Hindawi
Journal of Mathematics
Volume 2023, Article ID 4148088, 7 pages
https://doi.org/10.1155/2023/4148088

Research Article

$\mathscr{A}$-D3 Modules and $\mathscr{A}$-D4 Modules

Zhanmin Zhu

Department of Mathematics, Jiaxing University, Jiaxing, Zhejiang 314001, China

Correspondence should be addressed to Zhanmin Zhu; zhuzhanminzjxu@hotmail.com

Received 7 March 2023; Revised 17 July 2023; Accepted 19 July 2023; Published 4 September 2023

Academic Editor: Carmelo Antonio Finocchiaro

Let $\mathscr{A}$ be a class of some right R-modules that is closed under isomorphisms, and let M be a right R-module. Then M is called $\mathscr{A}$-D3 if, whenever N and K are direct summands of M with $M = N + K$ and $M/K \in \mathscr{A}$, then $N \cap K$ is also a direct summand of M; M is called an $\mathscr{A}$-D4 module, if whenever $M = B \oplus A$ where B and A are submodules of M and $A \in \mathscr{A}$, then every epimorphism $f\colon B \longrightarrow A$ splits. Several characterizations and properties of these classes of modules are investigated. As applications, some new characterizations of semisimple Artinian rings, quasi-Frobenius rings, von Neumann regular rings, semiregular rings, perfect rings, semiperfect rings, hereditary rings, semihereditary rings, and PP rings are given.

1. Introduction

A right R-module M is called direct projective [1] if for any submodule N of M with M/N is isomorphic to a direct summand of M, then N is a direct summand of M. In [2, 3], direct projective modules are also called D2 modules. A right R-module M is called a $D3$ module [2, 3] if, whenever N and K are direct summands of M with $= N + K$, $N \cap K$ is also a direct summand of M. In [4], Ding et al. generalized the concept of D3 modules to $D4$ modules. According to [4], a right R-module M is called a D4 module, if whenever A and B are submodules of M with $M = A \oplus B$ and $f\colon A \longrightarrow B$ is an epimorphism, then $\operatorname{Ker} f \subseteq^{\oplus} A$. $D3$ modules and $D4$ modules have several interesting characterizations and properties (see [2–4]). In [3, 4], some important rings such as semisimple Artinian rings, semiperfect rings, right perfect rings, and semiregular rings are characterized by $D3$ modules and $D4$ modules, respectively. It is natural to extend these classes of modules.

In this paper, we shall generalize the concepts of Di modules $(i = 2, 3, 4)$ to $\mathscr{A}$-Di modules $(i = 2, 3, 4)$, respectively, and give some interesting results on these modules. As applications, some new characterizations of several well-known classes of rings will be given in terms of $\mathscr{A}$-D4 modules. The concepts of $\mathscr{A}$-Di modules $(i = 2, 3, 4)$ are the dual concepts of $\mathscr{A}$-Ci modules $(i = 2, 3, 4)$ [5], respectively.

Throughout R is an associative ring with identity, all modules are unitary, unless otherwise specified, and $\mathscr{A}$ is a class of some right R-modules that is closed under isomorphisms. For a module M, we write $N \subseteq^{\oplus} M$ if N is a direct summand of M, and $N \ll M$ if N is a small submodule of M. We refer to [6] for the undefined notions in this paper.

2. $\mathscr{A}$-D2 Modules and $\mathscr{A}$-D3 Modules

Recall that a right R-module M is called pseudo-projective if for every submodule K of M, any epimorphism $\varphi\colon M \longrightarrow M/K$ lifted to an endomorphism of M, that is, there exists an endomorphism s of M such that $\varphi = \pi s$, where $\pi\colon M \longrightarrow M/K$ is the canonical homomorphism. We extend the concept of pseudo-projective modules as follows.

Definition 1. Let $\mathscr{A}$ be a class of some right R-modules, and let M and N be two right R-modules. Then M is called pseudo-$\mathscr{A}$-N-projective if, for every submodule K of N with $N/K \in \mathscr{A}$, every epimorphism $\varphi\colon M \longrightarrow N/K$ lifted to a homomorphism from M to N. M is called pseudo-$\mathscr{A}$-projective if it is pseudo-$\mathscr{A}$-M-projective.

Example 1

(1) Let $\mathscr{A}$ be the class of all right R-modules. Then M is pseudo-$\mathscr{A}$-projective if and only if it is pseudo-projective.

(2) Let $\mathscr{A}$ be the class of right R-modules that is isomorphic to a submodule of M. Then M is pseudo-$\mathscr{A}$-projective if and only if it is quasi-pseudo-principally projective [7, Definition 1].

(3) We call a right R-module M pseudo-S-projective (resp., pseudo-Inj-projective, pseudo-Flat-projective, pseudo-FP-projective, pseudo-F-projective, pseudo-P-projective, and pseudo-Soc-projective) if it is pseudo-$\mathscr{A}$-projective for the class $\mathscr{A}$ of all simple (resp., injective, flat, finitely presented, finitely generated, cyclic, and semisimple) right R-modules, and we call a right R-module M pseudo-I-projective (resp., pseudo-FI-projective and pseudo-PI-projective) if it is pseudo-$\mathscr{A}$-projective for the class $\mathscr{A}$ of all right R-modules that is isomorphic to a right ideal (resp., a finitely generated right ideal and a principal right ideal) of R.

Proposition 2. *Let $\mathscr{A}$ be a class of some right R-modules, M, N be two right R-modules, and N' be a factor module of N. If M is pseudo-$\mathscr{A}$-N-projective, then*

(1) Every direct summand of M is pseudo-$\mathscr{A}$-N-projective.

(2) M is pseudo-$\mathscr{A}$-N'-projective.

Proof

(1) Let $M = M_1 \oplus M_2$. Let π_1: $M \longrightarrow M_1$ be the the projection and ι_1: $M_1 \longrightarrow M$ be the injection. Then for every factor module $N/K \in \mathscr{A}$ of N and every epimorphism f of M_1 to N/K, let π: $N \longrightarrow N/K$ be the the canonical homomorphism. Since M is pseudo-$\mathscr{A}$-N-projective, there exists a homomorphism g: $M \longrightarrow N$ such that $\pi_1 = \pi g$. Thus, $g\iota_1$ is a homomorphism of M_1 to N and $= \pi(g\iota_1)$, and so M_1 is pseudo-$\mathscr{A}$-N-projective.

(2) It is obvious.

By Proposition 2, we have immediately the following corollary. □

Corollary 3. *Let $\mathscr{A}$ be a class of some right R-modules. Then every direct summand of a pseudo-$\mathscr{A}$-projective module is pseudo-$\mathscr{A}$-projective.*

Now we extend the concepts of D2 modules and D3 modules as follows.

Definition 4

(1) Let $\mathscr{A}$ be a class of some right R-modules that is closed under isomorphisms, and let M be a right R-module. Then M is called $\mathscr{A}$-D2 if, for every submodule $K \subseteq M$ with M/K isomorphic to a direct summand of M and $M/K \in \mathscr{A}$, K is a direct summand of M; M is called $\mathscr{A}$-D3 if, whenever N and K are direct summands of M with $M = N + K$ and $M/K \in \mathscr{A}$, $N \cap K$ is also a direct summand of M.

(2) A right R-module M is called S-Di (resp., Inj-Di, Flat-Di, FP-Di, F-Di, P-Di, and Soc-Di) if it is $\mathscr{A}$-Di for the class $\mathscr{A}$ of all simple (resp., injective, flat, finitely presented, finitely generated, cyclic, and semisimple) right R-modules; a right R-module M is called I-Di (resp., FI-Di and PI-Di) if it is $\mathscr{A}$-Di for the class $\mathscr{A}$ of all right R-modules that is isomorphic to a right ideal (resp., a finitely generated right ideal and a principal right ideal of R), $i = 2, 3$.

Theorem 5. *Let $\mathscr{A}$ be a class of some right R-modules that is closed under isomorphisms, and let M be a right R-module. Then the following conditions are equivalent:*

(1) M is an $\mathscr{A}$-D2 module.

(2) If A and B are direct summand of M and $A \in \mathscr{A}$, then any exact sequence $B \xrightarrow{f} A \longrightarrow 0$ splits.

Proof

$(1) \Longrightarrow (2)$. Let $M = B \oplus B'$. Then $M/(\mathrm{Ker} f \oplus B') = (B \oplus B')/(\mathrm{Ker} f \oplus B') \cong B/\mathrm{Ker} f \cong A \in \mathscr{A}$. Since M is $\mathscr{A}$-D2, $\mathrm{Ker} f \oplus B'$ is a direct summand of M. Hence, $\mathrm{Ker} f$ is a direct summand of B.

$(2) \Longrightarrow (1)$. Let $K \subseteq M$, $M/K \in \mathscr{A}$, and $M/K \overset{\sigma}{\cong} A \subseteq^{\oplus} M$. Then we have an exact sequence $M \xrightarrow{\sigma} \pi\, A \longrightarrow 0$, where π: $M \longrightarrow M/K$ is the canonical epimorphism. By (2), $\mathrm{Ker}(\sigma\pi) \subseteq^{\oplus} M$, i.e., $K \subseteq^{\oplus} M$. Therefore, M is $\mathscr{A}$-D2. □

Corollary 6. *Every direct summand of an $\mathscr{A}$-D2 module is $\mathscr{A}$-D2.*

Proof. It follows from Theorem 5. □

Theorem 7. *Let $\mathscr{A}$ be a class of some right R-modules that is closed under isomorphisms, and let M be a right R-module. Consider the following conditions:*

(1) M is pseudo-$\mathscr{A}$-projective.

(2) M is $\mathscr{A}$-D2.

(3) M is $\mathscr{A}$-D3.

Then we always have $(1) \Longrightarrow (2) \Longrightarrow (3)$.

Proof

$(1) \Longrightarrow (2)$. Let M_R be pseudo-$\mathscr{A}$-projective with $S = End(M_R)$. If K is a submodule of M, $M/K \in \mathscr{A}$, and $M/K \cong eM$, where $e^2 = e \in S$, then eM is pseudo-$\mathscr{A}$-M-projective by Proposition 2 (1) and hence M/K is also pseudo-$\mathscr{A}$-M-projective, and it shows that K is a direct summand of M. This proves (2).

(2) $\Longrightarrow$ (3). Let N and K be direct summands of M with $M = N + K$ and $M/K \in \mathscr{A}$. Let $M = K \oplus L$ for some submodule L of M. Then $N/(N \cap K) \cong (N + K)/K = M/K \cong L \in \mathscr{A}$, and so we have an exact sequence $N \xrightarrow{f} L \longrightarrow 0$ with $\mathrm{Ker} f = N \cap K$. Since M is $\mathscr{A}$-D2, by Theorem 5, we have that $N \cap K \subseteq^{\oplus} N$. This proves (3). □

Proposition 8. *Let $\mathscr{A}$ be a class of some right R-modules that is closed under factor modules, and let $\in \in \mathscr{A}$. Then M is a D2 module if and only if M is an $\mathscr{A}$-D2 module and M is a D3 module if and only if M is an $\mathscr{A}$-D3 module.*

Proof. The proof is obvious. □

Corollary 9

(1) *If M is a finitely generated module, then M is a D2 module if and only if it is a F-D2 module and M is a D3 module if and only if it is a F-D3 module.*

(2) *If M is a cyclic module, then M is a D2 module if and only if it is a P-D2 module and M is a D3 module if and only if it is a P-D3 module.*

Proposition 10. *A direct summand of an $\mathscr{A}$-D3 module is again an $\mathscr{A}$-D3 module.*

Proof. Let M be an $\mathscr{A}$-D3 module and $N \subseteq^{\oplus} M$. We prove that N is also $\mathscr{A}$-D3. Let B and C be two direct summands of N with $N = B + C$ and $N/C \in \mathscr{A}$. Write $N = B' \oplus C$ and $M = N' \oplus N$. Then $M = (N' \oplus C) \oplus B' = (N' \oplus C) + B$, and $M/(N' \oplus C) = (N' \oplus N)/(N' \oplus C) \cong N/C \in \mathscr{A}$. Since M is $\mathscr{A}$-D3, $(N' \oplus C) \cap B \subseteq^{\oplus} M$. Write $M = (N' \oplus C) \cap B \oplus K$. Then $N = M \cap N = ((N' \oplus C) \cap B \oplus K) \cap N = (N' \oplus C) \cap B \oplus (K \cap N) = (B \cap C) \oplus (K \cap N)$, as required. □

Theorem 11. *Let $\mathscr{A}$ be a class of some right R-modules that is closed under isomorphisms, and let M be a right R-module. Consider the following conditions:*

(1) *M is an $\mathscr{A}$-D3 module.*

(2) *If $B \subseteq^{\oplus} M, C \subseteq^{\oplus} M, M/C \in \mathscr{A}$ and $M = B + C$, then there exist $B_1 \subseteq B$ and $C_1 \subseteq C$ such that $M = B_1 \oplus C = B \oplus C_1$.*

(3) *If $B \subseteq^{\oplus} M, C \subseteq^{\oplus} M, B + C \subseteq^{\oplus} M$, and $M/C \in \mathscr{A}$, then $B \cap C \subseteq^{\oplus} M$.*

Then we always have (3) $\Longrightarrow$ (1) $\Longleftrightarrow$ (2).

Moreover, if $\mathscr{A}$ is closed under direct summands, then the above three conditions are equivalent.

Proof

(1) $\Longrightarrow$ (2). Let $B \subseteq^{\oplus} M, C \subseteq^{\oplus} M, M/C \in \mathscr{A}$ and $M = B + C$. Then by (1), $B \cap C \subseteq^{\oplus} M$, and so $M = (B \cap C) \oplus K$ for a submodule $K \subseteq M$. Write $B_1 = B \cap K, C_1 = C \cap K$. Then we have $B_1 \subseteq B$, $C_1 \subseteq C$, and $M = B + C = B + ((B \cap C) \oplus K) \cap C = B + ((B \cap C) \oplus (C \cap K)) = B \oplus (C \cap K) = B \oplus C_1$. In the same way, we have also that $M = B_1 \oplus C$.

(2) $\Longrightarrow$ (1). Let $B \subseteq^{\oplus} M, C \subseteq^{\oplus} M, M/C \in \mathscr{A}$ and $M = B + C$. Then by (2), we have $M = B_1 \oplus C = B \oplus C_1$ for some submodules $B_1 \subseteq B$ and $C_1 \subseteq C$. Since $C = C \cap M = C \cap (B \oplus C_1) = C_1 \oplus (B \cap C)$, we have $M = B_1 \oplus C = (B_1 \oplus C_1) \oplus (B \cap C)$, as required.

(3) $\Longrightarrow$ (1). It is clear.

Now suppose that $\mathscr{A}$ is closed under direct summands, we need to prove (1) $\Longrightarrow$ (3). Let $K = B + C$. Since $B \subseteq^{\oplus} M, C \subseteq^{\oplus} M$, we have $B \subseteq^{\oplus} K, C \subseteq^{\oplus} K$. Let $M = K \oplus K'$. Then $K/C \oplus (K' + C)/C = M/C \in \mathscr{A}$, and so $K/C \in \mathscr{A}$ by hypothesis. Since $K \subseteq^{\oplus} M$ and M is $\mathscr{A}$-D3, by Proposition 10, K is also $\mathscr{A}$-D3. So, $B \cap C \subseteq^{\oplus} K$, and hence $B \cap C \subseteq^{\oplus} M$. □

Lemma 12. *(see [8, Lemma 2.6 (1) (2)]). Let $M = B \oplus A$, $X \leq B$, and $f\colon X \longrightarrow A$. Then*

(1) *$X \oplus A = \langle f \rangle \oplus A$, where $\langle f \rangle = \{x - f(x) \mid x \in X\}$.*

(2) *$\mathrm{Ker} f = \langle f \rangle \cap B$.*

Theorem 13. *Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphism. If M is an $\mathscr{A}$-D3 module, $M = B \oplus A$ for some submodules B and A, where $A \in \mathscr{A}$, and $f\colon B \longrightarrow A$ is an R-homomorphism. Then*

(1) *If f is an epimorphism, then $\mathrm{Ker} f \subseteq^{\oplus} B$.*

(2) *If $\mathscr{A}$ is closed under direct summands and $\mathrm{Im} f \subseteq^{\oplus} A$, then $\mathrm{Ker} f \subseteq^{\oplus} B$.*

Proof

(1) By Lemma 12 (1), $M = \langle f \rangle \oplus A = \langle f \rangle \oplus \mathrm{Im} f$. Let $m \in M$ and write $m = b + f(b')$ where $b, b' \in B$. Then $m = b + f(b') = (b + b') + (-b' - f(-b')) \in B + \langle f \rangle$. This shows that $M = B + \langle f \rangle$. Since M is $\mathscr{A}$-D3 and $M/B \cong A \in \mathscr{A}$, $B \cap \langle f \rangle \subseteq^{\oplus} M$. Hence, by Lemma 12 (2), $\mathrm{Ker} f \subseteq^{\oplus} M$, and thus $\mathrm{Ker} f \subseteq^{\oplus} B$.

(2) Let $A = \mathrm{Im} f \oplus A'$. Then $M = (B \oplus \mathrm{Im} f) \oplus A'$, and so by Proposition 10, $B \oplus \mathrm{Im} f$ is an $\mathscr{A}$-D3 module. Since $\mathscr{A}$ is closed under direct summands and $A \in \mathscr{A}$, $\mathrm{Im} f \in \mathscr{A}$. By (1), we have that $\mathrm{Ker} f \subseteq^{\oplus} B$. □

3. A-D4 Modules

Now we extend the concept of D4 modules as follows.

Definition 14. Let $\mathscr{A}$ be a class of some right R-modules that is closed under isomorphisms. A right R-module M is called an $\mathscr{A}$-D4 module, if, whenever $M = B \oplus A$ where B and A are submodules of M and $A \in \mathscr{A}$, every epimorphism $f\colon B \longrightarrow A$ splits.

Definition 15. A right R-module M is called S-D4 (resp., Inj-D4, Flat-D4, FP-D4, F-D4, P-D4, and Soc-D4) if it is $\mathscr{A}$-D4 for the class $\mathscr{A}$ of all simple (resp., injective, flat, finitely presented, finitely generated, cyclic, and semisimple) right R-modules; a right R-module M is called I-D4 (resp., FI-D4 and PI-D4) if it is $\mathscr{A}$-D4 for the class $\mathscr{A}$ of all right

R-modules that is isomorphic to a right ideal (resp., a finitely generated right ideal and a principal right ideal) of R.

By Theorem 13 (1), it is easy to see that every $\mathscr{A}$-D3 module is $\mathscr{A}$-D4.

Theorem 16. *Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphisms, and let M be a right R-module. Then the following conditions are equivalent:*

(1) M is an $\mathscr{A}$-D4 module.

(2) If $M = B \oplus A$ where B and A are submodules of M with $A \in \mathscr{A}$ and $f\colon B \longrightarrow A$ is an epimorphism, then $\mathrm{Ker} f \subseteq^{\oplus} B$.

(3) If B and C are submodules of M with $M/B \in \mathscr{A}$, $M = B + C, B \subseteq^{\oplus} M$, and $M/B \cong M/C$, then $B \cap C$ is a direct summand of M.

(4) If B and C are direct summands of M with $M/B \in \mathscr{A}$, $M = B + C$, and $M/B \cong M/C$, then $B \cap C$ is a direct summand of M.

(5) If B and C are submodules of M with $M/B \in \mathscr{A}$, $M = B + C, B \subseteq^{\oplus} M$, and $M/B \cong M/C$, then C is a direct summand of M.

(6) If $M = B \oplus B' = C \oplus C' = B + C = B + C'$, where B, B', C, C' are submodules of M, and $M/B \in \mathscr{A}$, then $B \cap C$ is a direct summand of M.

(7) If B and C are direct summands of M with $M/B \in \mathscr{A}$, $M = B + C$, and $B \cong C$, then $B \cap C$ is a direct summand of M.

Proof

(1)$\Longleftrightarrow$(2); (3)$\Longrightarrow$(4). These are obvious.

(2)$\Longrightarrow$(3). Let B and C be submodules of M with $M/B \in \mathscr{A}$, $M = B + C, B \subseteq^{\oplus} M$, and $M/B \cong M/C$. Write $M = B \oplus A$ where $A \subseteq M$. Since $A \cong M/B \cong M/C = (B + C)/C \cong B/(B \cap C)$, we have an epimorphism $f\colon B \longrightarrow A$ with $\mathrm{Ker} f = B \cap C$. By (2), $B \cap C = \mathrm{Ker}\ f \subseteq^{\oplus} B$, and so $B \cap C \subseteq^{\oplus} M$.

(4)$\Longrightarrow$(2). Let $M = B \oplus A$ where B and A are submodules of M with $A \in \mathscr{A}$, and let $f\colon B \longrightarrow A$ be an epimorphism. Then $M = \langle f \rangle \oplus A = \langle f \rangle + B$ by Lemma 12 (1), and $\mathrm{Ker} f = \langle f \rangle \cap B$ by Lemma 12 (2). Thus, we have $\langle f \rangle \subseteq^{\oplus} M, B \subseteq^{\oplus} M, M = \langle f \rangle + B$ and $M/B \cong M/\langle f \rangle \cong A \in \mathscr{A}$. By (4), $\langle f \rangle \cap B \subseteq^{\oplus} M$, and so $\langle f \rangle \cap B \subseteq^{\oplus} B$, i.e., $\mathrm{Ker} f \subseteq^{\oplus} B$.

(3)$\Longrightarrow$(5). Let B and C be submodules of M with $M/B \in \mathscr{A}$, $M = B + C, B \subseteq^{\oplus} M$, and $M/B \cong M/C$. By (3), $B \cap C \subseteq^{\oplus} M$. Write $M = (B \cap C) \oplus K$ where $K \subseteq M$. Then by the modular law, $B = (B \cap C) \oplus (B \cap K)$ and $C = (B \cap C) \oplus (C \cap K)$. Thus, $M = B + C = [(B \cap C) \oplus (B \cap K)] + [(B \cap C) \oplus (C \cap K)] = (B \cap K) + [(B \cap C) \oplus (C \cap K)]$. Since $B \cap C \cap K = 0$, $(B \cap K) \cap [(B \cap C) \oplus (C \cap K)] = 0$. Therefore, $M = (B \cap K) \oplus [(B \cap C) \oplus (C \cap K)] = (B \cap K) \oplus C$, so C is a direct summand of M.

(5)$\Longrightarrow$(2). Let $M = B \oplus A$ where B and A are submodules of M with $A \in \mathscr{A}$, and let $f\colon B \longrightarrow A$ be an epimorphism. Then $B \subseteq^{\oplus} M$, $M/(\mathrm{Ker} f \oplus A) = (B \oplus A)/(\mathrm{Ker} f \oplus A) \cong B/\mathrm{Ker} f \cong A \in \mathscr{A}$, and $M = B + (\mathrm{Ker} f \oplus A)$. By (5), $(\mathrm{Ker} f \oplus A) \subseteq^{\oplus} M$, so $\mathrm{Ker} f \subseteq^{\oplus} M$, and thus $\mathrm{Ker} f \subseteq^{\oplus} B$.

(2)$\Longrightarrow$(6). We need to show that if $M = B \oplus B' = C \oplus C' = B + C = B + C'$, where B, B', C, C' are submodules of M, and $M/B \in \mathscr{A}$, then $B \cap C$ is a direct summand of M. Let $\pi_{B'}\colon M \longrightarrow B'$ and $\pi_C\colon M \longrightarrow C$ be the natural projections, and let $f = (\pi_{B'}\pi_C)_B\colon B \longrightarrow B'$. Then we have $\pi_{B'}(B + C) = \pi_{B'}(B + C'), \pi_C(B + C) = \pi_C(B + C')$, so $\pi_{B'}(C) = \pi_{B'}(C')$ and $C = \pi_C(B)$, and hence $B' = \pi_{B'}(M) = \pi_{B'}(C) + \pi_{B'}(C') = \pi_{B'}(C) + \pi_{B'}(C) = \pi_{B'}(C) = \pi_{B'}(\pi_C(B))$. This shows that f is epic. It is easy to check that $\mathrm{Ker} f = (B \cap C) \oplus (B \cap C')$. Note that $B' \cong M/B \in \mathscr{A}$, and by (2), $\mathrm{Ker} f \subseteq^{\oplus} B$, and therefore $B \cap C \subseteq^{\oplus} M$.

(6)$\Longrightarrow$(2). Let $M = B \oplus C$ where B and C are submodules of M with $M/B \in \mathscr{A}$, and let $f\colon B \longrightarrow C$ be an epimorphism. Then $M = \langle f \rangle \oplus C = \langle f \rangle + B$ by Lemma 12 (1). By (6), $\langle f \rangle \cap B \subseteq^{\oplus} M$. But $\mathrm{Ker} f = \langle f \rangle \cap B$ by Lemma 12 (2), so $\mathrm{Ker} f \subseteq^{\oplus} M$, and it shows that $\mathrm{Ker} f \subseteq^{\oplus} B$.

(2)$\Longrightarrow$(7). Let B and C be direct summands of M with $M/B \in \mathscr{A}$, $M = B + C$, and $B \overset{\sigma}{\cong} C$. Write $M = B \oplus B'$ where $B' \subseteq M$. Then $B' \in \mathscr{A}$, and the isomorphism $B' \cong M/B = (B + C)/B \cong C/(B \cap C)$ induces an epimorphism $f\colon C \longrightarrow B'$ with $\mathrm{Ker} f = B \cap C$. Then $f\sigma\colon B \longrightarrow B'$ is an epimorphism with $\mathrm{Ker}(f\sigma) = \sigma^{-1}(\mathrm{Ker} f) = \sigma^{-1}(B \cap C)$. Set $X = \sigma^{-1}(B \cap C)$. Then $\sigma(X) = B \cap C$, and by (2), $X \subseteq^{\oplus} B$. So, $B = X \oplus Y$ for some submodule Y of B. Now $C = \sigma(B) = \sigma(X \oplus Y) = \sigma(X) \oplus \sigma(Y) = (B \cap C) \oplus \sigma(Y)$. Note that C is a direct summand of M, and we have that $B \cap C$ is a direct summand of M.

(7)$\Longrightarrow$(2). Let $M = B \oplus A$ where B and A are submodules of M with $A \in \mathscr{A}$, and let $f\colon B \longrightarrow A$ be an epimorphism. Then $M = \langle f \rangle \oplus A = \langle f \rangle + B$ by Lemma 12 (1), and $\mathrm{Ker} f = \langle f \rangle \cap B$ by Lemma 12 (2). Since $\langle f \rangle \subseteq^{\oplus} M, M = B + \langle f \rangle$, $B \cong M/A \cong \langle f \rangle$, and $M/B \cong A \in \mathscr{A}$, by (7), $\langle f \rangle \cap B \subseteq^{\oplus} M$, so $\langle f \rangle \cap B \subseteq^{\oplus} B$, i.e., $\mathrm{Ker} f \subseteq^{\oplus} B$. □

Theorem 17. *Let $\mathscr{A}$ be a class of right R-modules that is closed under isomorphism and direct summands, and let M be a right R-module. Then the following conditions are equivalent:*

(1) M is an $\mathscr{A}$-D4 module.

(2) If $M = B \oplus A$ for some submodules B and A where $A \in \mathscr{A}$ and $f\colon B \longrightarrow A$ is an R-homomorphism such that $\mathrm{Im} f \subseteq^{\oplus} A$, then $\mathrm{Ker} f \subseteq^{\oplus} B$.

Proof

(1)$\Longrightarrow$(2). Let $M = B \oplus A$ where B and A are submodules of M with $A \in \mathscr{A}$, and let $f\colon B \longrightarrow A$ be an R-homomorphism with $\mathrm{Im} f \subseteq^{\oplus} A$. We need to show that $\mathrm{Ker} f \subseteq^{\oplus} B$. Write $A = A_1 \oplus \mathrm{Im} f$ where $A_1 \subseteq A$. Then by hypothesis, $M = B \oplus A = (B \oplus A_1) \oplus \mathrm{Im} f$ and $\mathrm{Im} f \in \mathscr{A}$. Let $\pi\colon B \oplus A_1 \longrightarrow B$ be the natural projection, and then $f\pi\colon B \oplus A_1 \longrightarrow \mathrm{Im} f$ is an epimorphism with

$\mathrm{Ker}(f\pi) = \mathrm{Ker} f \oplus A_1$. Since M is an $\mathscr{A}$-D4 module, by Theorem 16 (2). $\mathrm{Ker} f \oplus A_1 \subseteq^{\oplus} B \oplus A_1$, so $\mathrm{Ker} f \subseteq^{\oplus} M$, and hence $\mathrm{Ker} f \subseteq^{\oplus} B$, as required.

(2) $\Longrightarrow$ (1). It follows from Theorem 17 (2). □

Corollary 18. *If $\mathscr{A}$ is closed under isomorphisms and direct summands, M is an $\mathscr{A}$-D4 module, and $M \in \mathscr{A}$, then M is a D4 module.*

Proof. It follows from Theorem 17. □

Corollary 19. *Every cyclic (resp., finitely generated, finitely presented, semisimple, injective, and flat) P-D4 (resp., F-D4, FP-D4, Soc-D4, Inj-D4, and Flat-D4) module is a D4 module.*

Recall that a right R-module M is called simple-direct-projective [9, Proposition 2.1 (2), Definition 2.2] if $M = M_1 \oplus M_2$ with M_2 simple, and $f\colon M_1 \longrightarrow M_2$ is an R-homomorphism, then $\mathrm{Ker} f \subseteq^{\oplus} M_1$.

Example 2. By Theorem 16 (2), a module M is simple-direct-projective if and only if it is S-D4. By [9, Corollary 2.8 (2)] and Theorem 5, a module M is simple-direct-projective if and only if it is S-D2. So, in the case of $\mathscr{A}$ being the class of all simple right R-modules, $\mathscr{A}$-Di modules are the same for $i = 2, 3, 4$.

Proposition 20

(1) *A direct summand of an $\mathscr{A}$-D4 module is again an $\mathscr{A}$-D4 module.*

(2) *If $M \oplus M$ is an $\mathscr{A}$-D4 module, then M is an $\mathscr{A}$-D2 module.*

(3) *Let $M = B \oplus A$ be an $\mathscr{A}$-D4 module, $A \in \mathscr{A}$. If there exists an epimorphism $f\colon B \longrightarrow A$, then A is an $\mathscr{A}$-D2 module.*

Proof

(1) Let M be an $\mathscr{A}$-D4 module, $K \subseteq^{\oplus} M$, and $M = K \oplus N$. Suppose $K = B \oplus A, A \in \mathscr{A}$ and $f\colon B \longrightarrow A$ is an epimorphism. Then $M = (B \oplus N) \oplus A, A \in \mathscr{A}$, and $f\pi\colon B \oplus N \longrightarrow A$ is an epimorphism with $\mathrm{Ker}(f\pi) = \mathrm{Ker} f \oplus N$, where $\pi\colon B \oplus N \longrightarrow B$ is the natural projection. Since M is an $\mathscr{A}$-D4 module, $\mathrm{Ker}(f\pi) \subseteq^{\oplus} B \oplus N$, i.e., $(\mathrm{Ker} f \oplus N) \subseteq^{\oplus} B \oplus N$. Write $B \oplus N = (\mathrm{Ker} f \oplus N) \oplus L$. Then $M = (B \oplus N) \oplus A = \mathrm{Ker} f \oplus (N \oplus L \oplus A)$ and so $\mathrm{Ker} f \subseteq^{\oplus} B$ by the modular law. This follows that K is an $\mathscr{A}$-D4 module.

(2) Suppose that $M \oplus M$ is an $\mathscr{A}$-D4 module. Let $A \in \mathscr{A}$, $A, B \subseteq^{\oplus} M$, and $B \overset{f}{\longrightarrow} A \longrightarrow 0$ be exact. We need to prove that f splits. Since $A, B \subseteq^{\oplus} M$, $(B \oplus A) \subseteq^{\oplus} M \oplus M$. But $M \oplus M$ is an $\mathscr{A}$-D4 module, and by (1), $B \oplus A$ is also $\mathscr{A}$-D4, and so f splits, as required.

(3) Since M is an $\mathscr{A}$-D4 module and $f\colon B \longrightarrow A$ is an epimorphism, $\mathrm{Ker} f \subseteq^{\oplus} B$. Write $B = \mathrm{Ker} f \oplus C$. Then $C \cong A$. So $A \oplus A \cong C \oplus A \subseteq^{\oplus} M$. By (1), $A \oplus A$ is an $\mathscr{A}$-D4 module. So, by (2), A is an $\mathscr{A}$-D2 module. □

Theorem 21. *The following statements are equivalent for a ring R:*

(1) *Every $A \in \mathscr{A}$ is projective.*

(2) *Every right R-module is an $\mathscr{A}$-D4 module.*

(3) *Every $A \in \mathscr{A}$ is an $\mathscr{A}$-D4 module, and every direct sum of two $\mathscr{A}$-D4 modules is an $\mathscr{A}$-D4 module.*

Moreover, if every $A \in \mathscr{A}$ is n-generated, then the above conditions are equivalent to

(4) *Every 2n-generated right R-module is an $\mathscr{A}$-D4 module.*

Proof

(1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (4) are clear.

(3) $\Longrightarrow$ (1). Let $A \in \mathscr{A}$. Then there exists an epimorphism $f\colon P \longrightarrow A$, where P is projective. By hypothesis, $P \oplus A$ is an $\mathscr{A}$-D4 module, so f is split, and hence A is projective.

(4) $\Longrightarrow$ (1). It is similar to the proof of (3) $\Longrightarrow$ (1).

Recall that that a ring R is semisimple Artinian if and only if every simple module is projective, a ring R is a quasi-Frobenius ring if and only if every injective right R-module is projective, a ring R is right perfect if and only if every flat right R-module is projective, a ring R is von Neumann regular if and only if every finitely presented right R-module is projective, a ring R is right hereditary if every right ideal I of R is projective, a ring R is right semihereditary if every finitely generated right ideal I of R is projective, a ring R is called right PP if every principal right ideal I of R is projective. Based on these facts, by Theorem 21 and Corollary 18, we have the following corollaries. □

Corollary 22

(1) *A ring R is a semisimple Artinian ring if and only if every 2-generated right R-module is a simple-direct-projective module if and only if every direct sum of two simple-direct-projective modules is a simple-direct-projective module.*

(2) *A ring R is a quasi-Frobenius ring if and only if every right R-module is an Inj-D4 module if and only if every injective right R-module is a D4 module and every direct sum of two Inj-D4 modules is an Inj-D4 module.*

(3) *A ring R is a right perfect ring if and only if every right R-module is an Flat-D4 module if and only if every flat right R-module is a D4 module and every direct sum of two Flat-D4 modules is a Flat-D4 module.*

(4) *A ring R is a von Neumann regular ring if and only if every right R-module is an FP-D4 module if and only if every finitely presented right R-module is a D4 module and every direct sum of two FP-D4 modules is a FP-D4 module.*

(5) *A ring R is a right hereditary ring if and only if every right R-module is an I-D4 module if and only if every*

right ideal is a D4 module and every direct sum of two I-D4 modules is an I-D4 module.

(6) A ring R is a right semihereditary ring if and only if every right R-module is an FI-D4 module if and only if every finitely generated right ideal is a D4 module and every direct sum of two FI-D4 modules is an FI-D4 module.

(7) A ring R is a right PP ring if and only if every right R-module is a PI-D4 module if and only if every principal right ideal is a D4 module and every direct sum of two PI-D4 modules is an PI-D4 module.

Definition 23. An R-epimorphism $\varphi\colon P \longrightarrow M$ is called an $\mathscr{A}$-D4 cover of the right R-module M, if P is an $\mathscr{A}$-D4 module, and $\mathrm{Ker}\varphi \ll P$. If $\mathscr{A}$ is the class of all cyclic (resp., finitely generated) right R-modules, then an $\mathscr{A}$-D4 cover is called a P-D4 cover (resp., F-D4 cover).

Theorem 24. *The following statements are equivalent for a ring R:*

(1) R is semiperfect.

(2) Every finitely generated right R-module has a D4 cover.

(3) Every finitely generated right R-module has a P-D4 cover.

(4) Every 2-generated right R-module has a D4 cover.

(5) Every 2-generated right R-module has a P-D4 cover.

Proof

$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (5)$ and $(1) \Longrightarrow (4) \Longrightarrow (5)$ are clear.

$(5) \Longrightarrow (1)$. We need only to show that every cyclic right R-module M has a projective cover. Let $g\colon R \longrightarrow M$ be a epimorphism, $f\colon P \longrightarrow R \oplus M$ be a P-D4 cover of $R \oplus M$, and $\pi_1\colon R \oplus M \longrightarrow R$ be the natural projection. Then $\pi_1 f\colon P \longrightarrow R$ is epic. Since R is projective, there exists a homomorphism $\lambda\colon R \longrightarrow P$ such that $(\pi_1 f)\lambda = 1_R$, and so $P = \mathrm{Im}\lambda \oplus K$, where $K = \mathrm{Ker}(\pi_1 f)$. Let $\pi_2\colon R \oplus M \longrightarrow M$ be the natural projection and $h = \pi_2 f \mid K\colon K \longrightarrow M$. Then for any $m \in M$, there is $p = \lambda(r) + k$,where $r \in R$ and $k \in K$, such that $(0, m) = f(p) = f\lambda(r) + f(k)$, so $0 = \pi_1(0, m) = \pi_1 f\lambda(r) + \pi_1 f(k) = r + 0 = r$, and this follows that $(0, m) = f(p) = f(k)$ and hence $m = \pi_2 f(k) = h(k)$. Thus, h is epic. Moreover, it is easy to see that $\mathrm{Ker}h = K \cap \mathrm{Ker}f$. Next we show that $\mathrm{Ker}h \ll K$. To see this, let $X + \mathrm{Ker}h = K$ for some submodule $X \subseteq K$. We need to show that $X = K$. Since $P = \mathrm{Im}\lambda \oplus K = (\mathrm{Im}\lambda \oplus X) + \mathrm{Ker}h \subseteq (\mathrm{Im}\lambda \oplus X) + \mathrm{Ker}f \subseteq P$, we have that $P = (\mathrm{Im}\lambda \oplus X) + \mathrm{Ker}f$. But $\mathrm{Ker}f \ll P$, we infer that $P = \mathrm{Im}\lambda \oplus X$, and then $X = K$. Now, we show that K is projective. Since $h\colon K \longrightarrow M$ is an epimorphism and $g\colon R \longrightarrow M$ is a homomorphism, by the projectivity of R, there exists a homomorphism $\varphi\colon R \longrightarrow K$ such that $g = h\varphi$. It is easy to check that $K = \mathrm{Im}\varphi + \mathrm{Ker}h$, so $K = \mathrm{Im}\varphi$ is cyclic and φ is epic. Note that $R \oplus K \cong \mathrm{Im}\lambda \oplus K = P$ is P-D4, φ is split, and so K is projective. Thus, $h\colon K \longrightarrow M$ is a projective cover of M. Therefore, R is semiperfect.

Recall that a ring R is called semiregular [1] if, for any $a \in R$, there exists $e^2 = e \in aR$ such that $(1 - e)a \in J(R)$. By [1, Theorem 2.9], a ring R is semiregular if and only if every finitely presented right R-module has a projective cover. □

Theorem 25. *The following statements are equivalent for a ring R:*

(1) R is semiregular.

(2) Every finitely presented right R-module has an F-D4 cover.

Proof

$(1) \Longrightarrow (2)$. It is clear.

$(2) \Longrightarrow (1)$. We need only to show that every finitely presented right R-module M has a projective cover. Let $g\colon F \longrightarrow M$ be an epimorphism with F a finitely generated free right R-module. Then $F \oplus M$ is again finitely presented. If $f\colon P \longrightarrow F \oplus M$ is an F-D4 cover of $F \oplus M$ and $K = \mathrm{Ker}(\pi_1 f)$ where $\pi_1\colon F \oplus M \longrightarrow F$ is the natural projection, then we can use an argument that similar to the proof of Theorem 24 to show that $\pi_2 f \mid K\colon K \longrightarrow M$ is a projective cover of M, where $\pi_2\colon F \oplus M \longrightarrow M$ is the natural projection. Therefore, R is semiregular. □

*Example 3. L*et R be a von Neumann regular ring but not a semisimple Artinian ring, and let $\mathscr{A} = \{A\colon A \cong R/Ra \text{ for some } a \in R\}$. Then by Theorem 21, an $\mathscr{A}$-D4 module need not be a D4 module. Moreover, by Corollary 22, we can obtain a series of $\mathscr{A}$-D4 modules which are not D4 modules for some different classes of modules $\mathscr{A}$.

Question 26. Is there an $\mathscr{A}$-D4 module which is not an $\mathscr{A}$-D3 module?

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

This research was supported by the Natural Science Foundation of Zhejiang Province, China (LY18A010018).

References

[1] W. K. Nicholson, "Semiregular modules and rings," *Canadian Journal of Mathematics*, pp. 1105–1120, 1976.
[2] S. Mohamed, B. J. Müller, and S. Singh, "Quasi-dual-continuous modules," *Journal of the Australian Mathematical Society*, vol. 39, pp. 287–299, 1985.
[3] M. Yousif, I. Amin, and Y. Ibrahim, "D3-modules," *Communications in Algebra*, vol. 42, pp. 578–592, 2014.
[4] N. Q. Ding, Y. Ibrahim, M. F. Yousif, and Y. Q. Zhou, "D4-modules," *Journal of Algebra and Its Applications*, vol. 16, p. 25, 2017.
[5] Z. M. Zhu, "Generalizations of C3 modules and C4 modules," *Mathematical Reports*, vol. 25, no. 75, pp. 187–197, 2023.
[6] W. K. Nicholson and M. F. snd Yousif, "Quasi-frobenius rings," *Cambridge Tracts in Math*, 2003.
[7] T. C. Quynh, "On pseudo semi-projective modules," *Turkish Journal of Mathematics*, vol. 37, pp. 27–36, 2013.
[8] D. Keskin Tütüncü, S. H. Mohamed, and N. Orhan, "Mixed injective modules," *Glasgow Mathematical Journal*, vol. 52, pp. 111–120, 2010.
[9] Y. Ibrahim, M. T. Kosan, T. C. Quynh, and M. Yousif, "Simple-direct-projective modules," *Communications in Algebra*, vol. 44, pp. 5163–5178, 2016.