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Research Article

Well-Posedness in Variable-Exponent Function Spaces for the Three-Dimensional Micropolar Fluid Equations

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In this paper, we work on the Cauchy problem of the three-dimensional micropolar fluid equations. For small initial data, in the variable-exponent Fourier–Besov spaces, we achieve the global well-posedness result. The Littlewood–Paley decomposition method and the Fourier-localization technique are main tools to obtain the results. Moreover, the results discussed in our work show the Gevrey class regularity of solution to the Cauchy problem of micropolar fluid equations.

1. Introduction

In this paper, our aim is to study the Cauchy problem of the three-dimensional incompressible micropolar fluid equations:

$$\begin{cases} \omega_t - (\chi + \nu)\Delta\omega + \omega \cdot \nabla\omega + \nabla\pi - 2\chi\nabla \times \theta = 0, \\ \theta_t - \mu\Delta\theta + \omega \cdot \nabla\theta + 4\chi\theta - \kappa\nabla(\nabla \cdot \theta) - 2\chi\nabla \times \omega = 0, \\ \nabla \cdot \omega = 0, \\ \omega|_{t=0} = \omega_0, \nabla \cdot \omega_0 = 0, \\ \theta|_{t=0} = \theta_0, \end{cases} \tag{1}$$

where the $\omega(y,t)$ is the linear velocity of the fluid, the scalar $\pi(y,t)$ represents the pressure, and $\theta(y,t)$ is the field of microrotation and represents the angular velocity of the rotation of particles in the fluid. The symbols $\nabla \cdot \omega$ and $\nabla \times \omega$ represent the divergence and rotational of the field ω, respectively. The Newton kinematics viscosity is denoted by ν and χ represents the microrotational viscosity. The symbols κ and μ denote the angular viscosities. For convenience, we assume $\chi = \nu = 1/2$ and $\kappa = \mu = 1$.

The standard micropolar fluid equations, e.g., equation (1), were first investigated by Eringen [1]. This is a kind of fluid that possesses microrotational effects and microrotational inertia, which is thus classified as a non-Newtonian fluid. In the context of physical importance, a micropolar fluid could be a model of fluids where the particles are rigid and floating in a viscous medium without consideration for their deformation. It is significant to the researchers dealing with MHD fluid problems and other related phenomena because it can relate to many behaviors that occur in a wide variety of complicated fluids such as liquid crystals, animal blood, and suspensions that cannot be described properly by the Navier–Stokes (NS) equation. For more details related to its physical applications, we suggest the readers read [2] and the references therein.

Its significance in both physics and mathematics has contributed to extensive work in the area of mathematical analysis. Existence of the weak solution was established by Lukaszewicz [2]. Similarly, in 1997, Galdi and Rionero [3] published a note in which they considered equation (1) and achieved for the first time that the solution exists and is unique. Moreover, in their study, they have further proved that the corresponding solution is globally well-posed if the initial data are small and local well-posed for large initial data. Regarding the smooth solutions to the two-dimensional case of equation (1) with partial viscosity and full viscosity, Chen [4] worked on the global well-posedness and its uniqueness. The large-time phenomenon of the solution and global in time regularity were established by Dong et al. [5]. For more results related to the two-dimensional case, we refer to [6, 7]. Regarding the regularity criterion of the weak solution and the blown-up criterion, for the smooth solution, see [8, 9] and the references therein. The main difficulty of equation (1) is to deal with the linear coupling terms $\nabla \times \theta$ and $\nabla \times u$. These terms involve the curl operations on the microrotation field θ and velocity u, which make the equations more complicated and complex. The nonlinearity makes it harder to figure out how the system will behave and investigate it, which could lead to instability or chaotic behavior. Fereirra and Villamizar-Roa [10] overcame this difficulty and established the global existence of singular solution to the fractional case of equation (1) in the pseudomeasure space PM^a.

The PM^a spaces were studied in [11] to obtain the singular solutions to NS equations. The global existence for the solution of equation (1) in Besov spaces $\dot{\mathscr{B}}_{p,\infty}^{-1+(3/p)}$ with $1 \leq p < 6$ was established by Chen and Miao [12]. This work extended the result of Cannone-related NS equations, such as [13]. Zhu and Zhao recently obtained the decay results for equation (1) in critical Besov spaces [14] and Fourier – Besov spaces [15]. More recently, Nie and Zheng [16] have investigated the existence of global solution of equation (1) in Fourier – Herz spaces. These results motivate us to study the global well-posedness of equation (1) in the framework of variable-exponent Fourier – Besov spaces.

When $\theta = \chi = 0$, equation (1) corresponds to the three-dimensional incompressible NS equations. The scaling invariance is an important feature of the NS equations. The global existence of solutions for NS equations in the critical homogeneous Sobolev spaces $\dot{H}^{1/2}$ was proven by Fujita and Kato [17]. Chemin [18] established its existence and uniqueness in homogeneous Besov spaces $\dot{\mathscr{B}}_{p,\gamma}^{(3/p)-1}$ for $1 \leq p < \infty$ and $1 \leq \gamma \leq \infty$. The global unique solution in a more general space, BMO^{-1}, is obtained by Koch and Tataru [19]. Recently, for the fractional case of NS equations, Ru and Abidin [20] obtained the global well-posedness result under the smallness condition on initial data in variable-exponent Fourier – Besov spaces. The main purpose of this paper is to extend this result to the solution of equation (1). In this research article, we investigate the existence and analyticity of the global solution of the three-dimensional micropolar fluid equations in the framework of variable-exponent Fourier – Besov spaces. There exist some significant differences between variable-exponent Fourier–Besov spaces and Fourier–Besov spaces. Certain classical results, such as Young's inequality and the multiplier theorem, are not applicable within the context of variable-exponent Fourier–Besov spaces. Due to this setting, it becomes difficult to determine the well-posedness of equations within these spaces. This work primarily utilizes the tools outlined in Sections 2 and 3, together with Banach's contraction mapping concept, to study the global well-posedness of the micropolar fluid equations in variable-exponent Fourier–Besov spaces.

The constant exponent Fourier–Besov space can be traced back to the research of Konieczny and Yoneda [21], which focuses on the study of dispersion effect of Coriolis for the NS equations. Moreover, Iwabuchi [22] introduced Fourier–Herz spaces to study Keller–Segel system, which is a particular case of Fourier–Besov spaces.

The origin of the variable-exponent Lebesgue space $L^{q(\cdot)}$ can be followed back to Orlicz [23]. The researchers, such as Musielak, Nakano, and Zhikov, contributed to its further development. Section 1.2 of [24] provides a brief overview of the early development of the theory of variable-exponent function spaces. Kovaik and Rákosik [25], Cruz-Uribe [26], Diening [27], and Fan and Zhao [28] are considered the pioneers of modern theory of variable-exponent function space. For other details related to the variable-exponent Besov space and variable-exponent Fourier–Besov space, see [29–32] and the references therein. One of the primary motivations for the development of variable-exponent theory is the mathematical modelling of electrorheological fluids, which can be seen in [33]. In Section 3, we will give the definitions of various variable-exponent function spaces.

The following symbols are introduced for the convenience of description. Let Y, Z be Banach spaces; in this part and afterwards, we write $\|\cdot\|_{Y\cap Z} \coloneqq \|\cdot\|_Y + \|\cdot\|_Z$ and $\|(\cdot,\circ)\|_Y \coloneqq \|\cdot\|_Y + \|\circ\|_Y$. For the sake of convenience, we assume that $\kappa = \mu = 1$ and $\chi = \nu = 1/2$. This work is arranged in the following pattern: basic concepts of variable-exponent function space are given in Section 2. The equivalent integral form of equation (1) is presented in Section 3. In Section 4, we give the linear estimate, and in the last section, Section 5, we present the proof of Theorems 9 and 11.

2. Preliminaries

In this section, we give some basic definitions related to the variable-exponent function space [30] and some important propositions that are helpful to prove our main theorems.

Definition 1. We define the Lebesgue space in the framework of variable exponent $L^{q(y)}$ by the set

$$\left\{\omega : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ is measurable}, \int_{\mathbb{R}^n} |\omega(y)|^{q(y)} dy < +\infty\right\}, \quad (2)$$

with the following Luxemburg–Nakano norm:

$$\|\cdot\|_{L^{q(y)}} = \inf\left\{\zeta > 0\colon \int_{\mathbb{R}^n}\left(\frac{|\cdot|}{\zeta}\right)^{q(y)} dy \leq 1\right\}, \quad (3)$$

where $q(y) \in \mathscr{P}_0$ and $\mathscr{P}_0$ is the set of all measurable function $q\colon \mathbb{R}^n \longrightarrow \mathbb{R}^+$ such that

$$0 < q_- = \operatorname*{essinf}_{y\in\mathbb{R}^n} q(y), \operatorname*{esssup}_{y\in\mathbb{R}^n} q(y) = q_+ < \infty. \tag{4}$$

Furthermore, ($L^{q(\cdot)}(\mathbb{R}^n)$ with $\|\cdot\|_{L^{q(\cdot)}}$) is a Banach space. To differentiate between the variable exponent and the constant exponents, we indicate the variable exponents with $q(\cdot)$ and constant exponents with p.

Definition 2. Let $q(\cdot), \gamma(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$; the mixed sequence Lebesgue space $\ell^{\gamma(\cdot)}(L^{q(\cdot)})$ is the set of all sequences $\{h_j\}_{m\in\mathbb{Z}}$ of measurable functions belonging to $\mathbb{R}^n$ such that

$$\|\{h_m\}_{m\in\mathbb{Z}}\|_{\ell^{\gamma(\cdot)}(L^{q(\cdot)})} := \inf\left\{\mu > 0, \varrho_{\ell^{\gamma(\cdot)}(L^{q(\cdot)})}\left(\left\{\frac{h_m}{\mu}\right\}_{m\in\mathbb{Z}}\right) \le 1\right\} < \infty, \tag{5}$$

where

$$\varrho_{\ell^{\gamma(\cdot)}(L^{q(\cdot)})}(\{h_m\}_{m\in\mathbb{Z}}) = \sum_{m\in\mathbb{Z}} \inf\left\{\zeta_j : \int_{\mathbb{R}^n}\left(\frac{|h_m(y)|}{\zeta_j^{(1/\gamma(y))}}\right)^{q(y)} \mathrm{d}y \le 1\right\}. \tag{6}$$

Notice that $\varrho_{\ell^{\gamma(\cdot)}(L^{q(\cdot)})}(\{h_m\}_{m\in\mathbb{Z}}) = \sum_{m\in\mathbb{Z}}\||h_m|^{\gamma(\cdot)}\|_{L^{(q(\cdot)/\gamma(\cdot))}}$ with $\gamma_+ < \infty$.

For the sake of guarantee that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded by $L^{q(\cdot)}(\mathbb{R}^n)$, we suppose that the variable-exponent function $q(\cdot)$ satisfies the following conditions:

(1) If there exists a positive constant $c_{\log}$, such that

$$|q(y) - q(z)| \le \frac{c_{\log}}{\log\left(e + |y-z|^{-1}\right)}; \quad \text{for all } y, z \in \mathbb{R}^n \text{ and } y \ne z. \tag{7}$$

Then, $q(\cdot)$ is said to be locally log-Hölder continuous.

(2) If there exists some constant p_∞ independent of y, such that

$$\left|q(y) - p_\infty\right| \le \frac{C_p}{\log(e + |y|)}, \quad \text{for any } y \in \mathbb{R}^n. \tag{8}$$

Then, $q(\cdot)$ is said to be globally log-Hölder continuous.

Consider a set, consisting of all functions satisfying conditions 1 and 2, which is termed as the $K^{\log}(\mathbb{R}^n)$ space.

Here, we recall the Littlewood–Paley decomposition method. Suppose φ is the radial function satisfying non-negativity and smoothness with supp $\varphi := \{\xi \in \mathbb{R}^n \colon 3/4 \le |\xi| \le 8/3\}$ and for any $\xi \ne 0$, $\sum_{m\in\mathbb{Z}}\varphi_j(\xi) = 1$, where $\varphi_j(\cdot) = \varphi(2^{-j}\cdot)$.

Definition 3. Let $s(\cdot) \in K^{\log}(\mathbb{R}^n)$ and $q(\cdot), \gamma(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap K^{\log}(\mathbb{R}^n)$. We define the homogeneous variable-exponent Besov space by

$$\dot{\mathcal{B}}^{s(\cdot)}_{q(\cdot),\gamma(\cdot)} := \left\{\omega \in \frac{\mathcal{S}'}{D} \colon \|\omega\|_{\dot{\mathcal{B}}^{s(\cdot)}_{q(\cdot),\gamma(\cdot)}} < +\infty\right\}, \tag{9}$$

with

$$\|\omega\|_{\dot{\mathcal{B}}^{s(\cdot)}_{q(\cdot),\gamma(\cdot)}} := \left\|\left\{2^{ms(\cdot)}\dot{\Delta}_j\omega\right\}_{m\in\mathbb{Z}}\right\|_{\varrho_{\ell^{\gamma(\cdot)}(L^{q(\cdot)})}}, \tag{10}$$

where $\dot{\Delta}_j\omega = \varphi_j(\cdot)^\vee * \omega$ and D denotes the set of all polynomials.

Definition 4. Let $s(\cdot) \in K^{\log}(\mathbb{R}^n)$ and $q(\cdot), \gamma(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap K^{\log}(\mathbb{R}^n)$. We define the homogeneous variable-exponent Fourier–Besov space by

$$\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{q(\cdot),\gamma(\cdot)} := \left\{\omega \in \frac{\mathcal{S}'}{D} \colon \|\omega\|_{\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{q(\cdot),\gamma(\cdot)}} < +\infty\right\}, \tag{11}$$

with

$$\|\omega\|_{\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{q(\cdot),\gamma(\cdot)}} := \left\|\left\{2^{ms(\cdot)}\varphi_j\widehat{\omega}\right\}_{m\in\mathbb{Z}}\right\|_{\ell^{\gamma(\cdot)}(L^{q(\cdot)})}. \tag{12}$$

Definition 5. Suppose $T \in (0,\infty)$ and $p, r \in [1,\infty)$. Then, homogeneous Chemin–Lerner variable-exponent Fourier–Besov space can be defined as

$$\mathcal{L}^\rho\left(0,T;\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{q(\cdot),r}\right) := \left\{\omega \in \frac{\mathcal{S}'}{D} \colon \|\omega\|_{\mathcal{L}^\rho\left(0,T;\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{q(\cdot),r}\right)} < +\infty\right\}, \tag{13}$$

with

$$\|\omega\|_{\mathscr{L}^{\rho}\left(0,T;\mathscr{F}\dot{\mathscr{B}}^{s(\cdot)}_{q(\cdot),r}\right)} := \left\|\left\{\left\|2^{ms(\cdot)}\varphi_j\widehat{\omega}\right\|_{L^{\rho}_T L^{q(\cdot)}}\right\}_{m\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}. \quad (14)$$

Proposition 6. *The following conclusions hold in the variable-exponent function spaces:*

(i) *Hölder inequality [24]: suppose $r(\cdot), \gamma(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and q is a mapping from real n-tuple space to the set $[1,\infty)$ with $\operatorname{esssup}_{y\in\mathbb{R}^n} q(y) = q_+ < \infty$ such that $(1/q(y)) = (1/\gamma(y)) + (1/r(y))$. Then,*

$$\|\omega v\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \|\omega\|_{L^{\gamma(\cdot)}(\mathbb{R}^n)}\|v\|_{L^{r(\cdot)}(\mathbb{R}^n)}, \quad (15)$$

where $\omega \in L^{\gamma(\cdot)}(\mathbb{R}^n)$ and $v \in L^{r(\cdot)}(\mathbb{R}^n)$.

(ii) *Sobolev embeddings [30]: let $q_0(\cdot), q_1(\cdot), \gamma(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $s_0(\cdot), s_1(\cdot) \in L^\infty$ with $s_0(\cdot) \geq s_1(\cdot)$. If $(1/\gamma(\cdot))$ and $s_0(\cdot) - (d/q_0(\cdot)) = s_1(\cdot) - (d/q_1(\cdot))$ are locally log-Hölder continuous, then*

$$\dot{\mathscr{B}}^{s_0(\cdot)}_{q_0(\cdot),\gamma(\cdot)} \hookrightarrow \dot{\mathscr{B}}^{s_1(\cdot)}_{q_1(\cdot),\gamma(\cdot)}. \quad (16)$$

(iii) See [30]. *Let $q_0(\cdot), q_1(\cdot), \gamma_0(\cdot), \gamma_1(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $s_0(\cdot), s_1(\cdot) \in L^\infty$ with $s_0(\cdot) \geq s_1(\cdot)$. If $(1/\gamma_0(\cdot))$, $(1/\gamma_1(\cdot))$ and $s_0(\cdot) - (d/q_0(\cdot)) = s_1(\cdot) - (d/q_1(\cdot)) + \varepsilon(y)$ are locally log-Hölder continuous, where $\operatorname{essinf}_{y\in\mathbb{R}^n} \varepsilon(y) > 0$, then*

$$\dot{\mathscr{B}}^{s_0(\cdot)}_{q_0(\cdot),\gamma_0(\cdot)} \hookrightarrow \dot{\mathscr{B}}^{s_1(\cdot)}_{q_1(\cdot),\gamma_1(\cdot)}. \quad (17)$$

Proposition 7 (See [20]). *Suppose there exists a nonnegative constant s and $1 \leq \rho, \rho_1, \rho_2, p, \gamma, r, \varsigma \leq \infty$, $(1/\varsigma) = (1/r) + (1/p)$ and $(1/\rho) = (1/\rho_1) + (1/\rho_2)$. Then, we have*

$$\|ab\|_{\mathscr{L}^{\rho}\dot{\mathscr{B}}^{s}_{\varsigma,\gamma}} \lesssim \|a\|_{\mathscr{L}^{\rho_1}\dot{\mathscr{B}}^{s}_{p,\gamma}}\|b\|_{\mathscr{L}^{\rho_2}L^r} + \|b\|_{\mathscr{L}^{\rho_1}\dot{\mathscr{B}}^{s}_{p,\gamma}}\|a\|_{\mathscr{L}^{\rho_2}L^r}. \quad (18)$$

We will use the contraction mapping argument in critical Banach spaces to obtain the solution to equation (1).

Proposition 8 (See [34]). *Let g be a continuous bilinear mapping from $A \times A$ to A, where A is a Banach space and $\varepsilon > 0$ such that*

$$\varepsilon < \frac{1}{4\|g\|} \text{ with } \|g\| := \sup_{\|\omega\|,\|v\|\leq 1}\|g(\omega, v)\|. \quad (19)$$

For $y \in g(0,\varepsilon) \subseteq A$, where g is the ball with origin as a centre and radius ε, then

$$y = z + g(y, y), \quad (20)$$

where y obeys the property of uniqueness in the ball $g(0, 2\varepsilon)$.

3. Equivalent Integral Form of Equation (1)

Following [10], the corresponding linear equations to equation (25) are as follows:

$$\begin{cases} \partial_t\omega - \Delta\omega - \nabla\times\theta = 0, \\ \partial_t\theta - \Delta\theta + 2\theta - \nabla(\nabla\cdot\theta) - \nabla\times\omega = 0, \\ \omega|_{t=0} = \omega_0, \nabla\cdot\omega_0 = 0, \\ \theta|_{t=0} = \theta_0. \end{cases} \quad (21)$$

Let $G(t)$ denote the solution operator to the above linear equation (21), then

$$G(t)W_0 = \mathscr{F}^{-1}e^{-tH(\xi)}\widehat{W}_0, \quad (22)$$

where

$$H(\xi) = \begin{bmatrix} |\xi|^2 I & A(\xi) \\ A(\xi) & \left(|\xi|^2 + 2\right)I + B(\xi) \end{bmatrix}, \quad (23)$$

with

$$A(\xi) = i\begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix}, B(\xi) = \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{bmatrix}. \quad (24)$$

Applying $\mathbb{P}$, the Leray projector, to equation (1) to remove the pressure term π, we have the following:

$$\begin{cases} \partial_t\omega - \Delta\omega - \nabla\times\theta + \mathbb{P}(\omega\cdot\nabla\omega) = 0, \\ \partial_t\theta - \Delta\theta + \omega\cdot\nabla\theta + 2\theta - \nabla(\nabla\cdot\theta) - \nabla\times\omega = 0, \\ \omega|_{t=0} = \omega_0, \nabla\cdot\omega_0 = 0, \\ \theta|_{t=0} = \theta_0. \end{cases} \quad (25)$$

Let

$$W = \begin{pmatrix} \omega \\ \theta \end{pmatrix}, W_0 = \begin{pmatrix} \omega_0 \\ \theta_0 \end{pmatrix}, \quad \omega\otimes W = \begin{pmatrix} \omega\otimes\omega \\ \omega\otimes\theta \end{pmatrix}. \quad (26)$$

Furthermore, we define

$$\mathbb{P}(\nabla\cdot(\omega\otimes W)) := \begin{pmatrix} \mathbb{P}\nabla\cdot(\omega\otimes\omega) \\ \nabla\cdot(\omega\otimes\theta) \end{pmatrix}. \quad (27)$$

According to Duhamel principle, the solution to equation (25) can be deduced to the solution W of the following integral equation:

$$W(y,t) = G(t)W_0 - \int_0^t G(t-s)\mathbb{P}\nabla\cdot(\omega\otimes W)\mathrm{d}s. \quad (28)$$

4. Linear Estimate

For the sake of the proofs of the main theorems, we have to make a priori estimate by considering equation (28). Therefore, the following lemma is constructed.

Lemma 8. *Let* $q(\cdot), q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *for* $q_1(\cdot) \leq q(\cdot)$, $\beta(\cdot) \in C^{\log}(\mathbb{R}^n)$, $1 \leq \rho, \gamma \leq \infty$, $I = [0, T)$, $T \in (0, \infty]$, *and* $(\omega_0, \theta_0) \in \mathcal{F}\dot{\mathcal{B}}_{q(\cdot),\gamma}^{\beta(\cdot)+(3/q'(\cdot))}$, *then*

$$\|G(t)(\omega_0, \theta_0)\|_{\mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{B}}_{q_1(\cdot),\gamma}^{\beta(\cdot)+\left(3/q'(\cdot)\right)+(2/\rho)}\right)} \lesssim \|(\omega_0, \theta_0)\|_{\mathcal{F}\dot{\mathcal{B}}_{q(\cdot),\gamma}^{\beta(\cdot)+\left(3/q'(\cdot)\right)}}. \tag{29}$$

Proof. Let $p^* = (q(y)q_1(x)/q(y) - q_1(x))$, by using Proposition 6, and considering $q_1(\cdot) \leq q(\cdot)$, we obtain

$$\begin{aligned}
&\|G(t)W_0\|_{\mathcal{L}^\rho\left(I;\mathcal{F}\dot{\mathcal{B}}_{q_1(\cdot),\gamma}^{\beta(\cdot)+\left(3/q'(\cdot)\right)+(2/\rho)}\right)}\\
&\lesssim \left\|\left\|2^{m(\beta(\cdot)+(3/q(\cdot))+(2/\rho))}\varphi_i e^{-t|\cdot|^2}\widehat{W}_0\right\|_{\mathcal{L}^\rho\left(I;L^{q_1(\cdot)}\right)}\right\|_{l^\gamma}\\
&\lesssim \left\|\sum_{m=0,\pm1}\left\|2^{m\left(\beta(\cdot)+\left(3/q'(\cdot)\right)\right)}\varphi_i\widehat{W}_0\right\|_{L^{q(\cdot)}}\left\|2^{m\left(\left(3/q_1'(\cdot)\right)-\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_{m+k}e^{-t|\cdot|^2}\widehat{W}_0\right\|_{\mathcal{L}^\rho\left(I;L^{p^*}\right)}\right\|_{l^\gamma}\\
&\lesssim \left\|\sum_{m=0,\pm1}\left\|2^{m\left(\beta(\cdot)+\left(3/q'(\cdot)\right)\right)}\varphi_j\widehat{W}_0\right\|_{L^{q(\cdot)}}\left\|2^{m(2/\rho)}e^{-t2^{2(m+k)}}\right\|_{\mathcal{L}^\rho(I)}\left\|2^{3m\left(\left(1/q_1'(\cdot)\right)-\left(1/q'(\cdot)\right)\right)}\varphi_{m+k}\right\|_{L^{p^*}}\right\|_{l^\gamma}\\
&\lesssim \left\|\sum_{m=0,\pm1}\left\|2^{3m\left(\left(1/q_1'(\cdot)\right)-\left(1/q'(\cdot)\right)\right)}\varphi_{m+k}\right\|_{L^{p^*}}\left\|2^{m\left(\beta(\cdot)+\left(3/q'(\cdot)\right)\right)}\varphi_j\widehat{W}_0\right\|_{L^{q(\cdot)}}\right\|_{l^\gamma}\\
&\lesssim \left\|\left\|2^{m\left(\beta(\cdot)+\left(3/q'(\cdot)\right)\right)}\varphi_j\widehat{W}_0\right\|_{L^{q(\cdot)}}\right\|_{l^\gamma}\\
&= \|W_0\|_{\mathcal{F}\dot{\mathcal{B}}_{q_1(\cdot),\gamma}^{\beta(\cdot)+\left(3/q'(\cdot)\right)}}.
\end{aligned} \tag{30}$$

The above estimate was based on the following fact:

$$\begin{aligned}
&\sum_{m=0,\pm1}\left\|2^{3m\left(\left(1/q_1'(\cdot)\right)-\left(1/q'(\cdot)\right)\right)}\varphi_{m+k}\right\|_{L^{p^*}}\\
&= \sum_{m=0,\pm1}\left\|2^{3m\left(\left(1/q_1'(\cdot)\right)-\left(1/q'(\cdot)\right)\right)}\varphi_{m+k}\right\|_{L^{(q(\cdot)q_1(\cdot)/q(\cdot)-q_1(\cdot))}}\\
&= \sum_{m=0,\pm1}\inf\left\{\zeta>0\colon \int\left|\frac{2^{3m\left(\left(1/q_1'(\cdot)\right)-\left(1/q'(\cdot)\right)\right)}\varphi_{m+k}}{\zeta}\right|^{(q(\cdot)q_1(\cdot)/q(\cdot)-q_1(\cdot))}\mathrm{d}y\leq 1\right\}\\
&= \sum_{m=0,\pm1}\inf\left\{\zeta>0\colon \int\left|\frac{\varphi_{m+k}}{\zeta}\right|^{(q(\cdot)q_1(\cdot)/q(\cdot)-q_1(\cdot))}2^{-3m}\mathrm{d}y\leq 1\right\}\\
&= \sum_{m=0,\pm1}\inf\left\{\zeta>0\colon \int\left|\frac{\varphi_k}{\zeta}\right|^{(q(2^m\cdot)q_1(2^m\cdot)/q(2^m\cdot)-q_1(2^m\cdot))}\mathrm{d}y\leq 1\right\}\lesssim C.
\end{aligned} \tag{31}$$

5. Main Results

In this section, we give the main results related to the global well-posedness and Gevrey class regularity of the solution to equation (1).

Theorem 9. *Let* $q(\cdot) \in \mathcal{P}(\mathbb{R}^3) \cap K^{\log}(\mathbb{R}^3)$, $1 \leq \rho \leq \infty$, $1 \leq \gamma < 3$ *and* $2 \leq q(\cdot) \leq 6$. *Then, there exists a positive constant* ε_0 *such that for any initial data* (ω_0, θ_0) *in* $\mathcal{F}\dot{\mathcal{B}}_{q(\cdot),\gamma}^{-1+(3/q'(\cdot))}(\mathbb{R}^3)$ *with*

$$\left\|(\omega_0,\theta_0)\right\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)}} < \varepsilon_0, \tag{32}$$

there is a unique global solution (ω,θ) *in*

$$\mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)+(2/\rho)}\right)\cap\mathscr{L}^{\rho}\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)+(2/\rho)}\right)\cap\mathscr{L}^{\infty}\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right), \tag{33}$$

such that

$$\|(\omega,\theta)\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)+(2/\rho)}\right)\cap\mathscr{L}^{\rho}\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)+(2/\rho)}\right)\cap\mathscr{L}^{\infty}\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}\lesssim\left\|(\omega_0,\theta_0)\right\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)}}. \tag{34}$$

Proof. Define the set

$$B \coloneqq \mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)+(2/\rho)}\right)\cap\mathscr{L}^{\rho}\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)+(2/\rho)}\right)\cap\mathscr{L}^{\infty}\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right), \tag{35}$$

and consider the following mapping:

$$g\colon W(t)\longrightarrow G(t)W_0-\int_0^t G(t-s)\mathbb{P}\nabla\cdot(\omega\otimes W)\mathrm{d}s. \tag{36}$$

We need to prove that the mapping shown above is a contraction mapping. By the above mapping, we obviously have

$$\begin{aligned}
\|gW(t)\|_B &\le \left\|G(t)W_0\right\|_B+\left\|\int_0^t G(t-s)\mathbb{P}\nabla\cdot(\omega\otimes W)\mathrm{d}s\right\|_B\\
&=\left\|G\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\right\|_B+\left\|\int_0^t G(t)(t-s)\begin{pmatrix}\mathbb{P}\nabla\cdot(\omega\otimes\omega)\\ \nabla\cdot(\omega\otimes\theta)\end{pmatrix}\mathrm{d}s\right\|_B\\
&=I_1+I_2.
\end{aligned} \tag{37}$$

To estimate I_1, assume the hypothesis $q(\cdot)\ge 2$, and using Lemma 8, we have

$$\begin{aligned}
\left\|G(t)\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)+(2/\rho)}\right)} &\lesssim \left\|\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\right\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)}},\\
\left\|G(t)\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{2,\gamma}^{(1/2)+(2/\rho)}\right)} &\lesssim \begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\Bigg\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)}},\\
\left\|G(t)\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\right\|_{\mathscr{L}^{\infty}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)} &\lesssim \left\|\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\right\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)}}.
\end{aligned} \tag{38}$$

Hence, we have

$$\left\| G(t)\begin{pmatrix} \omega_0 \\ \theta_0 \end{pmatrix} \right\|_B \lesssim \left\| \begin{pmatrix} \omega_0 \\ \theta_0 \end{pmatrix} \right\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+(3/q'(\cdot))}} < \varepsilon_0. \tag{39}$$

To estimate I_2, utilizing Proposition 6, we have

$$\begin{aligned}
&\left\| \int_0^t G(t-s)\mathbb{P}\nabla\cdot(\omega\otimes\omega)\mathrm{d}s \right\|_{\mathscr{L}^{\rho_1}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+(3/q'(\cdot))+(2/\rho_1)}\right)} \\
&\lesssim \left\| \left\| \int_0^t 2^{m(-1+(3/q'(\cdot))+(2/\rho_1))}\varphi_j e^{-(t-s)|\cdot|^2}\operatorname{div}\widehat{(\omega\otimes\omega)}\mathrm{d}s \right\|_{L^{\rho_1}(\mathbb{R}_+;L^{q(\cdot)})} \right\|_{l^\gamma} \\
&\lesssim \left\| \left\| \int_0^t \left\| 2^{m((3/q'(\cdot))+(2/\rho_1))}\varphi_j e^{-(t-s)|\cdot|^2} \right\|_{L^{(6q(\cdot)/6-q(\cdot))}} \|\widehat{\omega\otimes\omega}\|_{L^6}\mathrm{d}s \right\|_{L^{\rho_1}(\mathbb{R}_+)} \right\|_{l^\gamma} \\
&\lesssim \left\| \left\| \int_0^t \left\| 2^{m((3/q'(\cdot))+(2/\rho_1))}\varphi_j e^{-(t-s)|\cdot|^2} \right\|_{L^{(6q(\cdot)/6-q(\cdot))}} \|\omega\otimes\omega\|_{L^{(6/5)}}\mathrm{d}s \right\|_{L^{\rho_1}(\mathbb{R}_+)} \right\|_{l^\gamma} \\
&\lesssim \left\| \left\| \int_0^t 2^{m((2/\rho_1)+(5/2))}e^{-(t-s)2^{2j}} \left\| 2^{-3m\cdot(6-q(\cdot)/6q(\cdot))}\varphi_j \right\|_{L^{(6q(\cdot)/6-q(\cdot))}} \left\| \dot{\Delta}_j(\omega\otimes\omega) \right\|_{L^{(6/5)}}\mathrm{d}s \right\|_{L^{\rho_1}(\mathbb{R}_+)} \right\|_{l^\gamma} \\
&\lesssim \left\| \left\| \int_0^t 2^{m((2/\rho_1)+(5/2))}e^{-(t-s)2^{2j}} \left\| \dot{\Delta}_j(\omega\otimes\omega) \right\|_{L^{(6/5)}}\mathrm{d}s \right\|_{L^{\rho_1}(\mathbb{R}_+)} \right\|_{l^\gamma} \\
&\lesssim \left\| \left\| 2^{m((2/\rho)+(1/2))} \left\| \dot{\Delta}_j(\omega\otimes\omega) \right\|_{L^{(6/5)}} \right\|_{L^\rho(\mathbb{R}_+)} \left\| e^{-t2^{2j}}2^{2j(1+(1/\rho_1)-(1/\rho))} \right\|_{\mathscr{L}^{(1+(1/\rho_1)-(1/\rho))^{-1}}(\mathbb{R}_+)} \right\|_{l^\gamma} \\
&\lesssim \left\| \left\| 2^{m((2/\rho)+(1/2))} \left\| \dot{\Delta}_j(\omega\otimes\omega) \right\|_{L^{(6/5)}} \right\|_{L^\rho(\mathbb{R}_+)} \right\|_{l^\gamma} \\
&\lesssim \|\omega\otimes\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{(6/5),\gamma}^{(2/\rho)+(1/2)}\right)}.
\end{aligned} \tag{40}$$

Using Proposition 7, the above inequality yields

$$\begin{aligned}
&\lesssim \|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)} \|\omega\|_{L^\infty(\mathbb{R}_+;L^3)} \\
&\lesssim \|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)} \|\omega\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}.
\end{aligned} \tag{41}$$

Next, we also have

$$\begin{aligned}
&\left\| \int_0^t G(t-s)\nabla\cdot(\omega\otimes\theta)\mathrm{d}s \right\|_{\mathscr{L}^{\rho_1}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+(3/q(\cdot))+(2/\rho_1)}\right)} \\
&\lesssim \|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)} \|\theta\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)} + \|\theta\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)} \|\omega\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}.
\end{aligned} \tag{42}$$

Therefore,

$$\begin{aligned}
&\left\| \int_0^t G(t)(t-s)\begin{pmatrix} \mathbb{P}\nabla\cdot(\omega\otimes\omega) \\ \nabla\cdot(\omega\otimes\theta) \end{pmatrix}\mathrm{d}s \right\|_{\mathscr{L}^{\rho_1}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+(3/q(\cdot))+(2/\rho_1)}\right)} \\
&\lesssim \|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)} \|\omega\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)} + \|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)} \|\theta\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)} \\
&\quad + \|\theta\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)} \|\omega\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}.
\end{aligned} \tag{43}$$

Similarly,

$$\begin{aligned}
&\left\|\int_0^t G(t)(t-s)\begin{pmatrix}\mathbb{P}\nabla\cdot(\omega\otimes\omega)\\ \nabla\cdot(\omega\otimes\theta)\end{pmatrix}\mathrm{d}s\right\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{2,\gamma}^{(1/2)+(2/\rho)}\right)}\\
&\lesssim \|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)}\|\omega\|_{\mathscr{L}^\infty(\mathbb{R}_+;L^3)}+\|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)}\|\theta\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}\\
&\quad+\|\theta\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)}\|\omega\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}\\
&\left\|\int_0^t G(t)(t-s)\begin{pmatrix}\mathbb{P}\nabla\cdot(\omega\otimes\omega)\\ \nabla\cdot(\omega\otimes\theta)\end{pmatrix}\mathrm{d}s\right\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}\\
&\lesssim \|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)}\|\omega\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}+\|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)}\|\theta\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}\\
&\quad+\|\theta\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)}\|\omega\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}.
\end{aligned}\tag{44}$$

In this way,

$$\begin{aligned}
&\left\|\int_0^t G(t)(t-s)\begin{pmatrix}\mathbb{P}\nabla\cdot(\omega\otimes\omega)\\ \nabla\cdot(\omega\otimes\theta)\end{pmatrix}\mathrm{d}s\right\|_{B}\\
&\lesssim \|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)}\|\omega\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}+\|\omega\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)}\|\theta\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}\\
&\quad+\|\theta\|_{\mathscr{L}^\rho\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(2/\rho)+(1/2)}\right)}\|\omega\|_{\mathscr{L}^\infty\left(\mathbb{R}_+;\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)}.
\end{aligned}\tag{45}$$

Therefore, we have

$$\begin{aligned}
\|gW(t)\|_B&\le \left\|G(t)W_0\right\|_B+\left\|\int_0^t G(t-s)\mathbb{P}\nabla\cdot(\omega\otimes W)\mathrm{d}s\right\|_B\\
&\le C_1\left\|\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\right\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)}}+3C_2\varepsilon^2.
\end{aligned}\tag{46}$$

Now taking $\varepsilon<(1/6(C_1\vee C_2))$ for any $\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\in\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+(3/q'(\cdot))}$ with

$$\left\|\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix}\right\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)}}<\frac{\varepsilon}{6(C_1\vee C_2)},\tag{47}$$

we can obtain

$$\begin{aligned}
\|gW(t)\|_B&<C_1\cdot\frac{\varepsilon}{6(C_1\vee C_2)}+3C_2\varepsilon^2\\
&<\frac{\varepsilon}{6}+3C_2\cdot\frac{1}{6(C_1\vee C_2)}\cdot\varepsilon\\
&\le\frac{\varepsilon}{6}+\frac{\varepsilon}{2}\\
&=\frac{2}{3}\varepsilon<\varepsilon,
\end{aligned}\tag{48}$$

where $C_1\vee C_2=\max\{C_1,C_2\}$. By using Proposition 8, one can easily obtain the global solution for under the condition of small initial data.

Next, we adopt the Gevrey class technique, which has its origin in the investigation of Foias and Temam [35] related to the NS equations to ensure that the solution to equation (1) is spatially analytic. This method has the obvious benefit of making it easier to get rid of cumbersome estimates of higher-order derivatives. For further details on this issue, see [35–38] and the references therein. In this work, we use the idea of Bae et al. [37] to establish the Gevrey class regularities for equation (1).

First, we give the following auxiliary lemma; we can easily obtain the spatial analyticity for the solution to (1). □

Lemma 10 (see [10]). *Let $0<s\le t<\infty$ and for any $x,y\in\mathbb{R}^n$, we have the following:*

$$t|x|^{(1/2)}-\frac{1}{2}\left(t^2-s^2\right)|x|-s|x-y|^{(1/2)}-s|y|^{(1/2)}\le\frac{1}{2}.\tag{49}$$

Theorem 11. *Let $q(\cdot)\in\mathscr{P}(\mathbb{R}^3)\cap K^{\log}(\mathbb{R}^3)$, $1\le\gamma<3$ and $2\le q(\cdot)\le 6$. Then, there exists a constant $\varepsilon_0'>0$ such that for the initial data (ω_0,θ_0) in $\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+(3/q'(\cdot))}(\mathbb{R}^3)$ with*

$\|(\omega_0, b_0)\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+(3/q'(\cdot))}} < \varepsilon_0'$, *the result established in Theorem 1 is analytic in a sense that*

$$\left\|\left(e^{\sqrt{t}|D|}\omega, e^{\sqrt{t}|D|}\theta\right)\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)+(2/\rho)}\right)\cap\mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{2,\gamma}^{(1/2)+(2/\rho)}\right)\cap\mathscr{L}^{\infty}\left(\mathbb{R}_+\mathscr{F}\dot{\mathscr{B}}_{2,\gamma}^{(1/2)}\right)} \lesssim \left\|(\omega_0,\theta_0)\right\|_{\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)}}, \tag{50}$$

where $e^{\sqrt{t}|D|}$ *defined as* $e^{\sqrt{t}|\xi|}$ *presents a Fourier multiplier.*

Proof. Assume $\widetilde{\omega}(y,t) = e^{\sqrt{t}|D|}\omega(y,t)$ and $\widetilde{\theta}(y,t) = e^{\sqrt{t}|D|}\theta(y,t)$, and from equation (28), we have

$$\begin{aligned}\widetilde{W} &= e^{\sqrt{t}|D|}W(x,t)\\ &= e^{\sqrt{t}|D|}G(t)\begin{pmatrix}\omega_0\\ \theta_0\end{pmatrix} - e^{\sqrt{t}|D|}\int_0^t G(t-s)\begin{pmatrix}\mathbb{P}\nabla\cdot(\omega\otimes\omega)\\ \nabla\cdot(\omega\otimes\theta)\end{pmatrix}\mathrm{ds}.\end{aligned} \tag{51}$$

We can easily obtain that

$$\begin{aligned}&\left\|\begin{pmatrix}\widetilde{\omega}\\ \widetilde{\theta}\end{pmatrix}\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)+(2/\rho)}\right)}\\
&\le \left\|\begin{pmatrix}e^{\sqrt{t}|D|}G(t)\omega_0\\ e^{\sqrt{t}|D|}G(t)\theta_0\end{pmatrix}\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)+(2/\rho)}\right)}\\
&\quad + \left\|e^{\sqrt{t}|D|}\int_0^t G(t-s)\begin{pmatrix}\mathbb{P}\nabla\cdot(\omega\otimes\omega)\\ \nabla\cdot(\omega\otimes\theta)\end{pmatrix}\mathrm{ds}\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;\mathscr{F}\dot{\mathscr{B}}_{q(\cdot),\gamma}^{-1+\left(3/q'(\cdot)\right)+(2/\rho)}\right)}\\
&\lesssim \left\|\left\|\begin{pmatrix}e^{\sqrt{t}|\xi|-t|\xi|^2}2^{m\left(-1+\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_j\widehat{\omega}_0\\ e^{\sqrt{t}|\xi|-t|\xi|^2}2^{m\left(-1+\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_j\widehat{\theta}_0\end{pmatrix}\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;L^{q(\cdot)}\right)}\right\|_{l^{\gamma}}\\
&\quad + \left\|\left\|2^{m\left(\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_j\int_0^t e^{-(1/2)(t-s)|\xi|^2}\int_{\mathbb{R}^3} e^{\sqrt{t}|\xi|-1/2(t-s)|\xi|^2-\sqrt{s}(|\xi-\eta|+|\eta|)}\begin{pmatrix}\widehat{\omega}(\xi-\eta,s)\otimes\widehat{\omega}(\eta,s)\\ \widehat{\omega}(\xi-\eta,s)\otimes\widehat{\theta}(\eta,s)\end{pmatrix}d\eta\, ds\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;L^{q(\cdot)}\right)}\right\|_{l^{\gamma}}\\
&\lesssim \left\|\left\|\begin{pmatrix}e^{-1/2t|\xi|^2}2^{m\left(-1+\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_j\widehat{\omega}_0\\ e^{-1/2t|\xi|^2}2^{m\left(-1+\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_j\widehat{\theta}_0\end{pmatrix}\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;L^{q(\cdot)}\right)}\right\|_{l^{\gamma}}\\
&\quad + \left\|\left\|2^{m\left(\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_j\int_0^t e^{-(1/2)(t-s)|\xi|^2}\int_{\mathbb{R}^3}\begin{pmatrix}\widehat{\omega}(\xi-\eta,s)\otimes\widehat{\omega}(\eta,s)\\ \widehat{\omega}(\xi-\eta,s)\otimes\widehat{\theta}(\eta,s)\end{pmatrix}d\eta\, ds\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;L^{q(\cdot)}\right)}\right\|_{l^{\gamma}}\\
&\lesssim \left\|\left\|\begin{pmatrix}e^{-1/2t|\xi|^2}2^{m\left(-1+\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_j\widehat{\omega}_0\\ e^{-1/2t|\xi|^2}2^{m\left(-1+\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_j\widehat{\theta}_0\end{pmatrix}\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;L^{q(\cdot)}\right)}\right\|_{l^{\gamma}}\\
&\quad + \left\|\left\|2^{m\left(\left(3/q'(\cdot)\right)+(2/\rho)\right)}\varphi_j\int_0^t e^{-(1/2)(t-s)|\xi|^2}\begin{pmatrix}\widehat{\omega\otimes\omega}\\ \widehat{\omega\otimes\theta}\end{pmatrix}\mathrm{ds}\right\|_{\mathscr{L}^{\rho}\left(\mathbb{R}_+;L^{q(\cdot)}\right)}\right\|_{l^{\gamma}},\end{aligned} \tag{52}$$

where we used $e^{\sqrt{t}|\xi|-(1/2)t|\xi|^2} = e^{-(1/2)(\sqrt{t}|\xi|-1)^2+(1/2)} \leq e^{(1/2)}$ and Lemma 10.

The rest of the proof follows a similar pattern established in the preceding proof of Theorem 9. Therefore, the remaining part is omitted.

Data Availability

The data used to support this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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