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Research Article

Boolean Algebras with Semigroup Operators: Free Product and Free Objects

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Two important algebraic structures are S-acts and Boolean algebras. Combining these two structures, one gets S-Boolean algebras, equipped with a compatible right action of a monoid S which is a special case of Boolean algebras with operators. In this article, we considered some category-theoretic properties of the category Boo-S of all S-Boolean algebras with action-preserving maps between them which also preserve Boolean operations. The purpose of the present article is to study certain categorical and algebraical concepts of the category Boo-S, such as congruences, indecomposable objects, coproducts, pushouts, and free objects.

1. Introduction and Preliminaries

Two important algebraic structures in many fields such as in computer science are S-acts and Boolean algebras and then a combination of these two structures called S-Boolean algebras, introduced in [1], are essential. In this article, some category-theoretic properties of the category Boo-S of all S-Boolean algebras, with action and operations preserving maps between them are considered. Let us explain our motivation and eagerness to do this article.

The study of categories in category $\mathscr{C}$ which is different from the category set of sets are always of interest, for example, topological groups, topological rings, topological semigroups or the category of general (universal) algebras in an arbitrary category (see [2]). In special, where $\mathscr{C}$ is the category Act-S of sets with an action of a monoid S on them has been considered in [1, 3–6]. Ebrahimi and Mahmoudi, in [1, 4, 5], have investigated the category Boo of Boolean algebras in the category Act-S. They have studied some of their properties such as internal injectivity and completion in Boo-S. Jónsson and Tarski in [7, 8] introduced the concept of Boolean algebras with operators. All S-Boolean algebras are special instances of Boolean algebras with operators that the set of these operators forms a monoid S. So, we are persuaded to get some categorical and algebraic structures and concepts such as congruences, coproducts, and free objects, which were not obtained in [1, 4, 5, 7, 8].

On the other hand, acts over a semigroup or monoid S, namely, S-acts, and also Boolean algebras are extended in many applications such as in theoretical computer science, algebraic automata theory, combinatorial problems, theory of machines, and graph theory. A comprehensive survey of S-acts was published by Kilp et al. in [9] and of Boolean Algebras by Koppelberg in [10] and Givant and Halmos in [11]. Recently, the study of the connection between actions and algebraic structures (for example, vector spaces, modules, S-acts, S-posets, and M-algebras) has been of interest for some authors (see for example [2, 3, 12, 13]). In [2], Ebrahimi introduced the concept of S-algebras which is the action of a monoid S on (universal) algebras. After that, the concepts of S-acts, S-poset, and soft S-act were introduced, respectively, in [3, 12, 13]. Inspired by these studies, in this article, we identified congruences and some limits and colimits such as products, coproducts, equalizers, pullbacks, and pushouts in the category Boo-S. Also the existence of free objects on a set X is shown and some adjoint situations are obtained. We should mention that the constructions in the category Boo-S are not mostly relevant to their counterparts in Act-S and Set.

Section 2 is devoted to definition and some elementary properties of S-Boolean algebras. In Section 3, we widely investigated congruences and simple, subdirectly irreducible, and indecomposable algebras in Boo-S. Coproducts and pushouts in Section 4 and free S-Boolean algebras in Section 5 are studied.

Throughout the article, S will denote a given monoid. A (right) S-act is a set A on which S acts unitarily from the right with the usual properties, that is, if there is an S-action $\mu\colon A \times S \longrightarrow A$, denoting $\mu(a,s)\colon = as$, such that $a(\mathrm{st}) = (\mathrm{as})t$ and $a1 = a$, where 1 denotes the identity of S. In fact, an S-act is a universal algebra $(A, (\mu_s)_{s\in S})$ where each $\mu_s\colon A \longrightarrow A, \mu_s(a) = as$ is a unary operation on A such that $\mu_s{}^\circ\mu_t = \mu_{ts}$ for each $s,t \in S$ and $\mu_1 = id_A$. Thus, all objects of the category Act-S form an equational class. On the other hand, considering S as a one-object category whose morphisms are the elements of S, the functor category Set^S is isomorphic to Act-S. Hence, since any functor category $\mathrm{Set}^{\mathscr{C}}$, for a small category $\mathscr{C}$, is a topos, the category Act-S is a topos (see [12]). An element a of an S-act A is called a fixed element if $as = a$ for all $s \in S$. The set of all fixed elements of A is denoted by $\mathrm{Fix}(A)$. Note that one can always adjoin a fixed element to A and get an act $A^0 = A \cup \{0\}$ with a fixed element 0. For two S-acts A and B, a map $f\colon A \longrightarrow B$ is called S-homomorphism (or S-map), if $f(\mathrm{as}) = f(a)s$ for each $a \in A, s \in S$. An equivalence relation θ on an S-act A is called an S-act congruence on A if $a\theta b$ implies $as\theta bs$ for all $a,b \in A$ and $s \in S$. If θ is a congruence on A, then the factor set $A/\theta = \{[a]_\theta \mid a \in A\}$, with the action given by $[a]_\theta s = [as]_\theta$, for $a \in A$ and $s \in S$, is clearly an S-act, called the factor act of A by θ. Considering S to be a one element monoid, the category Act-S is equivalent to the category Set and so the categorical constructions are obtained similarly for sets, for more information and the notions not mentioned here such as monomorphism, epimorphism, isomorphism, product, coproduct, equalizer, coequalizer, pullback, pushout, and free objects, about the category Act-S, see [9, 12]. Also, the interested reader is referred to [10, 11] for some required definitions and basic categorical ingredients of Boolean algebras needed in the sequel.

2. Boolean Algebras with Semigroup Operations

In this section, we give a brief account of some basic definitions and elementary properties about S-Boolean algebras needed in the sequel. As we say in Section 1, recalling the general notion of an algebra in a category, we discussed Boolean algebras in the category Act-S. More generally (see, [2]), an S-algebra A is an (ordinary) algebra of type τ which is also an S-act such that each operation $\lambda\colon A^n \longrightarrow A$ is an S-map, or equivalently, each S-action $\mu_s\colon A \longrightarrow A$ defined by $\mu_s(a) = as$ is an algebra homomorphism. Thus, the category Boo-S of right S-Boolean algebras is an example of S-algebras. In other words, we have the following definition:

Definition 1 (see [4]). Let S be a monoid. A right S-Boolean algebra is a (possibly empty) Boolean algebra $(A, \vee, \wedge, ', \bot, \top)$ which is also an S-act whose Boolean algebra operations are equivariant, that is, $(a\vee b)s = as\vee bs$, $(a\wedge b)s = as\wedge bs$, $(as)' = a's$, $\bot s = \bot$ and $\top s = \top$ for each $a,b \in A$ and $s \in S$.

Let S be a monoid. Then, every Boolean algebra A can be considered as a right S-Boolean algebra, by trivial action define as $xs = x$ for all $x \in A$ and all $s \in S$. Thus, the category Boo is a full subcategory of Boo-S.

Lemma 1. *Let a Boolean algebra $(A, \vee, \wedge, ', \bot, \top)$ be an S-act. The following are equivalent:*

(i) A is an S-Boolean algebra.

(ii) For every $a,b \in A$ and $s \in S$,

$$\begin{aligned} (a\vee b)s &= as\vee bs,\\ (as)' &= a's. \end{aligned} \tag{1}$$

(iii) For every $a,b \in A$ and $s \in S$,

$$\begin{aligned} (a\wedge b)s &= as\wedge bs,\\ (as)' &= a's. \end{aligned} \tag{2}$$

Proof. (i)⇒(ii) Clear. (ii)⇒(iii) Let $a,b \in L$ and $s \in S$. Then,

$$\begin{aligned} (a\wedge b)s &= \left(a'\vee b'\right)'s = \left(\left(a'\vee b'\right)s\right)' = \left(a's\vee b's\right)'\\ &= \left((bs)'\right)' = \mathrm{as}\wedge\mathrm{bs}. \end{aligned} \tag{3}$$

(iii)⇒(i) The proof of this part is similar to (ii)⇒(iii).

From now on we use Lemma 1, part (ii) or (iii) for the definition of S-Boolean algebra. Also, one can define an order on an S-Boolean algebra as follow: $a \le b$ if and only if $a\wedge b = a$. It is easy to check that if $a \le b$, then $as \le bs$.

By an S-Boolean algebra map (or homomorphism), we mean a map $f\colon A \longrightarrow B$ between right S-Boolean algebras which preserves binary operation $\vee$ (or $\wedge$), unary operation $'$ and the action. □

Remark 1. For a homomorphism $f\colon A \longrightarrow B$ and $a,b \in A$,

(i) $f(\bot_A) = \bot_B$ and $f(\top_A) = \top_B$

(ii) if $a \le b$, then $f(a) \le f(b)$

Let $\{A_i\}$ be a family of right S-Boolean algebras. The product of $\{A_i\}$ is their Cartesian product, with component wise action and operations. Clearly, the category Boo-S is product complete (i.e., for every family $\{A_i\}$ of S-boolean algebras, the product exists in Boo-S) and determined up to isomorphism. In particular, the terminal S-boolean algebra (the product of the empty family of S-Boolean algebras) is a one element object. Now, we are ready to explain a class of right S-Boolean algebras.

Example 1. Consider the Boolean algebra $A_0 = \{\bot, \top\}$ and the group $S = \{1, s_0\}$.

(1) Let $A_1 = \mathbf{2}^2 = \{\bot, a, a', \top\}$. It is not difficult to check that A_1 by the action satisfying $as_0 = a'$ and $a's_0 = a$ is an S-Boolean algebra.

(2) Let A_2 be the Boolean algebra with cardinality 2^3. Then, $A_2 \cong A_0 \times A_1$ which is an S-act by the action given by $(x, y)s = (xs, ys)$, for each $x \in A_0$ and $y \in A_1$. Using part (1), $((x, y) \vee (z, w))s = (x \vee z, \quad y \vee w)s = ((x \vee z)s, (y \vee w)s) = (xs \vee zs, ys \vee ws) = (xs, ys) \vee (zs, ws) = ((x, y)s) \vee ((z, w)s)$ and $(x, y)^{'}s = (x^{'}, y^{'})s = (x^{'}s, y^{'}s) = ((xs)^{'}, (ys)^{'}) = (xs, ys)^{'} = ((x, y)s)^{'}$. Thus, A_2 is a right S-Boolean algebra.

(3) Let A be a Boolean algebra with 2^n elements. Then, using part (1) and (2) by induction we can easily show that A is an S-Boolean algebra.

Example 2. For a S-Boolean algebra A, consider S to be the set of all Boolean endomorphisms on A. By the binary operation given by, for each $f, g \in S$, $(f * g)(a) = g(f(a))$, clearly S is a monoid. Define an action on A as follow, for each $a \in A$ and $f \in S$, $a.f = f(a)$. Now, it is not difficult to check that A is an S-Boolean algebra.

By definition, a sub-Boolean algebra of a Boolean algebra B is a subset A of B which is closed under $\vee$ and $'$. Also, a subset A of an S-act B is called an S-subact, if for each $a \in A$ and $s \in S$, $as \in A$. So, a subset A of an S-Boolean algebra B is said to be a sub S-Boolean algebra of B, if it is a sub Boolean algebra of B as well as an S-subact. Note that in which case binary and nullary operations are compatible with the S-action. For a subset $X \subseteq B$, a sub S-Boolean algebra of B generated by X, is a smallest sub S-Boolean algebra of B containing X. For an S-Boolean algebra B and $X \subseteq B$, denote $X^{'} = \{x^{'} \mid x \in X\}$. An element $y \in B$ is called an elementary product on X, if $y = x_1 s_1 \wedge x_2 s_2 \wedge \ldots \wedge x_n s_n$ in which $x_i \in X \cup X^{'}$ and $s_i \in S$. By the properties of right S-Boolean algebras we can easily show the following theorem:

Theorem 1. *For a right S-Boolean algebra B and a subset $X \subseteq B$, the subset*

$$\langle X \rangle = \{y_1 \vee y_2 \vee \ldots \vee y_k \mid y_i \text{is elementary produ cts on } X\}, \tag{4}$$

is the least sub S-Boolean algebra of B, called the sub S-Boolean algebra of B generated by X.

We consider the set of all covariant functors from the one-object category S to the category Boo and natural transformations between them, as a category which called functor category and denoted by Boo^{S}.

Theorem 2. *The category S-Boo of left S-Boolean algebras is isomorphic to the category* Boo^{S}.

Proof. Consider the functor ϕ: $\text{Boo}^{\text{S}} \longrightarrow$ S-Boo as follows: For each $F \in \text{Boo}^{\text{S}}$, $\phi(F) = F(S)$ with an action $sx = F(s)(x)$ and for any natural transformation τ: $F \longrightarrow G$ in S-Boo, we define $\phi(\tau)$ to be the only component τ_S of τ. Also, we consider the functor ψ: S-Boo $\longrightarrow \text{Boo}^{\text{S}}$, defined by $\psi(A)(S) = A$ and $\psi(A)(s) = \lambda_s$, for each $A \in$ S-Boo and $s \in S$. Note that each S-Boolean homomorphism f: $A \longrightarrow B$, $\psi(f)$: $\psi(A) \longrightarrow \psi(B)$ is the natural transformation whose only component is f. Now, it is not difficult to show that $\psi\phi = \text{id}_{\text{Boo}^{\text{S}}}$ and $\phi\psi = \text{id}_{\text{S-Boo}}$.

Also, clearly the category Boo-S is isomorphic to the category S^d-Boo (S^d is the dual monoid of S). In particular, if S is a commutative monoid, then the categories Boo-S and S-Boo are isomorphic, and hence the category Boo-S is isomorphic to the category Boo^{S} and this means that, as we said before, the category Boo-S is a topos. □

3. Congruences on S-Boolean Algebras

The action of a semigroup S on lattices, so called S-lattices, was defined by Luo, [14]. The author specially studied the S-lattice congruences of S-lattices. In this section, we introduced congruences of right S-Boolean algebras. Also, some characterizations of congruences generated by a subset X are investigated. Two kinds of congruence characterizations (Proposition 1 and Theorem 3) are given here. Then, relations between congruences and ideals are investigated. Finally, some results about simple, subdirectly irreducible, and indecomposable right S-Boolean algebras based on congruences are obtained.

Definition 2. An equivalence relation θ on an S-Boolean algebra A is said to be a congruence relation on A, if θ is a sub S-Boolean algebra of $A \times A$, or equivalently, for each $a, b, c \in A$ and $s \in S$, $a\theta b$ implies $(a \vee c)\theta(b \vee c)$, $a^{'}\theta b^{'}$, and $as\theta bs$.

Note that, in Definition 2, $a^{'}\theta b^{'}$ implies $(a^{'} \vee c^{'})\theta(b^{'} \vee c^{'})$. Now, Demorgan's low deduces that $(a \wedge c)\theta(b \wedge c)$. The set of all congruence relations on A is denoted by $\text{Con}(A)$. Also, each homomorphism f: $A \longrightarrow B$ induces the congruence relation, $\ker f$, on A, defined by $a_1 \ker(f) a_2$ if and only if $f(a_1) = f(a_2)$. As usual for a congruence $\rho \in \text{Con}(A)$ and $a \in A$, the congruence class of a is denoted by $[a]_\rho$, or $[a]$, and $(A/\rho) = \{[a]_\rho \mid a \in A\}$. By the action given by $[a]_\rho s = [as]_\rho$ and operations $[a]_\rho \vee [b]_\rho = [a \vee b]_\rho$, $[a]_\rho \wedge [b]_\rho = [a \wedge b]_\rho$, and $[a]_\rho^{'} = [a^{'}]_\rho$, (A/ρ) is an right S-Boolean algebras. It is easy to check that the canonical map π: $A \longrightarrow (A/\rho)$, defined by $\pi(a) = [a]_\rho$, is a homomorphism. Also, $a \leq b$ implies $[a]_\rho \leq [b]_\rho$. Note that $\rho(H)$ for $H \subseteq A \times A$ denotes the congruence generated by H (i.e., the smallest congruence on A containing H). We denote $H^{-1} = \{a, b \mid (b, a) \in H\}$. For every semigroup S without an identity one can adjoin an identity 1 by setting $1s = s = s1$ for all $s \in S$ and get an S-act denoted by S^1. By placing $1.1 = 1$, one can consider S^1 as a monoid.

Now, we are ready to explain the first congruence characterization.

Proposition 1. *Let $H \subseteq A \times A$ and $\rho = \rho(H)$. Then, for $a, b \in A$, one has $a\rho b$ if and only if either $a = b$ or for $1 \leq i \leq n, 1 \leq j \leq m_i$ there exist $(p_i, q_i) \in H \cup H^{'} \cup H^{-1} \cup (H^{-1})^{'}$, $s_i \in S^1$ and $a_{ij} \in A$, such that*

$$\begin{aligned}
a &= p_1s_1 * a_{11} * a_{12} * \cdots * a_{1m_1},\\
q_1s_1 * a_{11} * a_{12} * \cdots * a_{1m_1} &= p_2s_2 * a_{21} * a_{22} * \cdots * a_{2m_2},\\
q_2s_2 * a_{21} * a_{22} * \cdots * a_{2m_2} &= p_3s_3 * a_{31} * a_{32} * \cdots * a_{3m_3},\\
&\vdots\\
q_ns_n * a_{n1} * a_{n2} * \cdots * a_{nm_n} &= b,
\end{aligned} \tag{5}$$

where $* \in \{\vee, \wedge\}$, $m_i \in \mathbb{N} \cup \{0\}$, *and* $p_is_i * a_{i1} * a_{i2} * \ldots * a_{im_i} = (\ldots((((p_is_i * a_{i1}) * a_{i2}) * a_{i3}) * \ldots * a_{im_i})$.

Proof. It is not difficult to check that ρ is an equivalence relation on A. Let $s \in S$ and $a, b, c \in A$ such that $a\rho b$. The expression $(a\vee c)\rho(b\vee c)$ is obtained by adding $\vee c$ to the right side of all terms of the † chain. Using De Morgan's laws, we get $a\rho b$ and also since the action of S on the Boolean algebra operations of A is equivariant, we get $as\rho bs$. Now, we prove that ρ is an smallest congruence containing H. Let θ be a congruence containig H and $a\rho b$. So $a = b$ or the chain † holds. For each $1 \le i \le n$, since $p_i\theta q_i$ and θ is a congruence, $p_is_i\theta q_is_i$ and also $p_is_i * a_{i1}\theta q_is_i * a_{i1}$. By continuing this argument $p_is_i * a_{i1} * a_{i2} * \ldots * a_{im_i}\theta q_is_i * a_{i1} * a_{i2} * \ldots * a_{im_i}$. Thus, $a\theta b$ which implies $\rho \subseteq \theta$.

In what follows (the second congruence characterization), we shall often use a more explicit version of the previous proposition. □

Theorem 3. *Let A be a right S-Boolean algebra and* $H \subseteq A \times A$. *Consider the binary relation* θ *on A by for all* $x, y \in A$, $x\theta y$ *if and only if* $x = y$ *or there exist* $\{e_1, e_2, \ldots, e_n\} \subseteq A$ *and* $\{s_1, s_2, \ldots, s_{n-1}\} \subseteq S$ *such that*

$$\begin{array}{cc}
\multicolumn{2}{c}{x = e_1,}\\
e_1\wedge p_1s_1 = e_2\wedge p_1s_1, & e_1\vee q_1s_1 = e_2\vee q_1s_1,\\
e_2\wedge p_2s_2 = e_3\wedge p_2s_2, & e_2\vee q_2s_2 = e_3\vee q_2s_2,\\
\vdots, & \vdots\ (*),\\
e_{n-1}\wedge p_{n-1}s_{n-1} = e_n\wedge p_{n-1}s_{n-1}, & e_{n-1}\vee q_{n-1}s_{n-1} = e_n\vee q_{n-1}s_{n-1},\\
\multicolumn{2}{c}{e_n = y}
\end{array} \tag{6}$$

where $(p_i, q_i) \in H \cup H^{-1}$. *Then,* $\theta = \rho(H)$.

Proof. It is easy to check that θ is an equivalence relation. Suppose that $x\theta y$, $s \in S$, and $c \in A$. If $x = y$, then $x\vee c = y\vee c, x' = y'$, and $xs = ys$. Otherwise, there exist $\{e_1, e_2, \ldots, e_n\} \subseteq A$ and $\{s_1, s_2, \ldots, s_{n-1}\} \subseteq S$ such that $(*)$ holds. Clearly, $xs\theta ys$ and $(x\vee c)\theta(y\vee c)$.

It suffices to show that $x'\theta y'$. By the assumption, one gets the following equation:

$$\begin{array}{cc}
\multicolumn{2}{c}{x' = e',}\\
e_1'\vee p_1's_1 = e_2'\vee p_1's_1, & e_1'\wedge q_1's_1 = e_2'\wedge q_1's_1,\\
e_2'\vee p_2's_2 = e_3'\vee p_2's_2, & e_2'\wedge q_2's_2 = e_3'\wedge q_2's_2,\\
\vdots & \vdots\\
e_{n-1}'\vee p_{n-1}'s_{n-1} = e_n'\vee p_{n-1}'s_{n-1}, & e_{n-1}'\wedge q_{n-1}'s_{n-1} = e_n'\wedge q_{n-1}'s_{n-1},\\
\multicolumn{2}{c}{e_n' = y'.}
\end{array} \tag{7}$$

For each $1 \le i \le n-1$, $(e_i'\vee p_i's_i)\wedge p_is_i = (e_{i+1}'\vee p_i's_i)\wedge p_is_i$ deduces that $e_i'\wedge p_is_i = e_{i+1}'\wedge p_is_i$ and similarly $e_i'\vee q_is_i = e_{i+1}'\vee q_is_i$. So, we have the following equation:

$$\begin{array}{cc}
\multicolumn{2}{c}{x' = e',}\\
e_1'\wedge p_1s_1 = e_2'\wedge p_1s_1, & e_1'\vee q_1s_1 = e_2'\vee q_1s_1,\\
e_2'\wedge p_2s_2 = e_3'\wedge p_2s_2, & e_2'\vee q_2s_2 = e_3'\vee q_2s_2,\\
\vdots & \vdots\\
e_{n-1}'\wedge p_{n-1}s_{n-1} = e_n'\wedge p_{n-1}s_{n-1}, & e_{n-1}'\vee q_{n-1}s_{n-1} = e_n'\vee q_{n-1}s_{n-1},\\
\multicolumn{2}{c}{e_n' = y'.}
\end{array} \tag{8}$$

Thus, $x'\theta y'$ which deduces that θ is a congruence relation an A. Suppose that $(a, b) \in H$, consider $e_1 = a$, $e_2 = a\wedge b$, $e_3 = b$, $s_1 = s_2 = 1$, $p_1 = b$, $q_1 = a$, $p_2 = a$, and $q_2 = b$. So, $a\theta b$, and hence $H \subseteq \theta$. Now, let ρ be a congruence relation containing H and $x\theta y$. Thus, $x = y$ or the $(*)$ holds. For each $1 \le i \le n-1$, $e_i = e_i\vee(e_i\wedge p_is_i) = (e_i\vee(e_{i+1}\wedge p_is_i))\rho\ (e_i\vee(e_{i+1}\wedge q_is_i)) = (e_i\vee e_{i+1})\wedge(e_i\vee q_is_i) = ((e_i\vee e_{i+1})\wedge(e_{i+1}\vee q_is_i))\ \rho(e_{i+1}\vee(e_i\wedge p_is_i)) = e_{i+1}\vee(e_{i+1}\wedge p_is_i) = e_{i+1}$. Thus, $e_i\rho e_{i+1}$, and hence $x = e_1\rho e_2\rho\ldots\rho e_n = y$. Therefore, $\theta \subseteq \rho$ and then θ is the smallest congruence containing H. □

Corollary 1. *For a right S-Boolean algebra A and* $H \subseteq A \times A$, *if* $x\rho(H)y$, *then there exist* $n \in \mathbb{N} \cup \{0\}$, $\{s_1, s_2, \ldots, s_n\} \subseteq S$, *and* $(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n) \in H \cup H^{-1}$ *such that*

(i) $x\wedge p_1s_1\wedge\ldots\wedge p_ns_n = y\wedge p_1s_1\wedge\ldots\wedge p_ns_n$;

(ii) $x\vee q_1s_1\vee\ldots\vee q_ns_n = y\vee q_1s_1\vee\ldots\vee q_ns_n$.

Corollary 2. *Let* $H \subseteq A \times A$ *and* $K = \{a \in A \mid \exists b \in A \text{ s.t } (a, b) \in H \cup H^{-1}\}$, *the union of domain and image of H. If* $x\rho(H)y$, *then there exist* $a, b \in \langle K\rangle$ *such that* $x\wedge a = y\wedge a$ *and* $x\vee b = y\vee b$

Definition 3. An S-ideal of a right S-Boolean algebra A is a Boolean ideal of A which is an S-sub act as well. The set of all S-ideals of an S-Boolean algebra A is denoted by $ID_S(A)$.

Proposition 2. *Let I be an S-ideal in an S-Boolean algebra A and* $\rho_I = \rho(I \times I)$. *Then, the following are equivalent:*

(i) $x\rho_I y$.

(ii) $x\vee u = y\vee u$ *for some* $u \in I$.

(iii) $x\wedge v = y\wedge v$ for some $v \in I'$.

(iv) $x\Delta y \in I$, *where* $x\Delta y = (x\wedge y')\vee(x'\wedge y)$.

Proof

(i) ⇒ (ii) Let $x\rho_I y$. Using Corollary 1 in which $u = q_1s_1\vee\ldots\vee q_{n-1}s_{n-1} \in I$.

(ii) ⇒ (iii) $x\wedge u' = (x\vee u)\wedge u' = (y\vee u)\wedge u' = y\wedge u'$.

(iii) ⇒ (ii) $x\vee v' = (x\wedge v)\vee v' = (y\wedge v)\vee v' = y\vee v'$.

(ii) ⇒ (i) Consider $e_1 = x$, $e_2 = y$, $s_1 = 1$, $p_1 = \perp$ and $q_1 = u$. So, $x\rho_I y$.

(iv) ⇒ (ii) It is easy to check that $x\vee(x\triangle y) = x\vee y = y\vee(x\triangle y)$.

(iii) ⇒ (iv) $(x\triangle y)\wedge v = (x\wedge v)\triangle(y\wedge v) = (x\wedge v)\triangle(x\wedge v) = \perp_A$. Thus, $(x\triangle y)\leq v' \in I$, and therefore $(x\triangle y)\in I$. □

Corollary 3. *Let I be an S-ideal in an S-Boolean algebra A and $a\in I$. Then,*

(i) $I = [a]_{\rho_I}$.

(ii) ρ_I is the greatest congruence on A having I as a whole class.

Proof

(i) It is clear that $I\subseteq[a]_{\rho_I}$. For the converse, let $x\in[a]_{\rho_I}$. Then, there exists $u\in I$ such that $x\vee u = a\vee u\in I$. Thus, $x\in I$, and hence $[a]_{\rho_I} = I$.

(ii) Let $\theta\in Con(A)$, $I=[a]_\theta$, and $x\theta y$. Then, $(x\vee y)\theta(x\wedge y)$, which implies $x\triangle y = (x\wedge y)'\wedge(x\vee y)\theta(x\wedge y)'\wedge(x\wedge y) = \perp_A \in I = [a]_\theta$. So, $(x\triangle y)\theta a$, and hence $x\triangle y\in I$. Since $x\vee(x\triangle y) = y\vee(x\triangle y)$, by using Proposition 2 (ii), $x\rho_I y$. Thus, $\theta\subseteq\rho_I$.

Using Corollary 3, one has $[\perp_A]_{\rho_I} = I$. So, we have the following theorem. □

Theorem 4. *For an S-Boolean algebra A, there is a one to one correspondence between the sets $ID_S(A)$ and $Con_S(A)$.*

For an S-Boolean algebra A, the congruences $\Delta = \{(a,a)\mid a\in A\}$ and $\nabla = A\times A$ are called diagonal congruence and universal congruence, respectively. A congruence θ is called *trivial*, if it is universal or diagonal. An S-Boolean algebra A is called simple if $Con_S(A) = \{\Delta,\nabla\}$. As a consequence of Theorem 4, we have A is simple if and only if $ID_S(A) = \{\{\perp\}, A\}$.

In the following lemma, S-ideals generated by subsets of an S-Boolean algebra are constructed.

Lemma 2. *Let A be an S-Boolean algebra and $X\subseteq A$. The set $L(X) = \{a\in A\mid \exists n\in N, s_1,s_2,\ldots,s_n\in S, x_1,x_2,\ldots,x_n\in X$ s.t $a\leq x_1s_1\vee\ldots\vee x_ns_n\}$ is the smallest S-ideal containing X.*

Proof. Clearly, $X\subseteq L(X)$ and $x\leq y\in L(X)$ implies that $x\in L(X)$. Also, $L(X)$ is closed under $\vee$ and $\wedge$ which is an S-sub act of A, which is a subset of any S-ideal of A containing X, so it is the smallest S-ideal containing X.

Using Lemma 2, we have the following proposition: □

Proposition 3. *An S-Boolean algebra A is simple, if and only if for each $\perp\neq a\in A$ there exist $s_1,s_2,\ldots,s_n\in S$ such that $\vee_{i=1}^n as_i = \top$.*

In view of Proposition 3, we get the following example. It shows that for every Boolean algebra A one can consider a monoid S and an action of S on A, such that the induced S-Boolean algebra is simple.

Example 3. Let $(A,\vee,\wedge,',\perp,\top)$ be a Boolean algebra. The set of all Boolean endomorphisms on A endowed with composition of morphisms, denoted by $S = (\mathrm{Hom}_B(A,A),^\circ)$, is a monoid. It is not difficult to check that A by the left S-action, $f.a = f(a)$ for $a\in A$ and $f\in S$, is a left S-Boolean algebra. Using [11], Th. 20.12, for each $\perp\neq a\in A$, there exists a proper maximal ideal M of A containing a'. The map $f\colon A\longrightarrow A$ is defined by $f(x)=\perp$, if $x\in M$ and $f(x)=\top$ if $x\in A\setminus M$ is a Boolean endomorphism. So, for each $\perp\neq a\in A$, there exists an endomorphism f such that $f.a = f(a) = \top$. Now, Proposition 3 shows that A is a simple left S-Boolean algebra.

If ρ and θ are congruences on an S-Boolean algebra A, then the relational product $\rho^\circ\theta$ is the binary relation on A defined by $(a,b)\in\rho^\circ\theta$, if and only if there is $c\in A$ such that $(a,c)\in\rho$ and $(c,b)\in\theta$. Also, the pair ρ,θ is called a pair of factor congruences on A, if $\rho\cap\theta=\Delta$, $\rho\vee\theta=\nabla$, and $\rho^\circ\theta=\theta^\circ\rho$.

Definition 4

(i) An S-Boolean algebra A is said to be indecomposable if A is not isomorphic to a direct product of two nontrivial right S-Boolean algebras

(ii) An S-Boolean algebra A is called subdirectly irreducible, if A is trivial or there is a minimum congruence in $\mathrm{Con}(A)\backslash\{\Delta\}$ (i.e., the intersection of all nontrivial congruences is nontrivial)

By [15], Corollary II.7.7, an S-Boolean algebra A is indecomposable if and only if the only factor congruences on A are Δ and ∇. Unlike the case of Boolean algebras, in which every simple, subdirectly irreducible, and indecomposable algebras are equal to **2**, in the category Boo-S, these mentioned concepts are not necessary to be equal (example 2 and 3). Also, example 1 shows that, we can construct a simple S-Boolean algebra of each Boolean algebra.

Theorem 5. *An S-Boolean algebra A is indecomposable if and only if* $\mathrm{Fix}(A) = \{\perp,\top\}$.

Proof. Let A be an indecomposable S-Boolean algebra and $a\in A$ a fixed element such that $a\notin\{\perp,\top\}$. We define the binary relation θ and ρ as follows:

$$x\theta y\Leftrightarrow x\triangle y\leq a, \text{and } x\rho y\Leftrightarrow x\triangle y\leq a'. \tag{9}$$

Then, $\theta,\rho\in\mathrm{Con}(A)$. We only show the transitive property for θ. Let $x\triangle y\leq a$ and $y\triangle z\leq a$. So, $x\wedge z' = (x\wedge z')\wedge(y\vee y') = [x\wedge(y\vee y')]\wedge z' = [(x\wedge y)\wedge z']\vee[(x\wedge y')\wedge z']\leq (y\wedge z')\vee(x\wedge y') \leq (y\triangle z)\vee(x\triangle y)\leq a\vee a = a$, and similarly $x'\wedge z\leq a$. Thus, $x\triangle z\leq a$.

It is clear that $a\theta\perp$ and $a'\rho\perp$. Thus, $\theta\neq\Delta\neq\rho$. If $(x,y)\in\theta\cap\rho$, then $x\triangle y\leq a\wedge a' = \perp$ and hence $x=y$. Thus, $\theta\cap\rho=\Delta$. Now, we consider $\Theta=\theta\vee\rho$; the congruence generated by $\theta\cup\rho$. For each $x,y\in A$ we have the following equation:

$$x\theta(x\vee a)\theta(x\vee(y\wedge a))\rho((x\vee y)\wedge a)\rho(y\vee(x\wedge a))\theta y. \quad (10)$$

Thus, $x\Theta y$, by transitivity. So, $\Theta=\nabla$. Next, we showed that $\rho^\circ\theta=\theta^\circ\rho$. Let $(x,y)\in\theta^\circ\rho$. Then, there exists $c\in A$ such that $x\theta c\rho y$, and hence $x\triangle c\le a$ and $y\triangle c\le a'$. Consider $d=(x\wedge a)\vee(y\wedge a')$. So, $x\wedge d'\le a'$ and $x'\wedge d\le a'$, which implies $x\triangle d\le a'$ and similarly, $d\triangle y\le a$. Hence, $(x,y)\in\rho^\circ\theta$ and $\theta^\circ\rho\subseteq\rho^\circ\theta$. In a similar way, we concluded $\rho^\circ\theta\subseteq\theta^\circ\rho$. Therefore, ρ,θ is a pair of factor congruences on A, which is a contradiction. Thus, $\mathrm{Fix}(A)=\{\bot,\top\}$.

For the converse, let A be a decomposable S-Boolean algebra. So, there exists a pair of nontrivial factor congruences ρ,θ on A. By [15], Th. II.5.9, Corollary II.7.7, $\nabla=\rho\vee\theta=\rho^\circ\theta$. Thus, there exists nontrivial element $c\in A$ with $\bot\rho c\theta\top$. Hence, for each $s\in S$, $cs\rho\bot\rho c\theta\top\theta cs$ which deduces $(cs,c)\in\rho\cap\theta$. Thus, $cs=c$, a contradiction.

As we mentioned before, each indecomposable Boolean algebra is isomorphic to **2** (see [15]). Theorem 5 is a generalization for that results. Indeed, if we consider the Boolean algebra B, then by the note just after Definition 1, B is a right S-Boolean algebra with trivial action. Clearly, in this S-Boolean algebra each element is a fix element. So, Theorem 5 implies that B is an indecomposable S-Boolean algebra iff $B=\{\bot,\top\}$.

Using [15], Theorem. II.8.5, each simple algebra is subdirectly irreducible and each subdirectly irreducible algebra is indecomposable. In what follows, we introduce two examples to show the converse is not generally true.

Let X be a subset of a poset $(P\le)$. Then, $\downarrow X:=\{x\in P\mid x\le p,\exists p\in P\}$ and $\uparrow X:=\{x\in P\mid p\le x,\exists p\in P\}$. Moreover, if $X=\{p\}$, then we use $\downarrow p$ ($\uparrow p$) instead of $\downarrow\{p\}$ ($\uparrow\{p\}$), for simplicity. □

Example 4. There is a four elements subdirectly irreducible right S-Boolean algebras, which is not simple. Consider S an arbitrary semigroup and the Boolean algebra $A=\{\bot,a,a',\top\}$ by the action $as=\bot$ and $a's=\top$. It is not difficult to show that A has only one nontrivial S-ideal $I=\downarrow a$. So, by Theorem 4, A has only one nontrivial congruence. Thus, A is subdirectly irreducible which is not simple.

Example 5. If a Boolean algebra B has at least two atoms, then there is an action on B such that B is an indecomposable and not subdirectly irreducible S-Boolean algebra. Let $b\in B$ be an atom. The subsets $\uparrow b$ and $\downarrow b'$ are a partition of B. Consider two elements left zero semigroup $\{1,s\}$ with 1 an identity and define an action $xs=\begin{cases}\bot & \text{if } x\in\downarrow b'\\ \top & \text{if } x\in\uparrow b\end{cases}$. By this action B is a right S-Boolean algebra. Now, B is indecomposable by using Theorem 5. For each atom a, the subset $\downarrow a'$ is a right ideal of B. Let Λ be the set of all atoms of B. Since $\bigcap_{a\in\Lambda}\downarrow a'=\{\bot\}$, B is not subdirectly irreducible.

4. Coproduct

In [16], R. Lagrange studied the Amalgamation and epimorphisms in the category of $\mathscr{M}$-complete Boolean algebras. After that, Banaschewski [17] introduced the structure of strong amalgamations of Boolean algebras. In this section, we showed, based on [17], the coproduct (free product) of a family of right S-Boolean algebras exists and consequently, the pushout of two S-Boolean homomorphisms with same domain, exists as a qoutient of their coproduct.

Using [17], for Boolean algebras B and C, we consider the lattice $\mathscr{L}=\{U\subseteq B\times C\mid\downarrow U=U\}$ by the following requirements:

$$\begin{aligned}&(\vee S,c)\in U \text{ whenever } S\times\{c\}\subseteq U,\\&(b,\vee T)\in U \text{ whenever}\{b\}\times T\subseteq U,\end{aligned} \quad (12)$$

for all finite $S\subseteq B$ and $T\subseteq C$. Also, the maps

$$\begin{aligned}&\lambda\colon B\longrightarrow\mathscr{L}, x\longmapsto\downarrow(x,\top_C)\cup\downarrow(\top_B,\bot_C),\\&\mu\colon C\longrightarrow\mathscr{L}, y\longmapsto\downarrow(\top_B,y)\cup\downarrow(\bot_B,\top_C),\end{aligned} \quad (13)$$

are bounded lattice embeddings so that $\mathrm{Im}(\lambda)\cong B$ and $\mathrm{Im}(\mu)\cong C$ are sublattices of $\mathscr{L}$ consisting of complemented elements.

Coproduct of Boolean algebras B and C is a sublattice $\mathscr{M}$ of a lattice $\mathscr{L}$ generated by $\lambda(B)\cup\mu(C)$ in which both B and C are embedded.

Furthermore, for each $b\in B$ and $c\in C$, $\lambda(b)\wedge\mu(c)=\downarrow(b,c)\cup\downarrow(\top_B,\bot_C)\cup\downarrow(\bot_B,\top_C)$. So $\lambda(b)=\lambda(b)\wedge\mu(\top_C)$ and $\mu(c)=\lambda(\top_b)\wedge\mu(c)$. Also, if $b=\bot_B$ or $c=\bot_C$, then

$$\begin{aligned}\lambda(b)\wedge\mu(c)&=\lambda(\bot_B)\wedge\mu(\bot_C)\\&=\downarrow(\top_B,\bot_C)\cup\downarrow(\bot_B,\top_C)=\bot_{\mathscr{M}},\end{aligned} \quad (14)$$

and if b_1 and c_1 are not bottom and $\lambda(b_1)\wedge\mu(c_1)=\lambda(b_2)\wedge\mu(c_2)$, then b_2 and c_2 are not bottom as well as $b_1=b_2$ and $c_1=c_2$.

Now, if B and C are right S-Boolean algebras, for each $b\in B$, $c\in C$ which are not bottoms and $s\in S$, define $[\lambda(b)\wedge\mu(c)]s=\lambda(bs)\wedge\mu(cs)$ and if $b=\bot_B$ or $c=\bot_C$, then $[\lambda(b)\wedge\mu(c)]s=\bot_{\mathscr{M}}$. Also, for each $y=\vee_i(\wedge_j x_{ij})\in\mathscr{M}$ in which $x_{ij}\in\lambda(B)\cup\mu(C)$, we defined $ys=\vee_i(\wedge_j x_{ij}s)$. Using these actions, in the following, we deduced that $\mathscr{M}$ is an S-Boolean algebra which is a coproduct of B and C, and both λ and μ are S-Boolean monomorphisms.

The abovementioned actions are well defined. Assume that $s\in S$, $b_1,b_2\in B$, $c_1,c_2\in C$, and $\lambda(b_1)\wedge\mu(c_1)=\lambda(b_2)\wedge\mu(c_2)$. Let b_1 and c_1 are not bottom. Then, b_2 and c_2 are not bottom and $b_1=b_2$ and $c_1=c_2$. Thus, $b_1s=b_2s$ and $c_1s=c_2s$, which implies that $\lambda(b_1s)\wedge\mu(c_1s)=\lambda(b_2s)\wedge\mu(c_2s)$. If b_1 or c_1 are bottom, then b_2 or c_2 are bottom too and hence $\lambda(b_1s)\wedge\mu(c_1s)=\bot_{\mathscr{M}}=\lambda(b_2s)\wedge\mu(c_2s)$. Thus, the action $[\lambda(b)\wedge\mu(c)]s=\lambda(bs)\wedge\mu(cs)$ is well defined.

We will show the action $\vee_i(\wedge_j x_{ij})s=\vee_i(\wedge_j x_{ij}s)$ is well defined. Let $s\in S$ and $x=\vee_{i=1}^n(\wedge_{j=1}^{n_i}x_{ij})=\vee_{k=1}^m(\wedge_{l=1}^{m_k}y_{kl})=y$ in which $x_{ij},y_{kl}\in\lambda(B)\cup\mu(C)$. Thus, $x_{ij}=\lambda(b_{ij})\wedge\mu(c_{ij})$ such that $b_{ij}\in B$ and $c_{ij}\in C$. Using the fact that,

$(\lambda(b_1)\wedge\mu(c_1))\vee(\lambda(b_2)\wedge\mu(c_2)) = \lambda(b_1\vee b_2)\wedge\mu(c_1\vee c_2)$, we have the following equation:

$$\begin{aligned} x &= \vee_{i=1}^{n}\left(\wedge_{j=1}^{n_i} x_{ij}\right) \\ &= \vee_{i=1}^{n}\left(\wedge_{j=1}^{n_i}\left(\lambda\left(b_{ij}\right)\wedge\mu\left(c_{ij}\right)\right)\right) \\ &= \lambda\left(\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{n_i} b_{ij}\right)\right)\wedge\mu\left(\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{n_i} c_{ij}\right)\right), \end{aligned} \tag{15}$$

and in a similar way $y = \lambda(\vee_{k=1}^{m}(\wedge_{l=1}^{m_k} e_{kl})\wedge\mu(\vee_{k=1}^{m}(\wedge_{l=1}^{m_k} f_{kl})$. If one of the elements $\vee_{i=1}^{n}(\wedge_{j=1}^{n_i} b_{ij})$, $\vee_{i=1}^{n}(\wedge_{j=1}^{n_i} c_{ij})$, $\vee_{k=1}^{m}(\wedge_{l=1}^{m_k} e_{kl})$, and $\vee_{k=1}^{m}(\wedge_{l=1}^{m_k} f_{kl})$ is bottom, then $x = y = \perp_{\mathscr{M}}$. Now, let the mentioned elements are not bottoms. Then, $\vee_{i=1}^{n}(\wedge_{j=1}^{n_i} b_{ij}) = \vee_{k=1}^{m}(\wedge_{l=1}^{m_k} e_{kl})$ and $\vee_{i=1}^{n}(\wedge_{j=1}^{n_i} c_{ij}) = \vee_{k=1}^{m}(\wedge_{l=1}^{m_k} f_{kl})$ and by using the previous paragraph and the fact that B and C are right S-Boolean algebras, we have the following equation:

$$\begin{aligned} xs &= \lambda\left(\left(\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{n_i} b_{ij}\right)\right)s\right)\wedge\mu\left(\left(\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{n_i} c_{ij}\right)\right)s\right) \\ &= \lambda\left(\left(\bigvee_{k=1}^{m}\left(\bigwedge_{k=1}^{m_k} e_{kl}\right)\right)s\right)\wedge\mu\left(\left(\bigvee_{k=1}^{m}\left(\bigwedge_{k=1}^{m_k} f_{kl}\right)\right)s\right) = ys. \end{aligned} \tag{16}$$

Similar calculations show that $(x\vee y)s = xs\vee ys$ and $(xs)' = x's$. Therefore, $\mathscr{M}$ is an S-Boolean algebra.

Also $\mathscr{M}$ is an S-Boolean coproduct of B and C. For, let $f\colon B \longrightarrow A$ and $g\colon C \longrightarrow A$ be two S-Boolean maps. Since $\mathscr{M}$ is a coproduct in the category of Boolean algebras, there exists Boolean homomorphism $h\colon \mathscr{M} \longrightarrow A$ such that $h\lambda = f$ and $h\mu = g$. Consider $s \in S$ and $x = \vee_{i=1}^{n}(\wedge_{j=1}^{n_i} x_{ij})$ in which $x_{ij} \in \lambda(B)\cup\mu(C)$. If $x_{ij} \in \lambda(B)$, then $h(x_{ij})s = h\lambda(b_{ij})s = f(b_{ij})s = h\lambda(b_{ij}s) = h(x_{ij}s)$ and similarly for the case $x_{ij} \in \mu(C)$. So $h(x)s = [h(\vee_{i=1}^{n}(\wedge_{j=1}^{n_i} x_{ij}))]s = \vee_{i=1}^{n}(\wedge_{j=1}^{n_i} h(x_{ij}, s)) = h\vee_{i=1}^{n}(\wedge_{j=1}^{n_i} x_{ij}, s) = h(xs)$. Thus, h is an S-Boolean homomorphism. Also, h is unique.

As a corollary of the existence of coproducts, it is natural to consider the pushouts in the category of right S-Boolean algebras.

Lemma 3. *Pushout of the following diagram exists.*

Proof. Let $(\tau_i, A_1\coprod A_2)$ be the coproduct of the pair (A_1, A_2) and I be a right ideal of $A_1\coprod A_2$ generated by the set $\{\tau_1 h_1(c)\triangle\tau_2 h_2(c) \mid c \in C\}$. We consider $Q = A_1\coprod A_2/\rho_I$, in which ρ_I is a congruence generated by $I\times I$ and $p_i = \pi\tau_i\colon A_i \longrightarrow Q$, where π is a canonical map. Now, it is not difficult to show that (Q, p_1, p_2) is a pushout of the diagram. For $c \in C$, $\pi\tau_1 h_1(c)\triangle\pi\tau_2 h_2(c) = \pi(\tau_1 h_1(c)\triangle\tau_2 h_2(c)) = \perp_Q$. Thus, $p_1 h_1 = P_2 h_2$. To prove the universal property of pushouts, let $f_i\colon A_i \longrightarrow A$ be homomorphisms such that $f_1 h_1 = f_2 h_2$. So, there is a homomorphism $h\colon A_1\coprod A_2 \longrightarrow A$ satisfying $h\tau_i = h_i$. On the other hand, $h(\tau_1 h_1(c)\triangle\tau_2 h_2(c)) = h\tau_1 h_1(c)\triangle h\tau_2 h_2\ (c) = f_1 h_1(c)\triangle f_2 h_2\ (c) = f_1 h_1(c)\triangle f_1 h_1(c) = \perp_A$. Thus, $I\subseteq K(h) = \{x \in A_1\coprod A_2 \mid h(x) = \perp_A\}$, and hence there is a homomorphism $f\colon Q \longrightarrow A$ such that $f\pi = h$. It follows that $fp_i = f_i$. Moreover, f is uniquely determined, which implies that Q is a pushout of the diagram in Figure 1.

In [17], Banacschewski has shown that, in the category of Boolean algebras, the pushout of the previous diagram is a qoutient of the coproduct as $A_1\coprod A_2/\rho_J$ in which J is an ideal generated by the set $\{\tau_1 h_1(c)\wedge\tau_2 h_2(c') \mid c \in C\}$. It is not difficult to show that $I = J$, and hence $\rho_I = \rho_J$. Thus, $A_1\coprod A_2/\rho_I$ is also a coproduct of the diagram in Figure 1.

Finally, we showed that pushouts preserve monomorphisms, that is, in the pushout diagram of Lemma 3, p_2 is a monomorphism whenever h_1 is a monomorphism. Let $x \neq \perp_{A_2}$ and $p_2(x) = \perp_Q$. Then, $\pi\tau_2(x) = \pi(\perp_{A_1\coprod A_2})$ and by Corollary 3.7, there exists $u \in I$ such that $\tau_2(x)\vee u = \perp_{A_1\coprod A_2}\vee u = u$ which implies $\tau_2(x) \leq u$. Hence, $\tau_2(x)\wedge\tau_1(\top_{A_1}) = \tau_2(x) \in I$, and by [17], there exists $c \in C$ such that $\tau_2(x)\wedge\tau_1(\top_{A_1}) \leq \tau_2(h_2(c))\wedge\tau_1(h_1(c'))$. So, by [17], comparison principle, $x \leq h_2(c)$ and $\top_{A_1} \leq h_1(c')$ which deduces $\top_{A_1} = h_1(c')$. Thus, $h_1(c) = \perp_{A_1}$ and since h_1 is a monomorphism, $c = \perp_C$. Hence, $x \leq h_2(\perp_C) = \perp_{A_2}$, a contradiction, showing that p_2 is a monomorphism. ☐

5. Free S-Boolean Algebras and Adjoint Situations

By an act − free (Boolean − free) right S-Boolean algebra on a right S-act (Boolean algebra) X we mean an S-Boolean algebra $F(X)$ with an S-map (Boolean homomorphism) $\gamma\colon X \longrightarrow F$ which has the following universal property. For each S-Boolean algebra A and an S-map (Boolean homomorphism) $f\colon X \longrightarrow A$ there exists a unique right S-Boolean homomorphism $\overline{f}\colon F \longrightarrow A$ such that $\overline{f}\gamma = f$. In particular, a set − free right S-Boolean algebra on a set X is an act-free by taking the set X to be an S-act with trivial action (each element is fixed).

Lemma 4. *If X is a right S-act, the power set Boolean algebra $P(X)$ with the following action is a left S-Boolean algebra. For each $s \in S$ and $U \in P(X)$, $sU = \{x \in X \mid xs \in U\}$.*

Note that, by a similar argument, Lemma 4 is also true for a left S-act.

Lemma 5. *Let X be an infinite right S-act. Then, there exists an act-free S-Boolean algebra on X.*

Proof. Consider a map $\gamma\colon X \longrightarrow P(P(X))$ defined by, $\gamma(x) = \{U \mid x \in U \in P(X)\}$. By twice using Lemma 4, $P(P(X))$ is a right S-act. At first, we showed that γ is an S-act monomorphism. It is clear that γ is well defined. Note that, for each $x \in X$ and $s \in S$,

$$\begin{aligned} \gamma(xs) &= \{U \mid xs \in U \in P(X)\} \\ &= \{U \mid x \in sU \in P(X)\}, \\ \gamma(xs) &= \{W \in P(X) \mid sW \in \gamma(x)\} \\ &= \{W \in P(X) \mid x \in sW\}. \end{aligned} \tag{17}$$

If $\gamma(x_1) = \gamma(x_2)$, then $x_1 \in \{x_1\} \in \gamma(x_1) = \gamma(x_2)$ and hence $x_2 \in \{x_1\}$ which implies $x_1 = x_2$. Thus, γ is a right S-act monomorphism. Also, $\gamma(x)'s = \gamma(xs)'$, since

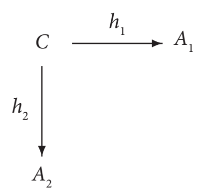

Figure 1: Pushout.

$$\begin{aligned}\gamma(x)'s &= \left\{W \in P(X) \mid sW \in \gamma(x)'\right\}\\ &= \{W \in P(X) \mid x \notin sW\}\\ &= \{W \in P(X) \mid xs \notin W\}\\ &= \gamma(xs)'.\end{aligned} \tag{18}$$

We showed that $\langle\gamma(X)\rangle$, the sub-S-Boolean algebra of $P(P(X))$ generated by $\gamma(X)$, is a free S-Boolean algebra generated by X. By Theorem 1, each $y \in \langle\gamma(X)\rangle$ is of the form $y = y_1 \vee y_2 \vee \ldots \vee y_k$ in which each $y_j = \gamma^{\alpha_{j1}}(x_{j1})s_{j1} \wedge \gamma^{\alpha_{j2}}(x_{j2})s_{j2} \wedge \ldots \wedge \gamma^{\alpha_{jn_j}}(x_{jn_j})s_{jn_j}$ such that $\gamma^{\alpha_{ji}}(x_{ji}) \in \left\{\gamma(x_{ji}), \gamma(x_{ji})'\right\}$. For an S-Boolean algebra A and an S-map $f\colon X \longrightarrow A$, we considered the map $\overline{f}\colon \langle\gamma(X)\rangle \longrightarrow A$, defined by $\overline{f}(y) = \vee_{j=1}^{k}(\wedge_{i_j=1}^{n_j} f^{\alpha_{ji_j}}(x_j i_j s_j i_j))$. To show $\overline{f}$ is well defined, we proved the following steps: □

Step 1. For every different elements $x_1, x_2, \ldots, x_n \in X\,(n \geq 1)$, we have $\gamma^{\alpha_1}(x_1) \wedge \cdots \wedge \gamma^{\alpha_n}(x_n) \neq \perp_{\gamma(X)} = \varnothing$. Otherwise, consider $\gamma^{\alpha_1}(x_1) \wedge \cdots \wedge \gamma^{\alpha_n}(x_n) = \perp_{\gamma(X)} = \varnothing$. If $\gamma^{\alpha_1}(x_1) = \gamma(x_1)', \ldots, \gamma^{\alpha_n}(x_n) = \gamma(x_n)'$, then $\gamma(x_1)' \wedge \cdots \wedge \gamma(x_n)' = \varnothing$ and hence $X = \{x_1, \ldots, x_n\}$, which is a contradiction. So, we can partite the set $\{x_1, \ldots, x_n\}$ in to two subsets $\{y_1, \ldots, y_m\}$ and $\{y_{m+1}, \ldots, y_n\}$ such that $\{y_1, \ldots, y_m\}$ is a nonempty set and $\gamma^{\alpha_i}(y_i) = \gamma(y_i)$, $1 \leq i \leq m$. Thus,

$$\begin{aligned}\gamma^{\alpha_1}(x_1) \wedge \cdots \wedge \gamma^{\alpha_n}(x_n) &= \gamma(y_1) \wedge \cdots \wedge \gamma(y_m) \wedge \gamma(y_{m+1})'\\ &\quad \wedge \cdots \wedge \gamma(y_n)' = \varnothing.\end{aligned} \tag{19}$$

Obviously $\{y_1, \ldots, y_m\} \in \gamma(y_1) \wedge \cdots \wedge \gamma(y_m) \wedge \gamma(y_{m+1})' \wedge \cdots \wedge \gamma(y_n)' = \varnothing$, which is a contradiction and hence $\gamma^{\alpha_1}(x_1) \wedge \cdots \wedge \gamma^{\alpha_n}(x_n) \neq \perp_{\gamma(X)} = \varnothing$. We concluded that if $\gamma^{\alpha_1}(x_1) \wedge \cdots \wedge \gamma^{\alpha_n}(x_n) = \varnothing$, then there exists $i, j \in \{1, \ldots, n\}$ such that $x_i = x_j$ and $\alpha_i \neq \alpha_j$.

Step 2. Let $y = z \in \langle\gamma(X)\rangle$. Then, $y \triangle z = \perp$ and hence $y \wedge z' = \perp$ and $z \wedge y' = \perp$. Applying Theorem 1, consider $y = \vee_{j=1}^{p} y_j$ in which each $y_j = \wedge_{i_j=1}^{n_j} \gamma^{\alpha_{ji_j}}(y_{ji_j})s_{ji_j} = \wedge_{i_j=1}^{n_j} \gamma^{\alpha_{ji_j}}(y_{ji_j}s_{ji_j})$ and $z = \vee_{l=1}^{q} z_l$ in which each $z_l = \wedge_{k_l=1}^{n_l} \gamma^{\alpha_{lk_l}}(z_{lk_l})s_{lk_l} = \wedge_{k_l=1}^{n_l} \gamma^{\alpha_{lk_l}}(z_{lk_l}s_{lk_l})$ and

$$\begin{aligned}z' &= \vee_{i_1=1}^{n_1} \vee_{i_2=1}^{n_2} \ldots \vee_{i_q=1}^{n_q} \left(\gamma^{\alpha_{1i_1}}\left(z_{1i_1}s_{1i_1}\right)' \wedge \gamma^{\alpha_{2i_2}}\left(z_{2i_2}s_{2i_2}\right)'\right.\\ &\quad \left. \wedge \cdots \wedge \gamma^{\alpha_{qi_q}}\left(z_{qi_q}s_{qi_q}\right)'\right).\end{aligned} \tag{20}$$

Since $\wedge_{j=1}^{p}(y_j \wedge z') = y \wedge z' = \perp$, we have $y_j \wedge z' = \perp$ for each $1 \leq j \leq p$. Thus,

$$\vee_{i_1=1}^{n_1} \vee_{i_2=1}^{n_2} \ldots \vee_{i_q=1}^{n_q} \left(y_j \wedge \left(\wedge_{t=1}^{q} \gamma^{\alpha_{ti_t}}\left(z_{ti_t}s_{ti_t}\right)'\right)\right) = \perp, \tag{21}$$

and hence for each $(i_1, \ldots, i_q) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_q}$, which $Z_n^* = \{1, 2, \ldots, n\}$, and $y_j = \wedge_{i_j=1}^{n_j} \gamma^{\alpha_{ji_j}}(y_{ji_j}s_{ji_j})$, we obtained the following equation:

$$\left(\wedge_{i_j=1}^{n_j} \gamma^{\alpha_{ji_j}}\left(y_{ji_j}s_{ji_j}\right)\right) \wedge \left(\wedge_{t=1}^{q} \gamma^{\alpha_{ti_t}}\left(z_{ti_t}s_{ti_t}\right)'\right) = \perp. \tag{22}$$

By Step 1, there exist $1 \leq i_j \leq n_j$ and $1 \leq t \leq q$ such that $y_{ji_j}s_{ji_j} = z_{ti_t}s_{ti_t}$ and $\alpha_{ji_j} \neq \alpha_{ti_t}$. Thus, $f^{\alpha_{ji_j}}(y_{ji_j}s_{ji_j}) \wedge f^{\alpha_{ti_t}}(z_{ti_t}s_{ti_t}') = \perp$ and hence $\overline{f}(y_j) \wedge (\wedge_{t=1}^{q} f^{\alpha_{ti_t}}(z_{ti_t}s_{ti_t})') = (\wedge_{i_j=1}^{n_j} f^{\alpha_{ji_j}}(y_{ji_j}s_{ji_j})) \wedge (\wedge_{t=1}^{q} f^{\alpha_{ti_t}}(z_{ti_t}s_{ti_t})') = \perp$. By distributive low, $\overline{f}(y_j) \wedge (\vee_{i_1=1}^{n_1} \vee_{i_2=1}^{n_2} \ldots \vee_{i_q=1}^{n_q} (\wedge_{t=1}^{q} f^{\alpha_{ti_t}}(z_{ti_t}s_{ti_t})')) = \overline{f}(y_j) \wedge \overline{f}(z)' = \perp$. So, $\overline{f}(y) \wedge \overline{f}(z)' = \perp$ and in a similar way $\overline{f}(y)' \wedge \overline{f}(z) = \perp$ which implies $\overline{f}(y) = \overline{f}(z)$. Therefore, $\overline{f}$ is well defined.

Routine calculations show that $\overline{f}$ is an S-Boolean homomorphism and unique under the universal property of free objects.

Proposition 4. *If X is a right S-act, then there exists an act-free S-Boolean algebra on X.*

Proof. If X is an infinite S-act, then we are done by using Lemma 5. Let X be a finite S-act. Consider X^+, the free word semigroup over X. The semigroup X^+ by the action, for each $x_1, x_2, \ldots, x_n \in X$ and $s \in S$, $(x_1x_2 \cdots x_n)s = (x_1s)(x_2s) \cdots (x_ns)$, is an S-act. Let A be an S-Boolean algebra and $f\colon X \longrightarrow A$ an S-map. The map $f_1\colon X^+ \longrightarrow A$ defined by $f_1(x_1x_2 \cdots x_n) = f(x_1) \wedge f(x_2) \wedge \cdots \wedge f(x_n)$ is an S-act homomorphism. So there exists a unique S-Boolean algebra homomorphism $f_2\colon F(X^+) \longrightarrow A$ such that $f_2\gamma = f_1$. Consider $F(X)$, the sub S-Boolean algebra of $F(X^+)$ generated by $\gamma(X)$ and $\overline{f}\colon F(X) \longrightarrow A$ defined by $\overline{f} = f_2|_{F(X)}$. It is clear that $\overline{f}\gamma|_X = f$. Let $g\colon F(X) \longrightarrow A$ be an S-Boolean homomorphism such that $g\gamma|_X = f$. Using the S-Boolean homomorphism defined in the proof of Lemma 5,

$$\begin{aligned}\overline{f}\left(\vee_{j=1}^{q}\left(\wedge_{i_j=1}^{n_j} \gamma^{\alpha_{ji_j}}\left(x_{ji_j}s_{ji_j}\right)\right)\right) &= f_2\left(\vee_{j=1}^{q}\left(\wedge_{i_j=1}^{n_j} \gamma^{\alpha_{ji_j}}\left(x_{ji_j}s_{ji_j}\right)\right)\right)\\ &= \left(\vee_{j=1}^{q}\left(\wedge_{i_j=1}^{n_j} f^{\alpha_{ji_j}}\left(x_{ji_j}s_{ji_j}\right)\right)\right),\end{aligned} \tag{23}$$

and on the other hand,

$$g\left(\vee_{j=1}^{q}\left(\wedge_{i_j=1}^{n_j}\gamma^{\alpha_{ji_j}}\left(x_{ji_j}s_{ji_j}\right)\right)\right)=\left(\vee_{j=1}^{q}\left(\wedge_{i_j=1}^{n_j}g\left(\gamma^{\alpha_{ji_j}}\left(x_{ji_j}s_{ji_j}\right)\right)\right)\right)$$
$$=\left(\vee_{j=1}^{q}\left(\wedge_{i_j=1}^{n_j}(g\gamma)^{\alpha_{ji_j}}\left(x_{ji_j}s_{ji_j}\right)\right)\right)$$
$$=\left(\vee_{j=1}^{q}\left(\wedge_{i_j=1}^{n_j}f^{\alpha_{ji_j}}\left(x_{ji_j}s_{ji_j}\right)\right)\right), \tag{24}$$

which means $\overline{f}=g$. Therefore, $F(X)$ is a free S-Boolean algebra on the S-act X.

Using the fact that, for each nonempty set X the S-act $X\times S$, with the action $(x,s)t=(x,st)$, is a set-free S-act on X, we have the following theorem: □

Theorem 6. *Let X be a set and $F(X)$ an act-free S-Boolean algebra on the S-act $X\times S$. Then, $F(X)$ is a set-free S-Boolean algebra on X.*

In what follows, we give adjoined pairs between the category Boo-S and categories Act-S, Boo, and Set. Applying Proposition 4 and Theorem 6, we have Theorem 7.

Theorem 7

(i) *The act-free functor F: Act-S* ⟶Boo–S is a left adj oint to the forgetful functor U: Boo – S ⟶ Act – S

(ii) *The set-free functor F : Set ⟶ Boo-S is a left adjoined to the forgetful functor U : Boo-S ⟶ Set*

Let X be a set and $F_B(X)$ a set-free Boolean algebra on X, which exists by [11]. Also consider $F(X)$ to be a set-free S-Boolean algebra on X. The following theorem shows that we can consider $F(X)$ as a Boolean-free S-Boolean algebra on $F_B(X)$.

Theorem 8. *For a set X, $F(X)$ is a Boolean-free S-Boolean algebra on $F_B(X)$.*

Proof. Suppose that $\gamma_1\colon X\longrightarrow F(X)$ and $\gamma_2\colon X\longrightarrow F_B(X)$ are set-free extensions. Since $F(X)$ is a Boolean algebra and $F_B(X)$ is set-free Boolean algebra, there is a Boolean homomorphism $f\colon F_B(X)\longrightarrow F(X)$ such that $f\gamma_2=\gamma_1$. Now, let B be an S-Boolean algebra and $g\colon F_B(X)\longrightarrow B$ a Boolean homomorphism. Since $F(X)$ is set-free on X, there exists $\overline{g}\colon F(X)\longrightarrow B$ such that $\overline{g}\gamma_1=g\gamma_2$ and hence $\overline{g}f=g$. Obviously, $\overline{g}$ is unique. Thus, $F(X)$ is a Boolean-free S-Boolean algebra on $F_B(X)$. □

Theorem 9. *For each $A\in$ Boo and $B\in$ Boo-S, there is a one to one correspondence between the sets Hom$(F(A),B)$ of S-Boolean algebra homomorphisms and Hom$(F_B(A),B)$ of Boolean homomorphisms.*

Proof. Suppose that $\gamma_1\colon A\longrightarrow F_B(A)$ and $\gamma_2\colon F_B(A)\longrightarrow F(A)$ are set free and Boolean-free extensions, respectively. The set map $\psi\colon \mathrm{Hom}(F(A),B)\longrightarrow \mathrm{Hom}(F_B(A),B)$, given by $\psi(f)=f\gamma_2$, is an inverse of the set map $\phi\colon \mathrm{Hom}(F_B(A),B)\longrightarrow \mathrm{Hom}(F(A),B)$, given by $\phi(g)=\overline{g}$, defined in Theorem 8. □

Data Availability

No data were used to support this study.

Disclosure

This study was a part of the Tafresh University.

Conflicts of Interest

The author declares no conflicts of interest.

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