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Research Article

Some Further Results Using Green's Function for *s*-Convexity

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For *s*-convex functions, the Hermite–Hadamard inequality is already well-known in convex analysis. In this regard, this work presents new inequalities associated with the left-hand side of the Hermite–Hadamard inequality for *s*-convexity by utilizing a novel technique based on Green's function. Also, Hölder, Young, and power-mean inequalities are used to obtain these new inequalities. Finally, some applications to special means of real numbers are provided. In conclusion, we think that the methodology used in this work will encourage more research in this field.

1. Introduction

Convexity is a fundamental idea in both applied and pure mathematics, acting as a powerful tool for evaluating functions and sets, proving inequalities, and modeling and resolving practical issues. In many areas of mathematics and beyond, this idea is essential for estimating integrals and setting boundaries. Relevant researchers can access some articles on convex functions from references [1–9].

Thus, we recall the elementary notation in convex analysis.

Definition 1. A set $\mathscr{I} \subset \mathbb{R}$ is said to be convex function if

$$\zeta\varpi + (1-\zeta)\phi \in \mathscr{I}, \tag{1}$$

for each $\varpi, \phi \in \mathscr{I}$ and $\zeta \in [0,1]$.

Definition 2. The mapping $\wp\colon \mathscr{I} \longrightarrow \mathbb{R}$ is said to be a convex function if the following inequality holds:

$$\wp(\zeta\varpi + (1-\zeta)\phi) \le \zeta\wp(\varpi) + (1-\zeta)\wp(\phi), \tag{2}$$

for all $\varpi, \phi \in \mathscr{I}$ and $\zeta \in [0,1]$. If $(-\wp)$ is convex, then $\wp$ is said to be concave. In terms of geometry, this indicates that if $\mathscr{B}$, $\mathscr{U}$, and $\mathscr{Z}$ are three separate locations on the graph of $\wp$, with $\mathscr{U}$ between $\mathscr{B}$ and $\mathscr{Z}$, then $\mathscr{U}$ is on or below the chord $\mathscr{B}\mathscr{Z}$.

In many different fields, convex functions are crucial. For instance, the concavity of $\wp$ is described in terms of declining returns for a production function $\varphi = \wp(L)$ in economics. If $\wp$ is convex, then increasing returns are shown. In addition, a convex function applied to the expected value of a random variable is always limited beyond the convex function's expected value, according to the theory of probability. It is possible to derive other inequalities, including the geometric-arithmetic mean inequality and Hölder's inequality, using this conclusion, known as Jensen's inequality. The idea of convexity has developed into a rich source of inspiration and a fascinating topic for scholars because of its widespread viewpoints, resilience, and plenty of applications. Mathematicians have developed incredible tools and numerical methods using the notion of convexity to deal with and resolve an enormous number of issues that emerge in the pure and applied sciences. This theory has a long and important history, and for more than a century, mathematics has focused on and concentrated on it. On the other hand, there are a lot of new issues in applied mathematics where the idea of convexity is insufficient to adequately characterize them in order to have beneficial consequences. Because of this, the idea of convexity has been expanded upon and developed in various research studies; see [10–16].

Utilizing various forms of convexity, some important inequalities have been observed. s-convexity is one of the several varieties of convexity. Hudzik and Maligranda in reference [17] took into account, among other things, the class of functions that are s-convex in the second sense. The following is the definition of this class: a function $\wp\colon [0,\infty) \longrightarrow \mathbb{R}$ is s-convex in the second sense if

$$\wp(\zeta\varpi + (1-\zeta)\phi) \le \zeta^s \wp(\varpi) + (1-\zeta)^s \wp(\phi) \tag{3}$$

holds for all $\varpi, \phi \in [0,\infty), \zeta \in [0,1]$ and for some fixed $s \in (0,1]$. The class of s-convexity is frequently denoted by the symbol K_s^2. It is obvious that $s = 1$ converts s-convexity into the typical convexity of functions defined on $[0,\infty]$.

The authors of the same paper, namely, [17], demonstrated that all functions from $K_s^2, s \in (0,1)$, are nonnegative if $\wp \in K_s^2$ implies $\wp([0,\infty)) \subseteq [0,\infty)$.

Example 3. (see [17]). Let $s \in (0,1)$ and $\ell, \hbar, \gamma \in \mathbb{R}$. We define function $\wp\colon [0,\infty) \longrightarrow \mathbb{R}$ as follows:

$$\wp(\zeta) = \begin{cases} \ell, & \zeta = 0, \\ \hbar\zeta^s + \gamma, & \zeta > 0. \end{cases} \tag{4}$$

It can be simply confirmed that

(i) If $\hbar \ge 0$ and $0 \le \gamma \le \ell$, then $\wp \in K_s^2$

(ii) If $\hbar > 0$ and $\gamma < 0$, then $\wp \notin K_s^2$

Remark 4. Throughout the article, the Hermite–Hadamard inequality will be denoted by $\mathscr{H} - \mathscr{H}$ inequality.

Recently, there are many studies on s-convexity in the literature. A few new general $\mathscr{H} - \mathscr{H}$ type inequalities for s-convex mappings were developed in [18] by Yildiz et al. The Hölder inequality, the power-mean integral inequality, and certain extensions connected to these inequalities were utilized to establish these inequalities. In addition, they compared some inequalities. In [19], a new definition for s-convex functions is given and some properties of this definition are investigated. In addition, extended versions of the previously well-known conclusions for harmonically convex functions, such as $\mathscr{H} - \mathscr{H}$, various $\mathscr{H} - \mathscr{H}$ refinements, and Ostrowski-type inequalities, are developed. In [20], the expression "extended s -convex functions" was introduced by the authors, who also developed some inequalities of the $\mathscr{H} - \mathscr{H}$ type for extended s-convex functions. The authors then used these newly discovered integral inequalities to deduce certain specific mean inequalities. In reference [21], the authors established an equation for a function whose third derivative is integrable, developed some novel integral inequalities of the $\mathscr{H} - \mathscr{H}$ type for extended s-convex mappings using the Hölder inequality, and then used these integral inequalities to produce inequalities for various kinds of special means. In reference [22], the authors established some new inequalities of the $\mathscr{H} - \mathscr{H}$ type for extended s-convex mappings and obtained new inequalities with respect to λ and μ using Lemma 2.1. Finally, utilizing the s-convexity for the Raina function, different inequalities are obtained with fractional integral operators in reference [23].

Convex mappings and sets have been improved and expanded in many disciplines of mathematics due to their robustness (as was described above); in particular, the convexity theory has been used to prove a number of inequalities that are prevalent in the literature. In the practical literature on mathematical inequalities, the $\mathscr{H} - \mathscr{H}$ type integral inequality is, to the best of our knowledge, a well-known, important, and incredibly helpful inequality. There are several classical inequalities that are closely associated with the classical $\mathscr{H} - \mathscr{H}$ type integral inequality, such as Simpson, Opial, Hardy, Hölder, Ostrowski, Minkowski, arithmetic-geometric, Young, and Gagliardo–Nirenberg inequalities. These inequalities are of pivotal significance. Following is a statement of this double inequality: assume that $\wp$ is a convex mapping on $[\varpi, \phi] \subset \mathbb{R}$, where $\varpi \neq \phi$. Therefore,

$$\wp\left(\frac{\varpi+\phi}{2}\right) \le \frac{1}{\phi-\varpi}\int_{\varpi}^{\phi} \wp(\xi)d\xi \le \frac{\wp(\varpi)+\wp(\phi)}{2}. \tag{5}$$

The reader who is interested can refer to references [24–27] for a number of recent findings pertaining to $\mathscr{H} - \mathscr{H}$ inequality.

In [28], the researchers proved a different form of $\mathscr{H} - \mathscr{H}$ inequality which holds for s-convex mappings in the second sense.

Theorem 5. *Suppose that $\wp\colon [0,\infty) \longrightarrow [0,\infty)$ is an s-convex function in the second sense, where $s \in [0,1)$, and let $\varpi, \phi \in [0,\infty)$, $\varpi < \phi$. If $\wp \in L[\varpi, \phi]$, then the following inequalities hold:*

$$2^{s-1}\wp\left(\frac{\varpi+\phi}{2}\right) \le \frac{1}{\phi-\varpi}\int_{\varpi}^{\phi} \wp(\xi)d\xi \le \frac{\wp(\varpi)+\wp(\phi)}{s+1}. \tag{6}$$

In the second inequality in [29], the constant $\alpha = 1/(s+1)$ is the best possibility.

The famous Young inequality is defined as follows:

$$\varpi\phi \le \frac{1}{p}\varpi^p + \frac{1}{q}\phi^q, \tag{7}$$

where ϖ and ϕ are nonnegative numbers, $p > 1$, and $(1/p) + (1/q) = 1$, in [30].

The reversed version of inequality (7) reads

$$\varpi\phi \ge \frac{1}{p}\varpi^p + \frac{1}{q}\phi^q, \quad \varpi, \phi > 0, 0 < \mathrm{p} < 1, \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1. \tag{8}$$

The definition of the Hölder inequality is as follows.

Let $p > 1$ and $(1/p) + (1/q) = 1$. If $\wp$ and κ are real functions defined on $[\varpi, \phi]$ such that $|\wp|^p$ and $|\kappa|^q$ are integrable functions on $[\varpi, \phi]$, then

$$\int_{\varpi}^{\phi} |\wp(x)\kappa(x)|dx \le \left(\int_{\varpi}^{\phi} |\wp(x)|^p dx\right)^{(1/p)} \left(\int_{\varpi}^{\phi} |\kappa(x)|^q dx\right)^{(1/p)}. \tag{9}$$

The well-known Hölder inequality, one of the most significant inequalities in analysis, was demonstrated in this way using inequality (7). It makes a significant contribution to many fields of applied and pure mathematics and is essential in helping to solve several issues in the social, cultural, and natural sciences.

The most popular form of Young's inequality, which is frequently used to demonstrate the well-known inequality for L_p functions, is as follows:

$$\varpi^{\zeta}\phi^{1-\zeta}\leq\zeta\varpi+(1-\zeta)\phi, \tag{10}$$

where $\varpi,\phi>0$ and $0\leq\zeta\leq 1$.

2. Preliminaries

This paper uses a relatively new approach based on Green's function to illustrate the $\mathscr{H}-\mathscr{H}$ inequalities for s-convex functions. First, we will start by giving the definition of Green's function.

Let $\varpi<\phi$ and the following four new Green's functions defined on $[\varpi,\phi]\times[\varpi,\phi]$ are defined by Mehmood et al. in [31] as follows:

$$\begin{aligned}\mathscr{G}_1(\lambda,\mu)&=\begin{cases}\varpi-\mu, & \varpi\leq\mu\leq\lambda,\\ \varpi-\lambda, & \lambda\leq\mu\leq\phi,\end{cases}\\ \mathscr{G}_2(\lambda,\mu)&=\begin{cases}\lambda-\phi, & \varpi\leq\mu\leq\lambda,\\ \mu-\phi, & \lambda\leq\mu\leq\phi,\end{cases}\\ \mathscr{G}_3(\lambda,\mu)&=\begin{cases}\lambda-\varpi, & \varpi\leq\mu\leq\lambda,\\ \mu-\varpi, & \lambda\leq\mu\leq\phi,\end{cases}\\ \mathscr{G}_4(\lambda,\mu)&=\begin{cases}\phi-\mu, & \varpi\leq\mu\leq\lambda,\\ \phi-\lambda, & \lambda\leq\mu\leq\phi.\end{cases}\end{aligned} \tag{11}$$

In [31], the authors developed the following Lemma, which we will utilize to demonstrate our main conclusions.

Lemma 6. *Let $\varpi<\phi$ and $\wp\colon[\varpi,\phi]\longrightarrow\mathbb{R}$ be a twice differentiable function and $\mathscr{G}_\kappa(\kappa=1,2,3,4)$ be the new Green's functions defined by (12). Then,*

$$\begin{aligned}\wp(\xi)&=\wp(\varpi)+(\xi-\varpi)\wp'(\phi)+\int_\varpi^\phi\mathscr{G}_1(\xi,\mu)\wp''(\mu)d\mu,\\ \wp(\xi)&=\wp(\phi)+(\phi-\xi)\wp'(\varpi)+\int_\varpi^\phi\mathscr{G}_2(\xi,\mu)\wp''(\mu)d\mu,\\ \wp(\xi)&=\wp(\phi)+(\phi-\varpi)\wp'(\phi)+(\xi-\varpi)\wp'(\varpi)+\int_\varpi^\phi\mathscr{G}_3(\xi,\mu)\wp''(\mu)d\mu,\\ \wp(\xi)&=\wp(\varpi)+(\phi-\varpi)\wp'(\varpi)-(\phi-\xi)\wp'(\phi)+\int_\varpi^\phi\mathscr{G}_4(\xi,\mu)\wp''(\mu)d\mu.\end{aligned} \tag{12}$$

Proof. By using the procedures of integration by parts in $\int_\varpi^\phi\mathscr{G}_1(\xi,\mu)\wp''(\mu)d\mu$, the above equation may be easily computed. The specifics of proof are thus left to readers who are interested. Likewise, a similar method can be used in other equations. □

Remark 7. Throughout this study, $\mathscr{G}=\mathscr{G}_1$ will be used.

There are many studies on Green's function in the literature. In these studies, different inequalities were obtained by using different methods as well as Green's function. For example, in [32], by utilizing Green's function, Jensen's inequality, convexity, and monotone functions, the authors developed the left Riemann–Liouville fractional $\mathscr{H}-\mathscr{H}$ type inequalities as well as the extended $\mathscr{H}-\mathscr{H}$ type inequalities. In [33], the $\mathscr{H}-\mathscr{H}$ inequalities have again been established using Green's function and convexity. In [34], the authors revisited the $\mathscr{H}-\mathscr{H}$ inequalities for the Riemann–Liouville fractional operators with the help of Green's function, and finally, in [35], Li et al. established $\mathscr{H}-\mathscr{H}$ inequalities for the left generalized fractional integral via Green's function.

The aim of this study is to obtain new integral inequalities with the use of Green's function for functions whose q-th power is s-convex and s-concave. In other words, it is to develop a new method using Green's function. In addition to the definition of Green's function, Young, Hölder, and power-mean inequalities were used to obtain these new identities. As a consequence, these inequalities are associated with the left-hand side of $\mathscr{H}-\mathscr{H}$ inequality. Finally, new propositions are given for special means.

3. Main Results

Theorem 8. *Let $\wp\colon[\varpi,\phi]\longrightarrow\mathbb{R}$ be a twice differentiable and $|\wp''|$ be a $s-$convex function in the second sense on $[\varpi,\phi]$ that satisfy the relation given in (13). If $\wp\in L[\varpi,\phi]$, then the following inequality holds:*

$$\begin{aligned}&\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_\varpi^\phi\wp(\xi)d\xi\right|\\ &\leq\frac{(\phi-\varpi)^2}{2^{s+2}}\frac{2^{s+2}-(s+3)}{(s+1)(s+2)(s+3)}\left\{\left|\wp''(\varpi)\right|+\left|\wp''(\phi)\right|\right\}\end{aligned}, \tag{13}$$

for some fixed $s\in(0,1]$.

Proof. Substituting $\xi=(\varpi+\phi/2)$ in (12), we have

$$\wp\left(\frac{\varpi+\phi}{2}\right)=\wp(\varpi)+\left(\frac{\phi-\varpi}{2}\right)\wp'(\phi)+\int_{\varpi}^{\phi}\mathcal{G}\left(\frac{\varpi+\phi}{2},\mu\right)\wp''(\mu)d\mu. \tag{14}$$

Integrating both sides of identity (13) with respect to ξ and multiplying by $(1/\phi-\varpi)$, we obtain

$$\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi=\wp(\varpi)+\left(\frac{\phi-\varpi}{2}\right)\wp'(\phi)+\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\int_{\varpi}^{\phi}\mathcal{G}(\xi,\mu)\wp''(\mu)d\mu\, d\xi. \tag{15}$$

Subtracting (15) from (14), we have

$$\begin{aligned}\wp\left(\frac{\varpi+\phi}{2}\right)&-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\\&=\int_{\varpi}^{\phi}\mathcal{G}\left(\frac{\varpi+\phi}{2},\mu\right)\wp''(\mu)d\mu-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\int_{\varpi}^{\phi}\mathcal{G}(\xi,\mu)\wp''(\mu)d\mu\, d\xi\\&=\int_{\varpi}^{\phi}\left[\mathcal{G}\left(\frac{\varpi+\phi}{2},\mu\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\mathcal{G}(\xi,\mu)d\xi\right]\wp''(\mu)d\mu.\end{aligned} \tag{16}$$

Now, from the property of integral and definition Green's function, we obtain

$$\begin{aligned}\wp\left(\frac{\varpi+\phi}{2}\right)&-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\\&=\int_{\varpi}^{\varpi+\phi/2}\left[\mathcal{G}\left(\frac{\varpi+\phi}{2},\mu\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\mathcal{G}(\xi,\mu)d\xi\right]\wp''(\mu)d\mu\\&\quad+\int_{\varpi+\phi/2}^{\phi}\left[\mathcal{G}\left(\frac{\varpi+\phi}{2},\mu\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\mathcal{G}(\xi,\mu)d\xi\right]\wp''(\mu)d\mu\\&=\int_{\varpi}^{\varpi+\phi/2}\left[(\varpi-\mu)-\frac{1}{\phi-\varpi}\left\{\int_{\varpi}^{\mu}(\varpi-\xi)d\xi+\int_{\mu}^{\phi}(\varpi-\mu)d\xi\right\}\right]\wp''(\mu)d\mu\\&\quad+\int_{\varpi+\phi/2}^{\phi}\left[\left(\frac{\varpi-\phi}{2}\right)-\frac{1}{\phi-\varpi}\left\{\int_{\varpi}^{\mu}(\varpi-\xi)d\xi+\int_{\mu}^{\phi}(\varpi-\mu)d\xi\right\}\right]\wp''(\mu)d\mu\\&=-\frac{1}{2(\phi-\varpi)}\left[\int_{\varpi}^{\varpi+\phi/2}(\mu-\varpi)^2\wp''(\mu)d\mu+\int_{\varpi+\phi/2}^{\phi}(\phi-\mu)^2\wp''(\mu)d\mu\right].\end{aligned} \tag{17}$$

Using the properties of absolute value and triangle inequality for integrals in (17), then we have

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{1}{2(\phi-\varpi)}\left[\int_{\varpi}^{\frac{\varpi+\phi}{2}}(\mu-\varpi)^2\left|\wp''(\mu)\right|d\mu+\int_{\frac{\varpi+\phi}{2}}^{\phi}(\phi-\mu)^2\left|\wp''(\mu)\right|d\mu\right]. \tag{18}$$

Let $\mu = \zeta\nu + (1-\zeta)\phi$ and $d\mu = (\varpi-\phi)d\zeta$ in inequality (18), then we clearly see that

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{1}{2(\phi-\varpi)}\left[\int_{0}^{\frac{1}{2}}\zeta^2(\phi-\varpi)^3\left|\wp''(\zeta\nu+(1-\zeta)\phi)\right|d\zeta + \int_{\frac{1}{2}}^{1}(1-\zeta)^2(\phi-\varpi)^3\left|\wp''(\zeta\nu+(1-\zeta)\phi)\right|\mathrm{d}\zeta\right]. \tag{19}$$

Since $|\wp''|$ is s-convex functions in the second sense on $[\varpi,\phi]$, we can write inequality (19) as follows:

$$\begin{aligned}\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| &\leq \frac{(\phi-\varpi)^2}{2}\left[\int_{0}^{\frac{1}{2}}\zeta^2\left\{\zeta^s\left|\wp''(\varpi)\right|+(1-\zeta)^s\left|\wp''(\phi)\right|\right\}d\zeta + \int_{\frac{1}{2}}^{1}(1-\zeta)^2\left\{\zeta^s\left|\wp''(\varpi)\right|+(1-\zeta)^s\left|\wp''(\phi)\right|\right\}\mathrm{d}\zeta\right]\\ &= \frac{(\phi-\varpi)^2}{2^{s+2}}\frac{2^{s+2}-(s+3)}{(s+1)(s+2)(s+3)}\left\{\left|\wp''(\varpi)\right|+\left|\wp''(\phi)\right|\right\},\end{aligned} \tag{20}$$

which is the desired inequality. □

Remark 9. If we choose $s=1$ in inequality (20), we obtain the following result presented in [36] (Corollary 11, inequality (2.9)).

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{48}\left\{\left|\wp''(\varpi)\right|+\left|\wp''(\phi)\right|\right\}. \tag{21}$$

Theorem 10. *Let* $\wp\colon [\varpi,\phi] \longrightarrow \mathbb{R}$ *be a twice differentiable and* $|\wp''|^q$ *be a* $s-$convex *function in the second sense on* $[\varpi,\phi]$. *If* $\wp \in L[\varpi,\phi]$, *then the inequality*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left\{\frac{1}{p(2p+1)4^p}+\frac{1}{q}\left[\frac{\left|\wp''(\varpi)\right|^q+\left|\wp''(\phi)\right|^q}{s+1}\right]\right\} \tag{22}$$

holds for some fixed $s\in(0,1]$ *and* $p,q>1$ *such that* $(1/p)+(1/q)=1$.

Proof. Utilizing the definition of Green's function, we have identity (17). From the properties of absolute value and triangle inequality for integrals in (17), we obtain inequality (18). By the change of variable in (18), we have

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left[\int_{0}^{(1/2)}\zeta^2\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|d\zeta + \int_{(1/2)}^{1}(1-\zeta)^2\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|\mathrm{d}\zeta\right]. \tag{23}$$

Using the Young inequality in (23), we get

$$\begin{aligned}\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| &\leq \frac{(\phi-\varpi)^2}{2}\left[\frac{1}{p}\int_{0}^{(1/2)}\zeta^{2p}d\zeta+\frac{1}{q}\int_{0}^{(1/2)}\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^q d\zeta\right.\\ &\quad\left.+\frac{1}{\mathrm{p}}\int_{(1/2)}^{1}(1-\zeta)^{2\mathrm{p}}\mathrm{d}\zeta+\frac{1}{\mathrm{q}}\int_{(1/2)}^{1}\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^{\mathrm{q}}\mathrm{d}\zeta\right].\end{aligned} \tag{24}$$

By using the s-convexity of $|\wp''|^q$ in (24), we have

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left[\frac{1}{p(2p+1)2^{2p+1}}+\frac{1}{q}\int_0^{(1/2)}\left\{\zeta^s\left|\wp''(\varpi)\right|^q+(1-\zeta)^s\left|\wp''(\phi)\right|^q\right\}d\zeta\right.$$
$$\left.\frac{1}{p(2p+1)2^{2p+1}}+\frac{1}{q}\int_{(1/2)}^{1}\left\{\zeta^s\left|\wp''(\varpi)\right|^q+(1-\zeta)^s\left|\wp''(\phi)\right|^q\right\}d\zeta\right]$$
$$=\frac{(\phi-\varpi)^2}{2}\left\{\frac{1}{p(2p+1)4^p}+\frac{1}{q}\left[\frac{\left|\wp''(\varpi)\right|^q+\left|\wp''(\phi)\right|^q}{s+1}\right]\right\}. \tag{25}$$

This completes the proof. □

Corollary 11. *Let all the assumptions of Theorem 10 be satisfied and since* $0<(1/p(2p+1)4^p)<(1/12)$, *for* $p>1$, *we obtain*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{24}+\frac{(\phi-\varpi)^2}{2q}\left[\frac{\left|\wp''(\varpi)\right|^{\varpi}+\left|\wp''(\phi)\right|^{\varpi}}{s+1}\right]. \tag{26}$$

Corollary 12. *If we choose* $s=1$ *in inequality (22), we get*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left\{\frac{1}{p(2p+1)4^p}+\frac{1}{q}\left[\frac{\left|\wp''(\varpi)\right|^q+\left|\wp''(\phi)\right|^q}{2}\right]\right\}$$
$$\leq \frac{(\phi-\varpi)^2}{24}+\frac{(\phi-\varpi)^2}{2q}\left[\frac{\left|\wp''(\varpi)\right|^q+\left|\wp''(\phi)\right|^q}{2}\right]. \tag{27}$$

For s-convex maps, the following theorem provides a new upper bound for the left-hand side of $\mathscr{H}-\mathscr{H}$ inequality.

Theorem 13. *Let* $\wp\colon[\varpi,\phi]\longrightarrow\mathbb{R}$ *be a twice differentiable and* $|\wp''|^q$ *be a* $s-$*convex function in the second sense on* $[\varpi,\phi]$. *If* $\wp\in L[\varpi,\phi]$, *then we have the following inequality:*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{(2p+1)4^{p+1}}\left\{\left[\frac{\left|\wp''(\varpi)\right|^q+(2^{s+1}-1)\left|\wp''(\phi)\right|^q}{(s+1)2^{s+1}}\right]^{(1/q)}\right.$$
$$\left.+\left[\frac{(2^{s+1}-1)\left|\wp''(\varpi)\right|^q+\left|\wp''(\phi)\right|^q}{(s+1)2^{s+1}}\right]^{(1/q)}\right\}, \tag{28}$$

for $p>1$ *and* $(1/p)+(1/q)=1$.

Proof. By using the same procedure in Theorem 8, we have inequality (19) as follows:

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left[\int_0^{(1/2)}\zeta^2\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|d\zeta\right. \tag{29}$$
$$\left.+\int_{(1/2)}^{1}(1-\zeta)^2\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|d\zeta\right].$$

Utilizing the Hölder inequality for $p>1$ and $(1/p)+(1/q)=1$ in (29), we get

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left[\left(\int_0^{(1/2)}\zeta^{2p}d\zeta\right)^{(1/p)}\left(\int_0^{(1/2)}\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^q d\zeta\right)^{(1/q)}\right. \tag{30}$$
$$\left.+\left(\int_{(1/2)}^{1}(1-\zeta)^{2p}d\zeta\right)^{(1/p)}\left(\int_{(1/2)}^{1}\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^q d\zeta\right)^{(1/p)}\right].$$

Because $|\wp''|^q$ is s-convex, we have

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{(2p+1)4^{p+1}}\left\{\left[\frac{|\wp''(\varpi)|^q+(2^{s+1}-1)|\wp''(\phi)|^q}{(s+1)2^{s+1}}\right]^{(1/q)} + \left[\frac{(2^{s+1}-1)|\wp''(\varpi)|^q+|\wp''(\phi)|^q}{(s+1)2^{s+1}}\right]^{(1/q)}\right\}. \tag{31}$$

Hence, the proof is done. □

Corollary 14. *Assume that Theorem 13's assumptions are all true and since* $0<(1/(2p+1)4^{p+1})<(1/48)$, *for* $p>1$, *we obtain*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{48}\left\{\left[\frac{|\wp''(\varpi)|^q+(2^{s+1}-1)|\wp''(\phi)|^q}{(s+1)2^{s+1}}\right]^{(1/q)} + \left[\frac{(2^{s+1}-1)|\wp''(\varpi)|^q+|\wp''(\phi)|^q}{(s+1)2^{s+1}}\right]^{(1/q)}\right\}. \tag{32}$$

Corollary 15. *Assume that Theorem 13's assumptions are all true. Then,*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{48}\left(\frac{1}{(s+1)2^{s+1}}\right)^{\frac{1}{q}}\left[(2^{s+1}-1)^{\frac{1}{q}}+1\right]\left\{|\wp''(\varpi)|+|\wp''(\phi)|\right\}. \tag{33}$$

Proof. Using the fact, we get

$$\sum_{j=1}^{l}\left(\varkappa_j+\delta_j\right)^{\tau} \leq \sum_{j=1}^{l}\varkappa_j^{\tau}+\sum_{j=1}^{l}\delta_j^{\tau}. \tag{34}$$

For $0<\tau<1, \varkappa_1, \varkappa_2, \ldots, \varkappa_l \geq 0$ and $\delta_1, \delta_2, \ldots, \delta_l \geq 0$ in (32), we obtain inequality (33). □

Corollary 16. *Let us choose s = 1 in ineqıality (28). Then, we obtain*

$$\begin{aligned}\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| &\leq \frac{(\phi-\varpi)^2}{(2p+1)4^{p+1}}\times\left\{\left[\frac{|\wp''(\varpi)|^q+3|\wp''(\phi)|^q}{8}\right]^{(1/q)}+\left[\frac{3|\wp''(\varpi)|^q+|\wp''(\phi)|^q}{8}\right]^{(1/q)}\right\}\\ &\leq \frac{(\phi-\varpi)^2}{48}\times\left\{\left[\frac{|\wp''(\varpi)|^q+3|\wp''(\phi)|^q}{8}\right]^{(1/q)}+\left[\frac{3|\wp''(\varpi)|^q+|\wp''(\phi)|^q}{8}\right]^{(1/q)}\right\}.\end{aligned} \tag{35}$$

Theorem 17. *Let* $\wp: [\varpi, \phi] \longrightarrow \mathbb{R}$ *be a twice differentiable and* $|\wp''|^q$ *be a* $s-$*convex function in the second sense on* $[\varpi, \phi]$. *If* $\wp \in L[\varpi, \phi]$ *and* $q \geq 1$, *then we obtain the following inequality:*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{16}\left(\frac{1}{3}\right)^{1-(1/q)}\left\{\left[\frac{\left|\wp''(\varpi)\right|^q}{(s+3)2^s}+\frac{2^{s+4}-s^2-7s-14}{(s+1)(s+2)(s+3)2^s}\left|\wp''(\phi)\right|^q\right]^{(1/q)} + \left[\frac{2^{s+4}-s^2-7s-14}{(s+1)(s+2)(s+3)2^s}\left|\wp''(\varpi)\right|^q+\frac{\left|\wp''(\phi)\right|^q}{(s+3)2^s}\right]^{(1/q)}\right\}. \tag{36}$$

Proof. Similarly, from the property of the Green function and inequality (19), we get

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left[\int_0^{(1/2)}\zeta^2\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|d\zeta + \int_{(1/2)}^{1}(1-\zeta)^2\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|d\zeta\right]. \tag{37}$$

Using the s-convexity and power-mean inequality in (37), we obtain

$$\begin{aligned}
&\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \\
&\leq \frac{(\phi-\varpi)^2}{2}\left[\left(\int_0^{(1/2)}\zeta^2 d\zeta\right)^{1-(1/q)}\left(\int_0^{(1/2)}\zeta^2\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^q d\zeta\right)^{(1/q)} + \left(\int_{(1/2)}^{1}(1-\zeta)^2 d\zeta\right)^{1-(1/q)}\left(\int_{(1/2)}^{1}(1-\zeta)^2\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^q d\zeta\right)^{(1/q)}\right] \\
&\leq \frac{(\phi-\varpi)^2}{2}\left(\frac{1}{24}\right)^{1-(1/q)}\left[\left(\int_0^{(1/2)}\zeta^2\left\{\zeta^s\left|\wp''(\varpi)\right|^q+(1-\zeta)^s\left|\wp''(\phi)\right|^q\right\}d\zeta\right)^{(1/q)} + \left(\int_{(1/2)}^{1}(1-\zeta)^2\left\{\zeta^s\left|\wp''(\varpi)\right|^q+(1-\zeta)^s\left|\wp''(\phi)\right|^q\right\}d\zeta\right)^{(1/q)}\right] \\
&= \frac{(\phi-\varpi)^2}{16}\left(\frac{1}{3}\right)^{1-(1/q)}\left\{\left[\frac{\left|\wp''(\varpi)\right|^q}{(s+3)2^s}+\frac{2^{s+4}-s^2-7s-14}{(s+1)(s+2)(s+3)2^s}\left|\wp''(\phi)\right|^q\right]^{(1/q)} + \left[\frac{2^{s+4}-s^2-7s-14}{(s+1)(s+2)(s+3)2^s}\left|\wp''(\varpi)\right|^q+\frac{\left|\wp''(\phi)\right|^q}{(s+3)2^s}\right]^{(1/q)}\right\},
\end{aligned} \tag{38}$$

which is our required inequality. □

Corollary 18. *If we choose $s=1$ in ineqıality (36), we have*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right|$$
$$\leq\frac{(\phi-\varpi)^2}{16}\left(\frac{1}{3}\right)^{1-(1/q)}\left\{\left[\frac{|\wp''(\varpi)|^q}{8}+\frac{5|\wp''(\phi)|^q}{24}\right]^{(1/q)}+\left[\frac{5|\wp''(\varpi)|^q}{24}+\frac{|\wp''(\phi)|^q}{8}\right]^{(1/q)}\right\}. \tag{39}$$

Theorem 19. *Let $\wp$: $[\varpi,\phi]\longrightarrow\mathbb{R}$ be a twice differentiable and $|\wp''|^q$ be a $s-$concave function in the second sense on $[\varpi,\phi]$. If $\wp\in L[\varpi,\phi]$, then we have the following inequality:*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right|\leq\frac{(\phi-\varpi)^2}{2^{4-s}}\left(\frac{1}{2p+1}\right)^{1/p}\left\{\left|\wp''\left(\frac{\varpi+3\phi}{4}\right)\right|+\left|\wp''\left(\frac{3\varpi+\phi}{4}\right)\right|\right\}, \tag{40}$$

for $p>1$ and $(1/p)+(1/q)=1$.

Proof. By using the same procedure in Theorem 13 and the Hölder inequality for $p>1$ and $(1/p)+(1/q)=1$ in (29), we obtain

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right|$$
$$\leq\frac{(\phi-\varpi)^2}{2}\left[\left(\int_0^{(1/2)}\zeta^{2p}d\zeta\right)^{(1/p)}\left(\int_0^{(1/2)}\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^q d\zeta\right)^{(1/q)}+\left(\int_{(1/2)}^1(1-\zeta)^{2p}d\zeta\right)^{(1/p)}\left(\int_{(1/2)}^1\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^q d\zeta\right)^{(1/q)}\right]. \tag{41}$$

Because $|\wp''|^q$ is s-concave, using inequality (6), we have

$$\int_0^{(1/2)}\left|\wp''(\zeta\rho+(1-\zeta)\phi)\right|^q d\zeta\leq 2^{s-1}\left|\wp''\left(\frac{\varpi+3\phi}{4}\right)\right|^q \tag{42}$$

and

$$\int_{(1/2)}^1\left|\wp''(\zeta\rho+(1-\zeta)\phi)\right|^q d\zeta\leq 2^{s-1}\left|\wp''\left(\frac{3\varpi+\phi}{4}\right)\right|^q. \tag{43}$$

From (41)–(43), we get

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right|\leq\frac{(\phi-\varpi)^2}{2^{4-s}}\left(\frac{1}{2p+1}\right)^{(1/p)}\left\{\left|\wp''\left(\frac{\varpi+3\phi}{4}\right)\right|+\left|\wp''\left(\frac{3\varpi+\phi}{4}\right)\right|\right\}, \tag{44}$$

which completes the proof. □

Corollary 20. *If we take $s=1$ in ineqıality (40), we have*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right|$$
$$\leq\frac{(\phi-\varpi)^2}{8}\left(\frac{1}{2p+1}\right)^{(1/p)}\left\{\left|\wp''\left(\frac{\varpi+3\phi}{4}\right)\right|+\left|\wp''\left(\frac{3\varpi+\phi}{4}\right)\right|\right\}$$
$$\leq\frac{(\phi-\varpi)^2}{8}\left\{\left|\wp''\left(\frac{\varpi+3\phi}{4}\right)\right|+\left|\wp''\left(\frac{3\varpi+\phi}{4}\right)\right|\right\}. \tag{45}$$

Theorem 21. *Let $\wp$: $[\varpi,\phi]\longrightarrow\mathbb{R}$ be a twice differentiable and $|\wp''|^q$ be a $s-$concave function in the second sense on $[\varpi,\phi]$. If $\wp\in L[\varpi,\phi]$, then the inequality*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right)-\frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right|\leq\frac{(\phi-\varpi)^2}{2}\left\{\frac{1}{p(2p+1)4^p}+\frac{2^{s-1}}{q}\left\{\left|\wp''\left(\frac{\varpi+3\phi}{4}\right)\right|^q+\left|\wp''\left(\frac{3\varpi+\phi}{4}\right)\right|^q\right\}\right\} \tag{46}$$

holds for some fixed $s \in (0,1]$ *and* $p, q > 1$ *such that* $(1/p) + (1/q) = 1$.

Proof. Let $p > 1$. Similarly, using inequality (19) and Young inequality, we obtain

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right) - \frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left[\frac{1}{p}\int_0^{(1/2)}\zeta^{2p}d\zeta + \frac{1}{q}\int_0^{(1/2)}\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^q d\zeta + \frac{1}{p}\int_{(1/2)}^{1}(1-\zeta)^{2p}d\zeta + \frac{1}{q}\int_{(1/2)}^{1}\left|\wp''(\zeta\varpi+(1-\zeta)\phi)\right|^q d\zeta\right]. \tag{47}$$

Since $|\wp''|^q$ is s-concave functions, from inequality (6), we can write inequalities (42) and (43). Therefore, we have

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right) - \frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left[\frac{1}{p(2p+1)4^p} + \frac{2^{s-1}}{q}\left\{\left|\wp''\left(\frac{\varpi+3\phi}{4}\right)\right|^q + \left|\wp''\left(\frac{3\varpi+\phi}{4}\right)\right|^q\right\}\right], \tag{48}$$

which concludes all of the proof. □

Corollary 22. *In Theorem 21, if we take* $s = 1$, *we obtain*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right) - \frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{2}\left[\frac{1}{p(2p+1)4^p} + \frac{1}{q}\left\{\left|\wp''\left(\frac{\varpi+3\phi}{4}\right)\right|^q + \left|\wp''\left(\frac{3\varpi+\phi}{4}\right)\right|^q\right\}\right]. \tag{49}$$

And since $0 < (1/p(2p+1)4^p) < (1/12)$, *for* $p > 1$, *then*

$$\left|\wp\left(\frac{\varpi+\phi}{2}\right) - \frac{1}{\phi-\varpi}\int_{\varpi}^{\phi}\wp(\xi)d\xi\right| \leq \frac{(\phi-\varpi)^2}{24} + \frac{(\phi-\varpi)^2}{2q}\left\{\left|\wp''\left(\frac{\varpi+3\phi}{4}\right)\right|^q + \left|\wp''\left(\frac{3\varpi+\phi}{4}\right)\right|^q\right\}. \tag{50}$$

4. Applications for Special Means

Now, let us look at the means for the random real numbers σ_1 and σ_2 $(\sigma_1 \neq \sigma_2)$.

(1) Arithmetic mean:

$$\mathscr{A}(\sigma_1, \sigma_2) = \frac{\sigma_1+\sigma_2}{2}, \sigma_1, \sigma_2 \in \mathbb{R}^+. \tag{51}$$

(2) Logarithmic mean:

$$\mathscr{L}(\sigma_1, \sigma_2) = \frac{\sigma_1-\sigma_2}{\ln|\sigma_1| - \ln|\sigma_2|}, |\sigma_1| \neq |\sigma_2|, \sigma_1, \sigma_2 \neq 0, \sigma_1, \sigma_2 \in \mathbb{R}^+. \tag{52}$$

(3) Generalized log − mean:

$$\mathscr{L}_n(\sigma_1, \sigma_2) = \left[\frac{\sigma_2^{n+1}-\sigma_1^{n+1}}{(n+1)(\sigma_2-\sigma_1)}\right]^{(1/n)} \quad n \in \mathbb{Z}\backslash\{-1,0\}, \sigma_1, \sigma_2 \in \mathbb{R}^+. \tag{53}$$

Now, we provide some applications to special means of real numbers based on the Section 3 results.

Proposition 23. *Let* $0 < \varpi < \phi$ *and* $s \in (0,1]$. *Then, we have*

$$\left|A^s(\varpi,\phi)-\mathscr{L}_s^s(\varpi,\phi)\right|\leq\frac{|s(s-1)|(\phi-\varpi)^2}{2^{s+2}}\frac{2^{s+2}-(s+3)}{(s+1)(s+2)(s+3)}\left\{|\varpi|^{s-2}+|\phi|^{s-2}\right\}. \tag{54}$$

Proof. The assumption follows from (13) applied to the s-convex function $\wp\colon [0,1]\longrightarrow[0,1]$, $\wp(\xi)=\xi^s$. □

Proposition 24. *Let* $0<\varpi<\phi$ *and* $s\in(0,1]$. *Then, we have*

$$\left|A^s(\varpi,\phi)-\mathscr{L}_s^s(\varpi,\phi)\right| \leq\frac{|s(s-1)|(\phi-\varpi)^2}{(2p+1)4^{p+1}}\left\{\left[\frac{|\varpi|^{q(s-2)}+(2^{s+1}-1)|\phi|^{q(s-2)}}{(s+1)2^{s+1}}\right]^{(1/\varpi)} +\left[\frac{(2^{s+1}-1)|\varpi|^{q(s-2)}+|\phi|^{q(s-2)}}{(s+1)2^{s+1}}\right]^{(1/\varpi)}\right\}. \tag{55}$$

Proof. The assumption follows from (28) applied to the s-convex function $\wp\colon [0,1]\longrightarrow[0,1]$, $\wp(\xi)=\xi^s$. □

5. Results and Discussion

If we use Green's functions $\mathscr{G}_2$, $\mathscr{G}_3$, and $\mathscr{G}_4$ for various convexities, we may obtain the same conclusions as in this article. The $\mathscr{H}-\mathscr{H}$ inequality for s-convex functions, as well as the $\mathscr{H}-\mathscr{H}$ inequality for fractional operators, preinvex, co-ordinate convex functions, and so on, may all be studied using Green's function or any other new Green's function. Readers who are interested in an exercise can utilize the other three Green's functions to produce the results that correspond to them.

6. Conclusion

Producing novel and special integral inequalities is the primary motivation for inequality theory, one of the most crucial areas of mathematical analysis. Researchers sometimes utilize novel function classes, sometimes new integral operators, and sometimes try to obtain modifications of a few well-known inequalities in different spaces for this aim. Especially in the last decade, many methods have been developed with different results, and many remarkable refinements, extensions, and generalizations have been obtained. Using these different methods, many different types of inequalities, lemmas, and different identities can be found in articles, and relevant researchers can access these studies in the literature. One of these important methods is the well-known Green's function.

In this study, a new method was developed using Green's function to prove the new results obtained. In addition, conclusions pertaining to the left-hand side of $\mathscr{H}-\mathscr{H}$ were obtained for derivatives of the q-th power of s-convex functions. Using the identity (2.11) (in [37]) and different types of convexity, researchers can derive both $\mathscr{H}-\mathscr{H}$ type and different type well-known inequalities. In this sense, we hope that this study will inspire researchers to obtain further results.

Data Availability

No data were used to support the findings of this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors have read and approved the final manuscript.

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