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Research Article

M-Hazy Module and Its Homomorphism Theorem

Donghua Huo and Hongyu Liu

School of Mathematics, Mudanjiang Normal University, Mudanjiang 157012, China

Correspondence should be addressed to Donghua Huo; i94donghua@163.com

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Based on a completely distributive lattice M, we propose a new fuzzification approach to a module, which leads to the concept of an M-hazy module. Different from the traditional fuzzification approach that defines a fuzzy algebra as a fuzzy subset of a classical algebra, we introduce an M-hazy module by fuzzifications of algebraic operations. Then, we investigate the fundamental properties of M-hazy modules and M-hazy submodules. In particular, we present the M-hazy module homomorphism theorem.

1. Introduction

In 1971, Rosenfeld [1] firstly introduced fuzzy sets to the algebra field. Fuzzy algebra has become a new field of abstract algebra, and many fruitful results were obtained.

There have been two methods of fuzzification of groups until now. On the one hand, it is fuzzifying the subset hood relation under the classical binary operations, such as fuzzy group [1], which is defined by fuzzy sets of the classical groups and fuzzifying groups [2], which are defined by the degree of groups. In 2017, Li and Shi [3] discussed L-fuzzy convexity induced by L-convex fuzzy sublattice degree. Wen et al. [4] defined L-fuzzy convexity induced by L-convex degree on vector spaces. In 2018, Zhong and Shi [5] obtained the characterizations of (L, M)-fuzzy topological degrees. With the progress of L-fuzzy convexity, L-fuzzy convexity space has become a hot topic. In [6], they apply Galois correspondence as a tool to the theory of lattice-valued convex structures. It mainly introduces the concept of lattice-valued interval operators and discusses its relationships with L-fuzzifying convex structures and L-convex structures. In [7], considering L being a continuous lattice and M being a completely distributive De Morgan algebra, several basic notions with respect to (L, M)-fuzzy convex structures in the sense of Shi and Xiu are introduced and their relationship with (L, M)-fuzzy convex structures are studied. In [8], the concepts of fuzzifying L-hull spaces and fuzzifying L-convex spaces are introduced and the resulting categories are shown to be isomorphic to that of L-fuzzifying convex spaces. As applications of these concepts, it is shown that the category of L-fuzzifying convex spaces can be embedded in the category of L-convex spaces as a simultaneously bireflective and bicoreflective subcategory. In [9], considering L being a continuous lattice and M being a completely distributive lattice, bases and subbases in (L, M)-fuzzy convex spaces are investigated. In an axiomatic approach, axiomatic bases and axiomatic subbases are proposed. It is shown that axiomatic bases and axiomatic subbases can be used to generate (L, M)-fuzzy convex structures, and some of their applications are investigated. In [10], each (L, L)-fuzzy subset can be regarded as an (L, L)-convex set to some degree in (L, M)-fuzzy convex structures. The notion of convexity preserving functions is also generalized to lattice-valued case. Moreover, under the framework of (L, M)-fuzzy convex structures, the concepts of quotient structures, substructures, and products are presented and their fundamental properties are discussed. At the same time, some scholars study the ideality and derivability of fuzzy sets [11–14].

On the other hand, it is axiomatic method by fuzzy binary operations, such as vague groups [15], which are defined by fuzzy equalities, smooth groups [16], which are defined by (strong) fuzzy functions, and fuzzy groups [17], which are defined by fuzzy binary operations. However, the special property under fuzzy situation is not obtained, so the degree between fuzzy binary operations is not considered. In 2019, Liu and Shi [18] introduced M-hazy groups by a new kind of fuzzy associative law on hazy operations. Some special properties under fuzzy situations are obtained. In 2020, Fan et al. [19] studied the M-hazy Γ-semigroup. At the same time, Mehmood et al. [20] proposed the notion of M-hazy ring and obtained some results. By the motivations of these newly proposed concepts through M-hazy operations, we provided a new approach to fuzzification of modules on an M-hazy ring.

Traditionally, the researchers introduced fuzzy algebras as a fuzzy subset of a classical algebra, such as fuzzy group and fuzzy ring. The greatest innovation in this paper is that we give the fuzzification on the algebraic operations. This idea is very different from the traditional way. The paper is organized as follows: Section 2 consists of fundamental notions about completely residuated lattices, M-hazy groups, M-hazy rings, and M-hazy operation. In Section 3, the concept of M-hazy module is defined and its fundamental properties are obtained. In Section 4, the concept of M-hazy submodule is introduced and its properties are discussed. In Section 5, the homomorphism of M-hazy module is introduced. Finally, the fundamental theorem of module homomorphism is proved. Section 6 concludes the paper.

2. Preliminaries

In this section, we will introduce triangular norm, completely residuated lattices, M-hazy operation, M-hazy groups, M-hazy rings, and so on. $(M, \vee, \wedge, *, \rightarrow, \perp, \top)$ represents a completely residuated lattice, and $\leq$ denotes the partial order of M. Let L be a nonempty set.

Definition 1 (see [21]). Assume $*: M \times M \longrightarrow M$ is a function. $*$ is defined to be a triangular norm (for short, t-norm) on M if the following conditions hold:

(1) $x * y = y * x$

(2) $(x * y) * z = x * (y * z)$

(3) $x \leq z,\ y \leq w$ implies $x * y \leq z * w$

(4) $x * 1 = x$ for all $x \in M$

Definition 2 (see [21]). Assume $\longrightarrow: M \times M \longrightarrow M$ is a function and $*$ is a t-norm in M. Then, $\longrightarrow$ is defined to be the residuum of $*$ if, for all x, y, $z \in M$,

$$x \leq y \longrightarrow z \Leftrightarrow x * y \leq z. \tag{1}$$

Definition 3 (see [22]). Assume $(M, \vee, \wedge, \perp, \top)$ is a bounded lattice, where $\perp$ represents the least element, and $\top$ the greatest element, $*$ is a t-norm on M and $\longrightarrow$ denotes the residuum of $*$. Then, $(M, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is said to be a residuated lattice.

A residuated lattice is defined to be a completely residuated lattice if the primary lattice is complete. Also, we define $x \leftrightarrow y = (x \longrightarrow y) \wedge (y \longrightarrow x)$. Proposition 1 shows properties of the implication operation.

Proposition 1 (see [23]). *Assume $(M, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a completely residuated lattice. Then, for every x, y, $z \in M$, $\{x_i\}_{i \in I}$, $\{y_i\}_{i \in I} \subseteq M$, the following statements are valid:*

(1) $x \longrightarrow y = \vee \{z \in M \mid x * z \leq y\}$

(2) $y \leq x \longrightarrow y$, $\top \longrightarrow x = x$

(3) $x \longrightarrow (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \longrightarrow y_i)$

(4) $(\bigvee_{i \in I} x_i) \longrightarrow y = \bigwedge_{i \in I} (x_i \longrightarrow y)$

For a nonempty set L, let 2^L denote the set of all subsets of L. A family $\{A_i \mid i \in \Omega\}$ is up-directed given for every $A_1, A_2 \in \{A_i \mid i \in \Omega\}$. There is an element $A_3 \in \{A_i \mid i \in \Omega\}$, such that $A_1 \subseteq A_3$ and $A_2 \subseteq A_3$.

A mapping $f\colon L \longrightarrow M$ is said to be an M-fuzzy set on L. The set of all M-fuzzy sets on L is represented by M^L. For $\lambda \in M$ and $A \in M^L$, an M-fuzzy set on L is defined as $(A * \lambda)(p) = A(p) * \lambda$. Let A, B be two nonempty sets and $\varphi\colon A \longrightarrow B$ be mapping. Define $\varphi^{\longrightarrow}\colon 2^A \longrightarrow 2^B$, $\varphi^{\leftarrow}\colon 2^B \longrightarrow 2^A$, $\varphi_M^{\longrightarrow}\colon M^A \longrightarrow M^B$ and $\varphi_M^{\leftarrow}\colon M^B \longrightarrow M^A$ as follows:

$$\begin{aligned} &\forall X \in 2^A, \varphi^{\rightarrow}(X) = \{\varphi(x) \mid x \in X\},\\ &\forall Y \in 2^B, \varphi^{\leftarrow}(Y) = \{x \mid \varphi(x) \in Y\},\\ &\forall Z \in M^A, \forall y \in B, \varphi_M^{\rightarrow}(Z)(y) = \bigvee_{\varphi(x)=y} Z(x),\\ &\forall W \in M^B, \forall x \in A, \varphi_M^{\leftarrow}(W)(x) = W(\varphi(x)). \end{aligned} \tag{2}$$

By the above definition, we can easily know that $\varphi_M^{\leftarrow} \circ \varphi_M^{\rightarrow}(X) \geqslant X$ and $\varphi_M^{\leftarrow} \circ \varphi_M^{\leftarrow}(Y) \leqslant Y$.

Definition 4 (see [18]). For two nonempty sets A and B, an M-fuzzy relation $f \in M^{A \times B}$ is called an M-fuzzy function (or mapping) from A to B if f satisfies the following two conditions:

(MF1) $\forall x \in A$, we have $\bigvee_{y \in B} f(x, y) \neq \perp$

(MF2) $\forall x \in A$, $f(x, y) * f(x, z) \neq \perp \Rightarrow y = z$

Definition 5 (see [18]). Assume $\circledast\colon L \times L \longrightarrow M^L$ is a function, then $\circledast$ is defined to be an M-hazy operation on L, if the conditions given below hold:

(MH1) $\forall x$, $y \in L$, we have $\bigvee_{p \in L} (x \circledast y)(p) \neq \perp$

(MH2) $\forall x$, y, p, $q \in L$, $(x \circledast y)(p) * (x \circledast y)(q) \neq \perp \Rightarrow p = q$

Definition 6 (see [18]). Assume $\circledast\colon L \times L \longrightarrow M^L$ is an M-hazy operation on a nonempty set L. Then, $(L, \circledast)$ is defined to be an M-hazy group (in short, MHG) if the following conditions hold:

(MG1) $\forall x$, y, z, p, $q \in L$, $(x \circledast y)(p) * (y \circledast z)(q) \leq \bigwedge_{r \in L} ((p \circledast z)(r) \leftrightarrow (x \circledast q)(r))$, i.e., the M-hazy associative law holds

(MG2) An element $o \in L$ is said to be the left identity element of L, if $o \circledast x = x_\top$ for all $x \in L$

(MG3) An element $y \in L$ is said to be the left inverse of x if, for each $x \in L$, $y \circledast x = o_\top$, and is denoted by x^{-1}

If the above $(L, \circledast)$ still satisfies the following condition:

(MG4) $x \circledast y = y \circledast x$ for all $x, y \in L$.

Then, $(L, \circledast)$ is an abelian MHG.

Proposition 2 (see [18]). *Let $(L, \circledast)$ be an M-hazy group; then, for all $x, y, z \in L$, we have*

$$\begin{aligned}(x \circledast y)(z) &= \left(x^{-1} \circledast z\right)(y)\\ &= \left(z \circledast y^{-1}\right)(x)\\ &= \left(y \circledast z^{-1}\right)\left(x^{-1}\right)\\ &= \left(z^{-1} \circledast x\right)\left(y^{-1}\right)\\ &= \left(y^{-1} \circledast x^{-1}\right)\left(z^{-1}\right).\end{aligned} \tag{3}$$

Definition 7 (see [20]). Let $+\colon R \times R \longrightarrow M^R$ and: $R \times R \longrightarrow M^R$ be the M-hazy addition operation and M-hazy multiplication operation on R, respectively. Then, $(R, +, \bullet)$ is defined to be an M-hazy ring (in short, MHR) if the below conditions hold:

(MHR1) $(R, +)$ is an abelian MHG

(MHR2) $(R, \cdot)$ is an M-hazy semigroup

(MHR3) $\forall x, y, z, p, q, r \in R$, $(x \cdot y)(p) * (y + z)(q) * (x \cdot z)(r) \leq \wedge_{s\in G}((x \cdot q)(s) \leftrightarrow (p + r)(s))$

3. M-Hazy Modules

In this section, the M-hazy module is introduced and its characterizations are presented.

Definition 8. Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1 and let $(G, +)$ be an abelian M-hazy group. If there exists an M-hazy mapping $\circ\colon R \times G \longrightarrow M^G$, $(a, x) \longrightarrow a \circ x$, which satisfies for all $a, b \in R, x, y, p, q, r \in G$.

(MHM1) $(a \circ x)(p) * (x + y)(q) * (a \circ y)(r) \leq \wedge_{s\in G}((a \circ q)(s) \leftrightarrow (p + r)(s))$

(MHM2) $(a \circ x)(p) * (a \oplus b)(q) * (b \circ x)(r) \leq \wedge_{s\in G}((q \circ x)(s) \leftrightarrow (p + r)(s))$

(MHM3) $(a \odot b)(p) * (b \circ x)(q) \leq \wedge_{s\in G}((p \circ x)(s) \leftrightarrow (a \circ q)(s))$

(MHM4) $1 \odot x = x_\top$ for all $x \in G$.

Then, G is called an M-hazy left module on R (in short, M-hazy module).

Similarly, there exists an M-hazy mapping $\circ\colon G \times R \longrightarrow M^G$, $(x, a) \longrightarrow x \circ a$, which satisfies for all $a, b \in R, x, y, p, q, r \in G$.

(MHM1R) $(x \circ a)(p) * (x + y)(q) * (y \circ a)(r) \leq \wedge_{s\in G}((q \circ a)(s) \leftrightarrow (p + r)(s))$

(MHM2R) $(x \circ a)(p) * (a \oplus b)(q) * (x \circ b)(r) \leq \wedge_{s\in G}((x \circ q)(s) \leftrightarrow (p + r)(s))$

(MHM3R) $(a \odot b)(p) * (x \circ a)(q) \leq \wedge_{s\in G}((x \circ p)(s) \leftrightarrow (q \circ b)(s))$

(MHM4R) $x \odot 1 = x_\top$ for all $x \in G$

Then, G is called an M-hazy right module on R.

If $(R, \oplus, \odot)$ be an abelian M-hazy ring, then an M-hazy left module on R is an an M-hazy right module on R. The following discussion focuses on M-hazy left module. Because M-hazy right module is similar, so it is omitted.

Proposition 3. *Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1 and let $(G, +)$ be an abelian M-hazy group. $\circ\colon R \times G \longrightarrow M^G$, $(a, x) \longrightarrow a \circ x$ be an M-hazy mapping, then the following statements are equivalent for all $a \in R, x, y, p, q, r \in G$:*

(MHM1) $(a \circ x)(p) * (x + y)(q) * (a \circ y)(r) \leq \wedge_{s\in G}((a \circ q)(s) \leftrightarrow (p + r)(s))$

(MHM1a) $(a \circ x)(p) * (x + y)(q) * (a \circ y)(r) * (a \circ q)(s) \leq (p + r)(s)$, $(a \circ x)(p) * (x + y)(q) * (a \circ y)(r) * (p + r)(s) \leq (a \circ q)(s)$

(MHM1b) If $a \circ x = p_\lambda, x + y = q_\mu, a \circ y = r_\gamma$, *then* $(a \circ q) * \lambda * \mu * \gamma \leq (p + r)$ *and* $(p + r) * \lambda * \mu * \gamma \leq (a \circ q)$

(MHM1c) If $a \circ x = p_\lambda, x + y = q_\mu, a \circ y = r_\gamma, a \circ q = t_\alpha, p + r = u_\beta$, *then* $t = u, \alpha * \lambda * \mu * \gamma \leq \beta$ *and* $\lambda * \mu * \gamma * \beta \leq \alpha$

Proof. The proof is similar to the proof of Proposition 5 in [18] and is omitted. □ □

Similarly, we have the following two propositions:

Proposition 4. *Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1 and let $(G, +)$ be an abelian M-hazy group. $\circ\colon R \times G \longrightarrow M^G$, $(a, x) \longrightarrow a \circ x$ be an M-hazy mapping, then the following statements are equivalent for all $a, b, q \in R, x, p, r \in G$:*

(MHM2) $(a \circ x)(p) * (a \oplus b)(q) * (b \circ x)(r) \leq \wedge_{s\in G}((q \circ x)(s) \leftrightarrow (p + r)(s))$.

(MHM2a) $(a \circ x)(p) * (a \oplus b)(q) * (b \circ x)(r) * (q \circ x)(s) \leq (p + r)(s)$, $(a \circ x)(p) * (a \oplus b)(q) * (b \circ x)(r) * (p + r)(s) \leq (q \circ x)(s)$.

(MHM2b) If $a \circ x = p_\lambda, a \oplus b = q_\mu, a \circ y = r_\gamma$, *then* $(a \circ q) * \lambda * \mu * \gamma \leq (p + r)$ *and* $(p + r) * \lambda * \mu * \gamma \leq (a \circ q)$.

(MHM2c) If $a \circ x = p_\lambda, a \oplus b = q_\mu, a \circ y = r_\gamma, a \circ q = t_\alpha, p + r = u_\beta$, *then* $t = u, \alpha * \lambda * \mu * \gamma \leq \beta$ *and* $\lambda * \mu * \gamma * \beta \leq \alpha$.

Proposition 5. *Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1 and let $(G, +)$ be an abelian M-hazy group. $\circ\colon R \times G \longrightarrow M^G$, $(a, x) \longrightarrow a \circ x$ be an M-hazy mapping, then the* *following statements are equivalent for all $a, b, p \in R, x, q \in G$:*

(MHM3) $(a\odot b)(p) * (b\circ x)(q) \leqslant \wedge_{s\in G}((p\circ x)(s)\leftrightarrow(a\circ q)(s))$.

(MHM3a) $(a\odot b)(p) * (b\circ x)(q) * (p\circ x)(s) \leqslant (a\circ q)(s)$, $(a\odot b)(p) * (b\circ x)(q) * (a\circ q)(s) \leqslant (p\circ x)(s)$.

(MHM3b) *If* $a\odot b = p_\lambda, b\circ x = q_\mu$, *then* $(a\circ q) * \lambda * \mu \leqslant (p+r)$ *and* $(p+r) * \lambda * \mu \leqslant (a\circ q)$,

(MHM3c) If $a\odot b = p_\lambda, b\circ x = q_\mu, a\circ q = t_\alpha, p+r = u_\beta$, *then* $t = u, \alpha * \lambda * \mu \leqslant \beta$ *and* $\lambda * \mu * \beta \leqslant \alpha$.

In the following, we give Example 1.

Example 1. Let $R = \{0, u, v, w\}$ be a set and let $(M, *) = ([0,1], \wedge)$. The mapping $\oplus\colon R\times R \longrightarrow [0,1]^R$ are defined, as shown in Table 1, and $\odot\colon R\times R \longrightarrow [0,1]^R$ are defined, as shown in Table 2

We can get that $R_+ = (R, \oplus)$ is an M-hazy group. $(R, \oplus, \odot)$ is an M-hazy ring with identity o. We define $\circ\colon R\times R_+ \longrightarrow M^{R_+}$, as shown in Table 3.

It is easy to verify that $(MHM1)$, $(MHM2)$ and $(MHM4)$ are analogous to Example 3.3 in [20]. Now, we prove that $\circ$ satisfies $(MHM3)$.

Let $k = (a\odot b)(p) * (b\circ x)(q)$ and $t = \wedge_{s\in G}((p\circ x)(s)\leftrightarrow(a\circ q)(s))$ for every $a, b, p \in R, x, q \in R_+$, we get the following four cases.

Case 1. $a = u, b = u$.

(1) If $x = v$, then $(u\odot u)(p)\wedge(u\circ v)(q) = 0.4$

(2) If $x = w$, then $(u\odot u)(p)\wedge(u\circ w)(q) = 0.3$

(3) If $x \neq v, w$, then $(u\odot u)(p)\wedge(u\circ x)(q) = 0.5$

At this time, $\wedge_{s\in G}((p\circ x)(s)\leftrightarrow(u\circ q)(s)) = 1$, then $k < t$.

Case 2. $a = u, b \neq u$.

(1) If $b = o$, then $(u\odot o)(p)\wedge(o\circ x)(q) = 1$

(2) If $b = v$, then $(u\odot v)(p)\wedge(v\circ x)(q) = 0.4$

(3) If $b = w$, then $(u\odot w)(p)\wedge(w\circ x)(q) = 0.3$

At this time, $p \neq u, \wedge_{s\in G}((p\circ x)(s)\leftrightarrow(u\circ q)(s)) = 1$, then $k \leqslant t$.

Case 3. $a \neq u, b = u$.

(1) If $x = o$, then $(a\odot u)(p)\wedge(u\circ o)(q) = 1$

(2) If $x = u$, then $(a\odot u)(p)\wedge(u\circ u)(q) = 0.5$

(3) If $x = v$, then $(a\odot u)(p)\wedge(u\circ v)(q) = 0.4$

(4) If $x = w$, then $(a\odot u)(p)\wedge(u\circ w)(q) = 0.3$

At this time, $a \neq u, \wedge_{s\in G}((p\circ x)(s)\leftrightarrow(a\circ q)(s)) = 1$, then $k \leqslant t$.

Case 4. $a \neq u, b \neq u$.

$(a\odot b)(p) * (b\circ x)(q) = 1$ and $\wedge_{s\in G}((p\circ x)(s)\leftrightarrow(a\circ q)(s)) = 1$, then $k = t$.

According to the above analysis, R_+ is an M-hazy left module on R.

Table 1: Values of the [0, 1]-hazy operation $\oplus$.

$x\oplus y$	o	u	v	w
o	o_1	u_1	v_1	w_1
u	u_1	o_1	$w_{0.3}$	$v_{0.3}$
v	v_1	$w_{0.3}$	$u_{0.3}$	o_1
w	w_1	$u_{0.3}$	o_1	$u_{0.3}$

Table 2: Values of the [0, 1]-hazy operation $\odot$.

$x\odot y$	o	u	v	w
o	o_1	o_1	o_1	o_1
u	o_1	$u_{0.5}$	$v_{0.4}$	$w_{0.3}$
v	o_1	o_1	o_1	o_1
w	o_1	o_1	o_1	o_1

Table 3: Values of the [0, 1]-hazy mapping $\circ$.

$x^\circ y$	o	u	v	w
o	o_1	o_1	o_1	o_1
u	o_1	$u_{0.5}$	$v_{0.4}$	$w_{0.3}$
v	o_1	o_1	o_1	o_1
w	o_1	o_1	o_1	o_1

Proposition 6. *Let* $(R, \oplus, \odot)$ *be an* M*-hazy ring with identity 1 and* G *be an* M*-hazy module on* R*.* o, o' *be the additive identity element of* G *and* R*. For all* $a \in R, x, p \in G$*, we have*

(1) $(a\circ o)(o) \neq \perp$ *and* $(o'\circ x)(o) \neq \perp$.

(2) If $(a\circ x)(p) \neq \perp$, *then* $(a\circ(-x))(-p) \neq \perp$ *and* $((-a)\circ x)(-p) \neq \perp$.

(3) $(a\circ x_1)(p_1) * \cdots * (a\circ x_v)(p_v) * (\sum_{i=1}^{v} x_i)(q) \leqslant \wedge_{s\in G}((a\circ q)(s)\leftrightarrow(\sum_{i=1}^{v} p_i)(s))$.

$$\left(\sum_{i=1}^{v} a_i\right)(p) * (a_1\circ x)(q_1) * \cdots * (a_v\circ x)(q_v) \leqslant \bigwedge_{s\in G}\left((p\circ x)(s)\leftrightarrow\left(\sum_{i=1}^{v} q_i\right)(s)\right). \tag{4}$$

Proof

(1) Assume that $(a\circ o)(o) = \perp$ by (MHM1), $\forall a \in R, o \in G$, we have

$$\begin{aligned}&(a\circ o)(p) * (o+o)(o) * (a\circ o)(r)\\ &\leqslant \bigwedge_{s\in G}((a\circ o)(s)\leftrightarrow(p+r)(s)).\end{aligned} \tag{5}$$

When $r = s = p$, we have

$$(a\circ o)(p) = (a\circ o)(p) * (a\circ o)(p) * (a\circ o)(p) \leqslant (p+p)(p). \tag{6}$$

If $p \neq 0$, then $(p+p)(p) = (p-p)(p) = \perp$ by Proposition 2, which is in contradiction with $\wedge_{p\in G}(a\circ o)(p) \neq \perp$. Similarly, $(o' \circ x)(o) \neq \perp$.

(2) By (MHM1), $\forall a \in R, p, x \in G$, we have

$$(a\circ x)(p) * (x + (-x))(q) * (a\circ(-x))(r) \leqslant \bigvee_{p\in G}((a\circ q)(s) \leftrightarrow (p+r)(s)). \tag{7}$$

Let $q = s = 0$, then

$$(a\circ x)(p) * (a\circ(-x))(r) * (a\circ o)(o) \leqslant (p+r)(o). \tag{8}$$

We assume that $(a\circ(-x))(r) \neq \perp$ by (MH2). Since $\perp$ is prime, so we have $(p+r)(o) \neq \perp$. Also, $(p+(-p))(o) = \top$. This implies $p = -r$ by (MH2). Thus, $(a\circ(-x))(-p) \neq \perp$. Similarly, $((-a)\circ x)(-p) \neq \perp$.

(3) From (MHM1),

$$(a\circ x_1)(p_1) * \cdots * (a\circ x_v)(p_v) * \left(\sum_{i=1}^{v} x_i\right)(q) \leqslant \bigwedge_{s\in G}\left((a\circ q)(s) \leftrightarrow \left(\sum_{i=1}^{v} p_i\right)(s)\right). \tag{9}$$

From (MHM2),

$$\left(\sum_{i=1}^{v} a_i\right)(p) * (a_1\circ x)(q_1) * \cdots * (a_v\circ x)(q_v) \leqslant \bigwedge_{s\in G}\left((p\circ x)(s) \leftrightarrow \left(\sum_{i=1}^{v} q_i\right)(s)\right). \tag{10}$$

□

4. M-Hazy Submodule

In this section, we introduce the concept of M-hazy submodule.

Definition 9. Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1 and G is an M-hazy module on R. Let N be the nonempty subset of G. Then, N is said to be an M-hazy submodule of G, relative to the M-hazy mapping $\circ$ on N, N itself forms an M-hazy module on R.

Example 2. Consider the M-hazy submodule of R_+ in Example 1, let $N = \{o, u\}$, one can check that N is an M-hazy submodule of R_+.

Theorem 1. *Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1 and G be an M-hazy module on R. Let N be the nonempty subset of G. Then, N is an M-hazy submodule of G if and only if the following conditions holds:*

(1) $\forall k, l \in N$, we have $\vee_{p\in N}(k + (-l))(p) \neq \perp$

(2) $\forall a \in R$ and $\forall y \in N$, we have $\vee_{q\in N}(a\circ y)(q) \neq \perp$

Proof. From Theorem 4.4 in [18], the proof is simple and omitted. □

Proposition 7. *Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1 and G be an M-hazy module on R. Then, the intersection of a family of M-hazy submodule of G is an M-hazy submodule of G.*

Proof. Let Ω be an index set and N_i be an M-hazy submodule of. Let $L = \cap_{i\in\Omega} N_i$.

(1) Since $o \in N_i$ for each $i \in \Omega$ and L is a non-empty subset of G, so we have $o \in L$.

(2) For every $k, l \in L$, $a \in G$ and for every $i \in \Omega$, we obtain $\vee_{p\in N_i}(k + (-l))(p) \neq \perp$ and $\vee_{q\in N_i}(a\circ y)(q) \neq \perp$ by Theorem 1. Since $\vee_{q\in N_i}(a\circ y)(q) \neq \perp$, there exists $x_{ay} \in N_i$, such that $(a\circ y)(x_{ay}) \neq \perp$. This implies for all $i \in \Omega$, $x_{ay} \in N_i$. So we can obtain $x_{ay} \in \cap_{i\in\Omega} N_i$. Hence, $\vee_{q\in \cap_{i\in\Omega} N_i}(a\circ y)(q) \geqslant (a\circ y)(x_{ay}) \neq \perp$.

Similarly, $\vee_{p\in \cap_{i\in\Omega} N_i}(k + (-l))(p) \neq \perp$. Since $\vee_{p\in N_i}(k + (-l))(p) \neq \perp$, there exists $x_{kl} \in N_i$, such that $(k + (-l))(x_{kl}) \neq 0$. This implies for all $i \in \Omega$, $x_{kl} \in N_i$. So, we obtain $x_{kl} \in \cap_{i\in\Omega} N_i$. Hence, $\vee_{p\in \cap_{i\in\Omega} N_i}(k + (-l))(p) \geqslant (k + (-l))(x_{kl}) \neq \perp$. Hence, $L = \cap_{i\in\Omega} A_i$ is an M-hazy submodule of G. □ □

Proposition 8. *Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1 and G be an M-hazy module on R. Then, the union* *of a nonempty up-directed family of M-hazy submodule of G is an M-hazy submodule of G. In particular, the union of a nonempty chain of M-hazy submodule of G is an M of G.*

Proof. Let Ω be an index set and N_i be an M-hazy submodule of G, where $\{N_i \mid i \in \Omega\}$ is an up-directed subfamily of 2^G. Let $H = \cup_{i\in\Omega} N_i$.

(1) H is a nonempty subset of G.

(2) For every $k, l, y \in H$, there exists $i, j, k \in \Omega$, such that $k \in N_i, l \in N_j$ and $y \in N_k$. Since H is an up-directed family, then there exists $m \in \Omega$, such that $N_i \subseteq N_m, N_j \in N_m$ and $N_k \subseteq N_m$, this implies $k, l, y \in N_m$. As N_m is an M-hazy submodule of G, we obtain for all $a \in R$, $\vee_{p\in N_m}(k + (-l))(p) \neq \perp$, and $\vee_{q\in N_m}(a \circ y)(q) \neq \perp$ by Theorem 1. Hence, $\vee_{p\in H}(k + (-l))(p) \neq \perp$ and $\vee_{q\in H}(a \circ y)(q) \neq \perp$. Hence, $\cap_{i\in\Omega} N_i$ is an M-hazy submodule of G. □ □

5. Homomorphisms of the M-Hazy Module

Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1 and G be an M-hazy module on R. Let N be an M-hazy submodule of G. Then, N is an M-hazy subgroup so that N is an M-hazy normal subgroup. Let $\{x +_\perp y\} = \{a \in G \mid (x + y)(a) \neq \perp\}$ for all $x, y \in G$, then $(G, +_\perp)$ is a (classical) group from Proposition 5.7 in [18]. Because $A +_\perp B = \{x +_\perp y \mid x \in A, y \in B\}$ in (classical) group $(G, +_\perp)$, so we have $A +_\perp B = (A + B)^{(\perp)}$ from Proposition 5.8 in [18].

Now, we define relation r_+ on G such that $\forall x, y \in G, x r_+ y \Leftrightarrow x +_\perp N = y +_\perp N$. Then, it is easy to prove that r_+ is an equivalence relation on G. Let $\overline{x}$ denote equivalence class, which contains x in G. Thus, we can obtain a set $(G/r_+) = \{\overline{x} \mid x \in G\}$.

For all $\overline{x}, \overline{y} \in (G/r_+)$, we define a mapping $\tilde{+}\colon (G/r_+) \times (G/r_+) \longrightarrow M^{(G/r_+)}$ as follows:

$$(\overline{x} \tilde{+} \overline{y})(\overline{x +_\perp y}) = \bigvee_{m\in\overline{x}, n\in\overline{y}, r\in\overline{x +_\perp y}} (m + n)(r). \tag{11}$$

We know that $\tilde{+}$ is an M-hazy operation on (G/r_+) by [18].

Similarly, let $\{a \circ_\perp x\} = \{y \in G \mid (a \circ x)(y) \neq \perp\}$ for all $a \in R, x, y \in G$. $\{a \circ_\perp N\} = \{y \in G \mid (a \circ x)(y) \neq \perp\}$ for all $a \in R, x \in N, y \in G$. For all $a \in R, \overline{x} \in (G/r_+)$, we define a mapping $\Delta\colon R \times (G/r_+) \longrightarrow M^{(G/r_+)}$ as follows:

$$(a \Delta \overline{x})(\overline{a \circ_\perp x}) = \bigvee_{m\in\overline{x}, r\in\overline{a \circ_\perp x}} (a \circ m)(r). \tag{12}$$

Now, we prove that Δ is an M-hazy mapping.

(MH1) For all $a \in R, x \in \overline{x}$, it is obvious that $\overline{a \circ_\perp x} \in (G/r_+)$. If $(a\Delta\overline{x})(\overline{a \circ_\perp x}) = \perp$ for each $a \in R, x \in \overline{x}$, then for all $m \in \overline{x}, r \in \overline{a \circ_\perp x}$, we have $(a \circ x)(r) = \perp$. This shows $(a \circ x)(r) = \perp$. This contradicts with $r \in \overline{a \circ_\perp x}$.

(MH2) For all $\overline{x}, \overline{y}, \overline{z} \in (G/r_+)$, if $(a\Delta\overline{x})(\overline{y}) \wedge (a\Delta\overline{x})(\overline{z}) \neq \perp$, then we obtain

$$\left(\bigvee_{n\in\overline{x}, r\in\overline{y}} (a \circ n)(r)\right) \wedge \left(\bigvee_{m\in\overline{x}, s\in\overline{z}} (a \circ m)(s)\right) \neq \perp. \tag{13}$$

Then, there exists $n \in \overline{x}, r \in \overline{y}, m \in \overline{x}, s \in \overline{z}$, such that $(a \circ n)(r) \neq \perp$ and $(a \circ m)(s) \neq \perp$. This shows

$$\begin{aligned} r &= a \circ_\perp n \subseteq a \circ_\perp \overline{x} \\ &= a \circ_\perp (x +_\perp N) \\ &= (a \circ_\perp x) +_\perp N \\ &= \overline{a \circ_\perp x}. \end{aligned} \tag{14}$$

Similarly, $s \subseteq \overline{a \circ_\perp x}$. So, $\overline{y} = \overline{z}$.

Proposition 9. *Let $(R, \oplus, \odot)$ be an M-hazy ring with identity 1, G be an M-hazy module on R, and N be an M-hazy submodule of G. If*

(1) $\overline{x +_\perp y} \tilde{+} \overline{z} = \overline{x} \tilde{+} \overline{y +_\perp z}$

(2) $a\Delta\overline{x +_\perp y} = \overline{a \circ_\perp x} \tilde{+} \overline{a \circ_\perp y}$

(3) $(a \oplus b)\Delta\overline{x} = \overline{a \circ_\perp x} \tilde{+} \overline{b \circ_\perp x}$

(4) $(a \odot b)\Delta\overline{x} = a\Delta\overline{b \circ_\perp x}$

For all $a, b \in R, x, y, z \in G$, then $\overline{G} = (G/r_+)$ is an M-hazy (quotient) module under the M-hazy operation $\tilde{+}$ and M-hazy mapping Δ.

Proof. Because $\tilde{+}$ is an M-hazy operation on $\overline{G}$ and $\overline{x +_\perp y} \tilde{+} \overline{z} = \overline{x} \tilde{+} \overline{y +_\perp z}$, so $(\overline{G}, \tilde{+})$ is an M-hazy abelian group under $\tilde{+}$ by Proposition 5.10 in [18].

Next, we prove that $\overline{G}$ is an M-hazy module on R.

(MHM1) For all $a \in R, \overline{x}, \overline{y}, \overline{p}, \overline{q}, \overline{r}, \overline{s} \in \overline{G}$, we have

$$(a\Delta\overline{x})(\overline{p}) * (\overline{x} \tilde{+} \overline{y})(\overline{q}) * (a\Delta\overline{y})(\overline{r}) * (a\Delta\overline{q})(\overline{s}) = (a\Delta\overline{x})(\overline{p}) * (\overline{x} \tilde{+} \overline{y})(\overline{q}) * (a\Delta\overline{y})(\overline{r}) * (\overline{p} \tilde{+} \overline{r})(\overline{s}), \tag{15}$$

because $a\Delta\overline{x +_\perp y} = \overline{a \circ_\perp x} \tilde{+} \overline{a \circ_\perp y} \Leftrightarrow a\Delta\overline{q} = \overline{p} \tilde{+} \overline{r}$. By Proposition 3 (MHM1b), we obtain

$$(a\Delta\overline{x})(\overline{p}) * (\overline{x}\tilde{+}\overline{y})(\overline{q}) * (a\Delta\overline{y})(\overline{r}) \leqslant \bigwedge_{\overline{s}\in\overline{G}}((a\Delta\overline{q})(\overline{s})\leftrightarrow(\overline{p}\tilde{+}\overline{r})(\overline{s})). \tag{16}$$

(MHM2) For all $a, b, q \in R, \overline{x}, \overline{q}, \overline{r}\in\overline{G}$, we have

$$(a\Delta\overline{x})(\overline{p}) * (a\oplus b)(q) * (b\Delta\overline{x})(\overline{r}) * (q\Delta\overline{x})(\overline{s}) = (a\Delta\overline{x})(\overline{p}) * (a\oplus b)(q) * (b\Delta\overline{x})(\overline{r}) * (\overline{p}\tilde{+}\overline{r})(\overline{s}), \tag{17}$$

because $(a\oplus b)\Delta\overline{x} = \overline{a\circ_{\perp}x}\tilde{+}\overline{b\circ_{\perp}x}\Leftrightarrow q\Delta\overline{x} = \overline{p}\tilde{+}\overline{r}$. By Proposition 4 (MHM2b), we obtain

$$(a\Delta\overline{x})(\overline{p}) * (a\oplus b)(q) * (b\Delta\overline{x})(\overline{r}) \leqslant \bigwedge_{\overline{s}\in\overline{G}}((q\Delta\overline{x})(\overline{s})\leftrightarrow(\overline{p}\tilde{+}\overline{r})(\overline{s})). \tag{18}$$

(MHM3) For all $a, b, p \in R, \overline{x}, \overline{q} \in \overline{G}$, we have

$$(a\odot b)(p) * (b\Delta\overline{x})(\overline{q}) * (p\Delta\overline{x})(\overline{s}) = (a\odot b)(p) * (b\Delta\overline{x})(\overline{q}) * (a\Delta\overline{q})(\overline{s}), \tag{19}$$

because $(a\odot b)\Delta\overline{x} = a\Delta\overline{b\circ_{\perp}x}\Leftrightarrow p\Delta\overline{x} = a\Delta\overline{q}$. By Proposition 5 (MHM3b), we obtain

$$(a\odot b)(p) * (b\Delta\overline{x})(\overline{q}) \leqslant \bigwedge_{\overline{s}\in\overline{G}}((p\Delta\overline{x})(\overline{s})\leftrightarrow(a\Delta\overline{q})(\overline{s})). \tag{20}$$

(MHM4) For all $\overline{x}\in\overline{G}$, we have

$$(1\Delta\overline{x})(\overline{x}) = \bigvee_{x\in\overline{x}}(1\circ x)(x) \geqslant \left(1\circ x^{'}\right)\left(x^{'}\right) = \top. \tag{21}$$

So, $1\Delta\overline{x} = \overline{x}_{\top}$. Then, (G/r_{+}) is an M-hazy module on R. (G/r_{+}) is an M-hazy quotient module. □ □

Definition 10. Let G and $G^{'}$ be two M-hazy modules on R. Then, the mapping $\sigma: G \longrightarrow G^{'}$ is called an M-hazy module homomorphism if

(1) σ is M-hazy group homomorphism.

(2) $\forall a \in R, x \in G, \overrightarrow{\sigma_M}(a\circ_G x) = a\circ_{G^{'}}\sigma^{\longrightarrow}(x)$.

Where $\circ_G, \circ_{G^{'}}$ are $G, G^{'}$ module operation, respectively.

An injection, surjective, and bijective M-hazy module homomorphism is called an M-hazy module monomorphism, M-hazy module epimorphism, and M-hazy module isomorphism, respectively.

Proposition 10

(1) The identity mapping id_G: $(G, \circ_G) \longrightarrow (G, \circ_G)$ on an M-hazy module G is an M-hazy module homomorphism

(2) If σ: $(G, \circ_G) \longrightarrow (H, \circ_H)$ and τ: $(H, \circ_H) \longrightarrow (T, \circ_T)$ are two M-hazy module homomorphisms, then $\tau\sigma$: $(G, \circ_G) \longrightarrow (T, \circ_T)$ is an M-hazy module homomorphism

Proof. (1) is obvious. Now, we prove (2).

Firstly, from the proof of Proposition 5.2 in [18], we can know that $\tau\sigma$ is an M-hazy group homomorphism.

Secondly, for all $a \in R, x \in G$, we have

$$\begin{aligned}\overrightarrow{(\tau\sigma)_M}(a\circ_G x) &= \overrightarrow{\tau_M}(\overrightarrow{\sigma_M}(a\circ_G x))\\ &= \overrightarrow{\tau_M}(a\circ_G\sigma^{\longrightarrow}(x))\\ &= a\circ_G(\tau^{\longrightarrow}\sigma^{\longrightarrow}(x))\\ &= a\circ_G(\tau\sigma)^{\longrightarrow}(x)),\end{aligned} \tag{22}$$

so that $\tau\sigma$ is an M-hazy module homomorphism. □ □

Proposition 11. *Let σ: $(G, \circ_G) \longrightarrow (H, \circ_H)$ be an M-hazy module epimorphism. If A and B are M-hazy submodules of G and H, respectively. Then, the following statements are valid:*

(1) $\sigma^{\rightarrow}(A)$ is an M-hazy submodule of H

(2) $\sigma^{\leftarrow}(B)$ is an M-hazy submodule of G

Proof

(1) From Proposition 5.4 in [18], we know that $\sigma^{\rightarrow}(A)$ is an M-hazy subgroup of H. For all $a \in R, y \in \sigma^{\rightarrow}(A)$, there exists $x \in A$, such that $\sigma^{\rightarrow}(x) = y$ and

$$\begin{aligned}\bigvee_{\sigma(t)\in\sigma^{\longrightarrow}(A)}(a\circ y)(\sigma(t)) &= \bigvee_{\sigma(t)=\sigma^{\rightarrow}(A)}\left(a\circ\sigma^{\rightarrow}(x)\right)(\sigma(t))\\ &= \bigvee_{\sigma(t)\in\sigma^{\rightarrow}(A)}\sigma_M^{\rightarrow}(a\circ x)(\sigma(t))\\ &= \bigvee_{\sigma(t)\in\sigma^{\rightarrow}(A)}\bigvee_{\sigma(s)=\sigma(t)}(a\circ x)(s)\\ &= \bigvee_{\sigma(t)\in\sigma^{\rightarrow}(A)}(a\circ x)(t)\\ &\geqslant \bigvee_{t\in A}(a\circ x)(t)\\ &\neq \perp.\end{aligned}\tag{23}$$

(2) From Proposition 5.4 in [18], we know that $\sigma^{\leftarrow}(B)$ is an M-hazy subgroup of G. For all $a\in R, x\in\sigma^{\leftarrow}(B)$, there exists $y\in B$, such that $x=\sigma^{\leftarrow}(y)$ and

$$\begin{aligned}\bigvee_{\sigma(t)\in B}(a\circ y)(\sigma(t)) &= \bigvee_{\sigma(t)\in B}\left(a\circ\sigma^{\rightarrow}(x)\right)(\sigma(t))\\ &= \bigvee_{\sigma(t)\in B}\sigma_M^{\rightarrow}(a\circ x)(\sigma(t))\\ &= \bigvee_{\sigma(t)\in B}\bigvee_{\sigma(t)=\sigma(s)}(a\circ x)(s)\\ &= \bigvee_{\sigma(t)\in B}(a\circ x)(t)\\ &= \bigvee_{t\in\sigma^{\leftarrow}(B)}(a\circ x)(t)\\ &\neq \perp.\end{aligned}\tag{24}$$

Consequently, $\sigma^{\leftarrow}(B)$ is an M-hazy submodule of G. □ □

Definition 11. Let G and H be two M-hazy modules on R. σ: $(G,\circ_G)\longrightarrow(H,\circ_H)$ be an M-hazy module homomorphism and o' be the additive identity element of H. Then, the kernel of σ is determined by

$$\begin{aligned}\mathrm{Ker}\sigma &= \sigma^{\leftarrow}\left(o'\right)\\ &= \left\{x\in G\,|\,\sigma(x)=o'\right\}.\end{aligned}\tag{25}$$

Proposition 12. *Assume G and G' are two M-hazy modules on an M-hazy ring R. σ: $G\longrightarrow G'$ is an M-hazy module homomorphism. Then, $N=\mathrm{Ker}\sigma$ is an M-hazy submodule of G and ψ: $\overline{G}=(G/r_+)\longrightarrow\sigma(G)$ defined by $\psi(\overline{x})=\sigma(x)$ is an M-hazy module isomorphism.*

Proof. It is easy to see that $\left\{o'\right\}$ is an M-hazy submodule of H. Then, by Proposition 11, $\mathrm{Ker}\sigma$ is an M-hazy submodule of G.

From Proposition 5.11 in [18], we can know that ψ is an M-hazy group isomorphism. Next, we will prove ψ is an M-hazy module isomorphism. For all $a\in R, x\in G$, we have

$$\begin{aligned}\psi_M^{\rightarrow}(a\Delta\overline{x})(y) &= \bigvee_{\psi\left(\overline{a\circ_{G\perp}x}\right)=y}\left(a\circ_G\overline{x}\right)\left(\overline{a\circ_{G\perp}x}\right)\\ &= \bigvee_{\sigma\left(a\circ_{G\perp}x\right)=y}\left(a\circ_G x\right)\left(a\circ_{G\perp}x\right)\\ &= \sigma_M^{\rightarrow}\left(a\circ_G x\right)(y)\\ &= a\circ_{G'}\sigma^{\rightarrow}(x)(y)\\ &= a\circ_{G'}\psi^{\rightarrow}(x)(y).\end{aligned}\tag{26}$$

So, $\psi_M^{\rightarrow}(a\Delta\overline{x})=a\circ_{G'}\psi^{\rightarrow}(x)$. Consequently, $\overline{G}\cong\sigma(G)$. □

6. Conclusions

In this paper, we provided a new approach to the fuzzification of modules on the M-hazy ring. Traditionally, the researchers introduced fuzzy algebras as a fuzzy subset of classical algebras, such as fuzzy groups and fuzzy rings. Since Liu and Shi [18] finished M-hazy groups by using the M-hazy binary operation. Mehmood et al. [20, 24] extended this idea by defining M-hazy rings and obtaining their induced fuzzifying convexities. Based on these new concepts through M-hazy operations, we proposed a new approach to the fuzzification of the module on M-hazy ring R, which is the M-hazy module on M-hazy ring R. Also, using the completely residuated lattice-valued logic, some important properties of the M-hazy module, M-hazy submodule, and M-hazy module homomorphism are introduced. Finally, the fundamental theorem of module homomorphism is proved. The greatest innovation in this paper is that we gave fuzzification on the algebraic operations. This idea is very different from the traditional way.

In this paper, if R is a classical ring or G is a classical group, we can get similar conclusions, and the proof is omitted. The following research directions for future work could be very fruitful on other fuzzy algebra.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

D.H.H. was in charge of conceptualization, methodology, validation, software, formal analysis, investigation, resources, data curation, writing the original draft, writing review, editing, supervision, project administration, and funding acquisition. H.Y.L. was in charge of validation, resources, writing the original draft, and project administration.

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