Hindawi
Journal of Mathematics
Volume 2023, Article ID 4105842, 10 pages
https://doi.org/10.1155/2023/4105842

Research Article

Fixed-Point Results for Generalized Rational Contractions in Graphical *b*-Metric Spaces with Applications

Amer Hassan Albargi

Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Amer Hassan Albargi; ahalbargi23@gmail.com

Received 30 January 2023; Revised 19 August 2023; Accepted 4 September 2023; Published 1 November 2023

Academic Editor: Valerii Obukhovskii

The aim of this paper is to define generalized rational contractions in the setting of graphical *b*-metric spaces and obtain some fixed-point theorems. Our results are significant generalizations and extensions of some well-known results in the existing theory. We also supply a nontrivial example to show the validity of the obtained theorems. As applications, we obtain some results on rational expressions in the background of graphical metric spaces.

1. Introduction

In the context of fixed-point theory, the underlying space and contractive mapping play an important and crucial role. One of the pillars in this theory is the concept of metric space which was introduced by Frechet [1]. In this theory, the pioneer result is the well-known Banach contraction principle [2] which states that if $(\mathscr{M}, \mathfrak{d})$ is a complete metric space and $\mathscr{R}\colon \mathscr{M} \longrightarrow \mathscr{M}$ is a mapping satisfying the condition

$$\mathfrak{d}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \le \lambda \mathfrak{d}(\vartheta_1, \vartheta_2), \forall \vartheta_1, \vartheta_2 \in \mathscr{M}, \tag{1}$$

where $0 \le \lambda < 1$, then $\mathscr{R}$ has a unique fixed point.

Due to the simplicity and significance of the notion of metric space, it has been improved, extended, and generalized in various directions. The famous extension of the concept of metric spaces has been done by Bakhtin [3] which was formally defined by Czerwik [4] in 1993. Jachymski [5] replaced the order structure on a metric space with a graph structure and gave the graphic version of the Banach contraction principle. In 2017, Shukla et al. [6] introduced the concept of graphical metric spaces by proposing the graphical structure on metric spaces. Subsequently, Chuensupantharat et al. [7] combined the notions of *b*-metric spaces and graphical metric spaces and gave the concept of graphical *b*-metric spaces.

On the other hand, Fisher [8] and Dass and Gupta [9] introduced rational expression in the contractive condition and generalized the famous Banach contraction principle. Isik et al. [10] proved some fixed-point theorems for rational contractions endowed with a graph and investigated the solution of a system of integral equations as applications.

In this article, we introduce Fisher's graph contraction and Dass–Gupta's graph contraction in the setting of graphical *b*-metric spaces and obtain some fixed-point results. Based on this structure, we show that every Dass–Gupta's contraction is Dass–Gupta's graph contraction, but the converse is not generally true. Some nontrivial and significant examples are also provided, equipped with some worthy graphs to show the authenticity of established outcomes.

2. Preliminaries

Frechet [1] initiated the theory of metric space in the following manner.

$$\mathfrak{d}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \le \lambda \mathfrak{d}(\vartheta_1, \vartheta_2), \tag{2}$$

for all $\vartheta_1, \vartheta_2 \in \mathscr{M}$, there exists a unique point $\vartheta^* \in \mathscr{M}$ such that $\mathscr{R}\vartheta^* = \vartheta^*$.

Definition 1 (see [1]). Let $\mathscr{M} \neq \varnothing$ (nonempty set) and $\mathfrak{d}: \mathscr{M} \times \mathscr{M} \longrightarrow [0, \infty)$ be a function satisfying

(m1) $\mathfrak{d}(\vartheta, \theta) = 0$ if and only if $\vartheta = \theta$

(m2) $\mathfrak{d}(\vartheta, \theta) = \mathfrak{d}(\theta, \vartheta)$

(m3) $\mathfrak{d}(\vartheta, \varphi) \leq \mathfrak{d}(\vartheta, \theta) + \mathfrak{d}(\theta, \varphi)$

for all $\vartheta, \theta, \varphi \in \mathscr{M}$.

Then, $(\mathscr{M}, \mathfrak{d})$ is called a metric space.

Banach [2] obtained a fixed-point result in 1922 in the following way.

Theorem 2. *Let $(\mathscr{M}, \mathfrak{d})$ be a complete metric space and $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$. If there exists a nonnegative constant $\lambda \in [0, 1)$ such that*

In 1980, Fisher [8] presented the following result.

$$\mathfrak{d}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \leq \lambda_1 \mathfrak{d}(\vartheta_1, \vartheta_2) + \lambda_2 \frac{\mathfrak{d}(\vartheta_1, \mathscr{R}\vartheta_1)\mathfrak{d}(\vartheta_2, \mathscr{R}\vartheta_2)}{1 + \mathfrak{d}(\vartheta_1, \vartheta_2)}, \quad (3)$$

for all $\vartheta_1, \vartheta_2 \in \mathscr{M}$, there exists a unique point $\vartheta^* \in \mathscr{M}$ such that $\mathscr{R}\vartheta^* = \vartheta^*$.

Theorem 3. *Let $(\mathscr{M}, \mathfrak{d})$ be a complete metric space and $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$. If there exist nonnegative constants $\lambda_1, \lambda_2 \in (0, 1/2]$ with $\lambda_1 + \lambda_2 < 1$ such that*

Dass and Gupta [9] established a result in the following way.

$$\mathfrak{d}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \leq \lambda_1 \mathfrak{d}(\vartheta_1, \vartheta_2) + \lambda_2 \frac{[1 + \mathfrak{d}(\vartheta_1, \mathscr{R}\vartheta_1)]\mathfrak{d}(\vartheta_2, \mathscr{R}\vartheta_2)}{1 + \mathfrak{d}(\vartheta_1, \vartheta_2)}, \quad (4)$$

for all $\vartheta_1, \vartheta_2 \in \mathscr{M}$, there exists a unique point $\vartheta^* \in \mathscr{M}$ such that $\mathscr{R}\vartheta^* = \vartheta^*$.

Theorem 4. *Let $(\mathscr{M}, \mathfrak{d})$ be a complete metric space and $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$. If there exist nonnegative constants $\lambda_1, \lambda_2 \in (0, 1/2]$ with $\lambda_1 + \lambda_2 < 1$ such that*

Czerwik [4] presented the idea of b-metric space as follows.

Definition 5 (see [4]). Let $\mathscr{M} \neq \varnothing$, $s \geq 1$ be a constant and $\mathfrak{d}_b: \mathscr{M} \times \mathscr{M} \longrightarrow [0, \infty)$ be a function satisfying

(b1) $\mathfrak{d}_{\mathfrak{b}}(\vartheta, \theta) = 0$ if and only if $\vartheta = \theta$

(b2) $\mathfrak{d}_{\mathfrak{b}}(\vartheta, \theta) = \mathfrak{d}_{\mathfrak{b}}(\theta, \vartheta)$

(b3) $\mathfrak{d}_{\mathfrak{b}}(\vartheta, \varphi) \leq s[\mathfrak{d}_{\mathfrak{b}}(\vartheta, \theta) + \mathfrak{d}_{\mathfrak{b}}(\theta, \varphi)]$

for all $\vartheta, \theta, \varphi \in \mathscr{M}$.

Then, the pair $(\mathscr{M}, \mathfrak{d}_{\mathfrak{b}})$ is considered as a b-metric space.

Some concepts from graph theory given by Jachymski [5] will be presented here. Let Δ denotes the diagonal of $\mathscr{M} \times \mathscr{M}$, where $\mathscr{M}$ is any nonempty set. Let $G = (V(G), E(G))$ be a directed graph such that $V(G)$ is the set of its vertices, that is, $V(G)$ corresponding to $\mathscr{M}$ and $E(G)$ is the set of its edges, that is, $E(G)$ which contains all loops, i.e., $\Delta \subseteq E(G)$. If we alter the direction of the edges of G, the resultant graph is represented by G^{-1}. Moreover, the letter $\widetilde{G}$ represents a directed graph with symmetric edges. Specifically, we define

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}). \quad (5)$$

If $\hbar$ and ς are vertices in a graph G, then a path from $\hbar$ to ς in G of length m is a sequence $\{\theta_i\}_{i=0}^{m}$ of $m + 1$ vertices such that $\theta_0 = \hbar$, $\theta_m = \varsigma$, $(\theta_{n-1}, \theta_n) \in E(G)$, and $\forall i = 1, \cdots, m$. A graph G is connected if any two vertices of G have a path between them. Moreover, a graph G is weakly connected if there is a path between each two vertices in undirected graph G. We say $G^* = (V(G^*), E(G^*))$ is a subgraph of $G = (V(G), E(G))$ if $V(G^*) \subseteq V(G)$ and $E(G^*) \subseteq E(G)$.

Motivated by Shukla et al. [6], we represent

$$[\mathfrak{w}]_G^{\ell} = \{\hbar \in \mathscr{M}: \text{ there exists a direct path from } \mathfrak{w} \text{ to } \hbar \text{ with length } l\}. \quad (6)$$

Moreover, a relation P on $\mathscr{M}$ is such that

$$(\mathfrak{w}P\hbar)_G. \quad (7)$$

If there is a direct path in G from $\mathfrak{w}$ to $\hbar$, and $\varsigma \in (\mathfrak{w}P\hbar)_G$ if ς is in the path $(\mathfrak{w}P\hbar)_G$. If $\{\theta_n\}$ in $\mathscr{M}$ with $(\theta_n P \theta_{n+1})_G$, $\forall n \in \mathbb{N}$, then $\{\theta_n\}$ is said to be a G-termwise connected (shortly G-TWC) sequence.

Shukla et al. [6] introduced the notion of graphical metric space in the following way.

Definition 6 (see [7]). Let $\mathscr{M} \neq \varnothing$ and $\mathfrak{d}_{\mathfrak{gm}}: \mathscr{M} \times \mathscr{M} \longrightarrow [0, \infty)$ is a function satisfying

(gm1) $\mathfrak{d}_{\mathfrak{gm}}(\vartheta, \theta) = 0$ if and only if $\vartheta = \theta$

(gm2)dgm $(\vartheta, \theta) = \mathfrak{d}_{\mathfrak{gm}}(\theta, \vartheta)$

(gm3) $(\vartheta P \varphi)_G$ and $\theta \in (\vartheta P \varphi)_G$ implies $\mathfrak{d}_{\mathfrak{gm}}(\vartheta, \varphi) \leq \mathfrak{d}_{\mathfrak{gm}}(\vartheta, \theta) + \mathfrak{d}_{\mathfrak{gm}}(\theta, \varphi)$

for all $\vartheta, \theta, \varphi \in \mathscr{M}$. Then, the pair $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gm}})$ is said to be a graphical metric space.

Example 7. Every metric space $(\mathscr{M}, \mathfrak{d})$ is a graphical metric space with graph G, where $V(G) = X$ and $E(G) = \mathscr{M} \times \mathscr{M}$.

In 2019, Chuensupantharat et al. [7] gave the notion of graphical b-metric space as a generalization of b-metric space as follows:

Definition 8 (see [7]). Let $\mathscr{M} \neq \varnothing$ and $s \geq 1$ be a constant and $\mathfrak{d}_{\mathfrak{gb}}: \mathscr{M} \times \mathscr{M} \longrightarrow [0, \infty)$ be a function satisfying

(gb1) $\mathfrak{d}_{\mathfrak{gb}}(\vartheta, \theta) = 0$ if and only if $\vartheta = \theta$

(gb2) $\mathfrak{d}_{\mathfrak{gb}}(\vartheta, \theta) = \mathfrak{d}_{\mathfrak{gb}}(\theta, \vartheta)$

(gb3) $(\vartheta P \varphi)_G$ and $\theta \in (\vartheta P \varphi)_G$ implies $\mathfrak{d}_{\mathfrak{gb}}(\vartheta, \varphi) \leq s [\mathfrak{d}_{\mathfrak{gb}}(\vartheta, \theta) + \mathfrak{d}_{\mathfrak{gb}}(\theta, \varphi)]$

for all $\vartheta, \theta, \varphi \in \mathscr{M}$. Then, the pair $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ is said to be a graphical b-metric space with coefficient $s \geq 1$ on $\mathscr{M}$.

Remark 9. The notion of graphical b-metric space is a real generalization of graphical metric space because if we take $s = 1$ in the above definition, then we can get the notion of a graphical metric space.

Example 10 (see [7]). Let $\mathscr{M} = \{1, 2, 3, 4\}$ and $\mathfrak{d}_{\mathfrak{gb}}: \mathscr{M} \times \mathscr{M} \longrightarrow [0, \infty)$ is defined by

$$\mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \vartheta_2) = \begin{cases} 0, & \text{if } \vartheta_1 = \vartheta_2, \\ 3\eta, & \text{if } \vartheta_1, \vartheta_2 \in \{1, 2\} \text{ and } \vartheta_1 \neq \vartheta_2, \\ \eta, & \text{if } \vartheta_1 \text{ or } \vartheta_2 \notin \{1, 2\} \text{ and } \vartheta_1 \neq \vartheta_2, \end{cases} \tag{8}$$

where $\eta > 0$ is constant. Then, it is very simple to prove that $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ is a graphical b-metric space with coefficient $s = 3/2$, where $G = (V(G), E(G))$ and $V(G) = \mathscr{M}$ and $E(G)$ as shown in Figure 1.

Notice that it is not a graphical metric space because

$$\mathfrak{d}_{\mathfrak{gb}}(1, 2) = 3\eta > 2\eta \cdot \mathfrak{d}_{\mathfrak{gb}}(1, 3) + \mathfrak{d}_{\mathfrak{gb}}(3, 2). \tag{9}$$

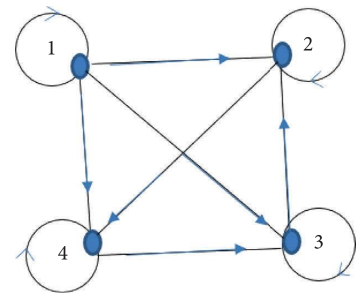

Figure 1: Graph depicting graphical b-metric space.

Definition 11. Let $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ be a graphical b-metric space, and $\{\theta_n\}$ be a sequence in $\mathscr{M}$, then

(i) $\{\theta_n\}$ is said to be a convergent sequence if there exists a point $\theta \in \mathscr{M}$ such that $\lim_{n \longrightarrow \infty} \mathfrak{d}_{\mathfrak{gb}}(\theta_n, \theta) = 0$

(ii) $\{\theta_n\}$ is said to be a Cauchy sequence if $\lim_{n,m \longrightarrow \infty} \mathfrak{d}_{\mathfrak{gb}}(\theta_n, \theta_m) = 0$

For more characteristics in the direction of graphic contractions, graphical metric spaces, and graphical b-metric spaces, we refer the readers to [10–21].

3. Results and Discussion

Let $\mathscr{H}_{\mathscr{G}}$ be a subgraph of graph G such that $\Delta \subseteq E(\mathscr{H}_{\mathscr{G}})$ and moreover suppose that $\mathscr{H}_{\mathscr{G}}$ is a weighted graph. Let $\{\vartheta_n\}$ be a sequence with initial point ϑ_0 in $\mathscr{M}$. Then, $\{\vartheta_n\}$ is said to be an $\mathscr{R}$-Picard sequence ($\mathscr{R}$-PS) for $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$ if $\vartheta_n = \mathscr{R}\vartheta_{n-1}$, for all $n \in \mathbb{N}$.

Now, we state a property (P) given by Shukla et al. [6] in the following way.

(P) A graph $\mathscr{H}_{\mathscr{G}} = (V(\mathscr{H}_{\mathscr{G}}), E(\mathscr{H}_{\mathscr{G}}))$ satisfies the property (P) if a G-termwise connected $\mathscr{R}$-PS $\{\vartheta_n\}$ converging in $\mathscr{M}$ guarantees that there is a limit $\theta \in \mathscr{M}$ of $\{\vartheta_n\}$ and $n_0 \in \mathbb{N}$ such that $(\vartheta_n, \theta) \in E(\mathscr{H}_{\mathscr{G}})$ or $(\theta, \vartheta_n) \in E(\mathscr{H}_{\mathscr{G}})$ for all $n > n_0$.

Now, we define Fisher-type graph contraction in a graphical b-metric space.

Definition 12. Let $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ be a graphical b-metric space and $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$. Then, $\mathscr{R}$ is said to be an Fisher-type graph contraction for $\mathscr{H}_{\mathscr{G}}$ on $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ if

(i) $\mathscr{H}_{\mathscr{G}}$ is graph preserving, i.e., for each $\vartheta_1, \vartheta_2 \in \mathscr{M}$ if $(\vartheta_1, \vartheta_2) \in\in E(\mathscr{H}_{\mathscr{G}})$ implies $(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \in E(\mathscr{H}_{\mathscr{G}})$,

(ii) There exist nonnegative constants λ_1, λ_2 with $\lambda_1 + \lambda_2 < 1/s$ such that for every $\vartheta_1, \vartheta_2 \in \mathscr{M}$ with $(\vartheta_1, \vartheta_2) \in E(\mathscr{H}_{\mathscr{G}})$, we have

$$\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \leq \lambda_1 \mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \vartheta_2) + \lambda_2 \frac{\mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \mathscr{R}\vartheta_1)\mathfrak{d}_{\mathfrak{gb}}(\vartheta_2, \mathscr{R}\vartheta_2)}{1 + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \vartheta_2)}. \tag{10}$$

Remark 13. Any Fisher contraction is Fisher graph contraction along with $\mathscr{H}_{\mathscr{G}} = G$ defined by $V(\mathscr{H}_{\mathscr{G}}) = \mathscr{M}$ and $E(\mathscr{H}_{\mathscr{G}}) = \mathscr{M} \times \mathscr{M}$.

(i) If $\mathscr{R}$ be a Fisher graph contraction with parameters λ_1, λ_2 and $\lambda_2 = 0$, then $\mathscr{R}$ is a Banach graph contraction for graphs in $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$.

(ii) Since every graphical b-metric space is graphical metric space and also every $(\mathscr{M}, \mathfrak{d})$ is $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gm}})$, thus our theorems in this paper are precise generalizations of some of the results regarding Fisher contraction (see e.g., [8]).

Theorem 14. *Let $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ be an $\mathscr{H}_{\mathscr{G}}$-complete graphical b-metric space such that $\mathfrak{d}_{\mathfrak{gb}}$ is continuous functional and let $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$ be a Fisher graph contraction. Suppose that these assertions hold:*

(i) $\mathscr{G}$ satisfies the property ($\mathscr{P}$)

(ii) There exists $\vartheta_0 \in \mathscr{M}$ with $\mathscr{R}\vartheta_0 \in [\vartheta_0]^r_{\mathscr{H}_{\mathscr{G}}}$ for some $r \in \mathbb{N}$

Then, there exists $\vartheta^* \in \mathscr{M}$ such that the $\mathscr{R}$-PS $\{\vartheta_n\}$ with initial point $\vartheta_0 \in \mathscr{M}$ is $\mathscr{H}_{\mathscr{G}}$-TWC and $\vartheta_n \longrightarrow \vartheta^*$ as $n \longrightarrow \infty$.

Proof. Let $\vartheta_0 \in \mathscr{M}$ be such that $\mathscr{R}\vartheta_0 \in [\vartheta_0]^r_{\mathscr{H}_{\mathscr{G}}}$, for some $r \in \mathbb{N}$. As $\{\vartheta_n\}$ is a $\mathscr{R}$-PS originating from ϑ_0, there is a path $(\theta_i)^r_{i=0}$ such that $\vartheta_0 = \theta_0$ and $\mathscr{R}\vartheta_0 = \theta_r$ and $(\theta_{i-1}, \theta_i) \in E(\mathscr{H}_{\mathscr{G}})$ for $i = 1, 2, \cdots, r$. Now, by assumption (i), we have $(\mathscr{R}\theta_{i-1}, \mathscr{R}\theta_i) \in E(\mathscr{H}_{\mathscr{G}})$ for $i = 1, 2, \cdots, r$. It yields that $(\mathscr{R}\theta_i)^r_{i=0}$ is a path from $\mathscr{R}\theta_0 = \mathscr{R}\vartheta_0 = \theta_1$ to $\mathscr{R}\theta_r = \mathscr{R}^2\vartheta_0 = \theta_2$ having length r and therefore $\vartheta_2 \in [\vartheta_1]^r_{\mathscr{H}_{\mathscr{G}}}$. Continuing in this way, we get $(\mathscr{R}^n\theta_i)_{i=0}^r$ is a path from $\mathscr{R}^n\theta_0 = \mathscr{R}^n\vartheta_0 = \theta_n$ to $\mathscr{R}^n\theta_r = \mathscr{R}^n\mathscr{R}\vartheta_0 = \vartheta_{n+1}$ of length r, and hence, $\vartheta_{n+1} \in [\vartheta_n]_{\mathscr{H}_{\mathscr{G}}}^r$, for all $n \in \mathbb{N}$. Hence, we have $\{\vartheta_n\}$ which is a $\mathscr{H}_{\mathscr{G}}$-TWC sequence. Thus, $(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i) \in E(\mathscr{H}_{\mathscr{G}})$ for $i = 1, 2, \cdots, r$ and $n \in \mathbb{N}$. Now, by (ii), we have

$$\begin{aligned}
\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i) &= \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}\left(\mathscr{R}^{n-1}\theta_{i-1}\right), \mathscr{R}\left(\mathscr{R}^{n-1}\theta_i\right)\right) \\
&\leq \lambda_1 \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right) \\
&\quad + \lambda_2 \frac{\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}\left(\mathscr{R}^{n-1}\theta_{i-1}\right)\right)\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_i, \mathscr{R}\left(\mathscr{R}^{n-1}\theta_i\right)\right)}{1 + \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right)} \\
&= \lambda_1 \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right) \\
&\quad + \lambda_2 \frac{\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right)\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i)}{1 + \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right)} \\
&\leq \lambda_1 \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right) + \lambda_2 \mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i),
\end{aligned} \tag{11}$$

which yields that

$$\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i) \leq \left(\frac{\lambda_1}{1-\lambda_2}\right)\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right). \tag{12}$$

Since $\lambda_1 + \lambda_2 < 1/s$, so taking $\lambda_1/1 - \lambda_2 = \gamma \in [0, 1/s)$, it follows from the above inequality that

$$\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i) \leq \gamma\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right), \tag{13}$$

where $\gamma \in [0, 1/s)$. Repeating in this way, we have

$$\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i) \leq \gamma^n\mathfrak{d}_{\mathfrak{gb}}(\theta_{i-1}, \theta_i). \tag{14}$$

As the sequence $\{\vartheta_n\}$ is a $\mathscr{H}_{\mathscr{G}}$-TWC sequence and $\mathscr{G}$ is a subgraph of G, so by using inequality (14) and the triangular inequality, we have

$$\begin{aligned}
\mathfrak{d}_{\mathfrak{gb}}(\theta_n, \theta_{n+1}) &= \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta_0, \mathscr{R}^{n+1}\vartheta_0\right) \\
&= \mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_0, \mathscr{R}^n\theta_r) \\
&\leq s\left[\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_0, \mathscr{R}^n\theta_1) + \mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_1, \mathscr{R}^n\theta_r)\right] \\
&\leq s\left[\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_0, \mathscr{R}^n\theta_1)\right] + s^2\left[\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_1, \mathscr{R}^n\theta_2)\right] + \cdots + s^n\left[\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_{r-1}, \mathscr{R}^n\theta_r)\right] \\
&\leq s\gamma^n\mathfrak{d}_{\mathfrak{gb}}(\theta_0, \theta_1) + s^2\gamma^n\mathfrak{d}_{\mathfrak{gb}}(\theta_1, \theta_2) + s^3\gamma^n\mathfrak{d}_{\mathfrak{gb}}(\theta_2, \theta_3) + \cdots + s^r\gamma^n\mathfrak{d}_{\mathfrak{gb}}(\theta_{r-1}, \theta_r) \\
&= s\gamma^n\sum_{k=1}^{r}s^{k-1}\mathfrak{d}_{\mathfrak{gb}}(\theta_{k-1}, \theta_k).
\end{aligned} \tag{15}$$

Setting $\aleph_b^r = \sum_{k=1}^r s^{k-1}\mathfrak{d}_{\mathfrak{gb}}(\theta_{k-1}, \theta_k)$ in inequality (15), we have

$$\mathfrak{d}_{\mathfrak{gb}}(\theta_n, \theta_{n+1}) \leq s\gamma^n(\aleph_b^r). \tag{16}$$

Again $\{\vartheta_n\}$ is a $\mathscr{H}_{\mathscr{G}}$-TWC sequence, so for all $m, n \in \mathbb{N}$, $(m > n)$, we have

$$\begin{aligned}\mathfrak{d}_{\mathfrak{gb}}(\vartheta_n, \vartheta_m) &\leq s\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta_n, \vartheta_{n+1}) + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n+1}, \vartheta_m)\right] \\ &\leq s\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta_n, \vartheta_{n+1})\right] + s^2\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n+1}, \vartheta_{n+2})\right] + s^2\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n+2}, \vartheta_m)\right] \\ &\leq s\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta_n, \vartheta_{n+1})\right] + s^2\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n+1}, \vartheta_{n+2})\right] + \cdots + s^{m-n}\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta_{m-1}, \vartheta_m)\right] \\ &= \sum_{k=p}^{m-1}\left[s^{k-n+1}\mathfrak{d}_{\mathfrak{gb}}(\vartheta_k, \vartheta_{k+1})\right] \\ &\leq s\sum_{k=n}^{m-1}\left[s^{k-n+1}\gamma^k \aleph_b^r\right] \\ &= s^2\gamma^n\left[\sum_{k=n}^{m-1}(s\gamma)^{k-n}\right]\aleph_b^r \\ &\leq s^2\gamma^n\left[\sum_{k=1}^{\infty}(s\gamma)^{k-1}\right]\aleph_b^r \\ &\leq s^2\gamma^n\left(\frac{1}{1-s\gamma}\right)\aleph_b^r.\end{aligned} \tag{17}$$

Since $\gamma \in [0, 1/s)$, so we obtain

$$\lim_{n,m \longrightarrow \infty} \mathfrak{d}_{\mathfrak{gb}}(\vartheta_n, \vartheta_m) = 0. \tag{18}$$

Thus, $\{\vartheta_n\}$ is a Cauchy sequence in $\mathscr{M}$. Also, since $\mathscr{M}$ is $\mathscr{H}_{\mathscr{G}}$-complete, so $\{\vartheta_n\}$ converges in $\mathscr{M}$ and by assumption, there exists $\vartheta^* \in \mathscr{M}$ and $n_0 \in \mathbb{N}$ such that $(\vartheta_n, \vartheta^*) \in\in E(\mathscr{G})$ or $(\vartheta^*, \vartheta_n) \in\in E(\mathscr{H}_{\mathscr{G}})$ for every $n > n_0$ and

$$\lim_{n \longrightarrow \infty} \mathfrak{d}_{\mathfrak{gb}}(\vartheta_n, \vartheta^*) = 0, \tag{19}$$

which shows that $\{\vartheta_n\}$ converges to ϑ^*. □

Theorem 15. *If assumptions given in Theorem 14 are satisfied and, in addition, we assume that the graph $\mathscr{H}_{\mathscr{G}}$ is weakly connected, then the fixed point of $\mathscr{R}$ is unique.*

Proof. By Theorem 14, we have $\{\vartheta_n\}$ is a $\mathscr{R}$-PS with initial point ϑ_0 converges to $\vartheta^* \in \mathscr{M}$. By assumption, $(\vartheta^* P\mathscr{R}\vartheta^*)_{\mathscr{H}_{\mathscr{G}}}$ or $(\mathscr{R}\vartheta^* P\vartheta^*)_{\mathscr{H}_{\mathscr{G}}}$ and thus, we obtain

$$\begin{aligned}\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \mathscr{R}\vartheta^*) &\leq s\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \vartheta_n) + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_n, \mathscr{R}\vartheta^*)\right] \\ &= s\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \vartheta_n) + \mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}\vartheta_{n-1}, \mathscr{R}\vartheta^*)\right].\end{aligned} \tag{20}$$

By (ii), we have

$$\begin{aligned}\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \mathscr{R}\vartheta^*) &\leq s\left[\begin{array}{c}\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \vartheta_n) + \lambda_1\mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta^*) \\ +\lambda_2\dfrac{\mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \mathscr{R}\vartheta_{n-1})\mathfrak{d}_{\mathfrak{gb}}\vartheta^*, \mathscr{R}\vartheta^*)}{1 + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta^*)}\end{array}\right] \\ &= s\left[\begin{array}{c}\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \vartheta_n) + \lambda_1\mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta^*) \\ +\lambda_2\dfrac{\mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta_n)\mathfrak{d}_{\mathfrak{gb}}\vartheta^*, \mathscr{R}\vartheta^*)}{1 + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta^*)}\end{array}\right].\end{aligned} \tag{21}$$

Letting $n \longrightarrow \infty$ and using the fact that $\lambda_1 + \lambda_2 < 1/s$, we have $\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \mathscr{R}\vartheta^*) = 0$. Thus, $\vartheta^* = \mathscr{R}\vartheta^*$ and therefore ϑ^* is a fixed point of $\mathscr{R}$. Now, we suppose that ϑ' is another fixed point of $\mathscr{R}$. Assume that $(\vartheta^* P\vartheta')_{\mathscr{H}_G}$, there exists $(\vartheta_j)_{j=0}^r$ in such a way that $\vartheta_0 = \vartheta^*$ and $\vartheta_r = \vartheta'$ with $(\vartheta_j, \vartheta_{j+1}) \in E(\mathscr{H}_{\mathscr{G}})$ for $j = 1, 2, \cdots, r$. Now, since $\mathscr{R}$ is Fisher graph contraction, so by using assumption (i) repeatedly, we have $(\mathscr{R}^n\vartheta_j, \mathscr{R}^n\vartheta_{j+1}) \in E(\mathscr{H}_{\mathscr{G}})$ for all $n \in \mathbb{N}$. Now, using (ii) as we did in Theorem 14, we have

$$\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta_j, \mathscr{R}^n\vartheta_{j+1}\right) \leq \gamma^n\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_j, \vartheta_{j+1}\right), \tag{22}$$

where $\gamma \in [0, 1/s)$. Now, by using the triangle inequality, we have

$$
\begin{aligned}
\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta^*, \mathscr{R}^n\vartheta'\right) &= \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta_0, \mathscr{R}^n\vartheta_r\right) \\
&\le s\left[\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta_0, \mathscr{R}^n\vartheta_1\right) + \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta_1, \mathscr{R}^n\vartheta_r\right)\right] \\
&\le s\sum_{k=1}^{r}\left[s^{k-1}\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta_{k-1}, \mathscr{R}^n\vartheta_k\right)\right] \\
&\le s\gamma^n\sum_{k=1}^{r}\left[s^{k-1}\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_{k-1}, \vartheta_k\right)\right].
\end{aligned}
\tag{23}
$$

Since $\vartheta^*, \vartheta' \in \mathrm{Fix}(\mathscr{R})$, so this implies that $\mathscr{R}^n\vartheta^* = \vartheta^*$ and $\mathscr{R}^n\vartheta' = \vartheta'$. Now, letting the limit as $n \longrightarrow \infty$, we obtain $\vartheta^* = \vartheta'$. Thus, ϑ^* is the unique fixed point. □

Corollary 16. *Let $(\mathscr{M}, \mathfrak{d})$ be an $\mathscr{H}_{\mathscr{G}}$-complete graphical b-metric space and let $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$. Suppose that there exists nonnegative constant $\lambda \in [0, 1)$ such that for every $\vartheta_1, \vartheta_2 \in \mathscr{M}$ with $(\vartheta_1, \vartheta_2) \in E(\mathscr{H}_{\mathscr{G}})$, we have*

$$\mathfrak{d}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \le \lambda\mathfrak{d}(\vartheta_1, \vartheta_2), \tag{24}$$

(i) $\mathscr{G}$ satisfies the property $(\mathscr{P})$

(ii) There exists $\vartheta_0 \in \mathscr{M}$ with $\mathscr{R}\vartheta_0 \in [\vartheta_0]^r_{\mathscr{H}_{\mathscr{G}}}$ for some $r \in \mathbb{N}$

Then, there exists $\vartheta^* \in \mathscr{M}$ such that the $\mathscr{R}$-PS $\{\vartheta_n\}$ with initial point $\vartheta_0 \in \mathscr{M}$ is $\mathscr{H}_{\mathscr{G}}$-TWC and $\vartheta_n \longrightarrow \vartheta^*$ as $n \longrightarrow \infty$.

Proof. Take $\lambda_1 = \lambda$ and $\lambda_2 = 0$ in Theorem 14. □

Remark 17. If $s = 1$ in Definition 8, then a graphical b-metric space reduced to a graphical metric space, so the following result is a direct consequence of Theorem 14.

Corollary 18. *Let $(\mathscr{M}, \mathfrak{d})$ be an $\mathscr{H}_{\mathscr{G}}$-complete graphical metric space and let $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$. Suppose that there exist nonnegative constants λ_1, λ_2 with $\lambda_1 + \lambda_2 < 1$ such that for every $\vartheta_1, \vartheta_2 \in \mathscr{M}$ with $(\vartheta_1, \vartheta_2) \in E(\mathscr{H}_{\mathscr{G}})$, we have*

$$\mathfrak{d}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \le \lambda_1\mathfrak{d}(\vartheta_1, \vartheta_2) + \lambda_2\frac{\mathfrak{d}(\vartheta_1, \mathscr{R}\vartheta_1)\mathfrak{d}(\vartheta_2, \mathscr{R}\vartheta_2)}{1 + \mathfrak{d}(\vartheta_1, \vartheta_2)}, \tag{25}$$

(i) $\mathscr{G}$ satisfies the property $(\mathscr{P})$

(ii) There exists $\vartheta_0 \in \mathscr{M}$ with $\mathscr{R}\vartheta_0 \in [\vartheta_0]^r_{\mathscr{H}_{\mathscr{G}}}$ for some $r \in \mathbb{N}$

Then, there exists $\vartheta^* \in \mathscr{M}$ such that the $\mathscr{R}$-PS $\{\vartheta_n\}$ with initial point $\vartheta_0 \in \mathscr{M}$ is $\mathscr{H}_{\mathscr{G}}$-TWC and $\vartheta_n \longrightarrow \vartheta^*$ as $n \longrightarrow \infty$.

Corollary 19 (see [6]). *Let $(\mathscr{M}, \mathfrak{d})$ be an $\mathscr{H}_{\mathscr{G}}$-complete graphical metric space and let $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$. Suppose that there exists nonnegative constant $\lambda \in [0, 1)$ such that for every $\vartheta_1, \vartheta_2 \in \mathscr{M}$ with $(\vartheta_1, \vartheta_2) \in\in E(\mathscr{H}_{\mathscr{G}})$, we have*

$$\mathfrak{d}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \le \lambda\mathfrak{d}(\vartheta_1, \vartheta_2), \tag{26}$$

(i) $\mathscr{G}$ satisfies the property $(\mathscr{P})$

(ii) There exists $\vartheta_0 \in \mathscr{M}$ with $\mathscr{R}\vartheta_0 \in [\vartheta_0]^r_{\mathscr{H}_{\mathscr{G}}}$ for some $r \in \mathbb{N}$

Then, there exists $\vartheta^* \in \mathscr{M}$ such that the $\mathscr{R}$-PS $\{\vartheta_n\}$ with initial point $\vartheta_0 \in \mathscr{M}$ is $\mathscr{H}_{\mathscr{G}}$-TWC and $\vartheta_n \longrightarrow \vartheta^*$ as $n \longrightarrow \infty$.

Proof. Take $\lambda_1 = \lambda$ and $\lambda_2 = 0$ in Corollary 18.

Now, we state Dass–Gupta graph contraction in the background of a graphical b-metric space. □

Definition 20. Let $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ be a graphical b-metric space and $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$. Then, $\mathscr{R}$ is said to be Dass–Gupta graph contraction for $\mathscr{H}_{\mathscr{G}}$ on $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ if

(i) $\mathscr{H}_{\mathscr{G}}$ is edge preserving, i.e., for each $\vartheta_1, \vartheta_2 \in \mathscr{M}$ if $(\vartheta_1, \vartheta_2) \in\in E(\mathscr{H}_{\mathscr{G}})$ implies $(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \in E(\mathscr{H}_{\mathscr{G}})$

(ii) There exist nonnegative constants λ_1, λ_2 with $\lambda_1 + \lambda_2 < 1/S$ such that for every $\vartheta_1, \vartheta_2 \in \mathscr{M}$ with $(\vartheta_1, \vartheta_2) \in E(\mathscr{H}_{\mathscr{G}})$, we have

$$
\begin{aligned}
\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) &\le \lambda_1\mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \vartheta_2) \\
&\quad + \lambda_2\frac{\left[1 + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \mathscr{R}\vartheta_1)\right]\mathfrak{d}_{\mathfrak{gb}}(\vartheta_2, \mathscr{R}\vartheta_2)}{1 + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \vartheta_2)}.
\end{aligned}
\tag{27}
$$

Example 21. Let $\mathscr{M} = \{0, 1, 2, 3, 4, 5, 6\}$ be equipped with $\mathfrak{d}_{\mathfrak{gb}}$ which is defined by

$$\mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \vartheta_2) = \begin{cases} |\vartheta_1 - \vartheta_2|^2, & \text{if } \vartheta_1 \ne \vartheta_2, \\ 0, & \text{if } \vartheta_1 = \vartheta_2. \end{cases} \tag{28}$$

Then, $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ is a graphical b-metric space with the coefficient $s = 2$. Define $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$ by $\mathscr{R}$

$$\mathscr{R}\vartheta = \begin{cases} 1, \text{ if } \vartheta \in \{0, 1\}, \\ 2, \text{ if } \vartheta \in \{2, 3, 4, 5, 6\}. \end{cases} \tag{29}$$

Now, taking $\mathscr{H}_{\mathscr{G}}$ such that $\mathscr{M} = V(\mathscr{H}_{\mathscr{G}})$ and

$$E(\mathscr{H}_{\mathscr{G}}) = \left\{ \begin{array}{l} (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (1,3), (1,4), (1,5), (1,6), \\ (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6) \end{array} \right\} \cup \Delta. \tag{30}$$

Then, $\mathscr{R}$ is a Dass and Gupta graph contraction for $\lambda_1 = 1/3$ and $\lambda_2 = 1/12$. It is shown in Figure 2.

Observe that, $\mathscr{R}$ is not a Dass–Gupta contraction, as

$$\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}1, \mathscr{R}2) = 1 > \lambda_1\mathfrak{d}_{\mathfrak{gb}}(1, 2) + \lambda_2\frac{\left[1 + \mathfrak{d}_{\mathfrak{gb}}(1, \mathscr{R}1)\right]\mathfrak{d}_{\mathfrak{gb}}(2, \mathscr{R}2)}{1 + \mathfrak{d}_{\mathfrak{gb}}(1, 2)}. \tag{31}$$

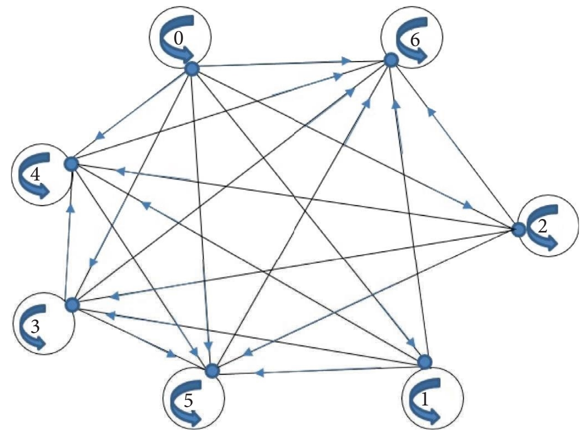

FIGURE 2: Graph associated with Dass–Gupta contraction.

Remark 22. Any Dass–Gupta contraction is Dass–Gupta graph contraction along with $\mathscr{H}_{\mathscr{G}} = G$ and defined by $V(\mathscr{H}_{\mathscr{G}}) = \mathscr{M}$ and $E(\mathscr{H}_{\mathscr{G}}) = \mathscr{M} \times \mathscr{M}$.

(i) If $\mathscr{R}$ is a Dass–Gupta graph contraction with parameters λ_1, λ_2 and $\lambda_2 = 0$, then $\mathscr{R}$ is a Banach graph contraction in $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$.

(ii) Since every $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ is a $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gm}})$ and also every $(\mathscr{M}, \mathfrak{d})$ is a $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gm}})$, thus our theorems in this paper are precise generalizations of some of the results regarding Dass–Gupta contraction (see e.g., [9]).

Now, we present our leading theorem regarding Dass–Gupta graph contraction in the following way.

Theorem 23. *Let $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ be an $\mathscr{H}_{\mathscr{G}}$-complete graphical b-metric space and let $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$ be a Dass and Gupta's graph contraction. Suppose that these assertions hold:*

(i) $\mathscr{G}$ satisfies the property $(\mathscr{P})$

(ii) There exists $\vartheta_0 \in \mathscr{M}$ with $\mathscr{R}\vartheta_0 \in [\vartheta_0]^r_{\mathscr{H}_{\mathscr{G}}}$ for some $r \in \mathbb{N}$

Then, there exists $\vartheta^* \in \mathscr{M}$ such that the $\mathscr{R}$-PS $\{\vartheta_n\}$ with initial point $\vartheta_0 \in \mathscr{M}$ is $\mathscr{H}_{\mathscr{G}}$-TWC and $\vartheta_n \longrightarrow \vartheta^*$ as $n \longrightarrow \infty$.

Proof. Let $\vartheta_0 \in \mathscr{M}$ be such that $\mathscr{R}\vartheta_0 \in [\vartheta_0]^r_{\mathscr{H}_{\mathscr{G}}}$, for some $r \in \mathbb{N}$. As $\{\vartheta_n\}$ is a $\mathscr{R}$-PS originating from ϑ_0, there is a path $(\theta_i)^r_{i=0}$ such that $\vartheta_0 = \theta_0$ and $\mathscr{R}\vartheta_0 = \theta_r$ and $(\theta_{i-1}, \theta_i) \in E(\mathscr{H}_{\mathscr{G}})$ for $i = 1, 2, \cdots, r$. Now, by assumption (i), we have $(\mathscr{R}\theta_{i-1}, \mathscr{R}\theta_i) \in\in E(\mathscr{H}_{\mathscr{G}})$ for $i = 1, 2, \cdots, r$. It yields that $(\mathscr{R}\theta_i)^r_{i=0}$ is a path from $\mathscr{R}\theta_0 = \mathscr{R}\vartheta_0 = \theta_1$ to $\mathscr{R}\theta_r = \mathscr{R}^2\vartheta_0 = \theta_2$ having length r and therefore $\vartheta_2 \in [\vartheta_1]^r_{\mathscr{H}_{\mathscr{G}}}$. Continuing in this way, we get $(\mathscr{R}^n\theta_i)^r_{i=0}$ is a path from $\mathscr{R}^n\theta_0 = \mathscr{R}^n\vartheta_0 = \theta_n$ to $\mathscr{R}^n\theta_r = \mathscr{R}^n\mathscr{R}\vartheta_0 = \vartheta_{n+1}$ of length r, and hence, $\vartheta_{n+1} \in [\vartheta_n]^r_{\mathscr{H}_{\mathscr{G}}}$, for all $n \in \mathbb{N}$. Hence, $\{\vartheta_n\}$ is a $\mathscr{H}_{\mathscr{G}}$-TWC sequence. Thus, $(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i) \in E(\mathscr{H}_{\mathscr{G}})$ for $i = 1, 2, \cdots, r$ and $n \in \mathbb{N}$. Now, by (ii), we have

$$\begin{aligned}
\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i) &= \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}\left(\mathscr{R}^{n-1}\theta_{i-1}\right), \mathscr{R}\left(\mathscr{R}^{n-1}\theta_i\right)\right) \\
&\leq \lambda_1 \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right) \\
&\quad + \lambda_2 \frac{\left[1 + \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}\left(\mathscr{R}^{n-1}\theta_{i-1}\right)\right)\right] \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_i, \mathscr{R}\left(\mathscr{R}^{n-1}\theta_i\right)\right)}{1 + \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right)} \\
&= \lambda_1 \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right) \\
&\quad + \lambda_2 \frac{\left[1 + \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right)\right] \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i\right)}{1 + \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right)} \\
&= \lambda_1 \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1}, \mathscr{R}^{n-1}\theta_i\right) + \lambda_2 \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_{i-1}, \mathscr{R}^n\theta_i\right),
\end{aligned} \tag{32}$$

which yields that

$$\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_{i-1},\mathscr{R}^n\theta_i\right)\leq\left(\frac{\lambda_1}{1-\lambda_2}\right)\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1},\mathscr{R}^{n-1}\theta_i\right). \tag{33}$$

Since $\lambda_1+\lambda_2<1/s$, so taking $\lambda_1/1-\lambda_2=\gamma\in[0,1/s)$, it follows from the above inequality that

$$\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_{i-1},\mathscr{R}^n\theta_i\right)\leq\gamma\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^{n-1}\theta_{i-1},\mathscr{R}^{n-1}\theta_i\right), \tag{34}$$

where $\gamma\in[0,1/s)$. Repeating in this way, we have

$$\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_{i-1},\mathscr{R}^n\theta_i\right)\leq\gamma^n\mathfrak{d}_{\mathfrak{gb}}\left(\theta_{i-1},\theta_i\right). \tag{35}$$

As the sequence $\{\vartheta_n\}$ is a $\mathscr{H}_{\mathscr{G}}$ -TWC sequence and $\mathscr{G}$ is a subgraph of G, so by using inequality (35) and the triangular inequality, we have

$$\begin{aligned}
\mathfrak{d}_{\mathfrak{gb}}\left(\theta_n,\theta_{n+1}\right)&=\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta_0,\mathscr{R}^{n+1}\vartheta_0\right)\\
&=\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_0,\mathscr{R}^n\theta_r\right)\\
&\leq s\left[\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_0,\mathscr{R}^n\theta_1\right)+\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_1,\mathscr{R}^n\theta_r\right)\right]\\
&\leq s\left[\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_0,\mathscr{R}^n\theta_1\right)\right]+s^2\left[\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_1,\mathscr{R}^n\theta_2\right)\right]+\cdots+s^n\left[\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\theta_{r-1},\mathscr{R}^n\theta_r\right)\right]\\
&\leq s\gamma^n\mathfrak{d}_{\mathfrak{gb}}\left(\theta_0,\theta_1\right)+s^2\gamma^n\mathfrak{d}_{\mathfrak{gb}}\left(\theta_1,\theta_2\right)+s^3\gamma^n\mathfrak{d}_{\mathfrak{gb}}\left(\theta_2,\theta_3\right)+\cdots+s^r\gamma^n\mathfrak{d}_{\mathfrak{gb}}\left(\theta_{r-1},\theta_r\right)\\
&=s\gamma^n\sum_{k=1}^{r}s^{k-1}\mathfrak{d}_{\mathfrak{gb}}\left(\theta_{k-1},\theta_k\right).
\end{aligned} \tag{36}$$

Setting $\aleph_b^r=\sum_{k=1}^r s^{k-1}\mathfrak{d}_{\mathfrak{gb}}\left(\theta_{k-1},\theta_k\right)$ in inequality (36), we have

$$\mathfrak{d}_{\mathfrak{gb}}\left(\theta_n,\theta_{n+1}\right)\leq s\gamma^n\left(\aleph_b^r\right). \tag{37}$$

Again $\{\vartheta_n\}$ is a $\mathscr{H}_{\mathscr{G}}$-TWC sequence, so for all $m,n\in\mathbb{N}$, $(m>n)$, we have

$$\begin{aligned}
\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_n,\vartheta_m\right)&\leq s\left[\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_n,\vartheta_{n+1}\right)+\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_{n+1},\vartheta_m\right)\right]\\
&\leq s\left[\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_n,\vartheta_{n+1}\right)\right]+s^2\left[\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_{n+1},\vartheta_{n+2}\right)\right]+s^2\left[\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_{n+2},\vartheta_m\right)\right]\\
&\leq s\left[\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_n,\vartheta_{n+1}\right)\right]+s^2\left[\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_{n+1},\vartheta_{n+2}\right)\right]+\cdots+s^{m-n}\left[\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_{m-1},\vartheta_m\right)\right]\\
&=\sum_{k=p}^{m-1}\left[s^{k-n+1}\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_k,\vartheta_{k+1}\right)\right]\\
&\leq s\sum_{k=n}^{m-1}\left[s^{k-n+1}\gamma^k\aleph_b^r\right]\\
&=s^2\gamma^n\left[\sum_{k=n}^{m-1}(s\gamma)^{k-n}\right]\aleph_b^r\\
&\leq s^2\gamma^n\left[\sum_{k=1}^{\infty}(s\gamma)^{k-1}\right]\aleph_b^r\\
&\leq s^2\gamma^n\left(\frac{1}{1-s\gamma}\right)\aleph_b^r.
\end{aligned} \tag{38}$$

Since $\gamma\in[0,1/s)$, so we obtain

$$\lim_{n,m\longrightarrow\infty}\mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_n,\vartheta_m\right)=0. \tag{39}$$

Thus, $\{\vartheta_n\}$ is a Cauchy sequence in $\mathscr{M}$. Also, since $\mathscr{M}$ is $\mathscr{H}_{\mathscr{G}}$-complete, so $\{\vartheta_n\}$ converges in $\mathscr{M}$ and by assumption, there exists $\vartheta^* \in \mathscr{M}$ and $n_0 \in \mathbb{N}$ such that $(\vartheta_n, \vartheta^*) \in E(\mathscr{G})$ or $(\vartheta^*, \vartheta_n) \in E(\mathscr{H}_{\mathscr{G}})$ for every $n > n_0$ and

$$\lim_{n \longrightarrow \infty} \mathfrak{d}_{\mathfrak{gb}}(\vartheta_n, \vartheta^*) = 0, \tag{40}$$

which shows that $\{\vartheta_n\}$ converges to ϑ^*. □

Theorem 24. *If assumptions given in Theorem 23 are satisfied and, in addition, we assume that the graph $\mathscr{H}_{\mathscr{G}}$ is weakly connected, then the fixed point of $\mathscr{R}$ is unique.*

Proof. By Theorem 23, we have $\{\vartheta_n\}$ is a $\mathscr{R}$-PS with initial point ϑ_0 converges to $\vartheta^* \in \mathscr{M}$. By assumption, $(\vartheta^* P \mathscr{R}\vartheta^*)_{\mathscr{H}_{\mathscr{G}}}$ or $(\mathscr{R}\vartheta^* P \vartheta^*)_{\mathscr{H}_{\mathscr{G}}}$ and thus, we obtain

$$\begin{aligned} \mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \mathscr{R}\vartheta^*) &\le s\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \vartheta_n) + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_n, \mathscr{R}\vartheta^*)\right] \\ &= s\left[\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \vartheta_n) + \mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}\vartheta_{n-1}, \mathscr{R}\vartheta^*)\right]. \end{aligned} \tag{41}$$

By (ii), we have

$$\begin{aligned} \mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \mathscr{R}\vartheta^*) &\le s\left[\begin{matrix} \mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \vartheta_n) + \lambda_1 \mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta^*) \\ +\lambda_2 \dfrac{\left[1 + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \mathscr{R}\vartheta_{n-1})\right]\mathfrak{d}_{\mathfrak{gb}}\vartheta^*, \mathscr{R}\vartheta^*)}{1 + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta^*)} \end{matrix}\right] \\ &= s\left[\begin{matrix} \mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \vartheta_n) + \lambda_1 \mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta^*) \\ +\lambda_2 \dfrac{\left[1 + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta_n)\right]\mathfrak{d}_{\mathfrak{gb}}\vartheta^*, \mathscr{R}\vartheta^*}{1 + \mathfrak{d}_{\mathfrak{gb}}(\vartheta_{n-1}, \vartheta^*)} \end{matrix}\right]. \end{aligned} \tag{42}$$

Taking the limit as $n \longrightarrow \infty$ and using the fact that $\lambda_1 + \lambda_2 < 1/s$, we have $\mathfrak{d}_{\mathfrak{gb}}(\vartheta^*, \mathscr{R}\vartheta^*) = 0$. Thus, $\vartheta^* = \mathscr{R}\vartheta^*$ and therefore ϑ^* is a fixed point of $\mathscr{R}$. Now, we suppose that ϑ' is another fixed point of $\mathscr{R}$. Assume that $(\vartheta^* P \vartheta')_{\mathscr{H}_G}$, there exists $(\vartheta_j)_{j=0}^r$ in such a way that $\vartheta_0 = \vartheta^*$ and $\vartheta_r = \vartheta'$ with $(\vartheta_j, \vartheta_{j+1}) \in E(\mathscr{H}_{\mathscr{G}})$ for $j = 1, 2, \cdots, r$. Now, since $\mathscr{R}$ is Dass–Gupta graph contraction, so by using assumption (i) repeatedly, we have $(\mathscr{R}^n\vartheta_j, \mathscr{R}^n\vartheta_{j+1}) \in\in E(\mathscr{H}_{\mathscr{G}})$ for $n \in \mathbb{N}$. Now, using (ii) as we did in Theorem 23, we have

$$\mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta_j, \mathscr{R}^n\vartheta_{j+1}\right) \le \gamma^n \mathfrak{d}_{\mathfrak{gb}}\left(\vartheta_j, \vartheta_{j+1}\right), \tag{43}$$

where $\gamma \in [0, 1/s)$. Now, by using the triangle inequality, we have

$$\begin{aligned} \mathfrak{d}_{\mathfrak{gb}}\left(\mathscr{R}^n\vartheta^*, \mathscr{R}^n\vartheta'\right) &= \mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\vartheta_0, \mathscr{R}^n\vartheta_r) \\ &\le s\left[\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\vartheta_0, \mathscr{R}^n\vartheta_1) + \mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\vartheta_1, \mathscr{R}^n\vartheta_r)\right] \\ &\le s\sum_{k=1}^{r}\left[s^{k-1}\mathfrak{d}_{\mathfrak{gb}}(\mathscr{R}^n\vartheta_{k-1}, \mathscr{R}^n\vartheta_k)\right] \\ &\le s\gamma^n\sum_{k=1}^{r}\left[s^{k-1}\mathfrak{d}_{\mathfrak{gb}}(\vartheta_{k-1}, \vartheta_k)\right]. \end{aligned} \tag{44}$$

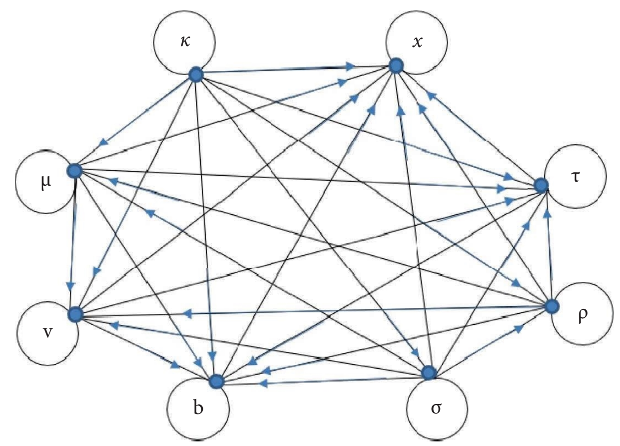

Figure 3: Weighted graph for $V'(\mathscr{H}_{\mathscr{G}})$, where $d_{\mathfrak{gb}}(\vartheta_1, \vartheta_2) =$ weight of edge $(\vartheta_1, \vartheta_2)$.

Since $\vartheta^*, \vartheta' \in \mathrm{Fix}(\mathscr{R})$, so this implies that $\mathscr{R}^n\vartheta^* = \vartheta^*$ and $\mathscr{R}^n\vartheta' = \vartheta'$. Now, letting the limit as $n \longrightarrow \infty$, we obtain $\vartheta^* = \vartheta'$. Thus, ϑ^* is a unique fixed point. □

Remark 25. If $s = 1$ in Definition 8, then the graphical b-metric space reduced to a graphical metric space, then following result is a direct consequence of Theorem 23.

Corollary 26. *Let $(\mathscr{M}, \mathfrak{d})$ be an $\mathscr{H}_{\mathscr{G}}$-complete graphical metric space and let $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$. Suppose that there exist nonnegative constants λ_1, λ_2 with $\lambda_1 + \lambda_2 < 1$ such that for every $\vartheta_1, \vartheta_2 \in \mathscr{M}$ with $(\vartheta_1, \vartheta_2) \in E(\mathscr{H}_{\mathscr{G}})$, we have*

$$\mathfrak{d}(\mathscr{R}\vartheta_1, \mathscr{R}\vartheta_2) \le \lambda_1 \mathfrak{d}(\vartheta_1, \vartheta_2) + \lambda_2 \frac{\left[1 + \mathfrak{d}(\vartheta_1, \mathscr{R}\vartheta_1)\right]\mathfrak{d}(\vartheta_2, \mathscr{R}\vartheta_2)}{1 + \mathfrak{d}(\vartheta_1, \vartheta_2)}, \tag{45}$$

(i) $\mathscr{G}$ satisfies the property $(\mathscr{P})$

(ii) There exists $\vartheta_0 \in \mathscr{M}$ with $\mathscr{R}\vartheta_0 \in [\vartheta_0]^r_{\mathscr{H}_{\mathscr{G}}}$ for some $r \in \mathbb{N}$

Then, there exists $\vartheta^* \in \mathscr{M}$ such that the $\mathscr{R}$-PS $\{\vartheta_n\}$ with initial point $\vartheta_0 \in \mathscr{M}$ is $\mathscr{H}_{\mathscr{G}}$-TWC and $\vartheta_n \longrightarrow \vartheta^*$ as $n \longrightarrow \infty$.

Example 27. Let $\mathscr{M} = \{1/3^n: n \in \mathbb{N}\} \cup \{0\}$ be equipped with $G = \mathscr{H}_{\mathscr{G}}$ such that $V(\mathscr{G}) = \mathscr{M}$ and

$$E(\mathscr{G}) = \Delta \cup \{(\vartheta_1, \vartheta_2) \in \mathscr{M} \times \mathscr{M}: (\vartheta_1 P \vartheta_2), \vartheta_2 \le \vartheta_1\}. \tag{46}$$

Define the graphical b-metric $\mathfrak{d}_{\mathfrak{gb}}$ by

$$\mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \vartheta_2) = \begin{cases} |\vartheta_1 - \vartheta_2|^2, & \text{if } \vartheta_1 \ne \vartheta_2, \\ 0, & \text{if } \vartheta_1 = \vartheta_2. \end{cases} \tag{47}$$

Clearly, $(\mathscr{M}, \mathfrak{d}_{\mathfrak{gb}})$ is a graphical b-metric space with the coefficient $s = 2$. Define $\mathscr{R}: \mathscr{M} \longrightarrow \mathscr{M}$ by

$$\mathscr{R}\vartheta = \frac{\vartheta}{3}, \tag{48}$$

for all $\vartheta \in \mathscr{M}$. There exists $\vartheta_0 = 1/3$ such that $\mathscr{R}(1/3) = 1/9 \in [1/3]^1_{\mathscr{H}_{\mathscr{G}}}$, that is, $((1/3)P(1/9))_{\mathscr{H}_{\mathscr{G}}}$ and the condition (27) is satisfied for $\lambda_1 = 1/3$ and $\lambda_2 = 1/12$. Hence,

$\mathscr{R}$ is Dass–Gupta graph contraction on $\mathscr{M}$. Hence, all assumptions of Theorem 23 are satisfied and 0 is the required fixed point of mapping $\mathscr{R}$.

Figure 3 exemplifies the weighted graph for $V^{/}(\mathscr{H}_{\mathscr{G}}) = \{\kappa, \sigma, \rho, \mu, \nu, \tau, \flat, \varkappa\} \subseteq V(\mathscr{H}_{\mathscr{G}})$, where the value of $\mathfrak{d}_{\mathfrak{gb}}(\vartheta_1, \vartheta_2)$ is equal to the weight of edge $(\vartheta_1, \vartheta_2)$ and $\{\kappa, \sigma, \rho, \mu, \nu, \tau, \flat, \varkappa\} = \{1/3, 1/3^2, 1/3^3, 1/3^4, 1/\ 3^5, 1/3^6, 1/3^7, 0\}$.

4. Conclusion

In this article, we have introduced the notion of Dass–Gupta graph contraction in the background of graphical b-metric spaces and established some fixed-point theorems. We also supplied some nontrivial examples to show the validity of obtained results. We hope that the obtained theorems in this article will make new relations for those people who are employing in graphical b-metric spaces.

The prospective work in this direction will focus on finding the common fixed points of self-mappings and set valued mappings in the context of graphical b-metric spaces. Differential and integral inclusions can be investigated as applications of these results.

Data Availability

The required data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declared that there are no conflicts of interest.

References

[1] M. Frechet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, vol. 22, pp. 1–72, 1906.

[2] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[3] I. A. Bakhtin, "The contraction mapping principle in almost metric spaces," *Funct. Anal.*, vol. 30, pp. 26–37, 1989.

[4] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Math. Inform. Univ. Ostra.*, vol. 1, pp. 5–11, 1993.

[5] J. Jachymski, "The contraction principle for mappings on a metric space with a graph," *Proceedings of the American Mathematical Society*, vol. 136, no. 4, pp. 1359–1373, 2007.

[6] S. Shukla, S. Radenović, and C. Vetro, "Graphical metric space: a generalized setting in fixed point theory," *RACSAM*, vol. 111, no. 3, pp. 641–655, 2017.

[7] N. Chuensupantharat, P. Kumam, V. Chauhan, D. Singh, and R. Menon, "Graphic contraction mappings via graphical b-metric spaces with applications b-metric spaces with applications," *Bull. Malays. Math. Sci. Soc.*, vol. 42, no. 6, pp. 3149–3165, 2019.

[8] B. Fisher, "Mappings satisfying a rational inequality," *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.)*, vol. 24, pp. 247–251, 1980.

[9] B. K. Dass and S. Gupta, "An extension of Banach contraction principle through rational expression," *Indian Journal of Pure and Applied Mathematics*, vol. 6, pp. 1455–1458, 1975.

[10] H. Isik, N. Hussain, and K. Ma, "Generalized rational contractions endowed with a graph and an application to a system of integral equations," *Journal of Computational Analysis and Applications*, vol. 22, no. 6, pp. 1158–1175, 2017.

[11] M. Younis, D. Singh, M. Asadi, and V. Joshi, "Results on contractions of Reich type in graphical b-metric spaces with applications b-metric spaces with applications," *Filomat*, vol. 33, pp. 5723–5735, 2019.

[12] M. Younis, D. Singhb, and A. Petrusel, "Applications of graph Kannan mappings to the damped spring-mass system and deformation of an elastic beam," *Discrete Dynamics in Nature and Society*, vol. 2019, Article ID 1315387, 9 pages, 2019.

[13] X. Li, F. Khan, G. Ali, L. Gul, and M. Sarwar, "Hypergraphical metric spaces and fixed point theorems," *Mathematical Problems in Engineering*, vol. 2021, Article ID 9961734, 7 pages, 2021.

[14] L. Chen, N. Yang, Y. Zhao, and Z. Ma, "Fixed point theorems for set-valued G-contractions in a graphical convex metric space with applicationsG-contractions in a graphical convex metric space with applications," *J. Fixed Point Theory Appl*, vol. 22, no. 4, p. 88, 2020.

[15] S. Shukla, N. Mlaiki, and H. Aydi, "On (G, G′)-Prešić–Ćirić Operators in Graphical Metric SpacesG,G/-Preši ć-Ćirić operators in graphical metric spaces," *Mathematics*, vol. 7, no. 5, p. 445, 2019.

[16] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge UK, 2001.

[17] P. Debnath, N. Konwar, and S. Radenović, *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences*, Springer, Singapore, 2021.

[18] P. Debnath, H. M. Srivastava, P. Kumam, and B. Hazarika, *Fixed Point Theory and Fractional Calculus: Recent Advances and Applications*, Springer, Singapore, 2022.

[19] J. R. Roshan, N. Shobkolaei, S. Sedghi, V. Parvaneh, and S. Radenović, "Common fixed point theorems for three maps in discontinuous Gbmetric spaces," *Acta Mathematica Scientia*, vol. 34, no. 5, pp. 1643–1654, 2014.

[20] Z. Mustafa, J. R. Roshan, V. Parvaneh, and Z. Kadelburg, "Fixed point theorems for weakly T-Chatterjea and weakly T-Kannan contractions in b-metric spacesT-Chatterjea and weakly T-Kannan contractions in b-metric spaces," *Journal of Inequalities and Applications*, vol. 2014, no. 1, p. 46, 2014.

[21] Z. Mustafa, V. Parvaneh, M. M. M. Jaradat, and Z. Kadelburg, "Extended Rectangular b-Metric Spaces and Some Fixed Point Theorems for Contractive Mappings b-metric spaces and some fixed point theorems for contractive mappings," *Symmetry*, vol. 11, no. 4, p. 594, 2019.