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Research Article

Weighted Extropy for Concomitants of Upper k-Record Values Based on Huang–Kotz Morgenstern of Bivariate Distribution

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In this paper, the marginal distribution of concomitants of k−record values (CKR) based on the Huang–Kotz Farlie–Gumbel–Morgenstern (HK-FGM) family of bivariate distributions is derived. In addition, we obtained the joint distribution of CKR for this family. Also, we obtained the hazard rate, reversed hazard rate, and residual life functions of CKR using the HK-FGM family. The weighted extropy and the weighted cumulative past extropy (WCPJ) are acquired for CKR under the HK-FGM family. In addition, we look into the issue of estimating the WCPJ by combining the empirical method with the concurrent use of KR in the HK-FGM family. Finally, we analyzed real-world data for illustration purposes, and the outcomes are rather striking.

1. Introduction

Let $G_T(t)$ be a continuous distribution function (DF) with a probability density function (PDF) $g_T(t)$ for a series of i.i.d. random variables (RVs) $\{T_i, i \geq 1\}$. An observation T_j is called an upper record value if $T_j > T_i$ for every $i < j$. It is no longer adequate to use the model of record values when waiting times between two record values are considered. Numerous situations can benefit from record values, including industrial stress tests, weather data analyses, sporting events, and oil and mining surveys. As mentioned above, many of the instances are related to informational and reliability measures in record values, see Makouei et al. [1] because record data are scarce in practical contexts and each subsequent record is predicted to wait for an infinite time, statistical inference based on records is difficult. In these circumstances, the largest second or third values typically play a significant role. It is possible to avoid these issues by considering the k−record values (KR) model, as described by Berred [2] and Fashandi and Ahmadi [3]. The PDF of the nth upper KR is given by Dziubdziela and Kopociński [4] as

$$L_{n,k}(t) = \frac{k^n}{\Gamma(n)}\left(-\log\left(\overline{G}_T(t)\right)\right)^{n-1}\left(\overline{G}_T(t)\right)^{k-1} g_T(t), \quad (1)$$

where $\Gamma(.)$ is the gamma function and $\overline{G}_T(t) = 1 - G_T(t)$. In addition, the joint PDF (JPDF) of the nth and the mth upper KR, $T_{n,k}$, and $T_{m,k}$, respectively, is given by

$$\begin{aligned} L_{m,n,k}(t_1, t_2) &= \frac{k^n}{\Gamma(n)\Gamma(m-n)}\left(-\log\left(\overline{G}_T(t_1)\right)\right)^{n-1}\left(\overline{G}_T(t_2)\right)^{k-1} \\ &\times \left(-\log\frac{\overline{G}_T(t_2)}{\overline{G}_T(t_1)}\right)^{m-n-1}\frac{g_T(t_1)g_T(t_2)}{\overline{G}_T(t_1)}, \; t_1 \leq t_2. \end{aligned} \quad (2)$$

The use of families of bivariate distributions with specified marginals is recommended when prior knowledge exists as marginal distributions. Huang and Kotz [5] introduced the Huang–Kotz Farlie–Gumbel–Morgenstern (HK-FGM) family as an expansion of the traditional FGM family of bivariate distributions. The PDF for this model is provided by

$$g_{T,Z}(t,z) = g_T(t)g_Z(z)\left[1 + \alpha\left(1 - (1+c)G_T^c(t)\right)\left(1 - (1+c)G_Z^c(z)\right)\right], c > 0, \tag{3}$$

where $G_T(t)$ and $G_Z(z)$ are the marginal DFs of two RVs T and Z, respectively. The admissible range of association parameter α is $-c^{-2} \leq \alpha \leq c^{-1}$ and the range for correlation coefficient is $-(c+2)^{-2}\min(1,c^2) \leq \rho \leq 3c(c+2)^{-2}$. See Elgawad et al. [6] for more details about this family. This family has been the focus of a great deal of research from several perspectives. Elgawad et al. [6]; Barakat et al. [7], and Hussieny and Syam [8] are three examples of these investigations.

If only the sequence of KR of the first component T is of interest to the investigator, the second component is referred to as its concomitant. The most striking application of concomitants arises in industry and biological selection problems. Using concomitants in reliability models has been shown to be useful in various industry real-life situations by Eryilmaz [9]. There are several practical experiments that deal with KR and their concomitants, e.g., those of Alawady et al. [10] and Chacko and Shy Mary [11]. The PDF of the concomitant $Z_{[n,k]}$ (the nth upper concomitant of $T_{n,k}$) is given by

$$g_{[n,k]}(z) = \int_0^\infty g_{Z|T}(Z|t)L_{n,k}(t)\mathrm{d}t, \tag{4}$$

where $g_{Z|T}(Z|t)$ is the conditional PDF of Z given T. Moreover, the JPDF of concomitants $Z_{[n,k]}$ and $Z_{[m,k]}$ $(n < m)$ is given by

$$g_{[n,m,k]}(z_1,z_2) = \int_0^\infty\int_0^\infty g_{Z|T}\left(z_1|t_1\right)g_{Z|T}\left(z_2|t_2\right)L_{m,n,k}(t_1,t_2)\mathrm{d}t_2\mathrm{d}t_1. \tag{5}$$

For an absolutely continuous nonnegative RV T, extropy has been presented by Lad et al. [12] as a new indicator of uncertainty, specified by

$$J(T) = -\frac{1}{2}\int_0^\infty \left[g_T(t)\right]^2\mathrm{d}t = -\frac{1}{2}\int_0^1 g_T\left(G_T^{-1}(u)\right)\mathrm{d}u. \tag{6}$$

It is clear that $J(T) \leq 0$. Kelbert et al. [13] defined the weighted extropy (WJ) as

$$J^w(T) = -\frac{1}{2}\int_0^\infty t\left[g_T(t)\right]^2\mathrm{d}t. \tag{7}$$

Recently, Kazemi et al. [14] proposed the weighted cumulative past extropy (WCPJ) as

$$\xi^w(T) = -\frac{1}{2}\int_0^\infty t\left[G_T(t)\right]^2\mathrm{d}t. \tag{8}$$

Almaspoor et al. [15] have investigated the extropy measurements for CKR in the FGM family. Also, Husseiny et al. [16] have explored some properties of the extropy measure in concomitants of records from the Sarmanov family of bivariate DF. While Qiu and Jia [17] examined extropy estimators utilised in uniformity testing, Qiu and Jia [18] examined residual extropy utilising order statistics. An investigation of the extropy properties of mixed systems was conducted by Qiu et al. [19].

As a whole, the paper follows the following structure. Section 2 provides marginal DFs, moment generating functions (MGFs), and moments of the CKR as a function of the HK-FGM family. Moreover, the joint DF (JDF) of the bivariate CKR for this family is derived. In addition, the hazard rate, reversed hazard rate, and mean residual life functions for $Z_{[n,k]}$ based on the HK-FGM family are obtained. In Section 3, the WJ and WCPJ are obtained. Also, we investigate the problem of estimating the WCPJ by using the empirical technique in conjunction with the CKR based on the HK-FGM family. Finally, we analyzed real-world data for illustration purposes, and the results are quite impressive.

2. CKR Based on HK-FGM

In this section, based on the HK-FGM family, we obtain marginal DFs, MGFs, and moments for the CKR. On the basis of the HK-FGM family, we also derive the JPDF for the bivariate CKR. As well as hazard rates, reversed hazard rates, and residual life functions based on the HK-FGM family, the mean residual life functions are studied for $Z_{[n,k]}$.

2.1. Marginal DF of CKR. The following theorem represents the PDF of $Z_{[n,k]}$ in a useful way. To indicate that T is distributed as G, we use the notation $T \sim G$.

Theorem 1. *Let $V \sim G_Z^{c+1}$. Then,*

$$g_{[n,k]}(Z) = \left(1 + \delta_{n,k:c}^{(\alpha)}\right)g_Z(z) - \delta_{n,k:c}^{(\alpha)}g_V(z), \tag{9}$$

where $\delta_{n,k:c}^{(\alpha)} = \alpha\left[1 - (1+c)I_c\right]$ and $I_c = \sum_{i=0}^{\aleph(c)}(-1)^i\binom{c}{i}(k/i+k)^n$, $\aleph(t) = \infty$, if t is non-integer and $\aleph(t) = t$, if t is integer.

Proof. Consider the following integration

$$I_c = \frac{k^n}{\Gamma(n)} \int_0^\infty G_T^c(t)\left(-\log\left(\overline{G}_T(t)\right)\right)^{n-1}\left(\overline{G}_T(t)\right)^{k-1} g_T(t)\mathrm{d}t. \tag{10}$$

Using the transformation $(-\log(\overline{G}_T(t))) = v$, we obtain

$$I_c = \frac{k^n}{\Gamma(n)} \int_0^\infty \left(1 - e^{-v}\right)^c v^{n-1} e^{-vk}\mathrm{d}v. \tag{11}$$

After using the binomial expansion, we obtain

$$I_c = \frac{k^n}{\Gamma(n)} \sum_{i=0}^{\aleph(c)} (-1)^i \binom{c}{i} \int_0^\infty v^{n-1} e^{-v(i+k)}\mathrm{d}v = \sum_{i=0}^{\aleph(c)} (-1)^i \binom{c}{i} \left(\frac{k}{i+k}\right)^n. \tag{12}$$

Now, by using equations (1), (3), and (5), we obtain

$$g_{[n,k]}(z) = g_Z(z)\left\{1 + \alpha\left(1 - (1+c)I_c\right)\left(1 - (1+c)G_Z^c(z)\right)\right\}. \tag{13}$$

The proof is now completed. □

Remark 2. Assuming $k = 1$ in Theorem 1, which covers record values mostly, we obtain the result of Barakat et al. [7].

Relying on equation (9), the MGF of CKR based on HK-FGM family is given by

$$M_{Z_{[n,k]}}(w) = \left(1 + \delta_{n,k:c}^{(\alpha)}\right) M_Z(w) - \delta_{n,k:c}^{(\alpha)} M_V(w), \tag{14}$$

where $M_Z(w)$ and $M_V(w)$ are the MGFs of the RVs Z and V, respectively. Thus, by using equation (9) or (14), the mth moment of CKR based on HK-FGM family is given by

$$\mu_{Z_{[n,k]}}^{(m)} = \left(1 + \delta_{n,k:c}^{(\alpha)}\right)\mu_Z^{(m)} - \delta_{n,k:c}^{(\alpha)}\mu_V^{(m)}, \tag{15}$$

where $\mu_Z^{(m)} = E[Z^m]$ and $\mu_V^{(m)} = E[V^m]$.

2.2. JDF of CKR

Following are the theorems we used to determine the JPDF $g_{[n,m,k]}(z_1, z_2)$ of concomitants of $Z_{[n,k]}$ and $Z_{[m,k]}$ in HK-FGM.

Theorem 3. *Let $V \sim G_Z^{c+1}$. Then,*

$$\begin{aligned} g_{[n,m,k]}(z_1, z_2) = {} & \left(1 + \delta_{n,m,k:c}^{(1)} + \delta_{n,m,k:c}^{(2)} + \delta_{n,m,k:c}\right) g_Z(z_1) g_Z(z_2) - \left(\delta_{n,m,k:c}^{(1)} + \delta_{n,m,k:c}\right) g_Z(z_2) g_V(z_1) \\ & - \left(\delta_{n,m,k:c}^{(2)} + \delta_{n,m,k:c}\right) g_Z(z_1) g_V(z_2) + \left(\delta_{n,m,k:c}\right) g_V(z_1) g_V(z_2), \end{aligned} \tag{16}$$

where

$$\begin{aligned} \delta_{n,m,k:c} &= \alpha\left(\delta_{n,m,k:c}^{(1)} + \delta_{n,m,k:c}^{(2)} - \delta_{n,m,k:c}^{(3)}\right), \\ \delta_{n,m,k:c}^{(1)} &= \alpha\left[1 - (1+c)k^{2n-m} \sum_{i=0}^{\aleph(c)} \frac{(-1)^i \binom{c}{i}}{(i+k)^n}\right], \\ \delta_{n,m,k:c}^{(2)} &= \alpha\left[1 - (1+c)k^n \sum_{j=0}^{\aleph(c)} \frac{(-1)^j \binom{c}{j}}{(j+k)^m}\right], \\ \delta_{n,m,k:c}^{(3)} &= \alpha\left[1 - (1+c)^2 k^n \sum_{i=0}^{\aleph(c)} \sum_{j=0}^{\aleph(c)} \frac{(-1)^{i+j} \binom{c}{i}\binom{c}{j}}{(i+j+k)^n (j+k)^{m-n}}\right]. \end{aligned} \tag{17}$$

Proof. By using equations (2), (3), and (5), we obtain

$$
\begin{aligned}
g_{[n,m,k]}(z_1,z_2) = {} & \frac{k^n}{\Gamma(n)\Gamma(m-n)}\int_0^\infty\int_{t_1}^\infty \left[g_Z(z_1)\left(1+\alpha\left(1-(1+c)G_T^c(t_1)\right)\left(1-(1+c)G_Z^c(z_1)\right)\right)\right] \\
& \times\left[g_Z(z_2)\left(1+\alpha\left(1-(1+c)G_T^c(t_2)\right)\left(1-(1+c)G_Z^c(z_2)\right)\right)\right]\left(-\log\left(\overline{G}_T(t_1)\right)\right)^{n-1} \\
& \times\left(\overline{G}_T(t_2)\right)^{k-1}\left(-\log\frac{\overline{G}_T(t_2)}{\overline{G}_T(t_1)}\right)^{m-n-1}\frac{g_T(t_1)g_T(t_2)}{\overline{G}_T(t_1)}\mathrm{d}t_2\mathrm{d}t_1.
\end{aligned}
\tag{18}
$$

After a little algebra, we obtain

$$
\begin{aligned}
\delta_{n,m,k:c}^{(1)} = {} & \frac{\alpha k^n}{\Gamma(n)\Gamma(m-n)}\int_0^\infty\int_{t_1}^\infty\left(1-(1+c)G_T^c(t_1)\right)\left(-\log\left(\overline{G}_T(t_1)\right)\right)^{n-1}\left(\overline{G}_T(t_2)\right)^{k-1} \\
& \times\left(-\log\frac{\overline{G}_T(t_2)}{\overline{G}_T(t_1)}\right)^{m-n-1}\frac{g_T(t_1)g_T(t_2)}{\overline{G}_T(t_1)}\mathrm{d}t_2\mathrm{d}t_1.
\end{aligned}
\tag{19}
$$

Taking the transformation $u=-\log\left(\overline{G}_T(t_1)\right)$ and $v=-\log\left(\overline{G}_T(t_2)\right)$, we obtain

$$
\delta_{n,m,k:c}^{(1)} = \alpha\left[1-(1+c)k^{2n-m}\sum_{i=0}^{\aleph(c)}\frac{(-1)^i\binom{c}{i}}{(i+k)^n}\right].
\tag{20}
$$

Similarly,

$$
\begin{aligned}
\delta_{n,m,k:c}^{(2)} &= \frac{\alpha k^n}{\Gamma(n)\Gamma(m-n)}\int_0^\infty\int_{t_1}^\infty\left(1-(1+c)G_T^c(t_2)\right)\left(-\log\left(\overline{G}_T(t_1)\right)\right)^{n-1}\left(\overline{G}_T(t_2)\right)^{k-1} \\
&\quad\times\left(-\log\frac{\overline{G}_T(t_2)}{\overline{G}_T(t_1)}\right)^{m-n-1}\frac{g_T(t_1)g_T(t_2)}{\overline{G}_T(t_1)}\mathrm{d}t_2\mathrm{d}t_1 \\
&= \alpha\left[1-(1+c)k^n\sum_{j=0}^{\aleph(c)}\frac{(-1)^j\binom{c}{j}}{(j+k)^m}\right], \\
\delta_{n,m,k:c} &= \frac{\alpha^2 k^n}{\Gamma(n)\Gamma(m-n)}\int_0^\infty\int_{t_1}^\infty\left(1-(1+c)G_T^c(t_1)\right)\left(1-(1+c)G_T^c(t_2)\right)\left(-\log\left(\overline{G}_T(t_1)\right)\right)^{n-1} \\
&\quad\times\left(\overline{G}_T(t_2)\right)^{k-1}\left(-\log\frac{\overline{G}_T(t_2)}{\overline{G}_T(t_1)}\right)^{m-n-1}\frac{g_T(t_1)g_T(t_2)}{\overline{G}_T(t_1)}\mathrm{d}t_2\mathrm{d}t_1 \\
&= \alpha\left(\delta_{n,m,k:c}^{(1)}+\delta_{n,m,k:c}^{(2)}-\delta_{n,m,k:c}^{(3)}\right),
\end{aligned}
\tag{21}
$$

where

$$\delta^{(3)}_{n,m,k:c}=\frac{\alpha k^n(1-(1+c)^2}{\Gamma(n)\Gamma(m-n)}\int_0^\infty\int_{t_1}^\infty G^c_T(t_1)G^c_T(t_2)\left(-\log\left(\overline{G}_T(t_1)\right)\right)^{n-1}\left(\overline{G}_T(t_2)\right)^{k-1}$$
$$\times\left(-\log\frac{\overline{G}_T(t_2)}{\overline{G}_T(t_1)}\right)^{m-n-1}\frac{g_T(t_1)g_T(t_2)}{\overline{G}_T(t_1)}\mathrm{d}t_2\mathrm{d}t_1$$
$$=\alpha\left[1-(1+c)^2k^n\sum_{i=0}^{\aleph(c)}\sum_{j=0}^{\aleph(c)}\frac{(-1)^{i+j}\binom{c}{i}\binom{c}{j}}{(i+j+k)^n(j+k)^{m-n}}\right]. \tag{22}$$

The proof is completed. □

Remark 4. For $k=1$ (record case), Theorem 3 yields the results of Barakat et al. [7].

The JMGF of $Z_{[n,k]}$ and $Z_{[m,k]}, n<m$, based on HK-FGM family is given by

$$M_{[n,m,k]}(w_1,w_2)=\left(1+\delta^{(1)}_{n,m,k:c}+\delta^{(2)}_{n,m,k:c}+\delta_{n,m,k:c}\right)M_Z(w_1)M_Z(w_2)-\left(\delta^{(1)}_{n,m,k:c}+\delta_{n,m,k:c}\right)M_V(w_1)M_Z(w_2)$$
$$-\left(\delta^{(2)}_{n,m,k:c}+\delta_{n,m,k:c}\right)M_Z(w_1)M_V(w_2)+\left(\delta_{n,m,k:c}\right)M_V(w_1)M_V(w_2). \tag{23}$$

2.3. Reliability Concepts for CKR Based on the HK-FGM Family. In this subsection, we derived the failure rate, reversed hazard rate, and mean residual life functions for CKR for any arbitrary DFs based on the HK-FGM family of bivariate distributions. The failure rate (hazard rate) function of $Z_{[n,k]}$ is defined as

$$h_{[n,k]}(z)=\frac{g_{[n,k]}(z)}{\overline{G}_{[n,k]}(z)}$$
$$=\frac{g_Z(z)+\delta^{(\alpha)}_{n,k:c}\left(g_Z(z)-g_V(z)\right)}{\overline{G}_Z(z)+\delta^{(\alpha)}_{n,k:c}\left(\overline{G}_Z(z)-\overline{G}_V(z)\right)}. \tag{24}$$

Also, the reversed hazard rate function is given by

$$\Lambda_{[n,k]}(z)=\frac{g_{[n,k]}(z)}{G_{[n,k]}(z)}$$
$$=\frac{g_Z(z)+\delta^{(\alpha)}_{n,k:c}\left(g_Z(z)-g_V(z)\right)}{G_Z(z)+\delta^{(\alpha)}_{n,k:c}\left(G_Z(z)-G_V(z)\right)}. \tag{25}$$

The mean residual life function for $Z_{[n,k]}$ can be expressed as follows:

$$\Omega_{[n,k]}(w)=E\left(Z_{[n,k]}-w|Z_{[n,k]}>w\right)=\frac{1}{\overline{G}_{[n,k]}(z)}\int_w^\infty(z-w)g_{[n,k]}(z)\mathrm{d}z$$
$$=\frac{1}{\overline{G}_Z(z)+\delta^{(\alpha)}_{n,k:c}\left(\overline{G}_Z(z)-\overline{G}_V(z)\right)}\left(\int_w^\infty(z-w)\left(\left(1+\delta^{(\alpha)}_{n,k:c}\right)g_Z(z)-\delta^{(\alpha)}_{n,k:c}g_V(z)\right)\right)\mathrm{d}z \tag{26}$$
$$=\frac{\left(1+\delta^{(\alpha)}_{n,k:c}\right)m_Z(w)\overline{G}_Z(w)-\delta^{(\alpha)}_{n,k:c}m_V(w)\overline{G}_V(w)}{\overline{G}_Z(z)+\delta^{(\alpha)}_{n,k:c}\left(\overline{G}_Z(z)-\overline{G}_V(z)\right)},$$

where $m_Z(w)$ is the mean residual life of Z and $m_V(w)$ is the mean residual life of V.

3. Measures of Extropy for CKR Based on the HK-FGM Family

In this section, we study the WJ and WCPJ for CKR based on the HK-FGM family of bivariate DF. We consider the extended Weibull (EW) family of distributions, which developed by Gurvich et al. [20] as a case study for family. According to the EW distribution, the DF is as follows:

$$G_T(t) = 1 - \exp(-\tau H(t;\overline{\omega})), t > 0, \tau > 0, \tag{27}$$

where $H(t;\overline{\omega}))$ is differentiable, nonnegative, continuous, and monotone increasing when t depends on the parameter vector $\overline{\omega}$. Also, $H(t;\overline{\omega}) \longrightarrow 0^+$ as $t \longrightarrow 0^+$ and $H(t;\overline{\omega}) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. This DF is denoted by EW $(\tau,\overline{\omega})$ and has the following PDF:

$$g_T(t) = \tau h(t;\overline{\omega})\exp(-\tau H(t;\overline{\omega})), t > 0, \tag{28}$$

where $h(t;\overline{\omega})$ is the derivative of $H(t;\overline{\omega})$ with respect to t. A number of important models are included in the EW DF, including uniform, Weibull, generalized exponential, Rayleigh, and Pareto. For more details about this family see, Jafari et al. [21].

3.1. Weighted Extropy of CKR. If $Z_{[n,k]}$ is the CKR from HK-FGM, then the WJ of $Z_{[n,k]}$ is

$$\begin{aligned} J^w(Z_{[n,k]}) &= -\frac{1}{2}\int_0^\infty z(g_{[n,k]}(z))^2 \mathrm{d}z \\ &= -\frac{1}{2}\int_0^\infty z\left[\left(1+\delta^{(\alpha)}_{n,k:c}\right)g_Z(z) - \delta^{(\alpha)}_{n,k:c}g_V(z)\right]^2 \mathrm{d}z \\ &= -\frac{1}{2}\left[\left(1+\delta^{(\alpha)}_{n,k:c}\right)^2\int_0^\infty z g_Z^2(z)\mathrm{d}z + \left(\delta^{(\alpha)}_{n,k:c}\right)^2\int_0^\infty z g_V^2(z)\mathrm{d}z - 2\delta^{(\alpha)}_{n,k:c}\left(1+\delta^{(\alpha)}_{n,k:c}\right)\int_0^\infty z g_Z(z)g_V(z)\mathrm{d}z\right] \\ &= \left(1+\delta^{(\alpha)}_{n,k:c}\right)^2 J^w(Z) + \left(\delta^{(\alpha)}_{n,k:c}\right)^2 J^w(V) + \delta^{(\alpha)}_{n,k:c}\left(1+\delta^{(\alpha)}_{n,k:c}\right)\mathrm{E}(Zg_V(Z)), \end{aligned} \tag{29}$$

where $J^w(Z)$ is the WJ of Z and $J^w(V)$ is the WJ of V.

Proposition 5. *Let $Z_{[n,k]}$ be the CKR, $\widetilde{Z} = aZ_{[n,k]} + b$, then $J^w(\widetilde{Z}) = J^w(Z_{[n,k]}) + b/aJ(Z_{[n,k]})$*

Proof. From $\widetilde{Z} = aZ_{[n,k]} + b$, we have $g_{\widetilde{Z}}(z) = 1/ag_{[n,k]}(z-b/a)$, $z > b$ and so

$$\begin{aligned} J^w(\widetilde{Z}) &= -\frac{1}{2}\int_b^\infty \frac{1}{a^2}z\left(g_{[n,k]}\left(\frac{z-b}{a}\right)\right)^2 dz \\ &\quad -\frac{1}{2}\int_0^\infty \frac{az+b}{a}z(g_{[n,k]}(z))^2 dz \\ &= -\frac{1}{2}\int_0^\infty z(g_{[n,k]}(z))^2 dz - \frac{1}{2}\frac{b}{a}\int_0^\infty (g_{[n,k]}(z))^2 dz \\ &= J^w(Z_{[n,k]}) + \frac{b}{a}J(Z_{[n,k]}), \end{aligned} \tag{30}$$

where $J(Z_{[n,k]})$ is the extropy of $Z_{[n,k]}$. □

Remark 6. Assume that T and Z are EW based on HK-FGM (HK-FGM-EW). Then, the WJ of CKR is given by

$$\begin{aligned} J^w_{EW}(Z_{[n,k]}) &= \left(1+\delta^{(\alpha)}_{n,k:c}\right)^2 J^w_{EW}(Z) + \left(\delta^{(\alpha)}_{n,k:c}\right)^2 J^w_{EW}(V) \\ &\quad + \delta^{(\alpha)}_{n,k:c}\left(1+\delta^{(\alpha)}_{n,k:c}\right)E_{EW}(Zg_V(Z)), \end{aligned} \tag{31}$$

where

$$J_{EW}^{w}(Z) = -\frac{1}{2}\int_{0}^{\infty} z\left[\tau h(t;\overline{\omega})\exp(-\tau H(t;\overline{\omega}))\right]^{2}\mathrm{d}z,$$
$$J_{EW}^{w}(V) = -\frac{1}{2}\int_{0}^{\infty} (c+1)^{2} z\left[\tau h(t;\overline{\omega})\exp(-\tau H(t;\overline{\omega}))(1-\exp(-\tau H(t;\overline{\omega})))^{c}\right]^{2}\mathrm{d}z, \tag{32}$$
$$E_{EW}(Zg_{V}(z)) = \int_{0}^{\infty} (c+1)z\left[\tau h(t;\overline{\omega})\exp(-\tau H(t;\overline{\omega}))\right]^{2}(1-\exp(-\tau H(t;\overline{\omega})))^{c}\mathrm{d}z.$$

Example 1. Based on Remark 6, by choosing $H(t;\overline{\omega}) = t$ and $\tau = \theta$, we have T and Z are exponentially distributed as HK-FGM (HK-FGM-ED) with DF as

$$G_{T,Z}(t,z) = (1-\exp(-\theta_1 t))(1-\exp(-\theta_2 z))\left[1+\alpha\left(1-(1-\exp(-\theta_1 t))^{c}\right)\left(1-(1-\exp(-\theta_2 t))^{c}\right)\right], t, z, \theta_1, \theta_2 > 0. \tag{33}$$

Then, we have

$$\begin{aligned} J^{w}(Z) &= -\frac{1}{2}\int_{0}^{\infty} z\theta_2^{2}\exp(-2\theta_2 z)\mathrm{d}z \\ &= -\frac{1}{8}, \\ J^{w}(V) &= -\frac{1}{2}\int_{0}^{\infty} (c+1)^{2} z\theta_2^{2}\exp(-2\theta_2 z) \\ &\quad\cdot (1-\exp(-\theta_2 z))^{2c}\mathrm{d}z \\ &= -\frac{(c+1)^{2}(H_{2c+2}-1)}{4\left(2c^{2}+3c+1\right)}, \\ E(Zg_{V}(z)) &= \int_{0}^{\infty} (c+1)\theta_2^{2} z\exp(-2\theta_2 z) \\ &\quad\cdot (1-\exp(-\theta_2 z))^{c}\mathrm{d}z \\ &= \frac{H_{c+2}-1}{c+2}, \end{aligned} \tag{34}$$

where H_n denotes the generalized harmonic numbers, which is calculated by $H_n = \sum_{i=1}^{n} 1/i$. Finally, the $J^{w}(Z_{[n,k]})$ based on HK-FGM-ED can be written as

$$\begin{aligned} J^{w}\left(Z_{[n,k]}\right) = {} & -\frac{\left(1+\delta_{n,k:c}^{(\alpha)}\right)^{2}}{8} - \left(\delta_{n,k:c}^{(\alpha)}\right)^{2}\frac{(c+1)^{2}(H_{2c+2}-1)}{4\left(2c^{2}+3c+1\right)} \\ & + \delta_{n,k:c}^{(\alpha)}\left(1+\delta_{n,k:c}^{(\alpha)}\right)\frac{H_{c+2}-1}{c+2}. \end{aligned} \tag{35}$$

Example 2. Based on Remark 6, by choosing $H(t;\overline{\omega}) = -\log(a - t/a)$ and $\tau = 1$, we have T and Z from the HK-FGM with uniform marginals DF as

$$G_{T,Z}(t,z) = \frac{tz}{ab}\left[1+\alpha\left(1-\left(\frac{t}{a}\right)^{c}\right)\left(1-\left(\frac{z}{b}\right)^{c}\right)\right], \quad 0 < t < a, 0 < z < b. \tag{36}$$

Then, we have

$$\begin{aligned} J^{w}(Z) &= -\frac{1}{2}\int_{0}^{b} z\frac{1}{b^{2}}\mathrm{d}z \\ &= -\frac{1}{4}, \\ J^{w}(V) &= -\frac{1}{2}\int_{0}^{b} (c+1)^{2}\frac{1}{b^{2}}z\left(\frac{z}{b}\right)^{2c}\mathrm{d}z \\ &= -\frac{c+1}{4}, \\ E(Zg_{V}(z)) &= \int_{0}^{b} (c+1)\frac{1}{b^{2}}z\left(\frac{z}{b}\right)^{c}\mathrm{d}z \\ &= \frac{c+1}{c+2}. \end{aligned} \tag{37}$$

The $J^{w}(Z_{[n,k]})$ based on HK-FGM-uniform distribution can be written as

$$\begin{aligned} J^{w}\left(Z_{[n,k]}\right) = {} & -\frac{\left(1+\delta_{n,k:c}^{(\alpha)}\right)^{2}}{4} - \frac{\left(\delta_{n,k:c}^{(\alpha)}\right)^{2}(c+1)}{4} \\ & + \delta_{n,k:c}^{(\alpha)}\left(1+\delta_{n,k:c}^{(\alpha)}\right)\frac{c+1}{c+2}. \end{aligned} \tag{38}$$

Example 3. Based on Remark 6, by choosing $H(t;\overline{\omega}) = t^2$ and $\tau = 1/2\sigma^2$, we have a HK-FGM bivariate Rayleigh distribution (HK-FGM-RD) for T and Z with DF

$$G_{T,Z}(t,z) = \left(1 - \exp\left(-\frac{t^2}{2\sigma_1^2}\right)\right)\left(1 - \exp\left(-\frac{z^2}{2\sigma_2^2}\right)\right)\left[1 + \alpha\left(1 - \left(1 - \exp\left(-\frac{t^2}{2\sigma_1^2}\right)\right)^c\right) \times \left(1 - \left(1 - \exp\left(-\frac{z^2}{2\sigma_2^2}\right)\right)^c\right)\right]. \tag{39}$$

Then, we have

$$\begin{aligned} J^w(Z) &= -\frac{1}{2}\int_0^\infty z\left(\frac{z}{\sigma_2^2}\right)^2 \exp\left(-\frac{z^2}{\sigma_2^2}\right)\mathrm{d}z \\ &= -\frac{1}{4}, \\ J^w(V) &= -\frac{1}{2}\int_0^\infty (c+1)^2 z\left(\frac{z}{\sigma_2^2}\right)^2 \exp\left(-\frac{z^2}{\sigma_2^2}\right)\left(1 - \exp\left(-\frac{z^2}{2\sigma_2^2}\right)\right)^{2c}\mathrm{d}z \\ &= (-1)^{2c+1}(c+1)^2\Gamma(-2(c+1))\Gamma(2c+1), \\ E\left(Zg_V(z)\right) &= \int_0^\infty (c+1)z\left(\frac{z}{\sigma_2^2}\right)^2 \exp\left(-\frac{z^2}{\sigma_2}\right)\left(1 - \exp\left(-\frac{z^2}{2\sigma_2^2}\right)\right)^c \mathrm{d}z \\ &= -\frac{2\pi(-1)^c \csc(\pi c)}{c+2}. \end{aligned} \tag{40}$$

The $J^w(Z_{[n,k]})$ based on HK-FGM-RD can be written as

$$J^w\left(Z_{[n,k]}\right) = -\frac{\left(1 + \delta_{n,k:c}^{(\alpha)}\right)^2}{4} - \left(\delta_{n,k:c}^{(\alpha)}\right)^2 (-1)^{2c+1}(c+1)^2\Gamma(-2(c+1))\Gamma(2c+1) - \delta_{n,k:c}^{(\alpha)}\left(1 + \delta_{n,k:c}^{(\alpha)}\right)\frac{2\pi(-1)^c \csc(\pi c)}{c+2}. \tag{41}$$

Figures 1(a) and 1(b) show WJ in $Z_{[n,k]}$ from HK-FGM-ED for various values of n and k at $c = 2, 3$. These properties can be derived from Figure 1:

(1) With fixed c and k, the value of WJ increases as n, $(n \leq 4)$ increases. Stability occurs in the values of WJ when $n > 4$, see Figure 1(a).

(2) As k increases, the value of WJ decreases with fixed c, and n,. Especially, when $k > 3$, see Figure 1(b)

Table 1 displays the WJ of $Y_{[n,k]}$ from HK-FGM-ED at $c = 2$, $c = 4$. From Table 1, the following properties can be extracted:

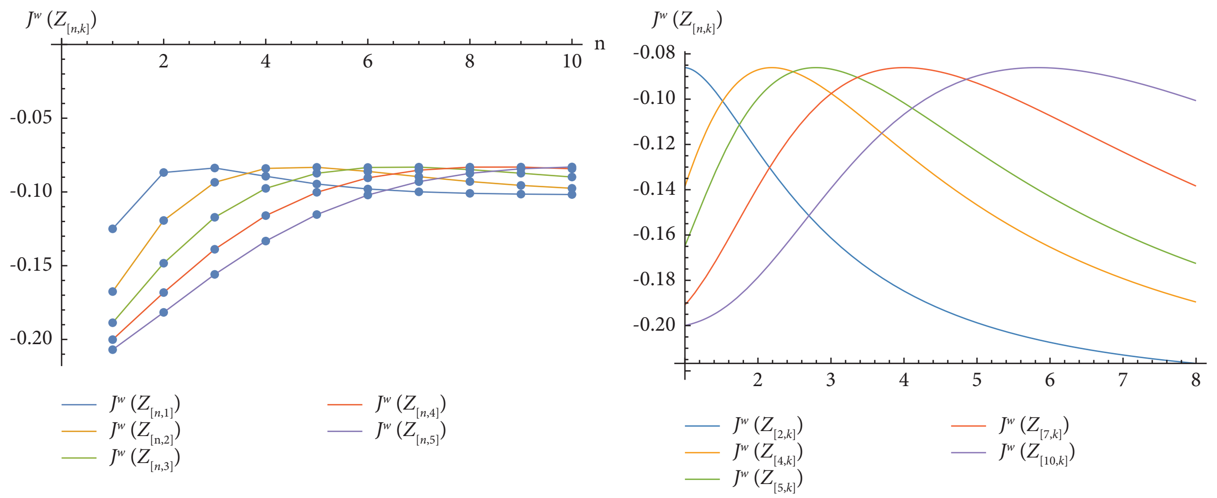
(i) For $n > 1$, and $c = 2$, the value of $J^w(Z_{[n,k]})$ decreases as the value of n increases at $k = 1$, and the value of $J^w(Z_{[n,k]})$ increases as the value of n increases at $k = 3, 6$

(ii) For $n > 1$, and $c = 4$, the value of $J^w(Z_{[n,k]})$ decreases as the value of n increases at for $(\alpha < 0)$, and the value of $J^w(Z_{[n,k]})$ increases as the value of n increases at $k = 3, 6$ and the value of $J^w(Z_{[n,k]})$ decreases as the value of n increases at $k = 1$ for $(\alpha > 0)$

Table 2 displays the WJ of $Y_{[n,k]}$ from HK-FGM-copula at $c = 2$ and $c = 4$. From Table 2, the following properties can be extracted:

(i) For $n > 1$, and $c = 2$ and $c = 4$, the value of $J^w(Z_{[n,k]})$ increases as the value of n increases for $(\alpha < 0)$, and the value of $J^w(Z_{[n,k]})$ decreases as the value of n increases for $(\alpha > 0)$

3.2. Weighted Cumulative Past Extropy of CKR. If $Z_{[n,k]}$ is the CKR from HK-FGM, then the WCPJ of $Z_{[n,k]}$ is



(a) (b)

FIGURE 1: WJ in $Z_{[n,k]}$ from HK-FGM-ED. (a) $c=2$ and $k=1$, 2, 3, 4, and 5. (b) $c=3$ and $k=2$, 4, 5, 7, and 10.

TABLE 1: $J^w(Z_{[n,k]})$ from HK-FGM-ED.

		$c=2$						$c=4$			
k	n	$\alpha=-0.2$	$\alpha=-0.1$	$\alpha=0.1$	$\alpha=0.3$	k	n	$\alpha=0.06$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.2$
1	1	−0.125	−0.125	−0.125	−0.125	1	1	−0.125	−0.125	−0.125	−0.125
1	2	−0.123519	−0.123762	−0.127234	−0.134688	1	2	−0.126538	−0.124918	−0.126421	−0.132825
1	3	−0.12464	−0.123492	−0.129163	−0.145456	1	3	−0.129147	−0.124888	−0.131323	−0.154189
1	4	−0.125994	−0.123517	−0.130442	−0.153205	1	4	−0.131591	−0.124887	−0.13653	−0.176204
1	5	−0.126943	−0.123585	−0.131188	−0.157884	1	5	−0.133336	−0.124893	−0.140422	−0.192487
1	6	−0.127502	−0.123636	−0.131595	−0.160479	1	6	−0.134413	−0.124899	−0.142873	−0.202697
1	7	−0.127807	−0.123666	−0.13181	−0.161855	1	7	−0.135025	−0.124904	−0.144278	−0.208541
1	8	−0.127967	−0.123682	−0.13192	−0.162567	1	8	−0.135356	−0.124906	−0.145042	−0.211713
1	9	−0.128049	−0.123691	−0.131976	−0.16293	1	9	−0.13553	−0.124907	−0.145444	−0.213381
1	10	−0.128091	−0.123695	−0.132005	−0.163114	1	10	−0.135619	−0.124908	−0.145652	−0.214243
3	1	−0.129321	−0.12681	−0.123893	−0.123786	3	1	−0.124971	−0.125083	−0.126825	−0.130873
3	2	−0.126479	−0.125677	−0.124448	−0.123718	3	2	−0.124888	−0.125049	−0.12587	−0.127596
3	3	−0.124553	−0.124767	−0.125253	−0.125818	3	3	−0.12497	−0.125005	−0.125059	−0.125131
3	4	−0.123646	−0.124151	−0.126194	−0.129616	3	4	−0.125364	−0.124964	−0.124948	−0.125588
3	5	−0.123506	−0.123783	−0.127156	−0.134285	3	5	−0.126099	−0.12493	−0.125755	−0.129745
3	6	−0.123828	−0.12359	−0.128057	−0.139114	3	6	−0.127102	−0.124907	−0.127375	−0.137099
3	7	−0.124373	−0.123507	−0.128853	−0.143638	3	7	−0.128251	−0.124893	−0.129526	−0.146482
3	8	−0.124984	−0.123486	−0.129526	−0.147616	3	8	−0.129428	−0.124887	−0.131901	−0.156651
3	9	−0.125572	−0.123497	−0.13008	−0.150969	3	9	−0.130542	−0.124885	−0.134251	−0.166613
3	10	−0.126092	−0.123523	−0.130524	−0.153712	3	10	−0.131539	−0.124886	−0.136416	−0.175726
6	1	−0.131006	−0.127431	−0.123711	−0.124562	6	1	−0.125031	−0.125096	−0.127255	−0.132392
6	2	−0.129492	−0.126874	−0.123871	−0.123846	6	2	−0.124995	−0.125088	−0.127003	−0.131498
6	3	−0.127891	−0.126257	−0.12412	−0.123491	6	3	−0.124943	−0.125075	−0.126584	−0.13003
6	4	−0.126446	−0.125663	−0.124457	−0.123731	6	4	−0.124897	−0.125056	−0.12606	−0.128231
6	5	−0.125279	−0.125136	−0.124869	−0.124626	6	5	−0.124889	−0.125034	−0.125533	−0.126501
6	6	−0.124425	−0.124695	−0.125339	−0.12612	6	6	−0.12495	−0.12501	−0.125115	−0.12527
6	7	−0.12387	−0.124341	−0.125845	−0.128096	6	7	−0.125105	−0.124986	−0.1249	−0.124894
6	8	−0.123573	−0.124068	−0.126369	−0.130418	6	8	−0.125365	−0.124963	−0.124948	−0.125593
6	9	−0.123486	−0.123864	−0.126893	−0.13295	6	9	−0.125732	−0.124944	−0.125282	−0.127443
6	10	−0.123557	−0.123718	−0.127404	−0.135576	6	10	−0.126195	−0.124927	−0.125893	−0.13039

Table 2: $J^w(Z_{[n,k]})$ from HK-FGM-copula.

		$c=2$						$c=4$			
k	n	$\alpha=-0.2$	$\alpha=-0.1$	$\alpha=0.1$	$\alpha=0.3$	k	n	$\alpha=0.06$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.2$
1	1	−0.25	−0.25	−0.25	−0.25	1	1	−0.25	−0.25	−0.25	−0.25
1	2	−0.215278	−0.230903	−0.272569	−0.328125	1	2	−0.228286	−0.254388	−0.303757	−0.379474
1	3	−0.200471	−0.220604	−0.288659	−0.393767	1	3	−0.216354	−0.25816	−0.364414	−0.55175
1	4	−0.194522	−0.215355	−0.298457	−0.436806	1	4	−0.211325	−0.260788	−0.413701	−0.701456
1	5	−0.192058	−0.212704	−0.303946	−0.46179	1	5	−0.209463	−0.262409	−0.446778	−0.805077
1	6	−0.190977	−0.211365	−0.306883	−0.475391	1	6	−0.208803	−0.263338	−0.466595	−0.868108
1	7	−0.190476	−0.210688	−0.308413	−0.482539	1	7	−0.208565	−0.263846	−0.477685	−0.903651
1	8	−0.190235	−0.210346	−0.309197	−0.48622	1	8	−0.208472	−0.264115	−0.483636	−0.922802
1	9	−0.190117	−0.210174	−0.309596	−0.488095	1	9	−0.208433	−0.264255	−0.48675	−0.93284
1	10	−0.190058	−0.210087	−0.309797	−0.489042	1	10	−0.208416	−0.264326	−0.488352	−0.938011
3	1	−0.2899	−0.268725	−0.233725	−0.208525	3	1	−0.268906	−0.247192	−0.226327	−0.212449
3	2	−0.26562	−0.257593	−0.242843	−0.229833	3	2	−0.261311	−0.248247	−0.234163	−0.222095
3	3	−0.244299	−0.247115	−0.252953	−0.259063	3	3	−0.251288	−0.249787	−0.247897	−0.245848
3	4	−0.227886	−0.238342	−0.26286	−0.292187	3	4	−0.240979	−0.25161	−0.267479	−0.288011
3	5	−0.21608	−0.231402	−0.271874	−0.325451	3	5	−0.231886	−0.253518	−0.291594	−0.347444
3	6	−0.207889	−0.226071	−0.279676	−0.356269	3	6	−0.224658	−0.255357	−0.318148	−0.418685
3	7	−0.202309	−0.222039	−0.286193	−0.383272	3	7	−0.219319	−0.257031	−0.344968	−0.494727
3	8	−0.198531	−0.219011	−0.291497	−0.406016	3	8	−0.215578	−0.258491	−0.370313	−0.569305
3	9	−0.195969	−0.216746	−0.295732	−0.424629	3	9	−0.213055	−0.259723	−0.393066	−0.637995
3	10	−0.19422	−0.215052	−0.299065	−0.439543	3	10	−0.211398	−0.260737	−0.412696	−0.698341
6	1	−0.302615	−0.274314	−0.229672	−0.200973	6	1	−0.271811	−0.24681	−0.223813	−0.210333
6	2	−0.291227	−0.269315	−0.233282	−0.207636	6	2	−0.270133	−0.247029	−0.225235	−0.211457
6	3	−0.27827	−0.263477	−0.237837	−0.217457	6	3	−0.267165	−0.247426	−0.227955	−0.214114
6	4	−0.265309	−0.257445	−0.242974	−0.230177	6	4	−0.263	−0.248005	−0.23225	−0.219395
6	5	−0.253245	−0.251612	−0.248408	−0.245287	6	5	−0.257927	−0.248746	−0.238322	−0.22856
6	6	−0.242513	−0.246197	−0.253923	−0.262126	6	6	−0.252323	−0.249619	−0.246268	−0.242711
6	7	−0.233255	−0.241302	−0.25935	−0.280003	6	7	−0.246562	−0.250587	−0.256059	−0.262529
6	8	−0.225439	−0.236957	−0.264567	−0.298275	6	8	−0.240956	−0.251615	−0.26753	−0.288131
6	9	−0.218941	−0.23315	−0.269492	−0.3164	6	9	−0.23573	−0.252668	−0.280406	−0.319091
6	10	−0.213595	−0.229841	−0.274071	−0.333951	6	10	−0.231027	−0.253718	−0.294333	−0.354554

$$
\begin{aligned}
\xi^w\left(Z_{[n,k]}\right) &= -\frac{1}{2}\int_0^\infty z\left(G_{[n,k]}(z)\right)^2 \mathrm{d}z \\
&= -\frac{1}{2}\int_0^\infty z\left[\left(1+\delta_{n,k:c}^{(\alpha)}\right)G_Z(z) - \delta_{n,k:c}^{(\alpha)} G_V(z)\right]^2 \mathrm{d}z \\
&= -\frac{1}{2}\left[\left(1+\delta_{n,k:c}^{(\alpha)}\right)^2 \int_0^\infty z G_Z^2(z)\mathrm{d}z + \left(\delta_{n,k:c}^{(\alpha)}\right)^2 \int_0^\infty z G_V^2(z)dz - 2\delta_{n,k:c}^{(\alpha)}\left(1+\delta_{n,k:c}^{(\alpha)}\int_0^\infty z G_Z(z) G_V(z)\mathrm{d}z\right)\right] \\
&= \left(1+\delta_{n,k:c}^{(\alpha)}\right)^2 \xi^w(Z) + \left(\delta_{n,k:c}^{(\alpha)}\right)^2 \xi^w(V) \\
&\quad + \delta_{n,k:c}^{(\alpha)}\left(1+\delta_{n,k:c}^{(\alpha)}\right)\mathrm{E}\left(U^{c+2} Q(u) q(u)\right),
\end{aligned} \tag{42}
$$

where $\xi^w(Z)$ is the WCPJ of Z, $\xi^w(V)$ is the WCPJ of V, U is a uniformly RV on (0, 1), and $Q(u) = G_Z^{-1}(u)$ is the quantile function (QF). The QF density is defined as $q(u) = 1/g_Z(Q(u))$, where $q(u)$ is the derivative of $Q(u)$ with respect to u, i.e., $q(u) = Q'(u)$.

Proposition 7. *Let $Z_{[n,k]}$ be the CKR from HK-FGM, $\widetilde{Z} = a Z_{[n,k]} + b$, $a > 0$, $b > 0$ then $\xi^w(\widetilde{Z}) = a^2 \xi^w(Z_{[n,k]}) + ab\xi(Z_{[n,k]})$*

Remark 8. Assume that T and Z are EW based on HK-FGM (HK-FGM-EW). Then, the WCPJ of CKR is given by

$$
\xi_{EW}^w\left(Z_{[n,k]}\right) = \left(1+\delta_{n,k:c}^{(\alpha)}\right)^2 \xi_{EW}^w(Z) + \left(\delta_{n,k:c}^{(\alpha)}\right)^2 \xi_{EW}^w(V) + \delta_{n,k:c}^{(\alpha)}\left(1+\delta_{n,k:c}^{(\alpha)}\right) E_{EW}\left(U^{c+2} Q(u) q(u)\right), \tag{43}
$$

where

$$\xi^{w}_{EW}(Z) = -\frac{1}{2}\int_0^\infty z\left[1-\exp\left(-\tau H(t;\overline{\omega})\right)\right]^2 dz,$$
$$\xi^{w}_{EW}(V) = -\frac{1}{2}\int_0^\infty z\left[1-\exp\left(-\tau H(t;\overline{\omega})\right)\right]^{2(c+1)} dz,$$
$$E_{EW}\left(U^{c+2}Q(u)q(u)\right) = \int_0^\infty z\left[1-\exp\left(-\tau H(t;\overline{\omega})\right)\right]^{c+2} dz. \tag{44}$$

Example 4. Based on Remark 6, by choosing $H(t;\overline{\omega}) = -\log(b-t/b-a)$ and $\tau=1$, we have T and Z follow the HK-FGM with uniform marginals DF as

$$G_{T,Z}(t,z) = \frac{t-a_1}{b_1-a_1}\frac{z-a_2}{b_2-a_2}\left[1+\alpha\left(1-\left(\frac{t-a_1}{b_1-a_1}\right)^c\right)\left(1-\left(\frac{z-a_2}{b_2-a_2}\right)^c\right)\right], a_1<t<b_1, a_2<z<b_2. \tag{45}$$

Then, we have

$$\xi^{w}(Z) = -\frac{a_2^2+2a_2b_2+3b_2^2}{24},$$
$$\xi^{w}(V) = \frac{(a_2-b_2)(a_2+b_2(2c+3))}{4(c+2)(2c+3)}, \tag{46}$$
$$E\left(U^{c+2}Q(u)q(u)\right) = -\frac{(a_2-b_2)(a_2+b_2(c+3))}{(c+3)(c+4)}.$$

$\xi^{w}(Z_{[n,k]})$, in the case of HK-FGM-uniform distribution, can be written as follows:

$$\xi^{w}\left(Z_{[n,k]}\right) = -\left(1+\delta^{(\alpha)}_{n,k:c}\right)^2\frac{a_2^2+2a_2b_2+3b_2^2}{24} + \left(\delta^{(\alpha)}_{n,k:c}\right)^2\frac{(a_2-b_2)(a_2+b_2(2c+3))}{4(c+2)(2c+3)} - \delta^{(\alpha)}_{n,k:c}\left(1+\delta^{(\alpha)}_{n,k:c}\right)\frac{(a_2-b_2)(a_2+b_2(c+3))}{(c+3)(c+4)}. \tag{47}$$

Example 5. Based on Remark 6, by choosing $H(t;\overline{\omega}) = -\log(1-(t/\beta)^\lambda)$ and $\tau=1$, we have T and Z follow the HK-FGM bivariate power distribution with DF as

$$G_{T,Z}(t,z) = \left(\frac{t}{\beta_1}\right)^{\lambda_1}\left(\frac{z}{\beta_2}\right)^{\lambda_2}\left[1+\alpha\left(1-\left(\frac{t}{\beta_1}\right)^{\lambda_1 c}\right)\left(1-\left(\frac{z}{\beta_2}\right)^{\lambda_2 c}\right)\right], 0<t<\beta_1, 0<z<\beta_2. \tag{48}$$

Then,

$$\xi^{w}(Z) = -\frac{\beta_2^2}{4(\lambda_2+1)},$$
$$\xi^{w}(V) = -\frac{\beta_2^2}{4((c+1)\lambda_2+1)}, \tag{49}$$
$$E\left(U^{c+2}Q(u)q(u)\right) = \frac{\beta_2^2}{\lambda_2(c+2)+2}.$$

Table 3 displays the WCPJ of $Y_{[n,k]}$ from HK-FGM-copula at $c=2$, $c=4$. From Table 3, the following properties can be extracted:

(i) For $c=2$ and $c=4$, the value of $\xi^{w}(Z_{[n,k]})$ decreases as the value of n increases for $(\alpha<0)$, and the value of $\xi^{w}(Z_{[n,k]})$ increases as the value of n increases for $(\alpha>0)$

Table 4 displays the WCPJ of $Y_{[n,k]}$ from HK-FGM with power distribution at $c=2, \lambda_2=1$, and $c=4$, $\lambda_2=2$. From Table 4, the following properties can be extracted:

Table 3: $\xi^w(Z_{[n,k]})$ in HK-FGM copula.

		$c = 2$						$c = 4$			
k	n	$\alpha = -0.2$	$\alpha = -0.1$	$\alpha = 0.1$	$\alpha = 0.3$	k	n	$\alpha = 0.06$	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.2$
1	1	−0.125	−0.125	−0.125	−0.125	1	1	−0.125	−0.125	−0.125	−0.125
1	2	−0.139468	−0.132089	−0.1182	−0.105469	1	2	−0.134872	−0.123403	−0.109645	−0.0956616
1	3	−0.149229	−0.136729	−0.114043	−0.0944459	1	3	−0.14336	−0.1221	−0.0980462	−0.0756499
1	4	−0.155003	−0.139426	−0.111725	−0.0886285	1	4	−0.149272	−0.121226	−0.0907486	−0.0642505
1	5	−0.158189	−0.140901	−0.110487	−0.0856227	1	5	−0.152921	−0.1207	−0.086541	−0.058177
1	6	−0.159881	−0.14168	−0.109841	−0.0840837	1	6	−0.155011	−0.120403	−0.0842276	−0.0550125
1	7	−0.160758	−0.142083	−0.109508	−0.0833002	1	7	−0.156153	−0.120242	−0.0829924	−0.0533775
1	8	−0.161208	−0.142289	−0.109339	−0.0829031	1	8	−0.156758	−0.120157	−0.0823459	−0.0525373
1	9	−0.161435	−0.142394	−0.109253	−0.0827027	1	9	−0.157073	−0.120112	−0.082012	−0.0521078
1	10	−0.161551	−0.142447	−0.10921	−0.0826017	1	10	−0.157234	−0.12009	−0.0818414	−0.0518894
3	1	−0.113742	−0.119269	−0.130935	−0.143419	3	1	−0.118682	−0.126074	−0.13602	−0.147653
3	2	−0.120156	−0.12256	−0.127476	−0.132538	3	2	−0.121055	−0.125666	−0.131763	−0.138762
3	3	−0.126957	−0.125976	−0.12403	−0.122107	3	3	−0.124521	−0.12508	−0.125801	−0.126605
3	4	−0.133373	−0.129136	−0.120964	−0.113192	3	4	−0.128624	−0.124403	−0.119113	−0.113418
3	5	−0.139037	−0.131882	−0.118391	−0.105992	3	5	−0.132915	−0.123712	−0.112521	−0.100932
3	6	−0.143826	−0.134174	−0.116305	−0.100353	3	6	−0.137054	−0.123062	−0.10653	−0.0900839
3	7	−0.147757	−0.136035	−0.114651	−0.0960099	3	7	−0.14082	−0.122483	−0.101373	−0.08117
3	8	−0.150913	−0.137519	−0.113357	−0.0926978	3	8	−0.144104	−0.121988	−0.0970938	−0.0741052
3	9	−0.153408	−0.138684	−0.112356	−0.090186	3	9	−0.146876	−0.121577	−0.0936331	−0.0686326
3	10	−0.155357	−0.13959	−0.111586	−0.0882872	3	10	−0.149158	−0.121243	−0.0908841	−0.0644526
6	1	−0.110783	−0.117726	−0.132607	−0.148816	6	1	−0.117822	−0.126224	−0.137599	−0.150993
6	2	−0.113422	−0.119103	−0.131114	−0.14399	6	2	−0.118315	−0.126138	−0.13669	−0.149068
6	3	−0.116672	−0.120781	−0.129328	−0.138313	6	3	−0.119209	−0.125983	−0.135062	−0.145636
6	4	−0.120246	−0.122606	−0.127429	−0.132393	6	4	−0.120511	−0.125759	−0.132727	−0.14076
6	5	−0.123936	−0.124467	−0.125535	−0.12661	6	5	−0.122178	−0.125474	−0.129799	−0.134717
6	6	−0.127595	−0.126293	−0.123717	−0.121182	6	6	−0.124143	−0.125143	−0.126438	−0.127886
6	7	−0.131124	−0.128035	−0.122019	−0.11622	6	7	−0.126322	−0.124781	−0.122818	−0.120661
6	8	−0.134457	−0.129665	−0.120462	−0.111767	6	8	−0.128633	−0.124401	−0.119098	−0.113387
6	9	−0.137554	−0.131167	−0.119053	−0.107819	6	9	−0.131003	−0.124018	−0.115409	−0.106335
6	10	−0.140395	−0.132535	−0.117791	−0.104352	6	10	−0.133366	−0.12364	−0.111851	−0.0996951

Table 4: $\xi^w(Z_{[n,k]})$ in HK-FGM with power distribution.

		$c = 2$ and $\lambda_2 = 1$						$c = 4$ and $\lambda_2 = 2$			
k	n	$\alpha = -0.2$	$\alpha = -0.1$	$\alpha = 0.1$	$\alpha = 0.3$	k	n	$\alpha = 0.06$	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.2$
1	1	−0.125	−0.125	−0.125	−0.125	1	1	−0.0833333	−0.0833333	−0.0833333	−0.0833333
1	2	−0.139468	−0.132089	−0.1182	−0.105469	1	2	−0.090872	−0.0821168	−0.0716815	−0.0611704
1	3	−0.149229	−0.136729	−0.114043	−0.0944459	1	3	−0.0973787	−0.081125	−0.0629549	−0.0463646
1	4	−0.155003	−0.139426	−0.111725	−0.0886285	1	4	−0.101923	−0.0804606	−0.0575055	−0.0381221
1	5	−0.158189	−0.140901	−0.110487	−0.0856227	1	5	−0.104732	−0.0800605	−0.0543808	−0.033819
1	6	−0.159881	−0.14168	−0.109841	−0.0840837	1	6	−0.106343	−0.0798346	−0.0526689	−0.0316105
1	7	−0.160758	−0.142083	−0.109508	−0.0833002	1	7	−0.107223	−0.0797122	−0.0517567	−0.0304805
1	8	−0.161208	−0.142289	−0.109339	−0.0829031	1	8	−0.10769	−0.0796475	−0.0512798	−0.0299031
1	9	−0.161435	−0.142394	−0.109253	−0.0827027	1	9	−0.107933	−0.079614	−0.0510337	−0.0296089
1	10	−0.161551	−0.142447	−0.10921	−0.0826017	1	10	−0.108058	−0.0795968	−0.0509079	−0.0294595
3	1	−0.113742	−0.119269	−0.130935	−0.143419	3	1	−0.078527	−0.0841522	−0.091751	−0.100678
3	2	−0.120156	−0.12256	−0.127476	−0.132538	3	2	−0.0803307	−0.0838406	−0.0884944	−0.0938512
3	3	−0.126957	−0.125976	−0.12403	−0.122107	3	3	−0.0829686	−0.0833942	−0.0839435	−0.0845564
3	4	−0.133373	−0.129136	−0.120964	−0.113192	3	4	−0.0860965	−0.0828784	−0.078855	−0.0745353
3	5	−0.139037	−0.131882	−0.118391	−0.105992	3	5	−0.0893752	−0.0823523	−0.073856	−0.0651192
3	6	−0.143826	−0.134174	−0.116305	−0.100353	3	6	−0.0925424	−0.0818574	−0.0693309	−0.057011
3	7	−0.147757	−0.136035	−0.114651	−0.0960099	3	7	−0.0954294	−0.0814169	−0.0654502	−0.0504134
3	8	−0.150913	−0.137519	−0.113357	−0.0926978	3	8	−0.0979501	−0.0810403	−0.0622418	−0.0452375
3	9	−0.153408	−0.138684	−0.112356	−0.090186	3	9	−0.10008	−0.0807274	−0.0596552	−0.0412687
3	10	−0.155357	−0.13959	−0.111586	−0.0882872	3	10	−0.101835	−0.0804732	−0.0576064	−0.0382665
6	1	−0.110783	−0.117726	−0.132607	−0.148816	6	1	−0.0778739	−0.0842663	−0.0929604	−0.103248
6	2	−0.113422	−0.119103	−0.131114	−0.14399	6	2	−0.0782488	−0.0842007	−0.0922641	−0.101766

Table 4: Continued.

		$c=2$ and $\lambda_2=1$						$c=4$ and $\lambda_2=2$			
k	n	$\alpha=-0.2$	$\alpha=-0.1$	$\alpha=0.1$	$\alpha=0.3$	k	n	$\alpha=0.06$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.2$
6	3	−0.116672	−0.120781	−0.129328	−0.138313	6	3	−0.0789276	−0.0840825	−0.0910172	−0.0991272
6	4	−0.120246	−0.122606	−0.127429	−0.132393	6	4	−0.0799166	−0.0839117	−0.0892313	−0.0953835
6	5	−0.123936	−0.124467	−0.125535	−0.12661	6	5	−0.081185	−0.0836949	−0.0869935	−0.0907532
6	6	−0.127595	−0.126293	−0.123717	−0.121182	6	6	−0.0826802	−0.0834425	−0.0844292	−0.0855341
6	7	−0.131124	−0.128035	−0.122019	−0.11622	6	7	−0.0843406	−0.0831662	−0.0816717	−0.0800313
6	8	−0.134457	−0.129665	−0.120462	−0.111767	6	8	−0.0861041	−0.0828771	−0.078843	−0.0745122
6	9	−0.137554	−0.131167	−0.119053	−0.107819	6	9	−0.0879135	−0.082585	−0.0760437	−0.0691842
6	10	−0.140395	−0.132535	−0.117791	−0.104352	6	10	−0.0897199	−0.0822978	−0.0733495	−0.064191

(i) For $c=2$ and $c=4$, the value of $\xi^w(Z_{[n,k]})$ decreases as the value of n increases for $(\alpha<0)$, and the value of $\xi^w(Z_{[n,k]})$ increases as the value of n increases for $(\alpha>0)$

3.3. Estimating of WCPJ for CKR Based on HK-FGM Family. This section uses empirical estimators to calculate the WCPJ for concomitant $Z_{[n,k]}$. Our next task is to estimate the WCPJ for concomitant using the empirical WCPJ. Let (T_i, Z_i), where $i=1,2,\ldots$, be a HK-FGM sequence. Using the relation (42), the empirical WCPJ of $Z_{[n,k]}$ may be calculated as follows:

$$\begin{aligned}
\widehat{\xi}^w\left(Z_{[n,k]}\right) &= -\frac{1}{2}\int_0^\infty z\left[\widehat{G}_{[n,k]}(z)\right]^2 \mathrm{d}z \\
&= -\frac{1}{2}\int_0^\infty z\left[\left(1+\delta_{n,k:c}^{(\alpha)}\right)\widehat{G}_Z(z) - \delta_{n,k:c}^{(\alpha)}\widehat{G}_V(z)\right]^2 \mathrm{d}z \\
&= -\frac{1}{2}\sum_{j=1}^{n-1}\int_{D_j}^{D_{j+1}} z\left(1+\delta_{n,k:c}^{(\alpha)}\right)^2\widehat{G}_Z^2(z) + \left(\delta_{n,k:c}^{(\alpha)}\right)^2\widehat{G}_V^2(z) - 2\delta_{n,k:c}^{(\alpha)}\left(1+\delta_{n,k:c}^{(\alpha)}\right)\widehat{G}_Z(z)\widehat{G}_V(z)\mathrm{d}z \\
&= -\frac{1}{4}\sum_{j=1}^{n-1}\left(D_{(j+1)}^2 - D_{(j)}^2\right)\left[\left(1+\delta_{n,k:c}^{(\alpha)}\right)^2\left(\frac{j}{n}\right)^2 + \left(\delta_{n,k:c}^{(\alpha)}\right)^2\left(\frac{j}{n}\right)^{2(c+1)} - 2\delta_{n,k:c}^{(\alpha)}\left(1+\delta_{n,k:c}^{(\alpha)}\right)\left(\frac{j}{n}\right)^{c+1}\right].
\end{aligned} \tag{50}$$

4. Application of Real Data

In Table 5, we present 31 annual observations (1980–2010) on exports of goods and services T and GDP growth Z as part of the economic dataset used by El-Sherpieny et al. [22] and Barakat et al. [23]. The first collection of these statistics was carried out both by the World Bank and the OECD. Considering that the correlation between the two datasets is 0.2709, those data are relevant to the FGM copula and its generalizations, including HK-FGM. As part of their research, El-Sherpieny et al. [22] used the maximum likelihood estimation (MLE) approach to compare three FGM families characterized by Weibull (FGM-WD), gamma, and generalized exponential (GE) marginals. As a result of this information, Barakat et al. [23] investigated some measures of information to assess this data. In this study (see Table 6), we estimate four parameters based on HK-FGM-WD and the MLE method $\mu_i, \beta_i, i=1,2$, in the Weibull DF, where WD is given by $G_W(w) = 1-\exp(-(w/\beta_i)^{\mu_i}), w>0$, besides the shape parameters α and c. Table 7 examines the WJ measure for the model estimated HK-FGM-WD at $\mu_2=0.207$ and $\beta_2=0.791$.

Table 5: Data of economics.

Years	T	Z
1980	30.51	10.01
1981	33.37	3.76
1982	27.03	9.91
1983	25.48	7.40
1984	22.35	6.09
1985	19.91	6.60
1986	15.73	2.65
1987	12.56	2.52
1988	17.32	7.93
1989	17.89	4.97
1990	20.05	5.70
1991	27.82	1.08
1992	28.40	4.43
1993	25.84	2.90
1994	22.57	3.97
1995	22.55	4.64
1996	20.75	4.99
1997	18.84	5.49
1998	16.21	4.04
1999	15.05	6.11
2000	16.20	5.37
2001	17.48	3.54
2002	18.32	2.37
2003	21.8	3.19
2004	28.23	4.09
2005	30.34	4.48
2006	29.95	6.85
2007	30.25	7.09
2008	33.04	7.16
2009	24.96	4.67
2010	21.35	5.15

Table 6: Parameters estimation for HK-FGM-WD.

MLE parameters estimation						
HK-FGM-WD	μ_1 1.0567	β_1 0.698	μ_2 0.207	β_2 0.791	α −0.265	c 0.729

Table 7: WJ of HK-FGM-WD at $\mu_2 = 0.207$ and $\beta_2 = 0.791$.

k	n	$J^w(Z_{[n,k]})$
2	1	−0.02629
2	3	−0.02547
2	5	−0.02520
2	8	−0.02506
2	10	−0.02503
4	1	−0.02666
4	3	−0.02589
4	5	−0.02552
4	8	−0.02525
4	10	−0.02516
4	15	−0.02506

Data Availability

The data used to support the findings of this study are available within the article.

Conflicts of Interest

The authors declare there are no conflicts of interest.

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