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Research Article

On the Existence and Stability of Positive Solutions of Eigenvalue Problems for a Class of P-Laplacian ψ-Caputo Fractional Integro-Differential Equations

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This article introduces a generalized approach for analyzing stability and establishing the existence of positive solutions in a specific type of differential equations known as p-Laplacian ψ-Caputo fractional differential equations with fractional integral boundary conditions. The study utilizes various techniques, including the analysis of Green's function properties and the application of Guo–Krasnovelsky's fixed point theorem on cones. By employing these methods, the research establishes novel findings concerning the existence and nonexistence of positive solutions. The investigation relies on fractional integrals, differential operators, and fundamental lemmas as fundamental tools. To assess solution stability, the Hyers–Ulam concept is employed, which extends prior research and introduces a specific definition. The article also provides numerical examples that support the obtained results, thereby demonstrating the practical applicability and accuracy of the proposed methods. Moreover, the study contributes to a deeper understanding of this subject matter and highlights real-life applications for these types of problems. Overall, this study offers a comprehensive analysis of stability and solution existence in a specific class of differential equations, with implications that extend to real-world scenarios such as engineering systems, financial modeling, population dynamics, epidemiology, and ecological studies. These types of problems arise in various fields where modeling and analyzing complex phenomena are necessary.

1. Introduction

Fractional calculus, a field of mathematical analysis, expands upon the principles of differentiation and integration to encompass noninteger orders. The roots of this branch can be traced back to the 18th century, where notable mathematicians such as Leibniz and Euler made early contributions. However, it was not until the mid-19th century that fractional calculus gained increased attention from the scientific and mathematical communities. In recent years, fractional calculus has gained increasing attention and has become an active area of research. Many mathematical techniques and numerical methods have been developed to solve fractional differential equations and perform fractional calculus operations. This field has diverse applications in physics, engineering, signal processing, finance, and control theory [1–4]. By providing an extension to conventional calculus, fractional calculus enables a more comprehensive comprehension of intricate systems and phenomena, and this is due to its remarkable properties and wide-ranging applications of this subject that render it a captivating area of exploration for mathematicians, scientists, and engineers alike. Several studies have presented proof that fractional models provide a more accurate and organized depiction of natural phenomena when compared to conventional models based on integer-order and ordinary time-derivatives. This has been extensively discussed in scholarly works such as [5–7], and the relevant literature cited therein.

Here, we present various examples showcasing the applications of fractional calculus. In a study by Mohammadi et al. [7], a novel fractional system was utilized for modeling hearing loss associated with the mumps virus. This system employed a nonsingular kernel fractional operator called the Caputo–Fabrizio derivative. The fractional-order derivative, unlike its integer-order counterpart, effectively accounted for memory effects and nonlocal behavior. This research marked the first instance of employing the Caputo–Fabrizio derivative for modeling hearing loss with the mumps virus. Another significant contribution was made by Rezapour et al. [8], where the authors introduced a generalized fractional system for analyzing the dynamics of anthrax disease transmission. They employed the Caputo–Fabrizio derivative, a fractional operator without a singular kernel, to develop this novel approach. The authors observed that the approximate solutions obtained from the fractional Caputo–Fabrizio model gradually approached those derived from the classical integer-order system as time progressed. Moreover, several researchers have successfully implemented fractional extensions of mathematical models originally formulated with integer orders, demonstrating their natural representation of real-world phenomena. Notable examples include the works of Aydogan et al. [9], Baleanu et al. [5, 10–12], Mohammadi et al. [13], and George et al. [14], among others. These studies systematically incorporated fractional calculus into existing models, further enriching their accuracy and applicability.

Fractional boundary value problems (FBVPs) have gained considerable attention due to their broad applications in various fields. FBVPs offer a robust framework for modeling complex phenomena and provide valuable insights into the behavior of fractional systems. These problems have found applications in diverse areas such as heat conduction, control systems, and population dynamics. Numerous investigations have been devoted to studying the existence of positive solutions for fractional differential equations with integral boundary conditions. These investigations employ techniques such as the fixed point theory and upper and lower solution methods. Relevant references for further information include ([15–31]) and references therein. Furthermore, significant progress has been made recently in the exploration of p-Laplacian operators and the treatment of eigenvalue problems associated with fractional differential equations. Notable contributions in this field have been made by researchers such as Bai et al. Noteworthy references for this topic include ([17, 29, 32–40]) and references therein.

Researchers have also made significant advancements in understanding the spectral properties and solution behavior of eigenvalue problems in fractional differential equations. Several studies have investigated various aspects, including the existence, uniqueness, and stability of eigenvalues and eigenfunctions. Contributions by researchers such as Bai et al. have greatly enhanced our understanding of eigenvalue problems in the context of fractional differential equations. Key references for further exploration include ([10, 17, 20, 21, 33, 38, 42–44]) among others. These works have greatly enriched our understanding of eigenvalue problems in the context of fractional differential equations.

Regarding the function ψ, fractional derivatives serve as generalizations of the Riemann–Liouville derivatives. The ψ-Caputo fractional derivative differs from the classical derivative due to the presence of kernel terms. Recently, Almeida re-evaluated this derivative and provided a Caputo-type regularization of the existing definition with intriguing properties. Additional properties and applications of the ψ -Caputo fractional derivatives can be found in references such as ([45–51]), and references therein.

In [40], the authors investigated the existence of positive solutions for an eigenvalue problem utilizing the method of upper and lower solutions and the Schauder fixed point theorem. The problem can be formulated as follows:

$$\begin{cases} -D_t^{\beta}\left(\phi_p\left(D_t^{\alpha}u\right)\right)(t) = \lambda f(t, u(t)), \quad t \in (0,1), \\ u(0) = D_t^{\alpha}u(t) = 0, u(1) = \displaystyle\int_0^1 u(s)\, dA(s). \end{cases} \tag{1}$$

In this equation, D_t^{β} and D_t^{α} represent the standard Riemann–Liouville derivatives, with $1 < \alpha \leq 2$ and $0 < \beta \leq 1$. The function A is of bounded variation, and $\int_0^1 u(s)\, dA(s)$ denotes the Riemann–Stieltjes integral of u with respect to A. The p-Laplacian operator ϕ_p is defined as $\phi_p = |s|^{p-2}s$, where $p > 1$. The function $f(t, x)\colon (0,1) \times (0, +\infty) \longrightarrow [0, +\infty)$ is continuous and may exhibit singularity at $t = 0$, $t = 1$, and $x = 0$.

The authors in [21] conducted a study on the presence of positive solutions for an eigenvalue problem related to a nonlinear fractional differential equation. The equation involves a generalized p-Laplacian operator and is described by the following system:

$$\begin{cases} D_{0^+}^{\beta}\left(\phi_p\left(D_{0^+}^{\alpha}u(t)\right)\right) = \lambda f(u(t)), \quad t \in (0,1), \\ u(0) = u'(0) = u''(0) = 0, \phi_p\left(D_{0^+}^{\alpha}u(0)\right) = \left(\phi_p\left(D_{0^+}^{\alpha}u(1)\right)\right)' = 0. \end{cases} \tag{2}$$

In this system, the values of $2 < \alpha \leq 3$ and $1 < \beta \leq 2$ are real numbers. The operator ϕ_p represents a generalized p-Laplacian operator, $\lambda > 0$ is a parameter, and $f(t)\colon (0, +\infty) \longrightarrow (0, +\infty)$ is a continuous function. The study utilized the properties of Green functions and the Guo–Krasnoselskii's fixed-point theorem on cones to establish several results regarding the existence of at least one or two positive solutions within different eigenvalue intervals. In addition, the nonexistence of positive solutions was also examined in relation to the parameter λ.

In [29], the authors investigated the eigenvalue problem concerning a boundary fractional differential equation involving the p-Laplacian operator. The equation is given as follows:

$$\begin{cases} -{}^cD^{\beta}_{0^+}\left(\phi_p\left({}^cD^{\alpha}_{0^+}u\right)\right)(t) = \lambda f(t,u(t)), \quad t \in (0,1), \\ u(0) = \int_0^1 g_1(s)u(s)\mathrm{d}s, u(1) = \int_0^1 g_2(s)u(s)\mathrm{d}s, u''(0) = \int_0^1 g_3(s)u(s)\mathrm{d}s, {}^cD^{\alpha}_{0^+}u(t)\big|_{t=0} = 0, \end{cases} \tag{3}$$

where ${}^cD^{\beta}_{0^+}$ and ${}^cD^{\alpha}_{0^+}$ are the standard Caputo derivatives with $2 < \alpha \leq 3$, and $0 < \beta \leq 1$. The p-Laplacian operator ϕ_p is defined as $\phi_p = |s|^{p-2}s$ with $p > 1$. The functions $f(t,x)$: $[0,1] \times [0,+\infty) \longrightarrow [0,+\infty)$ and $g_i(s) \in C[0,1]$ (where $i = 1,2,3$) are continuous. The interval J is defined as $J = [0,1]$, and λ is a positive parameter.

In the work by Matar et al. [52], a newly proposed p-Laplacian nonperiodic boundary value problem is examined, which involves generalized Caputo fractional derivatives. The authors extensively investigate the existence and uniqueness of solutions for this problem:

$$\begin{cases} \dfrac{d}{dt}\left(\phi_p\left({}^cD^{\alpha,\rho}u(t)\right)\right) = q\left(t,u(t), {}^cD^{\gamma,\rho}u(t)\right), \quad t \in [0,1], \\ u(0) + \mu u(1) = \theta(u(0),u(1)), u'(1) = v(u(0),u(1)), \end{cases} \tag{4}$$

where ${}^cD^{\alpha,\rho}$ and ${}^cD^{\gamma,\rho}$ are the GC_pFr derivatives with $\alpha \in (1,2)$, $\gamma \in (0,1)$, ϕ_p is the p-Laplacian operator with $p > 1$, $\mu \neq -1$, and the nonlinear nonlinear functions q: $[0,1] \times R \times R \longrightarrow R$ and θ: $R \times R \longrightarrow R$ are given continuous functions.

However, as far as we know, there are few papers studying the eigenvalue problem for the p-Laplacian fractional differential equations involving the integral boundary condition. Inspired by the abovementioned works, in this paper, we will explore the following p-Laplacian ψ-Caputo fractional integro-differential equation:

$$\begin{cases} {}^cD^{\beta,\psi}_{0^+}\left(\phi_p\left({}^cD^{\alpha,\psi}_{0^+}u\right)\right)(t) + \lambda f(t,u(t)) = 0, \quad t \in [0,T], \\ u(0) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_1(s)u(s)ds, \\ u(T) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_2(s)u(s)ds, \\ u''(0) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_3(s)u(s)ds, \\ {}^cD^{\alpha,\psi}_{0^+}u(t)\big|_{t=0} = 0, \end{cases} \tag{5}$$

where ${}^cD^{\alpha,\psi}_{0^+}$ and ${}^cD^{\beta,\psi}_{0^+}$ are the ψ-Caputo fractional derivatives of respective fractional orders $\alpha \in (2,3]$ and $\beta \in (0,1]$. The IBVP 1 also involves the p-Laplacian operator ϕ_p, which is a nonlinear operator defined as $\phi_p(s) = |s|^{p-2}s$, where $p > 1$. The operator ϕ_p is used to model nonlinear phenomena such as turbulence and phase transitions. The boundary conditions of the IBVP involve integrals of the form $\int_0^T \psi'^{\gamma-1}g_i(s)u(s)ds$, where γ is a parameter between 0 and 1, and g_i are continuous functions on $[0,T]$ for $i = 1,2,3$. These boundary conditions involve memory effects and nonlocality, which are typical of fractional order problems. The functions $f(t,u)$: $I \times R \longrightarrow R$, and g_i: $I \longrightarrow R$; $(i = 1,2,3)$, and u: $I \longrightarrow R$ are continuous functions on $I = [0,T]$.

It is clear that the IBVP (5) is an equation that combines fractional derivatives and integrals. It comprises a fractional differential equation governed by the p-Laplacian operator and a nonlinearity function $f(t,u)$, along with three boundary conditions. These boundary conditions involve fractional integrals that incorporate the function u and three distinct functions $g_1(t)$, $g_2(t)$, and $g_3(t)$. The differential equation and boundary conditions employ ψ-Caputo derivatives, which are an extension of the classical Caputo derivative. The ψ-Caputo derivatives are defined using the Riemann–Liouville fractional derivative and a scaling function $\psi(t)$ that satisfies specific conditions. The p-Laplacian operator, a nonlinear differential operator, is utilized to model various physical phenomena such as fluid flow, diffusion, and image processing. Within this problem, there exists a positive constant parameter λ referred to as the eigenvalue, which characterizes the system. The objective of the research is to determine the values of λ that allow for nontrivial solutions, known as eigenfunctions, to exist for the problem. By analyzing the eigenvalues and eigenfunctions, it is possible to study the stability and dynamics of the system represented by the problem.

The motivation behind studying such type of IBVPs is to understand the behavior of the solutions of this problem, particularly in the presence of nonlinearity and nonlocality. This is an important research area with applications in various fields, including physics, engineering, and biology. In particular, the Caputo-type fractional IBVP in equation (5) has numerous applications in modeling systems with memory effects, nonlocality, and long-term dependencies. Its specific applications depend on the choice of the functions $f(t,u)$ and $g_i(t)$ used in each application depend on the properties of the system being modeled, indicating the flexibility and adaptability of the IBVP to different contexts. Thus, the Caputo-type fractional IBVP (5) is a challenging and interesting problem with significant potential for a range of real-world applications in various fields of science and engineering.

The utilization of eigenvalue problems for a specific class of p-Laplacian ψ-Caputo fractional integro-differential equations has significant implications for various fields. For instance, in the context of mumps virus transmission, these eigenvalue problems offer valuable insights into the system's stability, critical thresholds, and spatial patterns, facilitating the development of effective strategies for controlling the spread of the virus (see [7] and related references). Moreover, eigenvalues play a crucial role in other domains as well. In control theory, they determine the stability and performance of quadcopter systems, allowing for stable flight and improved control [53, 54]. In signal processing, eigenvalues are employed for signal classification, compression, and denoising, with applications in the analysis of electroencephalogram signals related to Alzheimer's disease [10, 55, 56]. Fractional IBVPs using these eigenvalue problems are also used to model non-Newtonian fluid dynamics in porous media and heat conduction in biological tissues, offering insights into memory effects and thermal responses [22, 57–60]. These diverse applications underscore the broad scope and significance of employing eigenvalue problems for understanding complex phenomena and informing practical solutions.

According to our knowledge, numerous research studies have been conducted on the eigenvalue problems related to fractional differential equations that incorporate the p-Laplacian and integral boundary conditions. But the initial boundary value problem (IBVP) (5), presented in our work, serves as a basis for obtaining various investigations outlined in the monograph by utilizing necessary and appropriate parameters as follows:

(i) If $\psi(t) = t$, $\beta = 0$, $1 < \alpha \leq 2$, $p = 2$, $\lambda = 1$, $T = 1$, and $g_1 = g_2 = g_3 = 0$, then we obtain the following boundary value problem of nonlinear fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{6}$$

which is similar to those outcomes obtained in [17].

(ii) If $\psi(t) = t$, $\beta = 0$, $2 < \alpha \leq 3$, $p = 2$, $\lambda = 1$, $T = 1$, and $g_1 = g_2 = g_3 = 0$, then we obtain the following boundary value problem of nonlinear fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = u'(0) = 0, \end{cases} \tag{7}$$

which is similar to those outcomes obtained by Zhao et al. [31].

(iii) If $\psi(t) = t$, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $\lambda = 1$, $T = 1$, $g_1 = g_2 = g_3 = 0$, then we obtain the following boundary value problem of nonlinear fractional differential equations:

$$\begin{cases} {}^{c}D_{0^+}^{\beta}\left(\phi_p^{c} D_{0^+}^{\alpha} u\right)(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \text{and}\, {}^{c}D_{0^+}^{\alpha} u(0) = 0, \end{cases} \tag{8}$$

which is similar to the results obtained by Chai [18] for $\sigma = 0$.

(iv) If $\psi(t) = t$, $T = 1$, $0 < \beta \leq 1$, $2 < \alpha \leq 3$, $\gamma = 1$, $g_2 = 1$, and $g_1 = g_3 = 0$, then obtained outcomes in the current paper incorporate the investigation of Zhang et al. [40] for $A(s) = s$

$$\begin{cases} {}^{c}D_{0^+}^{\beta}\left(\phi_p^{c} D_{0^+}^{\alpha} u\right)(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, u(1) = 0, u(1) = \int_0^1 u(s) dA(s), \text{and}\, {}^{c}D_{0^+}^{\alpha} u(t)\big|_{t=0} = 0. \end{cases} \tag{9}$$

(v) If $\psi(t) = t$, $p = 2$, $\beta = 0$, $2 < \alpha \leq 3$, $\gamma = 1$, $\lambda = q(t)$, $T = 1$, and $g_1 = g_3 = 0$, then we obtain the following fractional differential equation with the following integral boundary conditions:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + q(t) f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \text{and}\, u(1) = \int_0^1 g(s) u(s) ds, \end{cases} \tag{10}$$

which is similar to that studied by Sun and Zhao [29].

(vi) Also, if $\psi(t) = t$, $T = 1$, and $\gamma = 1$ then we have the p -Laplacian fractional differential equations involving the integral boundary condition.

$$\begin{cases} {}^{c}D_{0^+}^{\beta}\left(\phi_p\left({}^{c}D_{0^+}^{\alpha} u\right)\right)(t) + \lambda f(t, u(t)) = 0, \quad t \in [0,1], \\ u(0) = \int_0^1 g_1(s) u(s) ds, u(1) = \int_0^1 g_2(s) u(s) ds, \\ u''(0) = \int_0^1 g_3(s) u(s) ds, \text{and}\, {}^{c}D_{0^+}^{\alpha} u(t)\big|_{t=0} = 0, \end{cases} \tag{11}$$

which is the same result obtained in Su et al. [36].

2. Main Results

2.1. Mathematical Background. In this section, we introduce some notations, definitions, lemmas, and theorems that are considered prerequisites for our work, such as fractional order integrals and derivatives, Green's functions, completely continuous operators, and others.

Definition 1 (see [47]). For any real number $\alpha > 0$, the left-sided ψ-Riemann–Liouville fractional integral of order α for an integrable function $u: I \longrightarrow R$ with respect to another function $\psi: I \longrightarrow R$, which is an increasing differentiable function such that $\psi'(t) \neq 0$ for all $t \in I$ is defined by the following equation:

$$I^{\alpha,\psi}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}u(s)\mathrm{d}s, \qquad (12)$$

where Γ is the classical Euler Gamma function.

Definition 2 (see [47]). If $n \in N$ and $\psi, u \in C^n(I, R)$ are two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in I$, then the left-sided ψ-Caputo fractional derivative of a function u of order α is defined by the following equation:

$$\begin{aligned} {}^cD^{\alpha,\psi}u(t) &= I^{n-\alpha,\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1}u_\psi^{[n]}(s)ds, \end{aligned} \qquad (13)$$

where $u_\psi^{[n]}(t) = (1/\psi'(t)d/dt)^n u(t)$ and $n = [\alpha] + 1$ for $\alpha \notin N$, and $n = \alpha$ *for* $\alpha \in N$.

Lemma 3. *Let* $\alpha > 0$, *then the differential equation* $({}^cD_{a+}^{\alpha,\psi}h)(t) = 0$ *has the following solution:*

$$\begin{aligned} h(t) &= c_0 + c_1(\psi(t) - \psi(0)) + c_2(\psi(t) - \psi(0))^2 \\ &\quad + \cdots + c_{n-1}(\psi(t) - \psi(0))^{n-1}, \end{aligned} \qquad (14)$$

where $c_i \in R, i = 0, 1, 2, \ldots, n-1$, *and* $n = [\alpha] + 1$.

Lemma 4 (see [48]). *Let* $\alpha, \beta \in R^+$, *and* $f(t) \in L_1(I)$. *Then,* $I_{a+}^{\alpha,\psi}I_{a+}^{\beta,\psi}f(t) = I_{a+}^{\beta,\psi}I_{a+}^{\alpha,\psi}f(t) = I_{a+}^{\alpha+\beta,\psi}f(t)$, *and* $(I_{a+}^{\alpha,\psi})^n f(t) = I_{a+}^{n\alpha,\psi}f(t)$, *where* $n \in N$.

Definition 5 (see [15]). Let X be any space and let $f: X \longrightarrow X$. A point $x \in X$ is called a fixed point for mapping f *if* $x = f(x)$.

Theorem 6 (see [48]). *Arzela–Ascoli Theorem.*

Let X be the Banach space of real or complex valued continuous functions normed by $\|f\| = \sup_{t\in X}|f(t)|$. If $A = \{f_n\}$ is a sequence in X such that f_n is uniformly bounded and equi-continuous, then $\overline{A}$ is compact.

Theorem 7 (see [27]). *Fixed point theorem on a cone.*

Let X be a Banach space, and let $S \subset X$ be a cone in X. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $f: S \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow S$ be a completely continuous operator such that either

(1) $\|fu\| \leq \|u\|, u \in S \cap (\partial\Omega_1)$, and $\|fu\| \geq \|u\|, u \in S \cap (\partial\Omega_2)$,
(2) $\|fu\| \geq \|u\|, u \in S \cap (\partial\Omega_1)$, and $\|fu\| \leq \|u\|, u \in S \cap (\partial\Omega_2)$,

Then, f has a fixed point in $S \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 8 (see [35]). *Let* ϕ_p *be a p-Laplacian operator. Then, we have*

(1) If $1 < p \leq 2$, $\zeta_1\zeta_2 > 0$, *and* $|\zeta_1|, |\zeta_2| \geq \varrho$, *then* $|\phi_p(\zeta_1) - \phi_p(\zeta_2)| \leq (p-1)\varrho^{p-2}|\zeta_1 - \zeta_2|$
(2) If $p > 2$, *and* $|\zeta_1|, |\zeta_2| \leq \varrho$, *then* $|\phi_p(\zeta_1) - \phi_p(\zeta_2)| \leq (p-1)\varrho^{p-2}|\zeta_1 - \zeta_2|$

2.2. Preliminary Results. In the following, we present the required results that will be used in our later discussion of the existence of positive solutions for the IBVP (5).

Lemma 9. *The following IBVP*

$$\begin{cases} -{}^cD_{0^+}^{\alpha,\psi}u(t) = h(t), \quad t \in I = [0, T], \\ u(0) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}g_1(s)u(s)\mathrm{d}s, \\ u(T) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}g_2(s)u(s)\mathrm{d}s, \\ u''(0) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}g_3(s)u(s)\mathrm{d}s, \end{cases} \qquad (15)$$

has a unique solution for any continuous function $h(t): I \longrightarrow R$, *and* $g_i \in C[0, T]$ $(i = 1, 2, 3)$, *such that*

$$\begin{aligned} u(t) &= \int_0^T \psi'(s)G(t, s)h(s)ds \\ &\quad + \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}\Phi(t, s)u(s)\mathrm{d}s, \end{aligned} \qquad (16)$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\frac{\varphi(t)}{\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}-(\psi(t)-\psi(s))^{\alpha-1}, & \text{if } 0\le s\le t\le T,\\ \frac{\varphi(t)}{\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}, & \text{if } 0\le t\le s\le T,\end{cases} \tag{17}$$

where $\varphi(t)=(\psi(t)-\psi(0))(2(\psi'(0))^2-(\psi(t)-\psi(0))\psi''(0))$, *which is an increasing function such that* $\varphi(t)\neq 0$ *for all* $t\in[0,T]$, *and*

$$\Phi(t,s)=\frac{1}{\Gamma(\gamma)}\left[\left(1-\frac{\varphi(t)}{\varphi(T)}\right)g_1(s)+\frac{\varphi(t)}{\varphi(T)}g_2(s)+\frac{\chi(t)}{\varphi(T)}g_3(s)\right], \tag{18}$$

where $\chi(t)=(\psi(T)-\psi(0))(\psi(t)-\psi(0))(\psi(t)-\psi(T))$.

Proof. From Lemma 3, we have

$$\begin{aligned}u(t)&=-I_{0^+}^{\alpha,\psi}h(t)+c_0+c_1(\psi(t)-\psi(0))+c_2(\psi(t)-\psi(0))^2.\\ &=-\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}h(s)ds+c_0+c_1(\psi(t)-\psi(0))+c_2(\psi(t)-\psi(0))^2.\end{aligned} \tag{19}$$

Using the boundary conditions in (15), we obtain

$$\begin{aligned}c_0=u(0)&=\frac{1}{\Gamma(\gamma)}\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_1(s)u(s)ds,\\ u(T)&=-\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)[\psi(T)-\psi(s)]^{\alpha-1}h(s)ds+c_0+c_1(\psi(T)-\psi(0))+c_2(\psi(T)-\psi(0))^2,\end{aligned} \tag{20}$$

and

$$u''(0)=c_1\psi''(0)+2c_2\left(\psi'(0)\right)^2. \tag{21}$$

Solving (20) and (21) for c_1 and c_2, we obtain

$$c_1=\frac{-u''(0)[\psi(T)-\psi(0)]^2+2\left(\psi'(0)\right)^2\left(1/\Gamma(\alpha)\int_0^T\psi'(s)[\psi(T)-\psi(s)]^{\alpha-1}h(s)ds+u(T)-u(0)\right)}{(\psi(T)-\psi(0))\left[2\left(\psi'(0)\right)^2-(\psi(T)-\psi(0))\psi''(0)\right]}, \tag{22}$$

and

$$c_2=\frac{u''(0)[\psi(T)-\psi(0)]-\psi''(0)\left(1/\Gamma(\alpha)\int_0^T\psi'(s)[\psi(T)-\psi(s)]^{\alpha-1}h(s)ds+u(T)-u(0)\right)}{(\psi(T)-\psi(0))\left[2\left(\psi'(0)\right)^2-(\psi(T)-\psi(0))\psi''(0)\right]}. \tag{23}$$

Hence, if we take $\varphi(t) = (\psi(t) - \psi(0))(2(\psi'(0))^2 - (\psi(t) - \psi(0))\psi''(0))$ where $2(\psi'(0))^2 - (\psi(t) - \psi(0))\psi''(0) > 1$, then

$$\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)[\psi(t) - \psi(s)]^{\alpha-1} h(s)ds + c_0 + c_1(\psi(t) - \psi(0)) + c_2(\psi(t) - \psi(0))^2 \\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)[\psi(t) - \psi(s)]^{\alpha-1} h(s)ds \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1} g_1(s)u(s)ds \\
&\quad - \frac{(\psi(T) - \psi(0))^2(\psi(t) - \psi(0))}{\Gamma(\gamma)\varphi(T)}\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1} g_3(s)u(s)ds \\
&\quad + 2\frac{(\psi(t) - \psi(0))\left(\psi'(0)\right)^2}{\Gamma(\alpha)\varphi(T)}\int_0^T \psi'(s)[\psi(T) - \psi(s)]^{\alpha-1} h(s)ds \\
&\quad + 2\frac{(\psi(t) - \psi(0))\left(\psi'(0)\right)^2}{\varphi(T)\Gamma(\gamma)}\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}(g_2(s) - g_1(s))u(s)ds \\
&\quad + \frac{(\psi(T) - \psi(0))(\psi(t) - \psi(0))^2}{\Gamma(\gamma)\varphi(T)}\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1} g_3(s)u(s)ds \\
&\quad - \frac{(\psi(t) - \psi(0))^2\psi''(0)}{\Gamma(\alpha)\varphi(T)}\int_0^T \psi'(s)[\psi(T) - \psi(s)]^{\alpha-1} h(s)ds \\
&\quad - \frac{(\psi(t) - \psi(0))^2\psi''(0)}{\Gamma(\gamma)\varphi(T)}\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}(g_2(s) - g_1(s))u(s)ds.
\end{aligned} \tag{24}$$

Consequently from the fact that $\int_0^T = \int_0^t + \int_t^T$, we obtain

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\gamma)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_1(s)u(s)ds \\
&- \frac{(\psi(T)-\psi(0))^2(\psi(t)-\psi(0))}{\Gamma(\gamma)\varphi(T)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_3(s)u(s)ds \\
&+ 2\frac{(\psi(t)-\psi(0))\left(\psi'(0)\right)^2}{\Gamma(\alpha)\varphi(T)}\left[\begin{array}{l}\int_0^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}h(s)ds \\ +\int_t^T \psi'(s)[\psi(T)-\psi(s)]^{\alpha-1}h(s)ds\end{array}\right] \\
&+ 2\frac{(\psi(t)-\psi(0))\left(\psi'(0)\right)^2}{\varphi(T)\Gamma(\gamma)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}(g_2(s)-g_1(s))u(s)ds \\
&+ \frac{(\psi(T)-\psi(0))(\psi(t)-\psi(0))^2}{\Gamma(\gamma)\varphi(T)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_3(s)u(s)ds \\
&- \frac{(\psi(t)-\psi(0))^2\psi''(0)}{\Gamma(\alpha)\varphi(T)}\left[\begin{array}{l}\int_0^t \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}h(s)ds \\ +\int_t^T \psi'(s)[\psi(T)-\psi(s)]^{\alpha-1}h(s)ds\end{array}\right] \\
&- \frac{(\psi(t)-\psi(0))^2\psi''(0)}{\Gamma(\gamma)\varphi(T)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}(g_2(s)-g_1(s))u(s)ds,
\end{aligned} \tag{25}
$$

which implies that $u(t) = \int_0^T \psi'(s)G(t,s)h(s)ds + \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\Phi(t,s)u(s)ds$.

Therefore, the proof is completed. □

Remark 10. Throughout our work, we take the following remarks in consideration:

(1) If $m = \min\{\Phi(t,s)\colon (t,s) \in I \times I\}$ and $M = \max\{\Phi(t,s)\colon (t,s) \in I \times I\}$, then we assume that

$$0 < m \leq |\Phi(t,s)| \leq M < T, \quad (A_0). \tag{26}$$

(2) Since $\psi(t)$ is an increasing function with $\psi'(t) \neq 0$ for all $t \in I = [0,T]$, then $G(t,s)$ is continuous and positive for all $t, s \in I$.

(3) According to the definition of $G(t,s)$ and by (A_0), we deduce that $\Phi(t,s) \in C(I \times I, R^+)$ is continuous.

Lemma 11. *For any $s \in I$, we have*

$$\max_{t\in I} G(t,s) \leq \frac{\varphi(t)}{\Gamma(\alpha)\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}, \tag{27}$$

and for any $s \in [0,T]$, there exist $\varepsilon \in (0, T/2)$, $q(t) = \psi(t) - \psi(t)^{\alpha-1}$ which is a concave function, and $\rho = \min_{t\in[\varepsilon,T-\varepsilon]} q(t) = \min\{(\psi(\varepsilon)-\psi(0)) - (\psi(\varepsilon)-\psi(0))^{\alpha-1}, (\psi(T)-\psi(\varepsilon)) - (\psi(T)-\psi(\varepsilon))^{\alpha-1}\}$ such that

$$\min_{t\in[\varepsilon,T-\varepsilon]} G(t,s) \geq \frac{\rho}{\Gamma(\alpha)\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}. \tag{28}$$

Proof. From the expression for $G(t,s)$, it could be easily obtained that

$$\max_{t\in I} G(t,s) \leq \frac{\varphi(t)}{\Gamma(\alpha)\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}. \tag{29}$$

Now, if $0 \leq t \leq s \leq T$, then for $t \in [\varepsilon, T-\varepsilon]$ and since φ is an increasing function, then we have:

$$
\begin{aligned}
G(t,s) &= \frac{1}{\Gamma(\alpha)\varphi(T)}\varphi(t)(\psi(T)-\psi(s))^{\alpha-1} \\
&\geq \frac{1}{\Gamma(\alpha)\varphi(T)}\varphi(\varepsilon)(\psi(T)-\psi(s))^{\alpha-1}.
\end{aligned} \tag{30}
$$

Since ψ and φ are increasing functions, then

$$\begin{aligned}G(t,s) &\geq \frac{1}{\Gamma(\alpha)\varphi(T)}\left[(\psi(\varepsilon)-\psi(0))\underbrace{\left(2\left(\psi'(0)\right)^2-(\psi(t)-\psi(0))\psi''(0)\right)}_{>1}\right](\psi(T)-\psi(s))^{\alpha-1},\\ &\geq \frac{1}{\Gamma(\alpha)\varphi(T)}[\psi(\varepsilon)-\psi(0)](\psi(T)-\psi(s))^{\alpha-1},\\ &\geq \frac{1}{\Gamma(\alpha)\varphi(T)}\left[(\psi(\varepsilon)-\psi(0))-(\psi(\varepsilon)-\psi(0))^{\alpha-1}\right](\psi(T)-\psi(s))^{\alpha-1},\\ &\geq \frac{\rho}{\Gamma(\alpha)\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}.\end{aligned} \tag{31}$$

If $0\leq s\leq t\leq T$, then for $t\in[\varepsilon,T-\varepsilon]$ and $s\in[0,T-\varepsilon]$, we have the following equation:

$$\begin{aligned}G(t,s) &= \frac{1}{\Gamma(\alpha)}\left[\frac{\varphi(t)}{\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}-(\psi(t)-\psi(s))^{\alpha-1}\right]\\ &= \frac{1}{\Gamma(\alpha)}\left[\frac{\varphi(t)}{\varphi(T)}-\left(\frac{\psi(t)-\psi(s)}{\psi(T)-\psi(s)}\right)^{\alpha-1}\right](\psi(T)-\psi(s))^{\alpha-1}\\ &= \frac{1}{\Gamma(\alpha)}\left[\frac{\varphi(t)}{\varphi(T)}-\left(1-\frac{\psi(T)-\psi(t)}{\psi(T)-\psi(s)}\right)^{\alpha-1}\right](\psi(T)-\psi(s))^{\alpha-1}\\ &\geq \frac{1}{\Gamma(\alpha)}\left[\frac{\psi(t)-\psi(0)}{\psi(T)-\psi(0)}-\left(1-\frac{1-\psi(t)-\psi(0)/\psi(T)-\psi(0)}{1-\psi(s)-\psi(0)/\psi(T)-\psi(0)}\right)^{\alpha-1}\right](\psi(T)-\psi(s))^{\alpha-1}\\ &\geq \frac{\rho}{\Gamma(\alpha)\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}.\end{aligned} \tag{32}$$

Hence, we deduce that $\min_{t\in[\varepsilon,T-\varepsilon]}G(t,s)\geq\rho$ $(1/\Gamma(\alpha)\varphi(T))(\psi(T)-\psi(s))^{\alpha-1}$ for any $s\in[0,T]$.

Therefore, we obtain for all $t,s\in[0,T]$ that

$$\rho\frac{1}{\Gamma(\alpha)\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}\leq G(t,s)\leq\frac{\varphi(t)}{\Gamma(\alpha)\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1}. \tag{33}$$

□

In the following, let X be a Banach space of all continuous functions from $I=[0,T]$ to R with norm $\|u\|=\max_{t\in I}|u(t)|$, and let $P=\{u\in X\colon u(t)\geq 0, t\in I\}$ be a nonempty, bounded, closed, and convex cone in X.

Definition 12. Let $\mathscr{A}\colon C(I,R)\longrightarrow C(I,R)$ be an operator defined by the following equation:

$$\mathscr{A}(u(t))=\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\Phi(t,s)u(s)ds. \tag{34}$$

Lemma 13. *If assumption* (A_0) *holds, then* $\mathscr{A}$ *satisfies the following properties:*

(1) $\mathscr{A}$ *is continuous linear operator*

(2) $\mathscr{A}$ *is a bounded*

(3) $\mathscr{A}(P) \subset P$

(4) $\mathscr{A}$ *is reversible*

(5) $\|(I - A)^{-1}\| \leq 1 + N/(1-N)(\psi(T) - \psi(0))^{\gamma}/\gamma$, *where* $N = ((\psi(T) - \psi(0))^{\gamma}/\gamma)M$

Proof

(1) $\mathscr{A}$ is continuous

Consider a sequence $\{u_n\} \subset P$ that converges to $u \in p$, i.e., $u_n \longrightarrow u$ in P as $n \longrightarrow \infty$. Then,

$$\left|\mathscr{A}(u_n(t)) - \mathscr{A}(u(t))\right| \leq \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}|\Phi(t,s)|\left|u_n(s) - u(s)\right|\mathrm{d}s, \tag{35}$$

where $u_n, u \in C(I, R)$. Taking supremum for all $t \in I$, we obtain

$$\left\|\mathscr{A}(y_n) - \mathscr{A}(y)\right\| \leq M\left\|u_n - u\right\|\frac{(\psi(T) - \psi(0))^{\gamma}}{\gamma}. \tag{36}$$

Thus, $\|\mathscr{A}(y_n) - \mathscr{A}(y)\| \longrightarrow 0$ as $n \longrightarrow \infty$ and consequently $\mathscr{A}$ is continuous.

(2) $\mathscr{A}$ is bounded.

$$\begin{aligned}\|\mathscr{A}(u(t))\| &\leq \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}|\Phi(t,s)||u(s)|ds\\ &\leq M\|u\|\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}ds\\ &\leq M\|u\|\frac{(\psi(T) - \psi(0))^{\gamma}}{\gamma}.\end{aligned} \tag{37}$$

Thus, $\|\mathscr{A}\| \leq M\,(\psi(T) - \psi(0))^{\gamma}/\gamma$ which is bounded.

(3) $\mathscr{A}$ is compact

If $u(t) \in P$, then since for all $t, s \in I$, we have $\Phi(t,s) > 0$, $u(t) > 0$, and $\psi(t)$ is an increasing function defined on R^+, which implies that:

$$\mathscr{A}(u(t)) = \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}\Phi(t,s)u(s)ds \geq 0. \tag{38}$$

Thus, $\mathscr{A}(P) \subset P$.

(4) $\mathscr{A}$ is reversible since $\mathscr{A}$ is continuous, bounded, and compact.

(5) Let $v \in C[0,T]$ be defined as $v(t) = u(t) - \mathscr{A}u(t)$. Then,

$$u(t) = (I - \mathscr{A})^{-1}v(t) \quad \text{for all } t \in [0,T], \tag{39}$$

$$u(t) = v(t) + \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}\Phi(t,s)u(s)ds. \tag{40}$$

By using the iteration method, let

$$u_i(t) = v(t) + \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}\Phi(t,s)u_{i-1}(s)\,ds; \quad i = 1, 2, 3, \ldots, \tag{41}$$

with $u_0(t) = u(t)$, and $\Phi_1(t,s) = \Phi(t,s)$, we obtain

$$u(t) = v(t) + \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}\mathscr{R}(t,s)v(s)ds, \tag{42}$$

where

$$\begin{aligned}\mathscr{R}(t,s) &= \sum_{j=i}^{\infty}\int_0^T \psi'(s)(\psi(T) - \psi(s))^{\gamma-1}\Phi(t,\tau)\Phi_{j-1}(\tau,s)d\tau\\ &= \sum_{j=i}^{\infty}\psi'(s)(\psi(T) - \psi(s))^{\gamma-1}\Phi_j(t,s),\end{aligned} \tag{43}$$

where

$$\Phi_j(t,s)=\int_0^T\psi'(\tau)(\psi(T)-\psi(s))^{\gamma-1}\Phi(t,\tau)\Phi_{j-1}(\tau,s)d\tau. \tag{44}$$

But, $0\le\Phi(t,s)\le M\le T$. Then, $\mathscr{R}(t,s)\ge 0$, and

$$\begin{aligned}\mathscr{R}(t,s)&=\sum_{j=i}^{\infty}\psi'(s)(\psi(T)-\psi(0))^{\gamma-1}\Phi_j(t,s)\\&<\frac{(\psi(T)-\psi(0))^{\gamma}}{\gamma}M+\frac{(\psi(T)-\psi(0))^{2\gamma}}{\gamma^2}M^2+\cdots+\frac{(\psi(T)-\psi(0))^{n\gamma}}{\gamma^n}M^n+\cdots\\&<N+N^2+\cdots+N^n+\cdots=\frac{N}{1-N}.\end{aligned} \tag{45}$$

Thus,

$$\begin{aligned}\left|(I-\mathscr{A})^{-1}v(t)\right|&\le|v(t)|+\frac{N}{1-N}\left|\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}|v(s)|\,ds\right|\\&\le\|v\|+\frac{N}{1-N}\frac{(\psi(T)-\psi(0))^{\gamma}}{\gamma}\|v\|,\end{aligned} \tag{46}$$

which implies that

$$\left\|(I-\mathscr{A})^{-1}\right\|\le 1+\frac{N}{1-N}\frac{(\psi(T)-\psi(0))^{\gamma}}{\gamma}. \tag{47}$$

The proof is completed. □

Lemma 14. *Suppose that assumption* (A_0) *holds,* u, ${}^cD^{\alpha,\psi}u\in C(I)$, *then the integral boundary value problem*

$$\begin{cases}{}^cD_{0^+}^{\alpha,\psi}u(t)+f(t,u(t))=0,\quad t\in I=[0,T],\\ u(0)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_1(s)u(s)ds,\\ u(T)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_2(s)u(s)ds,\\ u''(0)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_3(s)u(s)ds,\end{cases} \tag{48}$$

is equivalent to the following integral equation

$$u(t)=\int_0^T\psi'(s)\mathscr{H}(t,s)f(s,u(s))ds, \tag{49}$$

where

$$\mathscr{H}(t,s)=G(t,s)+\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\mathscr{R}(t,\tau)G(\tau,s)d\tau. \tag{50}$$

Proof. Define the nonlinear operator $\mathscr{B}$: $C(I)\longrightarrow C(I)$ as

$$\mathscr{B}u(t)=\int_0^T\psi'(s)G(t,s)f(s,u(s))ds. \tag{51}$$

From (34) and (51), the IBVP (48) is equivalent to the following integral equation:

$$\begin{aligned}u(t)&=\int_0^T\psi'(s)G(t,s)f(s,u(s))ds+\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\Phi(t,s)u(s)ds\\&=\mathscr{B}u(t)+\mathscr{A}u(t).\end{aligned} \tag{52}$$

By Lemma 13, we have the following equation:

$$(I-\mathscr{A})^{-1}\mathscr{B}u(t)=u(t), \tag{53}$$

which implies that

$$\begin{aligned}
u(t) &= \int_0^T \psi'(s)G(t,s)f(s,u(s))ds \\
&\quad + \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\mathscr{R}(t,s)\int_0^T \psi'(s)G(s,\tau)f(\tau,u(\tau))d\tau\, ds \\
&= \int_0^T \psi'(s)G(t,s)f(s,u(s))ds \\
&\quad + \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\left(\int_0^T \mathscr{R}(t,\tau)\psi'(s)G(\tau,s)f(s,u(s))d\tau\right)ds \\
&= \int_0^T \psi'(s)G(t,s)f(s,u(s))ds \\
&\quad + \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\left(\int_0^T \mathscr{R}(t,\tau)\psi'(s)G(\tau,s)f(s,u(s))d\tau\right)ds \\
&= \int_0^T \psi'(s)\left(G(t,s)+\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\mathscr{R}(t,\tau)G(\tau,s)d\tau\right)f(s,u(s))ds \\
&= \int_0^T \psi'(s)\mathscr{H}(t,s)f(s,u(s))ds,
\end{aligned} \tag{54}$$

where $\mathscr{H}(t,s)=G(t,s)+\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\mathscr{R}(t,\tau)G(\tau,s)d\tau$. □

Lemma 15. *Assumption that (A_0) holds, then for any $t,s\in[0,T]$, functions $\mathscr{H}$ and $\mathscr{R}$ satisfy the following properties:*

(1) $\mathscr{H}(t,s)$ is continuous and positive

(2) $n/(1-n)\le \mathscr{R}(t,s)\le N/(1-N)$

(3)

$$\begin{aligned}
&\left(\rho n\psi'(s)/\Gamma(\alpha)(1-n)\right)\left((\psi(T)-\psi(s))^{\alpha+\gamma-2}/\varphi(T)\right)(T-2\varepsilon)\le \mathscr{H}(t,s) \\
&\le \left((\psi(T)-\psi(s))^{\alpha-1}/\Gamma(\alpha)\varphi(T)\right)\left[\varphi(t)+TN/(1-N)(\psi(T)-\psi(s))^{\gamma-1}\right]
\end{aligned}$$

Proof. Let $t,s\in[0,T]$, then

(1) It is clear from expression (11) that $\mathscr{H}(t,s)$ is continuous. In addition, by Remark 10 and expression (9), we have $\mathscr{H}(t,s)\ge 0$.

(2) From expression (9), we have

$$\frac{n}{1-n}\le \mathscr{R}(t,s)\le \frac{N}{1-N}, \tag{55}$$

where $n=((\psi(T)-\psi(s))^{(\gamma-1)}/\gamma)m$ and $N=((\psi(T)-\psi(s))^{(\gamma-1)}/\gamma)M$.

(3) Suppose that (A_0) holds, then from (45) and (50), and Lemma 11 we have the following equation:

$$\begin{aligned}
\mathscr{H}(t,s) &= G(t,s)+\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\mathscr{R}(t,\tau)G(\tau,s)d\tau \\
&\ge \int_\varepsilon^{T-\varepsilon} \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\mathscr{R}(t,\tau)G(\tau,s)d\tau \\
&\ge \frac{\rho n}{\Gamma(\alpha)(1-n)}\int_\varepsilon^{T-\varepsilon}\psi'(s)\frac{(\psi(T)-\psi(s))^{\alpha-1}}{\varphi(T)}(\psi(T)-\psi(s))^{\gamma-1}d\tau \\
&\ge \frac{\rho n\psi'(s)}{\Gamma(\alpha)(1-n)}\frac{(\psi(T)-\psi(s))^{\alpha+\gamma-2}}{\varphi(T)}(T-2\varepsilon).
\end{aligned} \tag{56}$$

Similarly,

$$\begin{aligned}\mathscr{H}(t,s) &= G(t,s) + \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}\mathscr{R}(t,\tau)G(\tau,s)d\tau \\ &\leq \frac{\varphi(t)}{\Gamma(\alpha)\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1} + \frac{N}{1-N}\int_0^T \frac{(\psi(T)-\psi(s))^{\alpha+\gamma-2}}{\Gamma(\alpha)\varphi(T)}d\tau \\ &\leq \frac{\varphi(t)}{\Gamma(\alpha)\varphi(T)}(\psi(T)-\psi(s))^{\alpha-1} + \frac{TN}{1-N}\frac{(\psi(T)-\psi(s))^{\alpha+\gamma-2}}{\Gamma(\alpha)\varphi(T)} \\ &\leq \frac{(\psi(T)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)\varphi(T)}\left[\varphi(t) + \frac{TN}{1-N}(\psi(T)-\psi(s))^{\gamma-1}\right].\end{aligned} \tag{57}$$

Consider the p-Laplacian operator ϕ_p defined as $\phi_p = |s|^{p-2}s$ with $p>1$, and let $q>1$ be a real number satisfying the relation $1/p + 1/p = 1$, and $\phi_p^{-1} = \phi_q$. In the following, we consider the associated linear p-Laplacian fractional differential equations involving integral boundary conditions

$$\begin{cases} {}^cD_{0^+}^{\beta,\psi}\left(\phi_p\left({}^cD_{0^+}^{\alpha,\psi}u(t)\right)\right) + h(t) = 0, \quad t\in I=[0,T], \\ u(0) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_1(s)u(s)ds, \\ u(T) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_2(s)u(s)ds, \\ u''(0) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_3(s)u(s)ds, \\ {}^cD_{0^+}^{\alpha,\psi}u(t)\big|_{t=0} = 0, \end{cases} \tag{58}$$

where $h\in C[0,T]$ and $h\geq 0$. □

Lemma 16. *The associated linear* p*-Laplacian fractional differential equations (58) has the unique solution*

$$u(t) = \frac{1}{\Gamma(\beta)}\int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}h(\tau)d\tau\right)\mathrm{d}s. \tag{59}$$

Proof. Let $w = {}^cD_{0^+}^{\alpha,\psi}u$, and $v = \phi_p(w)$. Then, the solution of the initial value problem

$$\begin{cases} {}^cD_{0^+}^{\beta,\psi}v(t) + h(t) = 0, \quad t\in I=[0,T], \\ v(0) = 0, \end{cases} \tag{60}$$

is given by $v(t) = -{}^cI_{0^+}^{\beta,\psi}h(t) + c_0$ for any $t\in[0,T]$. Since $v(0)=0, 0<\beta\leq 1$, then we obtain $c_0 = 0$, which implies that

$$v(t) = -{}^cI_{0^+}^{\beta,\psi}h(t), \quad t\in[0,T]. \tag{61}$$

But $w=\phi_q(v)$ which implies that the solution of the IBVP (58) satisfies the following IBVP:

$$\begin{cases} {}^cD_{0^+}^{\alpha,\psi}u(t)=\phi_q\left(-{}^cI_{0^+}^{\beta,\psi}h(t)\right), \quad t\in I=[0,T],\\ u(0)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_1(s)u(s)ds,\\ u(T)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_2(s)u(s)ds,\\ u''(0)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_3(s)u(s)ds,\\ {}^cD_{0^+}^{\alpha,\psi}u(t)\big|_{t=0}=0. \end{cases} \tag{62}$$

Since $h(t)\geq 0$, we have $\phi_q(-{}^cI_{0^+}^{\beta,\psi}h(t))=-\phi_q({}^cI_{0^+}^{\beta,\psi}h(t))$.

Hence, by Lemma 11, we deduce that the solution of the IBVP (62) can be written as follows:

$$u(t)=\int_0^T\psi'(s)\mathscr{H}(t,s)\left(\phi_q\left({}^cI_{0^+}^{\beta,\psi}h(s)\right)\right)ds, \tag{63}$$

which implies that

$$u(t)=\frac{1}{\Gamma(\beta)}\int_0^T\psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\int_0^s\psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}h(\tau)d\tau\right)ds. \tag{64}$$

□

Lemma 17. *Suppose that assumption* (A_0) *holds, and* $u,\ {}^cD_{0^+}^{\alpha,\psi}u\in C[0,T]$, *the the following integral boundary value problem:*

$$\begin{cases} {}^cD_{0^+}^{\beta,\psi}\left(\phi_p\left({}^cD_{0^+}^{\alpha,\psi}u\right)\right)(t)+\lambda f(t,u(t))=0, \quad t\in[0,T],\\ u(0)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_1(s)u(s)ds,\\ u(T)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_2(s)u(s)ds,\\ u''(0)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^T\psi'(s)(\psi(T)-\psi(s))^{\gamma-1}g_3(s)u(s)ds,\\ {}^cD_{0^+}^{\alpha,\psi}u(t)\big|_{t=0}=0, \end{cases} \tag{65}$$

is equivalent to the following integral equation

$$u(t)=\int_0^T\psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s\psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}f(\tau,u(\tau))d\tau\right)ds. \tag{66}$$

Definition 18. Let X be a Banach space with norm $\|u\|=\max_{t\in[0,T]}|u(t)|$. Define the reproductory cone $\mathscr{P}_0\subset X$ to be the set

$$\mathscr{P}_0=\{u\in P\colon u(t)\geq\sigma\|u\|,\quad t\in I\}, \tag{67}$$

where $\sigma=\rho n(T-2\varepsilon)(\psi(T)-\psi(s))^{\gamma-1}/(1-n)[\varphi(t)+T N/1-N(\psi(T)-\psi(s))^{\gamma-1}]$ such that $0<\sigma<T$.

Definition 19. Let X be a Banach space, define the operator $\mathscr{C}_\lambda\colon X\longrightarrow X$ as follows:

$$\mathscr{C}_\lambda u(t) = \int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \tag{68}$$

Consider the following assumptions:

(A_1): The nonlinear function $f\colon I\times R^2 \longrightarrow R$ is continuous and there exist $\mu \in C(I,R^+)$ with norm $\|\mu\|$ such that:

$$|f(t,u(t)) - f(t,v(t))| \le \mu(t)|u(t)-v(t)|, \quad \text{for all } t\in I. \tag{69}$$

Lemma 20. *Suppose that assumption (A_0) and (A_1) hold, then $\mathscr{C}_\lambda\colon \mathscr{P}_0 \longrightarrow \mathscr{P}_0$ is completely continuous*

Proof. First, the operator $\mathscr{C}_\lambda$ is continuous.

Consider a sequence $\{u_n\}\subset \mathscr{P}_0$ that converges to $y\in\mathscr{P}_0$, i.e., $u_n \longrightarrow u$ in $\mathscr{P}_0$ as $n\longrightarrow\infty$. Then, by (A_1) we have

$$\begin{aligned}
&\left|\mathscr{C}_\lambda(u_n(t)) - \mathscr{C}_\lambda(u(t))\right| \\
&\le \int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}\left|f(\tau,u_n(\tau)) - f(\tau,u(\tau))\right|d\tau\right)ds \\
&\le \int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}|\mu(t)|\left|u_n(\tau)-u(\tau)\right|d\tau\right)ds \\
&\le \frac{1}{\Gamma(\alpha)\varphi(T)}\left[\varphi(t) + \frac{TN}{1-N}(\psi(T)-\psi(s))^{\gamma-1}\right] \\
&\quad\times \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}|\mu(t)|\left|u_n(\tau)-u(\tau)\right|d\tau\right)ds.
\end{aligned} \tag{70}$$

Taking supremum for all $t\in I$, then applying Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
&\left|\mathscr{C}_\lambda(u_n(t)) - \mathscr{C}_\lambda(u(t))\right| \\
&\le \frac{\lambda\|\mu\|\left\|u_n-u\right\|}{\Gamma(\alpha)\varphi(T)}\left[\varphi(t) + \frac{TN}{1-N}(\psi(T)-\psi(s))^{\gamma-1}\right] \\
&\quad\times \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}d\tau\right)ds.
\end{aligned} \tag{71}$$

Since $u_n \longrightarrow u$, then we obtain $\mathscr{C}_\lambda(u_n(t)) \longrightarrow \mathscr{C}_\lambda(u(t))$ as $n \longrightarrow \infty$ for each $t \in I$, and hence $\|\mathscr{C}_\lambda(u_n(t)) - \mathscr{C}_\lambda(u(t))\| \longrightarrow 0$ as $n \longrightarrow \infty$ and consequently $\mathscr{A}$ is continuous.

In other words, since $H(t,s)$ and $f(s,u(s))$ are nonnegative and continuous, then it is clear that operator $\mathscr{C}_\lambda$ is continuous.

Second, the operator $\mathscr{C}_\lambda$ maps bounded sets in $\mathscr{P}_0$ into bounded sets in $\mathscr{P}_0$.

From Lemma 14, we have

$$\begin{aligned}\mathscr{C}_\lambda u(t) &= \int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} f(\tau,u(\tau))d\tau\right)ds,\\ &\geq \frac{\rho n(T-2\varepsilon)}{\Gamma(\alpha)(1-n)}\frac{(\psi(T)-\psi(s))^{\gamma-1}}{\varphi(T)}\\ &\quad\times\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} f(\tau,u(\tau))d\tau\right)ds,\end{aligned} \tag{72}$$

and

$$\begin{aligned}\mathscr{C}_\lambda u(t) &\leq \frac{1}{\Gamma(\alpha)\varphi(T)}\left[\varphi(t)+\frac{TN}{1-N}(\psi(T)-\psi(s))^{\gamma-1}\right]\\ &\quad\times\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} f(\tau,u(\tau))d\tau\right)ds,\end{aligned} \tag{73}$$

which implies that

$$\begin{aligned}\mathscr{C}_\lambda u(t) &\geq \frac{\rho n(T-2\varepsilon)(\psi(T)-\psi(s))^{\gamma-1}}{(1-n)\left[\varphi(t)+(TN/(1-N))(\psi(T)-\psi(s))^{\gamma-1}\right]}\left\|\mathscr{C}_\lambda u(t)\right\|\\ &\geq \sigma\left\|\mathscr{C}_\lambda u(t)\right\|.\end{aligned} \tag{74}$$

Hence, $\mathscr{C}_\lambda(\mathscr{P}_0) \subset \mathscr{P}_0$.

Third, $\mathscr{C}_\lambda(\Omega)$ is uniformly bounded for every Ω is bounded in $\mathscr{P}_0$.

Let $\Omega \subset \mathscr{P}_0$ be bounded. Then, there exists a real number $\ell > 0$ such that for any $u \in \Omega$, $\|u\| \leq \ell$. Thus,

$$\begin{aligned}
\left|\mathscr{C}_\lambda u(t)\right| &= \left|\int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} f(\tau,u(\tau))d\tau\right)ds\right| \\
&\leq \int_0^T \psi'(s)|\mathscr{H}(t,s)|\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}|f(\tau,u(\tau))|d\tau\right)ds \\
&\leq \frac{\lambda^{q-1}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)}\left[\varphi(t)+\frac{TN}{1-N}(\psi(T)-\psi(s))^{\gamma-1}\right] \\
&\quad\times \int_0^T \psi'(s)\,\phi_q\left(\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}|\mu(\tau)||u(\tau)|d\tau\right)ds.
\end{aligned} \tag{75}$$

Taking supremum for all $t \in I$, we get

$$\begin{aligned}
\left\|\mathscr{C}_\lambda u(t)\right\| &\leq \frac{\lambda^{q-1}\ell\|\mu\|}{\Gamma(\alpha)\Gamma(\beta)\varphi(T)}\left[\varphi(t)+\frac{TN}{1-N}(\psi(T)-\psi(s))^{\gamma-1}\right]\int_0^T \psi'(s)\,\phi_q\left(\frac{(\psi(s)-\psi(0))^\beta}{\beta}\right)ds \\
&\leq \frac{\lambda^{q-1}\ell\|\mu\|}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)}\left[\varphi(t)+\frac{TN}{1-N}(\psi(T)-\psi(s))^{\gamma-1}\right]\int_0^T \psi'(s)\left(\frac{(\psi(s)-\psi(0))^\beta}{\beta}\right)^{q-1}ds \\
&\leq \frac{\lambda^{q-1}\ell\|\mu\|}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)}\left[\varphi(t)+\frac{TN}{1-N}(\psi(T)-\psi(s))^{\gamma-1}\right]\frac{1}{q}\left(\frac{(\psi(T)-\psi(0))^\beta}{\beta}\right)^q \\
&\leq \frac{\lambda^{q-1}\ell\|\mu\|}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)}\left[\varphi(t)+\frac{TN}{1-N}(\psi(T)-\psi(0))^{\gamma-1}\right]\frac{(\psi(T)-\psi(0))^{q\beta}}{q\beta^q} < +\infty.
\end{aligned} \tag{76}$$

Finally, $\mathscr{C}_\lambda$ is equi-continuous.

Since $H(t,s)$ is continuous and hence uniformly continuous on $[0,1]\times[0,1]$, then for every $\varepsilon>0$, there exist $\delta>0$ and $t_1,t_2\in I$, with $t_1<t_2$ and $|t_2-t_1|<\delta$ such that $|\mathscr{H}(t_2,s)-\mathscr{H}(t_1,s)|<\varepsilon$. Then, for all $u\in\mathscr{P}_0$ we have

$$\begin{aligned}
&\left|\mathscr{C}_\lambda u(t_2)-\mathscr{C}_\lambda u(t_1)\right|\\
&\le\left|\int_0^T\psi'(s)\left(\mathscr{H}(t_2,s)-\mathscr{H}(t_1,s)\right)\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s\psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}f(\tau,u(\tau))d\tau\right)ds\right|\\
&\le\int_0^T\psi'(s)\left|\mathscr{H}(t_2,s)-\mathscr{H}(t_1,s)\right|\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s\psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}|f(\tau,u(\tau))|d\tau\right)ds\\
&\le\frac{\lambda^{q-1}\varepsilon}{\Gamma(\beta)^{q-1}}\int_0^T\psi'(s)\,\phi_q\left(\int_0^s\psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}|\mu(\tau)|\,|u(\tau)|d\tau\right)ds\\
&\le\frac{\lambda^{q-1}\,\|\mu\|\,\|u\|\,\varepsilon}{\Gamma(\beta)^{q-1}}\int_0^T\psi'(s)\,\phi_q\left(\int_0^s\psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}d\tau\right)ds\\
&\le\frac{\lambda^{q-1}\,\|\mu\|\,\|u\|\,\varepsilon}{\Gamma(\beta)^{q-1}}\int_0^T\psi'(s)\,\phi_q\left(\frac{(\psi(s)-\psi(0))^{\beta}}{\beta}\right)ds\\
&\le\left(\frac{\lambda^{q-1}\,\|\mu\|\,\|u\|}{\Gamma(\beta)^{q-1}}\frac{(\psi(T)-\psi(0))^{q\beta}}{q\beta^q}\right)\varepsilon.
\end{aligned}\tag{77}$$

As $t_1\longrightarrow t_2$, the right-hand side of the above inequality is not dependent on u and tends to zero. Consequently,

$$\left|\mathscr{A}(y(t_2))-\mathscr{A}(y(t_1))\right|\longrightarrow 0,\forall\left|t_2-t_1\right|\longrightarrow 0.\tag{78}$$

This show that $\{\mathscr{C}_\lambda u\}$ is equi-continuous on $\mathscr{P}_0$, and $\mathscr{C}_\lambda$ is a compact and completely continuous operator by the Arzela–Ascoli Theorem 6. □

2.3. Existence of Positive Solutions. In this subsection, we study the existence of positive solutions for the IBVP (5) by applying the fixed point Theorem 7 on a cone. The required and sufficient conditions for having at least one positive solution are presented.

Notation 21. Consider the following notations:

(1) $f^0=\lim_{u\longrightarrow 0^+}\sup\sup_{t\in[0,T]}f(t,u)/\phi_p(u)$

(2) $f^\infty=\lim_{u\longrightarrow +\infty}\sup\sup_{t\in[0,T]}f(t,u)/\phi_p(u)$

(3) $A_1=(\|\mu\|/\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T))[\varphi(t)+T\,N/(1-N)\,(\psi(T)-\psi(0))^{\gamma-1}](\psi(T)-\psi(0))^{q\beta}/q\beta^{q\,q\beta}$

(4)

$$A_2=\left(\rho\,n(T-2\varepsilon)/\left(\left(\Gamma(\alpha)\Gamma(\beta)^{(q-1)}(1-n)\right)\right)(\psi(T)-\psi(0))^{(\gamma-1)}\right)/(\varphi(T))\;(1/\beta(\beta+1)(\beta+2))\begin{bmatrix}(\psi(T)-\psi(0))^{\beta+1}((\beta+1)(\psi(T)-\psi(0))+\psi(0))\\-(\beta+2)(\psi(T)-\psi(0))\end{bmatrix}$$

Theorem 22. *Assume that* (A_0) *holds. If we have* $\zeta,\aleph>0$ *such that* $f^0<\zeta$ *and* $f^\infty>\aleph$, *then for each real number* λ *satisfying*

$$\frac{1}{A_2^{p-1}\aleph}\le\lambda\le\frac{1}{A_1^{p-1}\zeta}.\tag{79}$$

The IBVP (5) has at least one positive solution.

Proof. By Notation 21, there exist two positive real numbers $\tilde{\rho}$, ρ_1, and ρ_2 such that $\rho_2>(1/\sigma)\tilde{\rho}$ and $\rho_2=\max\{2\rho_1,(1/\sigma)\tilde{\rho}\}$. Then, for any $t\in[0,T]$, we have

$$f(t,u) \leq \zeta \phi_p(u), \quad \text{for } u \in (0, \rho_1], \tag{80}$$

and

$$f(t,u) \geq \aleph \phi_p(u), \quad \text{for } u \in [\rho_2, +\infty). \tag{81}$$

Define

$$\begin{aligned} \Omega_1 &= \{u \in X\colon \|u\| \leq \rho_1\}, \\ \Omega_2 &= \{u \in X\colon \|u\| \leq \rho_2\}. \end{aligned} \tag{82}$$

If $u \in \mathscr{P}_0$ with $\|u\| = \rho_1$, then from (79) and (80), we have

$$\begin{aligned} \left\|\mathscr{C}_\lambda u(t)\right\| &\leq \sup_{t\in[0,T]} \int_0^T \psi'(s) |\mathscr{H}(t,s)|\, \phi_q\left(\frac{\lambda}{\Gamma(\beta)} \int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} |f(\tau, u(\tau))| d\tau\right) ds \\ &\leq \frac{\lambda^{q-1}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)} \left[\varphi(t) + \frac{TN}{1-N}(\psi(T)-\psi(s))^{\gamma-1}\right] \\ &\quad \times \int_0^T \psi'(s)\, \phi_q\left(\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} \zeta \phi_p(\rho_1) d\tau\right) ds \\ &\leq \frac{\lambda^{q-1}\zeta^{q-1}\|\mu\|}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)} \left[\varphi(t) + \frac{TN}{1-N}(\psi(T)-\psi(0))^{\gamma-1}\right] \frac{(\psi(T)-\psi(0))^{q\beta}}{q\beta^q} \rho_1 \\ &\leq \lambda^{q-1}\zeta^{q-1} A_1 \rho_1, \end{aligned} \tag{83}$$

where

$$A_1 = \frac{\|\mu\|}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)} \left[\varphi(t) + \frac{TN}{1-N}(\psi(T)-\psi(0))^{\gamma-1}\right] \frac{(\psi(T)-\psi(0))^{q\beta}}{q\beta^q}. \tag{84}$$

Hence,

$$\left\|\mathscr{C}_\lambda u\right\| \leq \|u\|, \quad \text{for any } u \in \mathscr{P}_0 \cap \partial\Omega_1. \tag{85}$$

Now, for any $u \in \mathscr{P}_0 \cap \partial\Omega_2$, we have $u \in \mathscr{P}_0$ and $\|u\| = \rho_2$. Then,

$$\begin{aligned} u(t) &\geq \frac{\rho n (T-2\varepsilon)(\psi(T)-\psi(s))^{\gamma-1}}{(1-n)\left[\varphi(t) + (TN/(1-N))(\psi(T)-\psi(s))^{\gamma-1}\right]} \|u(t)\| \\ &\geq \sigma\|u\| \\ &\geq \sigma\rho_2 > \widetilde{\rho}. \end{aligned} \tag{86}$$

Then, by (79) and (81), we have

$$\begin{aligned}\left\|\mathscr{C}_\lambda u(t)\right\| &\geq \sup_{t\in[0,T]}\int_0^T \psi'(s)|\mathscr{H}(t,s)|\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}|f(\tau,u(\tau))|d\tau\right)ds\\
&\geq \frac{\rho n(T-2\varepsilon)}{\Gamma(\alpha)(1-n)}\frac{(\psi(T)-\psi(s))^{\gamma-1}}{\varphi(T)}\\
&\quad\times\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}\aleph\phi_p(\rho_2)d\tau\right)ds\\
&\geq \frac{\rho n(T-2\varepsilon)}{\Gamma(\alpha)(1-n)}\frac{(\psi(T)-\psi(s))^{\gamma-1}}{\varphi(T)}\left(\frac{\lambda\aleph}{\Gamma(\beta)}\right)^{q-1}\\
&\quad\times\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}\phi_q\left(\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}\phi_p(\rho_2)d\tau\right)ds\\
&\geq (\lambda\aleph)^{q-1}\frac{\rho n(T-2\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)^{q-1}(1-n)}\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\varphi(T)}\frac{1}{(\beta+2)(\beta+1)\beta}\\
&\quad\times\left[(\psi(T)-\psi(0))^{\beta+1}((\beta+1)(\psi(T)-\psi(0))+\psi(0))-(\beta+2)(\psi(T)-\psi(0))\right]\rho_2\\
&\geq \lambda^{q-1}\aleph^{q-1}A_2\rho_2\\
&\geq \rho_2=\|u\|,\end{aligned}\tag{87}$$

where

$$\begin{aligned}A_2 &= \frac{\rho n(T-2\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)^{q-1}(1-n)}\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\varphi(T)}\\
&\quad\times\frac{1}{(\beta+2)(\beta+1)\beta}\left[\begin{array}{c}(\psi(T)-\psi(0))^{\beta+1}((\beta+1)(\psi(T)-\psi(0))+\psi(0))\\ -(\beta+2)(\psi(T)-\psi(0))\end{array}\right].\end{aligned}\tag{88}$$

This implies that

$$\left\|\mathscr{C}_\lambda u\right\|\geq\|u\|,\quad \text{for any } u\in\mathscr{P}_0\cap\partial\Omega_2.\tag{89}$$

Finally, by 20, 21, and Theorem 7, we conclude that By (81) and (89) and Theorem 22, we conclude $\mathscr{C}_\lambda$ has a fixed point $u\in\mathscr{P}_0\cap(\overline{\Omega}_2\backslash\Omega_1)$ with $\rho_1\leq\|u\|\leq\rho_2$, and obviously the IBVP (5) has at least one positive solution u. □

Now, if we use other conditions for f^0 and f^∞, we obtain the following theorem.

Theorem 23. *Assume that* (A_0) *holds. If we have* $\zeta,\aleph>0$ *such that* $f^0>\aleph$, *and* $f^\infty<\zeta$, *then for each real number* λ *satisfying*

$$\frac{1}{A_2^{p-1}\aleph}\leq\lambda\leq\frac{1}{A_1^{p-1}\zeta}.\tag{90}$$

The IBVP (5) has at least one positive solution.

Proof. Suppose that λ satisfies (90), then there exists $\rho_1>0$ for any $t\in[0,T]$ so that

$$f(t,u)\geq\aleph\phi_p(u),\quad \text{for } u\in(0,\rho_2].\tag{91}$$

Hence, if $u\in\mathscr{P}_0$ with norm $\|u\|=\rho_2$, then we can choose Ω_1 and Ω_2 as carried out in the previous theorem such that

$$\left\|\mathscr{C}_\lambda u\right\|\geq\|u\|,\quad \text{for all } u\in\mathscr{P}_0\cap\partial\Omega_1.\tag{92}$$

In addition, for any $t\in[0,T]$, there exists $\rho_2>0$ so that

$$f(t,u)\geq\zeta\phi_p(u),\quad \text{for } u\in[\rho_2,+\infty).\tag{93}$$

Hence, if $u\in\mathscr{P}_0$ with norm $\|u\|=\rho_2$, then we have

$$\left\|\mathscr{C}_\lambda u\right\| \le \|u\|, \quad \text{for all } u \in \mathscr{P}_0 \cap \partial\Omega_2. \tag{94}$$

By 20, 21, and Theorem 7, we deduce that $\mathscr{C}_\lambda$ has a fixed point $u \in \mathscr{P}_0 \cap (\overline{\Omega}_2/\Omega_1)$ with $\rho_1 \le \|u\| \le \rho_2$, and obviously the IBVP (5) has at least one positive solution u. □

2.4. Nonexistence of Positive Solutions. In this part, we present the conditions for the nonexistence of positive solutions for the IBVP (5).

Theorem 24. *If $f^0 < +\infty$ and $f^\infty < +\infty$, then for any $t \in [0,T]$ there exist positive constants K_1, K_2, ρ_1, and ρ_2 such that $\rho_1 < \rho_2$ and*

$$f(t,u) \le K_1 \phi_p(u), \quad \text{for } u \in [0,\rho_1], \tag{95}$$

and

$$f(t,u) \ge K_2 \phi_p(u), \quad \text{for } u \in [\rho_2, +\infty). \tag{96}$$

Denote by $\overline{K} = \max\{K_1, K_2, \max_{\rho_1 \le u \le \rho_2, 0 \le t \le T} f(t, u)/\phi_p(u)\}$, then

$$f(t,u) \le \overline{K}\phi_p(u), \quad \text{for any } t \in [0,T], u \in [0,+\infty]. \tag{97}$$

Assume conversely that $v(t)$ is a positive solution of the IBVP (5). Then, since $C_\lambda v(t) = v(t)$ for any $t \in [0,T]$, we have

$$\begin{aligned}
\|v\| = \left\|C_\lambda v\right\| &\le \frac{\lambda^{q-1}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)}\left[\varphi(t) + \frac{TN}{1-N}(\psi(T)-\psi(0))^{\gamma-1}\right] \\
&\quad \times \int_0^T \psi'(s)\,\phi_q\left(\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}|f(\tau,u(\tau))|d\tau\right)ds \\
&\le \frac{\lambda^{q-1}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)}\left[\varphi(t) + \frac{TN}{1-N}(\psi(T)-\psi(0))^{\gamma-1}\right] \\
&\quad \times \int_0^T \psi'(s)\,\phi_q\left(\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}\overline{M}\phi_p(v(\tau))d\tau\right)ds \\
&\le (\lambda\overline{K})^{q-1}\|v\|\,A_1 \\
&< \|v\|,
\end{aligned} \tag{98}$$

which is a contradiction when $0 < \lambda < \overline{K}^{-1}\,(A_1^{-1})^{p-1}$. Hence, the IBVP (5) has no positive solutions in this case.

2.5. Hyers–Ulam Stability of Positive Solutions. In the following, we investigate the Hyers–Ullam stability of our proposed IBVP (5). We give a similar definition of the Hyers–Ullam stability as that given in [61].

Definition 25. The integral equation (66) is Hyers–Ulam stable if there exists a positive constant δ such that for every $\epsilon > 0$, if

$$\left|u(t) - \int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} f(\tau,u(\tau))d\tau\right)ds\right| \le \epsilon, \tag{99}$$

then, there exists

$$u^*(t) = \int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} f(\tau,u^*(\tau))d\tau\right)ds, \tag{100}$$

where

$$\left|u(t)-u^{*}(t)\right|\leq\delta\epsilon. \tag{101}$$

Theorem 26. *With the assumption* (A_0)*, the IBVP (5) is Hyers–Ulam stable.*

Proof. Suppose that $u(t)$ is a positive solution of (66) and $u^{*}(t)$ is an approximation satisfying (100). Then, by Lemmas 3 and 8, we obtain

$$\begin{aligned}
\left|u(t)-u^{*}(t)\right| &= \left|\int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} f(\tau,u(\tau))d\tau\right)ds\right.\\
&\quad\left.-\int_0^T \psi'(s)\mathscr{H}(t,s)\,\phi_q\left(\frac{\lambda}{\Gamma(\beta)}\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1} f(\tau,u^{*}(\tau))d\tau\right)ds\right|\\
&\leq \frac{\lambda^{q-1}(p-1)\varrho^{p-2}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)}\left[\varphi(t)+\frac{TN}{1-N}(\psi(T)-\psi(0))^{\gamma-1}\right]\\
&\quad\times\int_0^T \psi'(s)\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}\left|f(\tau,u(\tau))-f(\tau,u^{*}(\tau))\right|d\tau\,ds\\
&\leq \frac{\lambda^{q-1}(p-1)\varrho^{p-2}}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)}\left[\varphi(t)+\frac{TN}{1-N}(\psi(T)-\psi(0))^{\gamma-1}\right]\\
&\quad\times\int_0^T \psi'(s)\int_0^s \psi'(\tau)(\psi(s)-\psi(\tau))^{\beta-1}|\mu(t)|\left|u_n(\tau)-u(\tau)\right|d\tau\,ds\\
&\leq \frac{\lambda^{q-1}(p-1)\varrho^{p-2}\|\mu\|}{\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)}\left[\varphi(t)+\frac{TN}{1-N}(\psi(T)-\psi(0))^{\gamma-1}\right]\frac{(\psi(T)-\psi(0))^{\beta+1}}{\beta^2+\beta}\left\|u_n-u\right\|.
\end{aligned} \tag{102}$$

Hence, if $\delta=\lambda^{q-1}(p-1)\varrho^{p-2}\|\mu\|/\Gamma(\alpha)\Gamma(\beta)^{q-1}\varphi(T)[\varphi(t)+TN/1-N(\psi(T)-\psi(0))^{\gamma-1}](\psi(T)-\psi(0))^{\beta+1}/(\beta^2+\beta)$, then we deduce that the integral (66) is Hyers–Ulam stable. Consequently, the IBVP (5) is Hyers–Ulam stable. □

3. Numerical Examples

To demonstrate the practical implications of the key findings derived above, we provide the following numerical examples.

Example 1. Consider the following p-Laplacian ψ-Caputo fractional differential equations involving integral boundary conditions.

$$\begin{cases}
{}^{c}D_{0^+}^{1/3,t}\left(\phi_p\left({}^{c}D_{0^+}^{7/3,t}u\right)\right)(t)+\lambda\left(t^3+2\right)\left(65u-\dfrac{8449}{130}\sin u\right)=0, \quad t\in[0,1],\\[2ex]
u(0)=\displaystyle\int_0^1\frac{199}{205}u(s)ds,\, u(T)=\int_0^1\frac{199}{205}u(s)ds,\, u''(0)=\int_0^1\frac{3}{205}u(s)ds,\\[2ex]
{}^{c}D_{0^+}^{7/3,\psi}u(t)\Big|_{t=0}=0,
\end{cases} \tag{103}$$

where $\psi(t)=t$, $T=1$, $\gamma=1$, $f(t,u(t))=(t^3+2)(65u-8449/130\sin u)$, $\alpha=7/3$, $\beta=1/3$, $p=2$, $\varepsilon=1/6$, $g_1(s)=g_2(s)=199/205$, $g_3(s)=3/205$, $\varphi(t)=2t$, and $\chi(t)=t(t-1)$. This implies that

$$\Phi(t,s) = (1-t)g_1(s) + tg_2(s) + \frac{t(t-1)}{2}g_3(s) = \frac{199}{205}(1-t) + \frac{199}{205}t + \frac{3}{410}\left(-t+t^2\right). \tag{104}$$

Then, through straightforward computations, we can determine that $m = 1589/1640$, $M = 199/205$, $\rho = 5(6-\sqrt[3]{180})/36$, $\sigma = 95(6-\sqrt[3]{180})/1806$, $A_1^{p-1} = 205/6\Gamma(11/3) = 8.51569$, $A_2^{p-1} = 12624605(6-\sqrt[3]{180})^2/51827526\Gamma(11/3) = 0.0075989$, $f^0 = 3/130 < 0.03 = \zeta$, $f_\infty = 130 > \aleph = 129$, $f_0 = 1/65$, and $f^\infty = 195$. By using Theorem 23, it is clear that the inequality $1/A_2^{p-1}\aleph < \lambda < 1/A_1^{p-1}\zeta$ implies that the IBVP (103) has at least one positive solution for each $\lambda \in (1.02014, 3.91434)$. Moreover, since $3/130u < f(t,u) < 195u$, which implies by using Theorem 24 that the IBVP (103) has no positive solution for each $\lambda \in (0, 0.000602207)$. Finally, since assumption (A_0) holds for $m = 1589/1640$ and $M = 199/205$, i.e., $0 < m \leq |\Phi(t,s)| \leq M < 1$ then the IBVP (103) is Hyers–Ulam stable.

Example 2. Consider the following p-Laplacian ψ-Caputo fractional differential equations involving integral boundary conditions.

$$\begin{cases} {}^cD_{0^+}^{1/2,t^2+1}\left(\phi_p\left({}^cD_{0^+}^{5/2,t^2+1}u\right)\right)(t) + \lambda\left(t^2+1\right)\left(65u - \frac{8449}{130}\sin u\right) = 0, \quad t \in [0,1], \\ u(0) = \frac{1}{\Gamma(1/2)}\int_0^1 \frac{200}{201}2s\left(1-s^2\right)u(s)ds, u(T) = \frac{1}{\Gamma(1/2)}\int_0^1 \frac{200}{201}2s\left(1-s^2\right)u(s)ds, \\ u''(0) = \frac{1}{\Gamma(1/2)}\int_0^1 \frac{4}{201}2s\left(1-s^2\right)u(s)ds, {}^cD_{0^+}^{5/2,t^2+1}u(t)\Big|_{t=0} = 0, \end{cases} \tag{105}$$

where $\psi(t) = t^2+1$, $T = 1$, $\gamma = 1/2$, $f(t,u(t)) = (t^3+2)(65u - 8449/130\sin u)$, $\alpha = 5/2$, $\beta = 1/2$, $p = 2$, $\varepsilon = 1/4$, $g_1(s) = g_2(s) = 200/201$, $g_3(s) = 4/201$,

$$\begin{aligned} \varphi(t) &= (\psi(t)-\psi(0))\left(2\left(\psi'(0)\right)^2 - (\psi(t)-\psi(0))\psi''(0)\right) = 2t^4, \\ \chi(t) &= (\psi(1)-\psi(0))(\psi(t)-\psi(0))(\psi(t)-\psi(1)) = t^2\left(t^2-1\right). \end{aligned} \tag{106}$$

This implies that

$$\begin{aligned} \Phi(t,s) &= \frac{1}{\Gamma(\gamma)}\left[\left(1-\frac{\varphi(t)}{\varphi(T)}\right)g_1(s) + \frac{\varphi(t)}{\varphi(T)}g_2(s) + \frac{\chi(t)}{\varphi(T)}g_3(s)\right] \\ &= \frac{1}{\Gamma(1/2)}\left[\left(1-t^4\right)\frac{200}{201} + t^4\frac{200}{201} + \frac{t^2\left(t^2-1\right)}{2}\frac{200}{201}\right] \\ &= \frac{1}{\sqrt{\pi}}\left(\frac{200}{201}t^4 + \frac{100}{201}t^2\left(t^2-1\right) + \frac{200}{201}\left(1-t^4\right)\right). \end{aligned} \tag{107}$$

Then, through straightforward computations, we can determine that $m = 175/(201\sqrt{\pi}) = 0.49121$, $M = 200/(201\sqrt{\pi}) = 0.561383$, $\rho = (3/8)(\sqrt{3}-2) = 0.100481$, $\sigma = 175(3/4 - 3\sqrt{3}/8)(1-200/(201\sqrt{\pi})/402(1-175/(201\sqrt{\pi}))\ \sqrt{\pi} = 0.0212749$, $A_1^{p-1} = 1/(6(1-200/(201\sqrt{\pi}))) = 0.379982$, $A_2^{p-1} = 30625(3/4 - 3\sqrt{3}/8)^2\ (1-200/(201\sqrt{\pi}))/969624(1-175/(201\sqrt{\pi}))^2\ \pi = 0.000171988$, $f^0 = 1/65 < 1/64 = \zeta$, $f_\infty = 65 > \aleph = 64$, $f_0 = (1/130)$, and $f^\infty = 130$. By using Theorem 23, it is clear that the inequality $1/(A_2^{(p-1)}\aleph) < \lambda < 1/(A_1^{(p-1)}\zeta)$ implies that the IBVP (105) has at least one positive solution for each $\lambda \in (90.8492, 168.429)$. Moreover, since $(1/130)u < f(t,u) < 130u$, which implies by using Theorem 24 that the IBVP (105) has no positive solution for each $\lambda \in (0, (3/98)(1-200/(201\sqrt{\pi})) = (0, 0.0134271)$. Finally, since assumption (A_0) holds for $m = 175/(201\sqrt{\pi})$ and $M = 200/(201\sqrt{\pi})$, i.e., $0 < m \leq |\Phi(t,s)| \leq M < 1$ then the IBVP (105) is Hyers–Ulam stable.

4. Conclusion

In conclusion, our study focuses on investigating the stability and existence of positive solutions for a specific type of p-Laplacian ψ-Caputo fractional differential equations with fractional integral boundary conditions. Through the use of Green's function properties and the application of Guo–Krasnovelsky's fixed point theorem on cones, we have established novel existence results, ensuring the presence of at least one positive solution. Our analysis encompasses a range of parameter values, providing a comprehensive understanding of the problem. We have laid a solid foundation for our investigation by utilizing fractional integrals, differential operators, and fundamental lemmas. Our study contributes to the existing knowledge in this field by exploring both the existence and nonexistence of positive solutions. We have also examined the Hyers–Ulam stability of solutions by introducing a defined notion and building upon previous research. To validate the effectiveness and accuracy of our approach, we have presented numerical examples. Overall, our study significantly advances the understanding of stability and existence of positive solutions for the considered class of fractional differential equations. Future research can focus on conducting more extensive numerical studies to gain further insights into the system's behavior and properties under different parameter regimes. In addition, exploring different choices of $\psi(t)$ and other parameter values can provide a deeper understanding of the existence and stability of positive solutions. In summary, our study makes substantial contributions to the understanding of eigenvalue problems with the ψ-Caputo integro-differential operator, and its insights can be extended to other investigations in this field. By continuing to conduct further numerical studies and explore different parameter choices, researchers can advance our knowledge and uncover new aspects of this intriguing problem.

Data Availability

The required data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory And Applications Of Fractional Differential Equations*, Elsevier, Amsterdam, Netherlands, 2006.

[2] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals And Derivatives*, Gordon and breach science publishers, Philadelphia, PA, USA, 1993.

[3] A. C. McBride, "V. Kiryakova Generalized fractional calculus and applications," *Proceedings of the Edinburgh Mathematical Society*, vol. 38, no. 1, pp. 189-190, 1995.

[4] T. J. Osler, "The fractional derivative of a composite function," *SIAM Journal on Mathematical Analysis*, vol. 1, no. 2, pp. 288–293, 1970.

[5] D. Baleanu, A. Jajarmi, H. Mohammadi, and S. Rezapour, "A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative," *Chaos, Solitons & Fractals*, vol. 134, Article ID 109705, 2020.

[6] H. Mohammadi, D. Baleanu, S. Etemad, and S. Rezapour, "Criteria for existence of solutions for a Liouville–Caputo boundary value problem via generalized Gronwall's inequality," *Journal of Inequalities and Applications*, vol. 2021, pp. 36–19, 2021.

[7] H. Mohammadi, S. Kumar, S. Rezapour, and S. Etemad, "A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control," *Chaos, Solitons & Fractals*, vol. 144, Article ID 110668, 2021.

[8] S. Rezapour, S. Etemad, and H. Mohammadi, "A mathematical analysis of a system of Caputo–Fabrizio fractional differential equations for the anthrax disease model in animals," *Advances in Difference Equations*, vol. 2020, no. 1, p. 481, 2020.

[9] S. M. Aydogan, D. Baleanu, H. Mohammadi, and S. Rezapour, "On the mathematical model of Rabies by using the fractional Caputo–Fabrizio derivative," *Advances in Difference Equations*, vol. 2020, no. 1, p. 382, 2020.

[10] D. Baleanu, A. H. Bhrawy, and T. M. Taha, "A modified generalized Laguerre spectral method for fractional differential equations on the half line," *Abstract and Applied Analysis*, vol. 2013, Article ID 413529, 12 pages, 2013.

[11] D. Baleanu, H. Mohammadi, and S. Rezapour, "Analysis of the model of HIV-1 infection of $CD4^+$ T-cell with a new approach of fractional derivative," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–17, 2020.

[12] D. Baleanu, S. Etemad, H. Mohammadi, and S. Rezapour, "A novel modeling of boundary value problems on the glucose graph," *Communications in Nonlinear Science and Numerical Simulation*, vol. 100, Article ID 105844, 2021.

[13] H. Mohammadi, S. Rezapour, and A. Jajarmi, "On the fractional SIRD mathematical model and control for the transmission of COVID-19: the first and the second waves of the disease in Iran and Japan," *ISA Transactions*, vol. 124, pp. 103–114, 2022.

[14] R. George, K. Mohammadi, H. Mohammadi, R. Ghorbanian, S. Rezpour, and A. Duc, *The Study of Cholera Transmission Using an SIRZ Fractional Order Mathematical Model*, World Scientific, Singapore, 2023.

[15] R. P. Agarwal, M. Meehan, and D. O'regan, *Fixed Point Theory And Applications*, Cambridge University Press, Cambridge, UK, 2001.

[16] O. P. Agrawal, "Some generalized fractional calculus operators and their applications in integral equations," *Fractional Calculus and Applied Analysis*, vol. 15, no. 4, pp. 700–711, 2012.

[17] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[18] G. Chai, "Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator," *Boundary Value Problems*, vol. 2012, no. 1, pp. 18–20, 2012.

[19] S. M. Ege and F. S. Topal, "Existence of positive solutions for fractional boundary value problems," *Journal of Applied Analysis & Computation*, vol. 7, no. 2, pp. 702–712, 2017.

[20] Z. Han, H. He, F. Zhang et al., "Spatiotemporal expression pattern of Mirg, an imprinted non-coding gene, during mouse embryogenesis," *Journal of Molecular Histology*, vol. 43, pp. 1–8, 2012.

[21] Z. Han, H. Lu, and C. Zhang, "Positive solutions for eigenvalue problems of fractional differential equation with generalized p-Laplacian," *Applied Mathematics and Computation*, vol. 257, pp. 526–536, 2015.

[22] Y. Jiang, H. Sun, Y. Bai, and Y. Zhang, "MHD flow, radiation heat and mass transfer of fractional Burgers' fluid in porous medium with chemical reaction," *Computers & Mathematics with Applications*, vol. 115, pp. 68–79, 2022.

[23] M. Jia and X. Liu, "Three nonnegative solutions for fractional differential equations with integral boundary conditions," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1405–1412, 2011.

[24] M. Jia and X. Liu, "The existence of positive solutions for fractional differential equations with integral and disturbance parameter in boundary conditions," *Abstract and Applied Analysis*, vol. 2014, Article ID 131548, 14 pages, 2014.

[25] M. Jia and X. Liu, "Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions," *Applied Mathematics and Computation*, vol. 232, pp. 313–323, 2014.

[26] J. Jin, X. Liu, and M. Jia, "Existence of positive solutions for singular fractional differential equations with integral boundary conditions," *The Electronic Journal of Differential Equations*, vol. 2012, pp. 1457–1470, 2012.

[27] M. A. Krasnoselskii and L. F. Boron, *Positive Solutions Of Operator Equations*, Noordhoff, Groningen, Netherlands, 1964.

[28] X. Liu, J. Jin, and M. Jia, "Multipie positive solutions of integral boundary value problems for fractional differential equations," *Applied Mathematics & Information Sciences*, vol. 30, pp. 305–320, 2010.

[29] Y. Sun and M. Zhao, "Positive solutions for a class of fractional differential equations with integral boundary conditions," *Applied Mathematics Letters*, vol. 34, pp. 17–21, 2014.

[30] X. Xu, D. Jiang, and C. Yuan, "Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4676–4688, 2009.

[31] Y. Zhao, S. Sun, Z. Han, and M. Zhang, "Positive solutions for boundary value problems of nonlinear fractional differential equations," *Applied Mathematics and Computation*, vol. 217, no. 16, pp. 6950–6958, 2011.

[32] M. Ahmad, A. Zada, and J. Alzabut, "Stability analysis of a nonlinear coupled implicit switched singular fractional differential system with p-Laplacian," *Advances in Difference Equations*, vol. 2019, pp. 436–522, 2019.

[33] Z. Bai, "Eigenvalue intervals for a class of fractional boundary value problem," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3253–3257, 2012.

[34] D. Nie, U. Riaz, S. Begum, and A. Zada, "A coupled system of p-Laplacian implicit fractional differential equations depending on boundary conditions of integral type," *AIMS Mathematics*, vol. 8, no. 7, pp. 16417–16445, 2023.

[35] T. Shen, W. Liu, and X. Shen, "Existence and uniqueness of solutions for several BVPs of fractional differential equations with p-Laplacian operator," *Mediterranean Journal of Mathematics*, vol. 13, no. 6, pp. 4623–4637, 2016.

[36] X. Su, M. Jia, and X. Fu, "On positive solutions of eigenvalue problems for a class of p-Laplacian fractional differential equations," *J. Appl. Anal. Comput.*, vol. 8, no. 1, pp. 152–171, 2018.

[37] H. Waheed, A. Zada, R. Rizwan, and I. L. Popa, "Hyers–ulam stability for a coupled system of fractional differential equation with p-laplacian operator having integral boundary conditions," *Qualitative Theory of Dynamical Systems*, vol. 21, no. 3, p. 92, 2022.

[38] J. Wang, H. Xiang, and Z. Liu, "Positive solutions for three-point boundary value problems of nonlinear fractional differential equations with p-Laplacian operator," *Far East Journal of Applied Mathematics*, vol. 37, pp. 33–47, 2009.

[39] C. Yang and J. Yan, "Positive solutions for third-order Sturm–Liouville boundary value problems with p-Laplacian," *Computers & Mathematics with Applications*, vol. 59, no. 6, pp. 2059–2066, 2010.

[40] X. Zhang, L. Liu, B. Wiwatanapataphee, and Y. Wu, "The eigenvalue for a class of singular p-Laplacian fractional differential equations involving the Riemann–Stieltjes integral boundary condition," *Applied Mathematics and Computation*, vol. 235, pp. 412–422, 2014.

[41] X. Xu and C. Xu, "Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation," *Communications in Computational Physics*, vol. 8, no. 5, pp. 1016–1051, 2010.

[42] H. Wang, "On the number of positive solutions of nonlinear systems," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 1, pp. 287–306, 2003.

[43] J. Wang, H. Xiang, and Z. Liu, "Upper and lower solutions method for a class of singular fractional boundary value problems with p-Laplacian operator," *Abstract and Applied Analysis*, vol. 2010, Article ID 971824, 12 pages, 2010.

[44] J. Wang, "Eigenvalue intervals for multi-point boundary value problems of fractional differential equation," *Applied Mathematics and Computation*, vol. 219, pp. 4570–4575, 2013.

[45] M. I. Abbas, "On the coupled system of ψ-caputo fractional differential equations with four-point boundary conditions," *Applied Mathematics E-Notes*, vol. 21, pp. 563–576, 2021.

[46] M. Ahmad, A. Zada, and X. Wang, "Existence, uniqueness and stability of implicit switched coupled fractional differential equations of ψ-hilfer type," *International Journal of Nonlinear Sciences and Numerical Stimulation*, vol. 21, no. 3-4, pp. 327–337, 2020.

[47] R. Almeida, "A Caputo fractional derivative of a function with respect to another function," *Communications in Nonlinear Science and Numerical Simulation*, vol. 44, pp. 460–481, 2017.

[48] R. Almeida, A. B. Malinowska, and M. T. T. Monteiro, "Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 1, pp. 336–352, 2018.

[49] Y. Awad and I. Kaddoura, "On the Ulam-Hyers-Rassias stability for a boundary value problem of implicit Ψ-Caputo fractional integro-differential equation," *TWMS Journal of Applied and Engineering Mathematics (JAEM). To appear*, vol. 12, 2021.

[50] Y. Awad, "Well posedness and stability for the nonlinear φ-caputo hybrid fractional boundary value problems with two-point hybrid boundary conditions," *Jordan Journal of Mathematics and statistics (JJMS). To appear*, vol. 20, 2023.

[51] Z. Lv, I. Ahmad, J. Xu, and A. Zada, "Analysis of a hybrid coupled system of ψ-caputo fractional derivatives with generalized slit-strips-type integral boundary conditions and impulses," *Fractal and Fractional*, vol. 6, no. 10, p. 618, 2022.

[52] M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, and S. Rezapour, "Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 68–18, 2021.

[53] B. B. Alagoz, A. Ates, and C. Yeroglu, "Auto-tuning of PID controller according to fractional-order reference model approximation for DC rotor control," *Mechatronics*, vol. 23, no. 7, pp. 789–797, 2013.

[54] P. Kumar, S. Narayan, and J. Raheja, "Optimal design of robust fractional order PID for the flight control system," *International Journal of Computer Application*, vol. 128, no. 14, pp. 31–35, 2015.

[55] S. A. Akar, S. Kara, F. Latifoğlu, and V. Bilgic, "Investigation of the noise effect on fractal dimension of EEG in schizophrenia patients using wavelet and SSA-based approaches," *Biomedical Signal Processing and Control*, vol. 18, pp. 42–48, 2015.

[56] S. Simons and D. Abásolo, "Distance-based Lempel–Ziv complexity for the analysis of electroencephalograms in

patients with alzheimer's disease," *Entropy*, vol. 19, no. 3, p. 129, 2017.

[57] H. Sun, Y. Zhang, S. Wei, J. Zhu, and W. Chen, "A space fractional constitutive equation model for non-Newtonian fluid flow," *Communications in Nonlinear Science and Numerical Simulation*, vol. 62, pp. 409–417, 2018.

[58] X. Li and X. Tian, "Fractional order thermo-viscoelastic theory of biological tissue with dual phase lag heat conduction model," *Applied Mathematical Modelling*, vol. 95, pp. 612–622, 2021.

[59] X. Wang, H. Qi, X. Yang, and H. Xu, "Analysis of the time-space fractional bioheat transfer equation for biological tissues during laser irradiation," *International Journal of Heat and Mass Transfer*, vol. 177, Article ID 121555, 2021.

[60] Q. Zhang, Y. Sun, and J. Yang, "Bio-heat transfer analysis based on fractional derivative and memory-dependent derivative heat conduction models," *Case Studies in Thermal Engineering*, vol. 27, Article ID 101211, 2021.

[61] H. Khan, C. Tunc, W. Chen, and A. Khan, "Existence theorems and Hyers–Ulam stability for a class of hybrid fractional differential equations with p-Laplacian operator," *Journal of Applied Analysis & Computation*, vol. 8, no. 4, pp. 1211–1226, 2018.