)}}-\frac{V_x^{(0)}V_{xx}^{(1)}}{\left(V_{xx}^{(0)}\right)^2}\right)+\frac{\rho_1g(z)}{\sigma(z)}\frac{V_{xz}^{(0)}}{V_{xx}^{(0)}}.\end{aligned} \tag{36}$$

Through (25) and (32), we have

$$\begin{aligned}\frac{V_x^{(0)}}{V_{xx}^{(0)}}&=-\frac{1}{r},\\\frac{V_{xz}^{(0)}}{V_{xx}^{(0)}}&=-(T-t)r\theta(z)\theta'(z),\end{aligned} \tag{37}$$

and

$$\frac{V_x^{(1)}}{V_{xx}^{(0)}}-\frac{V_x^{(0)}V_{xx}^{(1)}}{\left(V_{xx}^{(0)}\right)^2}=0. \tag{38}$$

Inserting equations (37) and (38) into (36), the approximate solution of f^* satisfies (33).

From these explicit expressions, we know the approximate strategy $f^{(0)} + \sqrt{\delta} f^{(1)}$ is independent of the initial price x. Moreover, it can be seen that time to maturity comes to play a role now, which is due to the fact that time matters when fluctuation on slow time scale exists in the dynamics. □

4. Accuracy of Approximate Value Function

Now, for the investment model in the previous section, we are going to prove the validity of our formal asymptotic of the value function V. First, we prove the accuracy of the approximate solution of linear PDE (15) which is a regular perturbation problem of the type.

Assumption 5. Before that, we list here and comment on the assumptions we make on the class of model we are considering.

(1) The operator $\mathcal{M}$ is the infinitesimal generator of a one dimensional diffusion process admitting moments of all order uniformly bounded in $t < T$

(2) $\mu(z)/\sigma(z)$ is bounded so that the diffusion process Z_t has moments of all order uniformly bounded in δ for $t \le T$

Lemma 6. *Under the assumptions listed above, for fixed $t < T$ and z, there is a constant C such that for any $\delta \ll 1$,*

$$\left|\varphi - \varphi^{(0)} - \sqrt{\delta}\,\varphi^{(1)}\right| \le \delta C. \tag{39}$$

Proof. The linear PDE (15) is rewritten as

$$\left(\delta\mathcal{M} + \sqrt{\delta}\,\mathcal{M}_1 + \mathcal{L}\right)\varphi = 0, \quad \varphi(T, z) = 1, \tag{40}$$

where $\mathcal{M}$ is defined in (10), and we define

$$\begin{aligned} \mathcal{M}_1 &= \rho_1 \mathrm{gr}\theta \frac{\partial}{\partial z}, \\ \mathcal{L} &= \frac{\partial}{\partial t} - \frac{\beta^2 r^2 - 2\alpha r - r^2\theta^2}{2q}. \end{aligned} \tag{41}$$

In previous section, we obtain the approximate solution of φ:

$$\varphi \approx \varphi^{(0)}(t, z) + \sqrt{\delta}\varphi^{(1)}(t, z). \tag{42}$$

With this choice of functions $\varphi^{(i)}, i = 0, 1$, the following equations are satisfied:

$$\mathcal{L}\varphi^{(0)} = 0, \tag{43}$$

$$\mathcal{L}\varphi^{(1)} + \mathcal{M}_1\varphi^{(0)} = 0. \tag{44}$$

Defining the residual,

$$R = \varphi - \left(\varphi^{(0)} + \sqrt{\delta}\,\varphi^{(1)}\right). \tag{45}$$

From equations (40), (43), and (44), one obtains

$$\left(\delta\mathcal{M} + \sqrt{\delta}\,\mathcal{M}_1 + \mathcal{L}\right)R = -\delta\mathcal{M}\varphi^{(0)} - \delta^{3/2}\mathcal{M}\varphi^{(1)} - \delta\mathcal{M}_1\varphi^{(1)} = \delta S_\delta(t, z), \tag{46}$$

where the source term $S_\delta(t, z)$ can simply be computed using the equations for $\mathcal{M}$, $\mathcal{M}_1$, $\varphi^{(0)}$, and $\varphi^{(1)}$. Using the terminal condition in (40) and the terminal values for $\varphi^{(i)}, i = 0, 1$, one obtains that the residual function R satisfies the terminal condition:

$$R(T, z) = 0. \tag{47}$$

Denoting by Z_t^δ the diffusion process with infinitesimal generator $\delta\mathcal{M} + \sqrt{\delta}\mathcal{M}_1$, the residual R, solution of PDE problems (46) and (47) is given by the Feynman–Kac formula:

$$R(t, z) = \delta E\left\{\int_t^T e^{-(1/2q)\int_t^s\left(\beta^2 r^2 - 2\alpha r - r^2\theta^2(Z_u)\right)\mathrm{du}} S_\delta\left(s, Z_s^\delta\right)\mathrm{ds} \mid Z_t^\delta = z\right\}. \tag{48}$$

Under our assumptions, one sees by direct computation that S_δ is at most polynomially growing in z, and one obtains $|R(t, z)| \le \delta C$, where C is a constant, that is,

$$\varphi = \varphi^{(0)}(t, z) + \sqrt{\delta}\varphi^{(1)}(t, z) + \mathcal{O}(\delta). \tag{49}$$

□

Theorem 7. *Under the assumptions of Lemma 6, for fixed t, x, and z, we obtain*

$$V(t, x, z) = d - \frac{l}{r}e^{-\mathrm{rx}}\left(1 - \sqrt{\delta}\ \frac{1}{2}(T - t)^2\rho_1 r^3 g(z)\theta(z)\theta'(z)\right)\cdot\left(\varphi^{(0)}\right)^q + \mathcal{O}(\delta). \tag{50}$$

Proof. According (14) and the result of Lemma 6, we have

$$\begin{aligned} V(t,x,z) &= d - \frac{l}{r}e^{-rx}\left(\varphi^{(0)} + \sqrt{\delta}\,\varphi^{(1)} + \mathcal{O}(\delta)\right)^q \\ &= d - \frac{l}{r}e^{-\mathrm{rx}}\left(\left(\varphi^{(0)}\right)^q + \sqrt{\delta}\,q\varphi^{(1)}\left(\varphi^{(0)}\right)^{q-1} + \mathcal{O}(\delta)\right) \\ &= d - \frac{l}{r}e^{-rx}\left(1 - \sqrt{\delta}\ \frac{1}{2}(T-t)^2\rho_1 r^3 g(z)\theta(z)\theta'(z)\right)\cdot\left(\varphi^{(0)}\right)^q + \mathcal{O}(\delta). \end{aligned} \tag{51}$$

□

5. An Illustrative Example and Numerical Computation

In this section, we consider the model adapted from [13] to show the feasibility and accuracy of the asymptotic approximation method. For the convenience of calculation, assume that the coefficients μ, σ, c, and g in equation (6) as

$$\begin{aligned} \mu(z) &= \rho_2\beta - k_1\sqrt{z}, \\ \sigma(z) &= \frac{1}{r}, \\ c(z) &= m - z, \\ g(z) &= k_2\sqrt{z}. \end{aligned} \tag{52}$$

Then, the model under consideration is

$$\begin{cases} \mathrm{dX}_t = \left[f_t\left(\rho_2\beta - k_1\sqrt{z}\right) + \alpha\right]\mathrm{dt} + \dfrac{f_t}{\mathrm{rdW}_t^{(1)}} + \beta\mathrm{dW}_t^{(2)}, \\ \mathrm{dZ}_t = \delta(m-z)\mathrm{dt} + \sqrt{\delta}\left(k_2\sqrt{z}\right)\mathrm{dW}_t^{(\delta)}. \end{cases} \tag{53}$$

The process Z is referred to as the "instantaneous precision" and the explicit solutions of value function and optimal strategy are available. The calculation here is straightforward and follows the road map laid out in [13]. The reduced linear PDE (15) for $\varphi(t,z)$ with dynamics (53) becomes

$$\varphi_t + \frac{1}{2}k_2^2\delta z\varphi_{zz} + \left(\delta(m-z) + \sqrt{\delta}\,\rho_1\mathrm{rk}_1k_2z\right)\varphi_z + \frac{1}{2q}\left(\beta^2r^2 - 2\alpha r - r^2k_1^2z\right)\varphi = 0, \tag{54}$$

with the terminal condition $\varphi(T,z) = 1$. A solution to this reduced problem is

$$\varphi(t,z) = e^{A(T-t)z + B(T-t)}, \tag{55}$$

where the function A satisfies the Riccati ODE

$$A' = \frac{1}{2}k_2^2\delta A^2 + \sqrt{\delta}\rho_1\mathrm{rk}_1k_2A - \delta A - \frac{1}{2q}r^2k_1^2,\ A(0) = 0, \tag{56}$$

where $B' = \delta\mathrm{mA} + 1/2q(\beta^2r^2 - 2\alpha r)$ with $B(0) = 0$. Assume the quadratic term on the right hand side has two real roots, which we denote by $a_\pm$; then, we have

$$A(T-t) = a_-\frac{1 - e^{-a(T-t)}}{1 - (a_-/a_+)e^{-a(T-t)}}, \tag{57}$$

where a is the square root of the discriminant of the quadratic. Inserting equation above into $B' = \delta\mathrm{mA} + 1/2q(\beta^2r^2 - 2\alpha r)$, $B(T-t)$ is given by

$$B(T-t) = \delta m\left[a_-(T-t) - \frac{2}{\delta k_2^2}\log\left(\frac{1 - (a_-/a_+)e^{-a(T-t)}}{1 - (a_-/a_+)}\right)\right] + \frac{1}{2q}(T-t)\left(\beta^2r^2 - 2\alpha r\right). \tag{58}$$

Therefore, we find the exact formula for the value function.

$$V(t,x,z) = d - \frac{l}{r}e^{-\mathrm{rx}}\cdot e^{\mathrm{qA}(T-t)z + \mathrm{qB}(T-t)}. \tag{59}$$

The first order approximate solutions of value function $V(t,x,z)$ and optimal strategy $f(t,x,z)$ are given in equations (18) and (33). With the exact and approximate formulas at hand, we demonstrate the numerical accuracy of approximate solutions of value function and optimal strategy. Combining the data in [13], the parameters in Figure 1 are chosen as $r = 0.75$, $d = 0$, $l = 1$, $x = 0$, $m = 17.4345$, $\rho_1 = 0.5241$, $\rho_2 = 0.1$, $k_1 = 0.18$, $k_2 = 0.3$, $\beta = 5$, $\alpha = 1.5$, $z = 20.4345$, and $T = 2$.

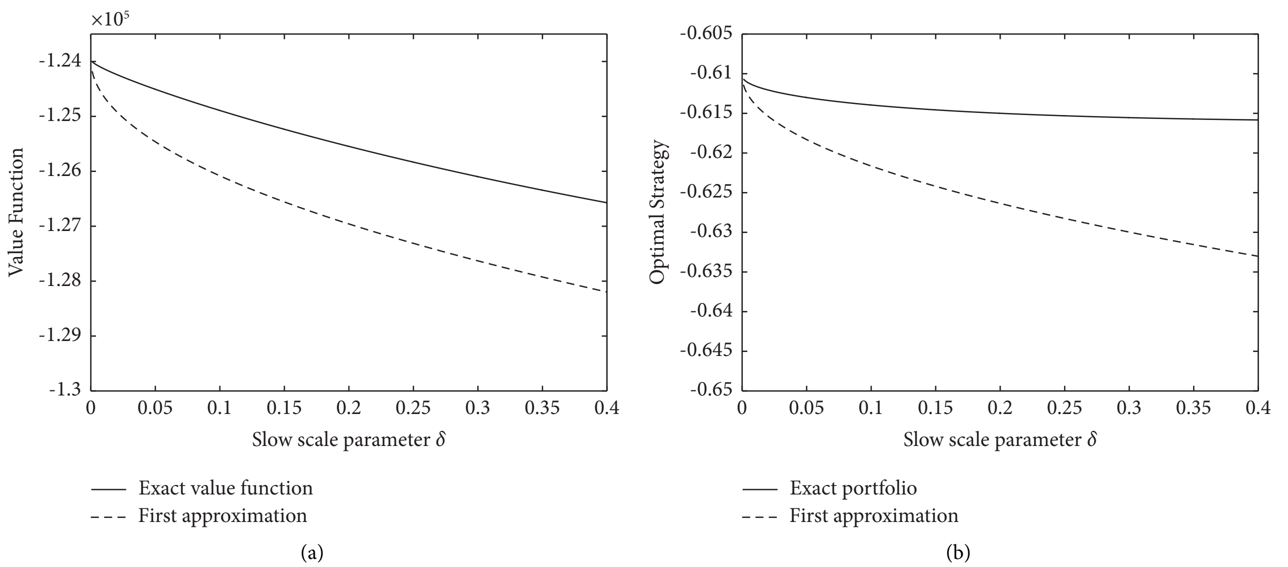

Figure 1: Value function (a) and optimal trading strategy (b) in the slow scale model for a range of scale parameter δ.

In Figure 1, the exact and approximate solutions of value function and optimal strategy are depicted over a range of the scale parameter δ up to the value 0.4 which is to be understood as monthly data. At that value of δ, the relative error of value function is about one percent and that of optimal strategy is about four percent, which show the tractability of the approximate solutions. Furthermore, the approximate solution presented in this article demonstrates superior financial explanatory capabilities. For example, through the approximated investment strategy (33), it becomes evident that as risk levels $\sigma(z)$ increase, the allocation of funds to risky assets decreases, while higher asset returns $\mu(z)$ lead to a greater allocation of funds to risky assets.

Remark 8. Compared with the classical Merton model with a slow factor in [12], the special model above contains a random risk process. When the parameter β is zero, the example in this section will reduce to the investment model in [12].

6. Conclusion and Future Work

In this paper, we construct a portfolio optimization model with a random risk process and a slow stochastic factor. We provide approximate solutions of the value function $V(t, x, z)$ and the optimal strategy f. Moreover, the accuracy of the value function are proved in Section 4, and an illustrate example is provided in Section 5.

There are a number of directions for us to extend and we mention a few. First, consider more stochastic factors in our portfolio optimization model, such as the fast mean-reverting factor and delay factor in [5, 12]. Second, we can focus on obtaining optimal investment strategies to minimize the probability of ruin in [2], not to maximize utility of terminal wealth. A third direction would be to predict asset returns, that is, parameters $\mu(z)$ and $\sigma(z)$ in our model through the value of $V(t, x, z)$ in the real world.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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