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Research Article

A Note on Multi-Euler–Genocchi and Degenerate Multi-Euler–Genocchi Polynomials

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Recently, the generalized Euler–Genocchi and generalized degenerate Euler–Genocchi polynomials are introduced. The aim of this note is to study the multi-Euler–Genocchi and degenerate multi-Euler–Genocchi polynomials which are defined by means of the multiple logarithm and generalize, respectively, the generalized Euler–Genocchi and generalized degenerate Euler–Genocchi polynomials. Especially, we express the former by the generalized Euler–Genocchi polynomials, the multi-Stirling numbers of the first kind and Stirling numbers of the second kind, and the latter by the generalized degenerate Euler–Genocchi polynomials, the multi-Stirling numbers of the first kind and Stirling numbers of the second kind.

1. Introduction

Carlitz began a study of degenerate versions of Bernoulli and Euler polynomials, namely, the degenerate Bernoulli and degenerate Euler polynomials (see [1, 2]). Recently, explorations for various degenerate versions of some special numbers and polynomials have regained interests of many mathematicians and yielded lots of fascinating and fruitful results. Indeed, this quest for degenerate versions even led to the development of degenerate umbral calculus [3], degenerate q-umbral calculus [4], and degenerate gamma function [5].

As it is well known, the Lah numbers arise in the expansion of the rising factorials in terms of the falling factorials and vice versa. They found their recent applications to the steganography method in telecommunication (see [6]) and the perturbative description of chromatic dispersion in optics (see [7]). In [8], as a generalization of the Lah numbers, the multi-Lah numbers are defined by means of the multiple logarithm and expressed in terms of the multi-Stirling numbers of the first kind and Stirling numbers of the second kind. Likewise, in this paper, the multi-Euler–Genocchi polynomials are defined with the help of the multiple logarithm and represented in terms of the multi-Stirling numbers of the first kind and Stirling numbers of the second kind. For these reasons, we think that the multi-Euler–Genocchi polynomials have potential applications in some real-world problems.

The Euler polynomials are defined by the following (see [9–11]):

$$\frac{2}{e^{t}+1}e^{xt}=\sum_{n=0}^{\infty}E_{n}(x)\frac{t^{n}}{n!}. \tag{1}$$

When $x=0$, $E_n=E_n(0)$, $(n\geq 0)$ are called the Euler numbers. The Genocchi polynomials are given by the following (see [9–14]):

$$\frac{2t}{e^{t}+1}e^{xt}=\sum_{n=0}^{\infty}G_{n}(x)\frac{t^{n}}{n!}. \tag{2}$$

When $x = 0$, $G_n = G_n(0)$ are called the Genocchi numbers. We note that $G_n \in \mathbb{Z}$, $(n \geq 0)$.

Recently, the generalized Euler–Genocchi polynomials are introduced as follows (see [15, 16]):

$$\frac{2t^r}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} A_n^{(r)}(x) \frac{t^n}{n!}, \tag{3}$$

where r is a nonnegative integer. Note that $A_n^{(0)}(x) = E_n(x)$ and $A_n^{(1)}(x) = G_n(x)(n \geq 0)$.

It is well known that the Stirling numbers of the first kind are given by the following (see [10, 11]):

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}, (k \geq 0). \tag{4}$$

As the inversion formula of (4), the Stirling numbers of the second kind are defined by the following (see [11]):

$$\frac{1}{k!}\left(e^t - 1\right)^k = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}, (k \geq 0). \tag{5}$$

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by the following (see [17, 18]):

$$e_\lambda^x(t) = (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \tag{6}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n \geq 1)$. In particular, for $x = 1$, $e_\lambda(t) = e_\lambda{}^1(t) = \sum_{n=0}^{\infty} (1)_{n,\lambda}/n! t^n$.

In [1, 2], Carlitz introduced the degenerate Euler polynomials given by

$$\frac{2}{e_\lambda(t) + 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} \mathscr{E}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{7}$$

Note that $\lim_{\lambda \longrightarrow 0} \mathscr{E}_{n,\lambda}(x) = E_n(x)$, $(n \geq 0)$. In view of (2), the degenerate Genocchi polynomials are defined by the following (see [19, 20]):

$$\frac{2t}{e_\lambda(t) + 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}(x) \frac{t^n}{n!}. \tag{8}$$

Recently, the generalized degenerate Euler–Genocchi polynomials are defined by the following (see [20]):

$$\frac{2t^r}{e_\lambda(t) + 1} e_\lambda^x(t) = \sum_{n=0}^{\infty} A_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}. \tag{9}$$

When $x = 0$, $A_{n,\lambda}^{(r)} = A_{n,\lambda}^{(r)}(0)$ are called the generalized degenerate Euler–Genocchi numbers. Note that $\lim_{\lambda \longrightarrow 0} A_{n,\lambda}^{(r)}(x) = A_n^{(r)}(x)$, $(n \geq 0)$.

For $k_1, k_2, \ldots, k_r \in \mathbb{Z}$, the multiple logarithm is defined by the following (see [8]):

$$Li_{k_1,k_2,\ldots,k_r}(t) = \sum_{0<n_1<n_2<\cdots<n_r} \frac{t^{n_r}}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}, (|t| < 1). \tag{10}$$

The multi-Bernoulli numbers are defined by the following (see [8]):

$$\frac{r! Li_{k_1,k_2,\ldots,k_r}\left(1 - e^{-t}\right)}{\left(e^t - 1\right)^r} = \sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)} \frac{t^n}{n!}. \tag{11}$$

From (10), we note that (see [18])

$$\begin{aligned} \frac{d}{dt} Li_{k_1,k_2,\ldots,k_r}(t) &= \frac{d}{dt} \sum_{0<n_1<n_2<\cdots<n_r} \frac{t_r^n}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}} \\ &= \frac{1}{t} Li_{k_1,\ldots,k_{r-1},k_r-1}(t). \end{aligned} \tag{12}$$

By (12), we get

$$Li_{\underbrace{1,1,\ldots,1}_{r-\text{times}}}(t) = \frac{1}{r!}(-\log(1-t))^r = \sum_{n=r}^{\infty} (-1)^{n-r} S_1(n,r) \frac{t^n}{n!}. \tag{13}$$

In light of (4) and (13), the multi-Stirling numbers of the first kind are defined by the following (see [8]):

$$Li_{k_1,k_2,\ldots,k_r}(t) = \sum_{n=r}^{\infty} S_1^{(k_1,k_2,\ldots,k_r)}(n,r) \frac{t^n}{n!}. \tag{14}$$

Note from (13) and (14) that

$$S_1^{\overbrace{(1,1,\ldots,1)}^{r-\text{times}}}(n,r) = (-1)^{n-r} S_1(n,r), (n,r \geq 0). \tag{15}$$

Goubi introduced the generalized Euler–Genocchi polynomials (of order α) in [16]. A degenerate version of those polynomials, namely, the generalized degenerate Euler–Genocchi polynomials, is investigated in [20]. Here, we study a "multi-version" of them, namely, the multi-Euler–Genocchi and degenerate multi-Euler–Genocchi polynomials. The generating function of the Stirling numbers of the first kind is given by the usual logarithm (see (4)). Naturally, the multi-Stirling numbers of the first kind (see (14)) are defined by means of the multiple logarithm (see (10)). In the same way, both the multi-Euler–Genocchi and the degenerate multi-Euler–Genocchi polynomials are defined by using the multiple logarithm, and the latter is a degenerate version of the former (see (16) and (27)).

The aim of this note is to study the multi-Euler–Genocchi polynomials and the degenerate multi-Euler–Genocchi polynomials which generalize, respectively, the generalized Euler–Genocchi polynomials (see (3)) and the generalized degenerate Euler–Genocchi polynomials (see (9)).

The outline of this note is as follows. In Theorem 1, the value of the multi-Euler–Genocchi polynomial at 1 is expressed in terms of the multi-Stirling numbers of the first kind and the Stirling numbers of the second kind. The multi-Euler–Genocchi polynomials are represented in terms of the generalized Euler–Genocchi polynomials, the multi-Stirling numbers of the first kind and the Stirling numbers of the second kind in Theorem 2. Likewise, the degenerate multi-Euler–Genocchi polynomials are expressed in terms of the generalized degenerate Euler–Genocchi polynomials, the multi-Stirling numbers of the first kind and the Stirling numbers of the second kind in Theorem 4. A distribution type formula is derived for the degenerate multi-Euler–Genocchi polynomials in Theorem 6. In the rest of this section, we recall the facts that are needed throughout this paper.

2. Multi-Euler–Genocchi and Degenerate Multi-Euler–Genocchi Polynomials

We consider the multi-Euler–Genocchi polynomials given by

$$\frac{2r!}{e^t+1}Li_{k_1,k_2,\ldots,k_r}\left(1-e^{-t}\right)e^{xt}=\sum_{n=0}^{\infty}A_n^{(k_1,k_2,\ldots,k_r)}(x)\frac{t^n}{n!}, \quad (16)$$

where r is a nonnegative integer. When $x=0$, $A_n^{(k_1,k_2,\ldots,k_r)}=A_n^{(k_1,k_2,\ldots,k_r)}(0)$ are called the multi-Euler–Genocchi numbers.

From (13) and (16), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}A_n^{\overbrace{(1,1,\ldots,1)}^{r\text{-times}}}(x)\frac{t^n}{n!}&=\frac{2r!}{e^t+1}Li_{\underbrace{1,1,\ldots,1}_{r\text{-times}}}\left(1-e^{-t}\right)e^{xt}\\&=\frac{2t^r}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}A_n^{(r)}(x)\frac{t^n}{n!}.\end{aligned} \quad (17)$$

Thus, by (17), we get

$$A_n^{\overbrace{(1,1,\ldots,1)}^{r\text{-times}}}(x)=A_n^{(r)}(x),\ (n\geq 0). \quad (18)$$

From (14) and (16), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}A_n^{(k_1,k_2,\ldots,k_r)}(1)\frac{t^n}{n!}\\&=r!\sum_{l=0}^{\infty}\left(\frac{1}{2}\right)^l(-1)^l\left(e^{-t}-1\right)^l\sum_{m=r}^{\infty}S_1^{(k_1,k_2,\ldots,k_r)}(m,r)(-1)^m\frac{1}{m!}\left(e^{-t}-1\right)^m\\&=r!\sum_{l=0}^{\infty}\sum_{m=r}^{\infty}\left(\frac{1}{2}\right)^lS_1^{(k_1,k_2,\ldots,k_r)}(m,r)(-1)^{l+m}\frac{(l+m)!}{m!}\frac{1}{(l+m)!}\left(e^{-t}-1\right)^{l+m}\\&=r!\sum_{l=0}^{\infty}\sum_{m=r}^{\infty}\left(\frac{1}{2}\right)^lS_1^{(k_1,k_2,\ldots,k_r)}(m,r)(-1)^{l+m}\frac{(l+m)!}{m!}\sum_{n=l+m}^{\infty}S_2(n,l+m)(-1)^n\frac{t^n}{n!}\\&=\sum_{n=r}^{\infty}\left(r!\sum_{k=r}^{n}\sum_{m=r}^{k}\left(\frac{1}{2}\right)^{k-m}(-1)^{n-k}S_1^{(k_1,k_2,\ldots,k_r)}(m,r)\frac{k!}{m!}S_2(n,k)\right)\frac{t^n}{n!}.\end{aligned} \quad (19)$$

Therefore, by comparing the coefficient on both sides of (19), we obtain the following theorem.

Theorem 1. *For n,r in nonnegative integer with $n\geq r$, we have*

$$A_n^{(k_1,k_2,\ldots,k_r)}(1)=r!\sum_{k=r}^{n}\sum_{m=r}^{k}\left(\frac{1}{2}\right)^{k-m}(-1)^{n-k}\frac{k!}{m!}S_1^{(k_1,k_2,\ldots,k_r)}(m,r)S_2(n,k), \quad (20)$$

and, for $0 \leq n < r$, *we have*

$$A_n^{(k_1,k_2,\ldots,k_r)}(1) = 0. \tag{21}$$

By (16), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} A_n^{(k_1,k_2,\ldots,k_r)}(x)\frac{t^n}{n!}
&= r!\frac{2}{e^t+1}e^{xt}\sum_{l=r}^{\infty} S_1^{(k_1,k_2,\ldots,k_r)}(l,r)(-1)^l\frac{1}{l!}\left(e^{-t}-1\right)^l \\
&= \frac{r!2e^{xt}}{e^t+1}\sum_{l=r}^{\infty} S_1^{(k_1,k_2,\ldots,k_r)}(l,r)(-1)^l\sum_{m=l}^{\infty} S_2(m,l)(-1)^m\frac{t^m}{m!} \\
&= \frac{r!2e^{xt}}{e^t+1}\sum_{m=r}^{\infty}\left(\sum_{l=r}^{m} S_1^{(k_1,k_2,\ldots,k_r)}(l,r)S_2(m,l)(-1)^{m-l}\right)\frac{t^m}{m!} \\
&= \frac{r!2t^r e^{xt}}{e^t+1}\sum_{m=0}^{\infty}\left(\sum_{l=r}^{m+r} S_1^{(k_1,k_2,\ldots,k_r)}(l,r)\frac{S_2(m+r,l)m!}{(m+r)!}(-1)^{m+r-l}\right)\frac{t^m}{m!} \\
&= \sum_{j=0}^{\infty} A_j^{(r)}(x)\frac{t^j}{j!}\sum_{m=0}^{\infty}\left(\sum_{l=r}^{m+r}\frac{S_1^{(k_1,k_2,\ldots,k_r)}(l,r)}{\binom{m+r}{r}}S_2(m+r,l)(-1)^{m+r-l}\right)\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}A_{n-m}^{(r)}(x)\sum_{l=r}^{m+r}\frac{S_1^{(k_1,k_2,\ldots,k_r)}(l,r)}{\binom{m+r}{r}}S_2(m+r,l)(-1)^{m+r-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{22}$$

Therefore, by comparing the coefficients on both sides of (22), we obtain the following theorem.

Theorem 2. *For* $n \geq 0$, *we have*

$$A_n^{(k_1,k_2,\ldots,k_r)}(x) = \sum_{m=0}^{n}\binom{n}{m}A_{n-m}^{(r)}(x)\sum_{l=r}^{m+r}\frac{S_1^{(k_1,k_2,\ldots,k_r)}(l,r)}{\binom{m+r}{r}}S_2(m+r,l)(-1)^{m+r-l}. \tag{23}$$

From (14), we note that

$$\begin{aligned}
Li_{k_1,k_2,\ldots,k_r}\left(1-e^{-t}\right) &= \sum_{k=r}^{\infty} S_1^{(k_1,k_2,\ldots,k_r)}(k,r)\frac{1}{k!}\left(1-e^{-t}\right)^k \\
&= \sum_{k=r}^{\infty} S_1^{(k_1,k_2,\ldots,k_r)}(k,r)\sum_{n=k}^{\infty}(-1)^{n-k}S_2(n,k)\frac{t^n}{n!} \\
&= \sum_{n=r}^{\infty}\left(\sum_{k=r}^{n} S_1^{(k_1,k_2,\ldots,k_r)}(k,r)(-1)^{n-k}S_2(n,k)\right)\frac{t^n}{n!}.
\end{aligned} \tag{24}$$

On the other hand, by (16), we get

$$\begin{aligned} Li_{k_1,k_2,\ldots,k_r}\left(1-e^{-t}\right) &= \frac{1}{2r!}\frac{2r!}{e^t+1}Li_{k_1,k_2,\ldots,k_r}\left(1-e^{-t}\right)\left(e^t+1\right) \\ &= \frac{1}{2r!}\sum_{n=0}^{\infty}\left(A_n^{(k_1,k_2,\ldots,k_r)}(1)+A_n^{(k_1,k_2,\ldots,k_r)}\right)\frac{t^n}{n!}. \end{aligned} \tag{25}$$

Therefore, by (24) and (25), we obtain the following theorem.

Theorem 3. *For $n, r \geq 0$, we have*

$$A_n^{(k_1,k_2,\ldots,k_r)}(1)+A_n^{(k_1,k_2,\ldots,k_r)} = \begin{cases} 2r!\displaystyle\sum_{k=r}^{n} S_1^{(k_1,k_2,\ldots,k_r)}(k,r)(-1)^{n-k}S_2(n,k), & \text{if } n \geq r, \\ 0, & \text{if } 0 \leq n < r. \end{cases} \tag{26}$$

Now, we consider the degenerate multi-Euler–Genocchi polynomials given by

$$\frac{2r!}{e_\lambda(t)+1}Li_{k_1,k_2,\ldots,k_r}\left(1-e^{-t}\right)e_\lambda^{\,x}(t) = \sum_{n=0}^{\infty}A_{n,\lambda}^{(k_1,k_2,\ldots,k_r)}(x)\frac{t^n}{n!}. \tag{27}$$

When $x=0$, $A_{n,\lambda}^{k_1,k_2,\ldots,k_r} = A_{n,\lambda}^{(k_1,k_2,\ldots,k_r)}(0)$ are called the degenerate multi-Euler–Genocchi numbers. Thus, by (27), we get

$$\begin{aligned} \sum_{n=0}^{\infty}A_{n,\lambda}^{(\overbrace{1,1,\ldots,1}^{r-\text{times}})}(x)\frac{t^n}{n!} &= \frac{2r!}{e_\lambda(t)+1}Li_{\underbrace{1,1,\ldots,1}_{r-\text{times}}}\left(1-e^{-t}\right)e_\lambda^{\,x}(t) \\ &= \frac{2t^r}{e_\lambda(t)+1}e_\lambda^{\,x}(t) = \sum_{n=0}^{\infty}A_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \end{aligned} \tag{28}$$

From (28), we have

$$A_{n,\lambda}^{(\overbrace{1,1,\ldots,1}^{r-\text{times}})}(x) = A_{n,\lambda}^{(r)}(x),\ (n \geq 0). \tag{29}$$

By (27), we get

$$\begin{aligned} &\sum_{n=0}^{\infty}A_{n,\lambda}^{(k_1,k_2,\ldots,k_r)}(x)\frac{t^n}{n!} \\ &= \frac{2r!}{e_\lambda(t)+1}e_\lambda^{\,x}(t)\sum_{m=r}^{\infty}S_1^{(k_1,k_2,\ldots,k_r)}(m,r)\frac{1}{m!}\left(1-e^{-t}\right)^m \\ &= \frac{2r!}{e_\lambda(t)+1}e_\lambda^{\,x}(t)\sum_{m=r}^{\infty}S_1^{(k_1,k_2,\ldots,k_r)}(m,r)\sum_{l=m}^{\infty}S_2(l,m)(-1)^{l-m}\frac{t^l}{l!} \\ &= \frac{2r!}{e_\lambda(t)+1}e_\lambda^{\,x}(t)\sum_{l=r}^{\infty}\left(\sum_{m=r}^{l}(-1)^{l-m}S_2(l,m)S_1^{(k_1,k_2,\ldots,k_r)}(m,r)\right)\frac{t^l}{l!} \\ &= \sum_{j=0}^{\infty}A_{j,\lambda}^{(r)}(x)\frac{t^j}{j!}\sum_{l=0}^{\infty}\left(\sum_{m=r}^{l+r}\frac{S_2(l+r,m)}{\binom{l+r}{r}}(-1)^{l-m-r}S_1^{(k_1,k_2,\ldots,k_r)}(m,r)\right)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}A_{n-l,\lambda}^{(r)}(x)\sum_{m=r}^{l+r}\frac{S_2(l+r,m)}{\binom{l+r}{l}}(-1)^{l-m-r}S_1^{(k_1,k_2,\ldots,k_r)}(m,r)\right)\frac{t^n}{n!}. \end{aligned} \tag{30}$$

Therefore, by comparing the coefficients on both sides of (30), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$A_{n,\lambda}^{(k_1,k_2,\ldots,k_r)}(x) = \sum_{l=0}^{n} \binom{n}{l} A_{n-l,\lambda}^{(r)}(x) \sum_{m=r}^{l+r} \frac{S_2(l+r,m)}{\binom{l+r}{l}} (-1)^{l-m-r} S_1^{(k_1,k_2,\ldots,k_r)}(m,r). \tag{31}$$

From (27), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} A_{n,\lambda}^{(k_1,k_2,\ldots,k_r)}(x) \frac{t^n}{n!} &= \frac{2r! Li_{k_1,k_2,\ldots,k_r}\left(1-e^{-t}\right)}{e_\lambda(t)+1} e_\lambda^{x}(t) \\ &= \sum_{k=0}^{\infty} A_{k,\lambda}^{(k_1,k_2,\ldots,k_r)} \frac{t^k}{k!} \sum_{m=0}^{\infty} (x)_{m,\lambda} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} A_{k,\lambda}^{(k_1,k_2,\ldots,k_r)}(x)_{n-k,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \tag{32}$$

Thus, by (32), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$A_{n,\lambda}^{(k_1,k_2,\ldots,k_r)}(x) = \sum_{k=0}^{n} \binom{n}{k} A_{k,\lambda}^{(k_1,k_2,\ldots,k_r)}(x)_{n-k,\lambda}. \tag{33}$$

Assume that $m \in \mathbb{N}$ with $m \equiv 1 \pmod 2$. Then, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} A_{n,\lambda}^{(k_1,k_2,\ldots,k_r)}(x) \frac{t^n}{n!} \\ &= \frac{2r!}{e_\lambda^{m}(t)+1} Li_{k_1,k_2,\ldots,k_r}\left(1-e^{-t}\right) \sum_{l=0}^{m-1} (-1)^l e_\lambda^{l+x}(t) \\ &= \frac{r!}{t^r} Li_{k_1,k_2,\ldots,k_r}\left(1-e^{-t}\right) \sum_{l=0}^{m-1} (-1)^l \frac{2t^r}{e_{\lambda/m}(mt)+1} e_{\lambda/m}^{l+x/m}(mt) \\ &= \frac{r!}{t^r} \sum_{k=r}^{\infty} S_1^{(k_1,k_2,\ldots,k_r)}(k,r) \sum_{j=k}^{\infty} S_2(j,k)(-1)^{j-k} \frac{t^j}{j!} \sum_{l=0}^{m-1} (-1)^l \sum_{i=0}^{\infty} A_{i,\lambda/m}^{(r)}\left(\frac{l+x}{m}\right) m^{i-r} \frac{t^i}{i!} \\ &= \frac{r!}{t^r} \sum_{j=r}^{\infty} \left(\sum_{k=r}^{j} S_1^{(k_1,k_2,\ldots,k_r)}(k,r)(-1)^{j-k} S_2(j,k) \right) \frac{t^j}{j!} \sum_{i=0}^{\infty} \left(\sum_{l=0}^{m-1} (-1)^l A_{i,\lambda/m}^{(r)}\left(\frac{l+x}{m}\right) m^{i-r} \right) \frac{t^i}{i!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=r}^{j+r} S_1^{(k_1,k_2,\ldots,k_r)}(k,r)(-1)^{j+r-k} S_2(j+r,k) \frac{r! j!}{(j+r)!} \right) \frac{t^j}{j!} \\ &\quad \times \sum_{i=0}^{\infty} \left(\sum_{l=0}^{m-1} (-1)^l A_{i,\lambda/m}^{(r)}\left(\frac{l+x}{m}\right) m^{i-r} \right) \frac{t^i}{i!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} \sum_{k=r}^{j+r} \sum_{l=0}^{m-1} S_1^{(k_1,k_2,\ldots,k_r)}(k,r)(-1)^{j+r-k} \right. \\ &\quad \left. \times \frac{S_2(j+r,k)}{\binom{j+r}{r}} (-1)^l A_{n-j,\lambda/m}^{(r)}\left(\frac{l+x}{m}\right) m^{n-j-r} \right) \frac{t^n}{n!}. \end{aligned} \tag{34}$$

Therefore, by comparing the coefficients on both sides of (34), we obtain the following theorem.

Theorem 6. *For $m \in \mathbb{N}$ with $m \equiv 1$ (mod 2), we have*

$$A_{n,\lambda}^{(k_1,k_2,\ldots,k_r)}(x) = \sum_{j=0}^{n}\binom{n}{j}\sum_{k=r}^{j+r}\sum_{l=0}^{m-1} S_1^{(k_1,k_2,\ldots,k_r)}(k,r)(-1)^{j+r-k} \times \frac{S_2(j+r,k)}{\binom{j+r}{r}}(-1)^l A_{n-j,\lambda/m}^{(r)}\left(\frac{l+x}{m}\right)m^{n-j-r}. \tag{35}$$

3. Conclusion

In addition to degenerate versions of many special numbers and polynomials, the degenerate gamma function, degenerate umbral calculus, and degenerate q-umbral calculus are introduced and a lot of interesting results about them are found in recent years.

In this note, we introduced the multi-Euler–Genocchi and degenerate multi-Euler–Genocchi polynomials which are multiversions of the generalized Euler–Genocchi and generalized degenerate Euler–Genocchi polynomials. Among other things, we expressed the former by the generalized Euler–Genocchi polynomials, the multi-Stirling numbers of the first kind and Stirling numbers of the second kind, and the latter by the generalized degenerate Euler–Genocchi polynomials, the multi-Stirling numbers of the first kind and Stirling numbers of the second kind.

It is one of our future projects to continue to study various degenerate versions of some special numbers and polynomials and those of certain transcendental functions and to find their applications to physics, science, and engineering as well as to mathematics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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