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Research Article

A (Discrete) Homotopy Theory for Geometric Spaces

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We define the concepts of homotopy and fundamental group for geometric spaces as a generalization of metric spaces, digital spaces, and graphs; then, we compare them with corresponding concepts in these spaces. Also, we state some properties of the fundamental group of geometric spaces and some theorems to calculate them.

1. Introduction and Preliminaries

"Homotopy equivalence relation" and "fundamental group" are two famous and helpful concepts in mathematics; these concepts and related concepts are initially used as a tool to describe topological spaces via the properties of the fundamental groups of them in the category of groups. Many benefits of this tool have motivated researchers to adapt it to other spaces, such as metric spaces [1], digital spaces [2–5], and graphs [6–9].

Since in modeling, a composition of some different spaces is needed, it will be helpful to adapt these tools for a space that includes graphs, digital spaces, and metric spaces. In digital images, there are two different sets of points, black points and wight points, and each of them can be considered by its own adjacency relation (see [10]). However, studying a subject in real life requires more than two sets and relations for classifications. A good example is a medical image of the body, which usually includes images of various components, such as bone, muscle, cartilage, and blood vessels. A string of colored light bulbs is a simple example that consists of colored bubbles, LEDs inside the bubbles, wires, LEDs inside the rope, and so on (Figure 1). A simple model of a string of colored light bubbles in two dimension by the geometric space is similar to e in Figure 2.

Another thing that motivates us to study geometric spaces is that defining a geometric space to model a subject in real life is more accessible than other spaces. For example, in the modeling of protein as the most important of cells in [11], the definition of the relevant topology seems to be not accurate, and in the continuation of the work, the modeling was completed with approximate numerical methods.

Freni [12] introduced the concept of "geometric space" when calculating the alpha relation in hypergroups (which was first called the "gamma relation" and is the smallest equivalence relation that makes an abelian group from a hypergroup). Afterwards, Torabi Ardakani and Pourhaghani [13], while extending the concept of the "topological hypergroup," added the concept of "good morphism" as a map between geometric spaces. In this paper, we adapt the tool of fundamental groups and related concepts for geometric spaces as a generalization of those of metric spaces, digital spaces, and graphs.

Recall from [12] that a *geometric space* is a pair $(S, \mathscr{B})$ such that S is a nonempty set whose elements are called *points* and $\mathscr{B}$ is a nonempty family of subsets of S whose elements are called *blocks*. If C is a subset of S, then it is called a $\mathscr{B}$*-part* of S if $B \cap C \neq \varnothing$ implies $B \subseteq C$ for every $B \in \mathscr{B}$. For a subset $X \subseteq S$, the intersection of all $\mathscr{B}$-parts of S containing X is denoted by $\Gamma(X)$. There are five examples of geometric spaces shown in Figure 2.

In Figure 2, the set of points of geometric space a is the black circle, and its blocks are four arcs of the black circle which are featured by colored closed curves. The set of points of the geometric space b is the brown circle, and its blocks are five arcs of the brown circle, which are featured by colored

FIGURE 1: A string of colored light bulbs.

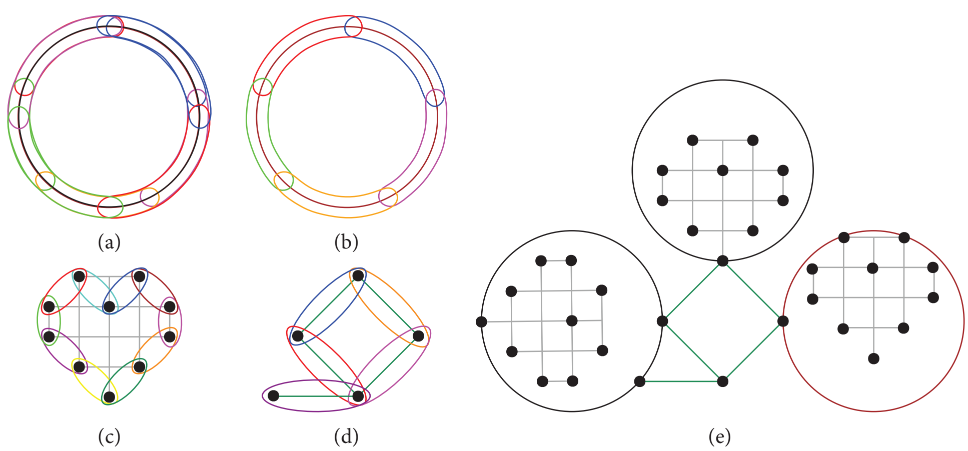


FIGURE 2: Some examples of geometric spaces: the sets of points of geometric spaces a and b are subsets of $\mathbb{R}^2$; the set of points of geometric space c is a subset of $\mathbb{Z}^2$; d is a graph; e is a composition of shapes $a, b, c,$ and d.

closed curves. The set of points of the geometric space c is the black points as a subset of $\mathbb{Z}^2$, and its blocks are featured by colored closed curves. The graph d is a geometric space, where the set of its points is the set of vertices of graph d, and each block of it is the vertices of both sides of the corresponding edge. The shape e is a composition of shapes $a, b, c,$ and d; its blocks are similar to the blocks of the corresponding shape in geometric spaces $a, b, c,$ and d.

The n-tuple $(B_1, B_2, \ldots, B_n)$ of blocks of a geometric space $(S, \mathscr{B})$ is called a polygonal if $B_i \cap B_{i+1} \neq \varnothing$ for each $1 \leq i < n$. By the concept of polygonal, Freni [12] defined the relation $\approx$ as follows.

$x \approx y \Leftrightarrow x = y$ or there exists a polygonal $(B_1, B_2, \ldots, B_n)$ such that $x \in B_1$ and $y \in B_n$.

The relation $\approx$ is an equivalence and coincides with the transitive closure of the following relation:

$$x \sim y \Leftrightarrow x = y \text{ or there exists } B \in \mathscr{B} \text{ such that } \{x, y\} \subseteq B. \tag{1}$$

Hence, $\approx$ is equal to $\cup_{n \geq 1} \sim^n$, where $\sim^n = \underbrace{\sim \circ \sim \circ \ldots \circ \sim}_{n \text{ times}}$ [12]. $x \sim^n y$ for $x, y \in S$ means that there exist some $z_1, \ldots, z_{n-1} \in S$ such that $x \sim z_1 \sim z_2 \sim \ldots \sim z_{n-1} \sim y$.

Recall from [13] that a map $f\colon (S_1, \mathscr{B}_1) \longrightarrow (S_2, \mathscr{B}_2)$ between the geometric spaces is called a good morphism if $x \sim y$ implies that $f(x) \sim f(y)$ for all $x, y \in S_1$.

Proposition 1 (see [13]). *Let $f\colon (S_1, \mathscr{B}_1) \longrightarrow (S_2, \mathscr{B}_2)$ be a good morphism between geometric spaces; then, $x \approx y$ yields $f(x) \approx f(y)$ for all $x, y \in S_1$.*

In this paper, we first define the concept of "homotopy" between good morphisms in geometric spaces and compare it with the concepts of "discrete homotopy" in metric spaces, "digital homotopy" in digital spaces, and "graph homotopy" in graphs; then, we present the concept of the "fundamental group" of geometric spaces and some properties of it; then, we state some theorems and propositions to find the fundamental groups of some geometric spaces up to isomorphism (Theorems 40 and 57, Proposition 47, and Corollaries 41, 51, and 62). In the final section, we present the relation between the discrete fundamental group, digital fundamental group, and graph homotopy group with the fundamental group of the corresponding geometric space.

2. Homotopic Geometric Spaces

In this section, we define the concept of homotopy between some maps in geometric spaces and state some of its properties. First, we introduce some concepts and properties, which are necessary to define the homotopy. To have some examples, we must recall some information from other objects in mathematics, such as metric spaces, digital spaces, and graph theory.

For every x in a geometric space $(S, \mathscr{B})$, the $\approx$-class of x is called the connected component of x in S. The geometric space S is called connected if it has one unique connected component (see [12]). In other words, a geometric space $(S, \mathscr{B})$ is called connected if and only if for each pair of different points $x, y \in S$, there exists a polygonal $(B_1, \ldots, B_n)$ of S such that $x \in B_1$ and $y \in B_n$. A component of the geometric space S is the maximal connected subset of S. Clearly, the connected component containing x is equal to $\Gamma(x) := \Gamma(\{x\})$ (see Proposition 2.1 in [9]).

In the following, we recall some concepts of digital spaces and some properties of them. For more information, see [14, 15].

Let $\mathbb{Z}$ be the set of integers, and let $n \in \mathbb{N}$. Two distinct points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2 \ldots, y_n)$ in $\mathbb{Z}^n$ are called *κ-adjacent* for $1 \leq \kappa \leq n$ if there exist at most κ distinct indices i such that $|x_i - y_i| = 1$ and $x_j = y_j$ for other indices j. *A(n) (n-dimensional) digital image*, denoted by (X, κ), is a subset $X \subseteq \mathbb{Z}_n$, where κ is an adjacency relationship on X. A digital image $X \subseteq \mathbb{Z}^n$ with adjacency relation κ is called κ-connected if and only if for every pair of different points $x, y \in X$, there is a set $\{z_1, \ldots, z_m\}$ of points in X such that $x = z_1$, $y = z_m$, and z_i and z_{i+1} are κ-adjacent for $i = 1, \ldots, m-1$. A κ-component of a digital image X is a maximal κ-connected subset of X. The function $f\colon (X_1, \kappa_1) \longrightarrow (X_2, \kappa_2)$ between digital images is called (κ_1, κ_2)-continuous if it preserves the adjacency; see [15].

Example 1

(1) Let (X, d) be a metric space, let $r \in \mathbb{R}$, and let $\mathscr{N}_r(X)$ be the family of maximal connected subsets $U \subseteq X$ such that $d(x, y) \leq r$ for all $x, y \in U$; then, $(X, \mathscr{N}_r(X))$ is a geometric space. Clearly, the connected component containing x is equal to $\Gamma(x)$ for all $x \in X$.

(2) Let (X, κ) be a(n) (n-dimensional) digital image, and let $\mathscr{A}(X)$ be the family of subsets $A \subseteq X$ such that a and b are κ-adjacent for all $a, b \in A$; then, $(X, \mathscr{A}(X))$ is a geometric space, and the adjacency relation κ in the digital image (X, κ) coincides with the relation $\sim$ in the corresponding geometric space $(X, \mathscr{A}(X))$. Clearly, the κ-component of X containing $x \in X$ is equal to $\Gamma(x)$ in the geometric space $(X, \mathscr{A}(X))$.

(3) Let (V, E) be a graph; then, (V, E) is a geometric space such that the relation "$\sim$" in geometric spaces coincides with adjacency in the graph. Clearly, the connected component containing x is equal to $\Gamma(x)$ for all $x \in V$.

Clearly, a geometric space $(S, \mathscr{B})$ is a digital image $(S, \sim)$ if $S \subseteq \mathbb{Z}^n$ for some $n \in \mathbb{N}$, and it is a graph if each $B \in \mathscr{B}$ has two elements.

Definition 2. Let $\mathbb{Z}$ be the set of integers. The (standard) geometric space, denoted by $\mathscr{Z}$, is the geometric space $(\mathbb{Z}, \mathscr{B})$, where $\mathscr{B} = \{\{m, m+1\} \mid m \in \mathbb{Z}\}$.

Recall from metric spaces that a function $f\colon (X_1, d_1) \longrightarrow (X_2, d_2)$ is called r-Lipschitz for $r > 0$ if $d_2(f(x), f(y)) \leq r d_1(x, y)$ for all $x, y \in X_1$.

Recall from the graph theory that a map $f\colon (V_1, E_1) \longrightarrow (V_2, E_2)$ is called a graph map if $f\colon V_1 \longrightarrow V_2$ is a map and $\{u, v\} \in E_1$ yields $f(u) = f(v)$ or $\{f(u), f(v)\} \in E_2$. Let $v_1 \in V_1$ and let $v_2 \in V_2$; then, a based graph map $f\colon (V_1, E_1, v_1) \longrightarrow (V_2, E_2, v_2)$ is a graph map $f\colon (V_1, E_1) \longrightarrow (V_2, E_2)$ such that $f(v_1) = v_2$.

According to Example 1, any metric space induces more than one geometric space and any digital image and any graph induce a geometric space. In the following proposition, we consider the relation between Lipschitz functions of metric spaces, continuous functions of digital images, and graph maps with good morphisms of corresponding geometric spaces.

Proposition 3

(1) Let $r > 0$ and let $f\colon (X_1, d_1) \longrightarrow (X_2, d_2)$ be an r-Lipschitz between metric spaces; then, for $r_1, r_2 > 0$ such that $r \leq r_2/r_1$, the induced map $\widetilde{f}(X_1, \mathscr{N}_{r_1}(X_1)) \longrightarrow (X_2, \mathscr{N}_{r_2}(X_2))$ defined by $\widetilde{f}(x) = f(x)$ for all $x \in X_1$ is a good morphism.

(2) The map $f\colon (X_1, \kappa_1) \longrightarrow (X_2, \kappa_2)$ between digital images is a (κ_1, κ_2)-continuous function if and only if the induced map $F\colon (X_1, \mathscr{A}(X_1)) \longrightarrow (X_2, \mathscr{A}(X_2))$ defined by $F(x) = f(x)$ for all $x \in X_1$ is a good morphism.

(3) The map $f\colon (V_1, E_1) \longrightarrow (V_2, E_2)$ is a graph map if and only if it is a good morphism.

Proof

(1) Since $\widetilde{f}(x) = f(x)$ for all $x \in X_1$, $\widetilde{f}$ is well defined. Let $x, y \in X_1$, and let $x \sim y$; then, there exists a block $\mathscr{N}_{r_1}(x) \in \mathscr{N}_{r_1}(X_1)$ such that $\{x, y\} \subseteq \mathscr{N}_{r_1}(x)$, so $d_1(x, y) \leq r_1$. Since f is an r-Lipschitz function, $d_2(f(x), f(y)) \leq r d_1(x, y) \leq r r_1 \leq r_2$; therefore, $f(y) \in \mathscr{N}_{r_2}(f(x)) \in \mathscr{N}_{r_2}(X_2)$; thus, $\widetilde{f}(x) \sim \widetilde{f}(y)$, which implies that $\widetilde{f}$ is a good morphism.

(2) Since $F(x) = f(x)$, F is well defined if and only if f is well defined. By Example 1, f preserves the adjacency relation if and only if F preserves the relation $\sim$ in the corresponding geometric spaces, which completes the proof.

(3) It is straightforward. □

Proposition 4 (see [16]). *Let $f\colon (S_1, \mathscr{B}_1) \longrightarrow (S_2, \mathscr{B}_2)$ and $g\colon (S_2, \mathscr{B}_2) \longrightarrow (S_3, \mathscr{B}_3)$ be good morphisms between geometric spaces; then, $g \circ f\colon (S_1, \mathscr{B}_1) \longrightarrow (S_3, \mathscr{B}_3)$ is a good morphism.*

Let $(S_i, \mathscr{B}_i)$ be a geometric space for $i = 1, \ldots, n$; then, $(\prod_{i=1}^n S_i, \prod_{i=1}^n \mathscr{B}_i)$ is a geometric space, where $\prod_{i=1}^n \mathscr{B}_i = \{\prod_{i=1}^n B_i \mid B_i \in \mathscr{B}_i, \text{for } i = 1, \ldots, n\}$ and $\Gamma(X) = \Gamma(\prod_{i=1}^n X_i) = \prod_{i=1}^n \Gamma(X_i)$ (see [13]).

Proposition 5. *Let $F\colon (S_1, \mathscr{B}_1) \times (S_2, \mathscr{B}_2) \longrightarrow (S_3, B_3)$ be a good morphism; then, $F_{x_1}\colon (S_2, \mathscr{B}_2) \longrightarrow (S_3, \mathscr{B}_3)$ by $F_{x_1}(y) = F(x_1, y)$ is a good morphism for all $x_1 \in S_1$ and $F_{y_2}\colon (S_1, \mathscr{B}_1) \longrightarrow (S_3, \mathscr{B}_3)$ by $F_{y_2}(x) = F(x, y_2)$ is a good morphism for all $y_2 \in S_2$.*

Proof. Let $x_1 \in S_1$. To show that F_{x_1} is a good morphism, let $y_1, y_2 \in S_2$ such that $y_1 \sim y_2$; then, $(x_1, y_1) \sim (x_1, y_2)$. Since F is a good morphism, $F_{x_1}(y_1) = F(x_1, y_1) \sim F(x_1, y_2) = F_{x_1}(y_2)$. Similarly, F_{y_2} is a good morphism for all $y_2 \in S_2$. □

Proposition 6. *Let $f\colon (S_1, \mathscr{B}_1) \longrightarrow (S_3, \mathscr{B}_3)$ and $g\colon (S_2, \mathscr{B}_2) \longrightarrow (S_4, \mathscr{B}_4)$ be good morphisms between geometric spaces; then, $(f, g)\colon S_1 \times S_2 \longrightarrow S_3 \times S_4$ is a good morphism.*

Proof. Let $(x_i, y_i) \in S_1 \times S_2$ for $i = 1, 2$, and let $(x_1, y_1) \sim (x_2, y_2)$; then, $x_1 \sim x_2$ and $y_1 \sim y_2$. Therefore, $f(x_1) \sim f(x_2)$ and $g(y_1) \sim g(y_2)$ since f and g are good morphisms; thus, $(f(x_1), g(y_1)) \sim (f(x_2), g(y_2))$; hence, $(f, g)(x_1, y_1) \sim (f, g)(x_2, y_2)$. □

Definition 7 (see [16]). Let $(S, \mathscr{B})$ be a geometric space with $\varnothing \neq K \subseteq S$; then, the geometric space $(K, \mathscr{C})$ is called a *geometric subspace* if $\varnothing \neq \mathscr{C} \subseteq \mathscr{B}$. The *induced geometric subspace* of K, denoted by $K \leqslant S$, is the geometric subspace $(K, \mathscr{B}(K))$, where $\mathscr{B}(K) = \{B \in \mathscr{B} \mid B \subseteq K\}$. If there is no ambiguity, we delete the word "geometric." The subspace $(\varnothing, \varnothing)$ is the induced subspace of $\varnothing$, and we call it the trivial subspace.

For example, $(\Gamma(x), \mathscr{B}[x])$ is an induced subspace of the geometric space $(S, \mathscr{B})$ for each $x \in S$, where $\mathscr{B}[x]$ is the family of blocks of $\mathscr{B}$ that are contained in $\Gamma(x)$.

Definition 8. Let $m, n \in \mathbb{Z}$ such that $m \leq n$. A countable interval $[m, n]_{\mathbb{Z}}$ is the induced geometric subspace of $\{m, m+1, \ldots, n\} \leqslant \mathscr{Z}$. For conveniently, the countable interval $[0, m]_{\mathbb{Z}}$ is denoted by $\mathscr{I}_m$.

Definition 9. Let $(S, \mathscr{B})$ be a geometric space, and let $x, y \in S$. A *path* from x to y in the geometric space S is a good morphism $f\colon \mathscr{I}_m \longrightarrow S$ for some $m \in \mathbb{Z}$, such that $f(0) = x$ and $f(m) = y$; moreover, $f(0)$ and $f(m)$ are, respectively, called initial and final points of f. If $f(0) = f(m)$, then f is called a *loop* based at $f(0)$. If f is a constant map, then it is called a trivial loop. Furthermore, $f\colon \mathscr{I}_m \longrightarrow S$ is called a simple path of length m when $f(i) \sim f(j)$ if and only if $j = i - 1$ or $j = i + 1$. If f is a loop, then f is called a simple loop.

Proposition 10. *Let $(S, \mathscr{B})$ be a geometric space, and let $x, y \in S$. There exists a path from x to y if and only if $x \approx y$.*

Proof. Let $x \approx y$; then, there exists a polygonal $(B_1, \ldots, B_n)$ of S such that $x \in B_1$ and $y \in B_n$. Define $f\colon \mathscr{I}_m \longrightarrow S$ by $f(0) = x$, $f(n) = y$, and $f(i) = z_i$ for $i = 1, \ldots, n-1$, where $z_i \in B_i \cap B_{i+1}$ is arbitrary. Clearly, f is a good morphism; thus, f is a path from x to y. In this case, f is called a *path corresponded to polygonal* $(B_1, \ldots, B_n)$.

Conversely, let $f\colon \mathscr{I}_m \longrightarrow S$ be a path from x to y; then, f is a good morphism; thus, $x = f(0) \sim f(1) \sim \cdots \sim f(m) = y$; hence, $x \approx y$.

Let $f\colon \mathscr{I}_m \longrightarrow S$ be a path from x to y in a geometric space $(S, \mathscr{B})$; then, $x = f(0) \sim f(1) \sim \cdots \sim f(m) = y$; hence, there exists $B_i \in \mathscr{B}$ such that $\{f(i-1), f(i)\} \subseteq B_i$ for $i = 1, \ldots, m$; thus, $(B_1, \ldots, B_m)$ is a polygonal; this polygonal is called corresponded to path f. □

Lemma 11 (Gluing lemma). *Let $(S_i, \mathscr{B}_i)$ be a geometric space for $i = 1, 2$, and let $K_1, K_2 \subseteq S_1$ such that for each $x, y \in K_1 \cup K_2$, if $x \sim y$, then there exists $B \in \mathscr{B}_1$ such that $\{x, y\} \subseteq B \in \mathscr{B}(K_i)$ for some $i = 1, 2$; then, the map $f\colon K_1 \cup K_2 \longrightarrow (S_2, \mathscr{B}_2)$ defined by the following:*

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in K_1, \\ f_2(x), & \text{if } x \in K_2, \end{cases} \quad (2)$$

is a good morphism if $f_i\colon (K_i, \mathscr{B}(K_i)) \longrightarrow (S_2, \mathscr{B}_2)$ is a good morphism for $i = 1, 2$ and $f_1(x) = f_2(x)$ for all $x \in K_1 \cap K_2$.

Proof. Let f_i be a good morphism for $i = 1, 2$. Since $f_1(x) = f_2(x)$ for all $x \in K_1 \cap K_2$, f is well defined. To show that f is a good morphism, let $x, y \in K_1 \cup K_2$ and $x \sim y$. By the hypotheses, there exists a block $B \in \mathscr{B}_1$ such that $\{x, y\} \subseteq B \in \mathscr{B}(K_i)$ for some $i = 1, 2$, so $f_i(x) \sim f_i(y)$. Since f_i is a good morphism, $f(x) \sim f(y)$. □

Definition 12. Let $(S_i, \mathscr{B}_i)$ be a geometric space for $i = 1, 2$, and let $f, g\colon S_1 \longrightarrow S_2$ be good morphisms. If for some $n \in \mathbb{N}$, there exists a good morphism $F\colon S_1 \times \mathscr{I}_n \longrightarrow S_2$ such that $F(x, 0) = f(x)$ and $F(x, n) = g(x)$ for all $x \in S_1$, then f and g are called *homotopic* and denoted by $F\colon f \simeq g$. The map F is called a *homotopy* between f and g. A good morphism f is called *null-homotopic* if f is homotopic to a constant map.

Recall from [1] that r-Lipschitz functions $f, g\colon (X_1, d_1) \longrightarrow (X_2, d_2)$ between metric spaces are called *r-discrete homotopic* if there exist $m \in \mathbb{N}$ and an r-Lipschitz function $F\colon X_1 \times \{0, \ldots, m\} \longrightarrow X_2$ such that $F(x, 0) = f(x)$ and $F(x, m) = g(x)$ for all $x \in X_1$, where $\{0, \ldots, m\}$ is equipped with the metric $d(a, b) = |a - b|$ for all

$a, b \in \{0, \ldots, m\}$ and the Cartesian product $X_1 \times \{0, \ldots, m\}$ is equipped with the l^1-metric.

Proposition 13. *Let $r > 0$, and let r-Lipschitz functions f, g: $(X_1, d_1) \longrightarrow (X_2, d_2)$ between metric spaces be r-discrete homotopic; then, the induced good morphisms $\tilde{f}, \tilde{g}$: $(X_1, \mathcal{N}_{r_1}(X_1)) \longrightarrow (X_2, \mathcal{N}_{r_2}(X_2))$ are homotopic for some $r_1, r_2 > 0$ such that $r \leq r_2/r_1 + 1$.*

Proof. Let f and g be r-discrete homotopic; then, there exist $m \in \mathbb{N}$ and an r-Lipschitz function F: $(X_1 \times \{0, \ldots, m\}, l^1) \longrightarrow (X_2, d_2)$ such that $F(x, 0) = f(x)$ and $F(x, m) = g(x)$ for all $x \in X_1$. Clearly, a suitable induced geometric space of $(\{0, \ldots, m\}, |\cdot|)$ is $\mathcal{I}_m$. Define $\tilde{F}$: $(X_1, \mathcal{N}_{r_1}(X_1)) \times \mathcal{I}_m \longrightarrow (X_2, \mathcal{N}_{r_2}(X_2))$ by $\tilde{F}(x, t) = F(x, t)$. By Proposition 3, $\tilde{F}$ is a good morphism and the hypothesis implies that $\tilde{F}(x, 0) = \tilde{f}(x)$ and that $\tilde{F}(x, m) = \tilde{g}(x)$, which completes the proof.

Recall from [2] that for each $a, b \in \mathbb{Z}$ with $a < b$, a *digital interval* $[a, b]_{\mathbb{Z}}$ is the digital image $(\{a, \ldots, b\}, 2)$. Two (κ_1, κ_2)-continuous functions f, g: $(X_1, \kappa_1) \longrightarrow (X_2, \kappa_2)$ between digital images are called *digital (κ_1, κ_2)-homotopic* if there exist $m \in \mathbb{N}$ and a function F: $(X_1, \kappa_1) \times [0, m]_{\mathbb{Z}} \longrightarrow (X_2, \kappa_2)$ that satisfy the following properties:

(1) $F(x, 0) = f(x)$ and $F(x, m) = g(x)$ for all $x \in X_1$

(2) The induced function F_x: $[0, m]_{\mathbb{Z}} \longrightarrow (X_2, \kappa_2)$ by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is $(2, \kappa_2)$-continuous for all $x \in X_2$

(3) The induced function F_t: $(X_1, \kappa_1) \longrightarrow (X_2, \kappa_2)$ by $F_t(x) = F(x, t)$ for all $x \in X_1$ is (κ_1, κ_2)-continuous for all $t \in [0, m]_{\mathbb{Z}}$

In this case, F is called *digital (κ_1, κ_2)-homotopy.* □

Proposition 14. *Let f, g: $(X_1, \kappa_1) \longrightarrow (X_2, \kappa_2)$ be (κ_1, κ_2)-continuous functions between digital images. If the induced good morphisms $\tilde{f}, \tilde{g}$: $(X_1, \mathcal{A}(X_1)) \longrightarrow (X_2, \mathcal{A}(X_2))$ by $\tilde{f}(x) = f(x)$ and $\tilde{g}(x) = g(x)$ for all $x \in X_1$ are homotopic, then f and g are digital (κ_1, κ_2)-homotopic.*

Proof. Let $\tilde{F}$: $(X_1, \mathcal{A}(X_1)) \times \mathcal{I}_m \longrightarrow (X_2, \mathcal{A}(X_2))$ be the homotopy map between $\tilde{f}$ and $\tilde{g}$. Define F: $(X_1, \kappa_1) \times [0, m]_{\mathbb{Z}} \longrightarrow (X_2, \kappa_2)$ by $F(x, t) = \tilde{F}(x, t)$ for all $x \in X_1$ and $t \in [0, m]_{\mathbb{Z}}$. By Proposition 5, the induced functions F_x and F_t preserve the adjacency relation for all $x \in X_1$ and $t \in [0, m]_{\mathbb{Z}}$, which completes the proof.

The inverse of the above proposition is not necessarily true because for $(x_1, t_1), (x_2, t_2) \in (X_1, \kappa_1) \times [0, m]_{\mathbb{Z}}$ such that $x_1 \neq x_2$ and $t_1 \neq t_2$, x_1 and x_2 are κ_1-adjacent, t_1 and t_2 are 2-adjacent, and $F(x_1, t_1)$ and $F(x_2, t_2)$ necessarily are not adjacent. □

Definition 15. Let f, g: $S_1 \longrightarrow S_2$ be good morphisms and let $K \subseteq S_1$; then, f and g are called *relative homotopic* with respect to K, denoted by $f \simeq_K g$, if there exists a homotopy map F: $S_1 \times \mathcal{I}_n \longrightarrow S_2$ such that $F(k, t) = f(k) = g(k)$ for all $k \in K$ and $t \in \mathcal{I}_n$; moreover, F is called *relative homotopy.*

Recall from the graph theory that the Cartesian product $(V_1, E_1) \times (V_2, E_2)$ is the graph with vertex set $V_1 \times V_2$ and edges $\{(u_1, u_2), (v_1, v_2)\}$ if either $u_1 = v_1$ and $\{u_2, v_2\} \in E_2$ or $u_2 = v_2$ and $\{u_1, v_1\} \in E_1$.

The Cartesian product of graphs necessarily does not induce the Cartesian product of the induced geometric spaces, because for $(v_1, t_1), (v_2, t_2) \in (V_1, E_1) \times I_m$ such that $v_1 \neq v_2$, $t_1 \neq t_2$, $\{v_1, v_2\} \in E_1$, and $\{t_1, t_2\} \in \{\{i-1, i\} | i = 1, \ldots, m\}$, but necessarily $\{(x_1, t_1), (x_2, t_2)\}$ is not an edge.

Recall from [7] that based graph maps f, g: $(V_1, E_1, v_1) \longrightarrow (V_2, E_2, v_2)$ are called *G-homotopic* if there exist $m \in \mathbb{N}$ and a graph map F: $(V_1, E_1) \times I_m \longrightarrow (V_2, E_2)$ such that $F(x, 0) = f(x)$ and $F(x, m) = g(x)$ for all $x \in V_1$ and $F(v_1, i) = v_2$ for all $i \in \{0, \ldots, m\}$, where I_m is the graph $(\{0, \ldots, m\}, \{\{i-1, i\} | i = 1, \ldots, m\})$.

Proposition 16. *Let f, g: $(V_1, E_1, v_1) \longrightarrow (V_2, E_2, v_2)$ be based graph maps. If the induced good morphisms $\tilde{f}, \tilde{g}$: $(V_1, E_1) \longrightarrow (V_2, E_2)$ defined by $\tilde{f}(x) = f(x)$ and $\tilde{g}(x) = g(x)$ for all $x \in V_1$ are relative homotopic with respect to $\{v_1\}$, then f and g are G-homotopic.*

Proof. Let $\tilde{F}$: $(V_1, E_1) \times \mathcal{I}_m \longrightarrow (V_2, E_2)$ be the homotopy map between $\tilde{f}$ and $\tilde{g}$. Define F: $(V_1, E_1) \times I_m \longrightarrow (V_2, E_2)$ by $F(x, t) = \tilde{F}(x, t)$ for all $x \in V_1$ and $t \in I_m$. Proposition 3 completes the proof. □

Theorem 17. *Let S_i be a geometric space for $i = 1, 2, 3$, let good morphisms f_1, f_2: $S_1 \longrightarrow S_2$ be homotopic under F: $S_1 \times \mathcal{I}_m \longrightarrow S_2$, and let good morphisms g_1, g_2: $S_2 \longrightarrow S_3$ be homotopic under G: $S_2 \times \mathcal{I}_n \longrightarrow S_3$; then, $g_1 \circ f_1$ is homotopic to $g_2 \circ f_2$.*

Proof. Define H: $S_1 \times \mathcal{I}_{n+m} \longrightarrow S_3$ by

$$H(x, t) = \begin{cases} g_1 \circ F(x, t), & \text{if } t \in [0, m]_{\mathcal{I}}, \\ G(f_2(x), t - m), & \text{if } t \in [m, m+n]_{\mathcal{I}}. \end{cases} \tag{3}$$

By Proposition 4, $g_1 \circ F$ is a good morphism, and by Proposition 6, $G(f_2(x), t - m)$ is a good morphism; moreover, $g_1 \circ F(x, m) = g_1 \circ f_2(x) = G(f_2(x), 0)$ for all $x \in S_1$, so by Lemma 11, H is a well-defined good morphism. Indeed $H(x, 0) = g_1 \circ F(x, 0) = g_1 \circ f_1(x)$ and $H(x, m+n) = G(f_2(x), n) = g_2 \circ f_2(x)$ for all $x \in S_1$, which completes the proof. □

Corollary 18. *Let S_i be a geometric space for $i = 1, 2$, let good morphisms f, g: $S_1 \longrightarrow S_2$ be homotopic under F: $S_1 \times \mathcal{I}_m \longrightarrow S_2$, and let $K \leqslant S_1$; then, $f|_K$ and $g|_K$ are homotopic.*

Proof. Let i: $K \longrightarrow S_1$ be the inclusion map; then, i is a good morphism and is homotopic to itself; hence, by Theorem 17, $f \circ i, g \circ i$: $K \longrightarrow S_2$ are homotopic; thus, $f|_K$ and $g|_K$ are homotopic. □

Proposition 19. *The homotopy relation between good morphisms from a geometric space S_1 to a geometric space S_2 is an equivalence relation.*

Proof. Let $f, g, h\colon S_1 \longrightarrow S_2$ be good morphisms between geometric spaces.

Reflexivity: clearly, $F\colon S_1 \times \mathcal{I}_m \longrightarrow S_2$ defined by $F(x,t) = f(x)$ is a good morphism; thus, $F\colon f \simeq f$. □

2.1. Symmetry. Let $F\colon S_1 \times \mathcal{I}_m \longrightarrow S_2$ be a homotopy map between f and g; then, by Proposition 4, $G\colon S_1 \times \mathcal{I}_n \longrightarrow S_2$ defined by $G(x,t) = F(x, n-t)$ is a good morphism; $G(x,0) = F(x,m) = g(x)$ and $G(x,m) = F(x,0) = f(x)$ for all $x \in S_1$; thus, $G\colon g \simeq f$.

2.2. Transitivity. Let f be homotopic to g under $F\colon S_1 \times \mathcal{I}_m \longrightarrow S_2$, and let g be homotopic to h under $G\colon S_1 \times \mathcal{I}_n \longrightarrow S_2$. Define $H\colon S_1 \times \mathcal{I}_{m+n} \longrightarrow S_2$ by

$$H(x,t) = \begin{cases} F(x,t), & \text{if } t \in [0,m]_{\mathbb{Z}}, \\ G(x,t-m), & \text{if } t \in [m, m+n]_{\mathbb{Z}}. \end{cases} \quad (4)$$

We know that $F(x,m) = g(x) = G(x,0)$, so by using Lemma 11, H is a well-defined good morphism, but $H(x,0) = F(x,0) = f(x)$ and $H(x, m+n) = G(x,n) = h(x)$ for all $x \in S_1$; thus, $H\colon f \simeq h$.

3. Fundamental Group

In this section, we define the concept of "fundamental group" of a geometric space and state some properties of it. First, we introduce some concepts and properties, which are necessary to define it.

Definition 20. Let $f\colon \mathcal{I}_m \longrightarrow (S,\mathcal{B})$ and $g\colon \mathcal{I}_n \longrightarrow (S,\mathcal{B})$ be two paths in geometric space $(S,\mathcal{B})$ with $f(m) = g(0)$. The product $f \star g\colon \mathcal{I}_{m+n} \longrightarrow (S,\mathcal{B})$ is defined as

$$f \star g(t) = \begin{cases} f(t), & \text{if } t \in [0,m]_{\mathbb{Z}}, \\ g(t-m), & \text{if } t \in [m, m+n]_{\mathbb{Z}}. \end{cases} \quad (5)$$

Lemma 11 immediately yields the following corollary.

Corollary 21. *The product of two paths in a geometric space is a path.*

Proposition 22. *The product of paths in a geometric space is associative.*

Proof. Let $f\colon \mathcal{I}_m \longrightarrow S$, $g\colon \mathcal{I}_n \longrightarrow S$, and $h\colon \mathcal{I}_l \longrightarrow S$ be paths in geometric space S, such that $f(m) = g(0)$ and $g(n) = h(0)$; then, by the definition of the product of two paths, $f \star (g \star h)$ and $(f \star g) \star h$ are equal to $f \star g \star h\colon \mathcal{I}_{m+n+l} \longrightarrow S$ defined by

$$f \star g \star h(t) = \begin{cases} f(t), & \text{if } t \in [0,m]_{\mathbb{Z}}, \\ g(t-m), & \text{if } t \in [m, m+n]_{\mathbb{Z}}, \\ h(t-(m+n)), & \text{if } t \in [m+n, m+n+l]_{\mathbb{Z}}. \end{cases} \quad (6)$$

To define the homotopy between paths, we need the following definition. □

Definition 23. Let $f\colon \mathcal{I}_m \longrightarrow S$ and $g\colon \mathcal{I}_n \longrightarrow S$ be two paths in a geometric space S. The path g is called a *trivial extension* of f if $n \geq m$ and there exist sets of paths $\{f_1, f_2, \ldots, f_p\}$ and $\{g_1, g_2, \ldots, g_q\}$ in S such that $p \leq q$, $f = f_1 \star f_2 \star \cdots \star f_p$, and $g = g_1 \star g_2 \star \cdots \star g_q$, and there exists a sequence $1 \leq i_1 < i_2 < \cdots < i_p \leq q$ such that $g_{i_j} = f_j$ for $1 \leq j \leq p$ and g_i is the trivial loop for all $i \in \{1, \ldots, q\} \backslash \{i_1, \ldots, i_p\}$.

Clearly, any path is a trivial extension of itself. Let $f\colon \mathcal{I}_m \longrightarrow S$ be a path in a geometric space S, and let $n \in \mathbb{N}$; then, $g\colon \mathcal{I}_{m+n} \longrightarrow S$ by

$$g(t) = \begin{cases} f(t), & \text{if } t \in [0,m]_{\mathbb{Z}}, \\ f(m), & \text{if } t \in [m+1, m+n], \end{cases} \quad \text{for all } t \in \mathcal{I}_{m+n}, \quad (7)$$

is a trivial extension of f.

Definition 24. Two paths $f\colon \mathcal{I}_m \longrightarrow S$ and $g\colon \mathcal{I}_n \longrightarrow S$ in a geometric space S are called *path homotopic* and denoted by $f \simeq_p g$ if there exists a trivial extension $f'\colon \mathcal{I}_r \longrightarrow S$ of f and a trivial extension $g'\colon \mathcal{I}_r \longrightarrow S$ of g such that there exists a relative homotopy map $F\colon \mathcal{I}_r \times \mathcal{I}_l \longrightarrow S$ between f' and g' with respect to $K = \{0, r\}$. In this case, F is called a *path homotopy*.

Proposition 25. *Let $f\colon \mathcal{I}_m \longrightarrow S$ be a path in a geometric space $(S,\mathcal{B})$ from x to y, and let $(B_1, \ldots, B_m)$ be a polygonal corresponded to the path f with $B_i \subseteq B_{i-1}$ (or $B_i \subseteq B_{i+1}$, resp.) for some $1 \leq i \leq m$; then, $(B_1, \ldots, B_{i-1}, B_{i-1}, B_{i+1}, \ldots, B_m)$ (or $(B_1, \ldots, B_{i-1}, B_{i+1}, B_{i+1}, \ldots, B_m)$, resp.) is a polygonal corresponded to the path f.*

Proof. Since $B_i \subseteq B_{i-1}$ (or $B_i \subseteq B_{i+1}$ resp.), $f(i) \in B_i \cap B_{i+1} \subseteq B_{i-1} \cap B_{i+1}$ (or $f(i-1) \in B_{i-1} \cap B_i \subseteq B_{i-1} \cap B_{i+1}$, resp.), so $(B_1, \ldots, B_{i-1}, B_{i-1}, B_{i+1}, \ldots, B_m)$ (or $(B_1, \ldots, B_{i-1}, B_{i+1}, B_{i+1}, \ldots, B_m)$, resp.) is a polygonal corresponded to the path f. □

Definition 26. Let $(B_1, \ldots, B_n)$ be a polygonal of geometric space $(S,\mathcal{B})$. If $B_i \subseteq B_{i+1}$ and $B_{i+1} \subseteq B_i$ for all $1 \leq i \leq n-1$, then it is called the *reduced polygonal*. If all blocks B_i of a polygonal $(B_1, \ldots, B_n)$ with $B_i \subseteq B_{i+1}$ for $1 \leq i \leq n-1$ or $B_i \subseteq B_{i-1}$ for $2 \leq i \leq n$ are omitted, then its *reduced polygonal* is obtained.

Lemma 27. *Let $f\colon \mathcal{I}_m \longrightarrow S$ be a path in a geometric space $(S,\mathcal{B})$ from x to y, and let $(B_1, \ldots, B_m)$ be a polygonal corresponded to the path f. Let $(B_{i_1}, B_{i_2}, \ldots, B_{i_n})$ for some $n \leq m$ be the reduced polygonal of $(B_1, \ldots, B_m)$. If $g\colon \mathcal{I}_{i_n} \longrightarrow S$ is a path corresponded to $(B_{i_1}, B_{i_2}, \ldots, B_{i_n})$ from x to y, then f and g are path homotopic.*

Proof. If $n = m$, then we are done. Let $n < m$. Define $g'\colon \mathcal{I}_m \longrightarrow S$ as follows: $g'(0) = x$, $g'(m) = y$, and for $1 \leq j \leq m-1$:

$$g'(j) = \begin{cases} g(j+1), & \text{if } B_j \subseteq B_{j+1}, \\ g(j-1), & \text{if } B_j \subseteq B_{j-1}, \\ g(j), & \text{otherwise.} \end{cases} \quad (8)$$

Clearly, g' is an extension of the path g. Define $F: \mathcal{I}_m \times \mathcal{I}_1 \longrightarrow S$ by $F(s,0) = f(s)$ and $F(s,1) = g'(s)$. By the structure of g and g', $g'(i) \sim f(i)$ for $0 \le i \le m$, so F is a good morphism. Indeed $f(0) = x = g'(0)$ and $f(m) = y = g'(m)$, so F is a path homotopy between f and g'; hence, $f \simeq_p g$.

The above lemma immediately yields the following proposition. □

Proposition 28. *Let $f: \mathcal{I}_m \longrightarrow S$ and $g: \mathcal{I}_n \longrightarrow S$ be paths in a geometric space $(S,\mathcal{B})$ from x to y. If reduced polygonals corresponded to f and g are equal, then f and g are path homotopic.*

By Proposition 19, immediately we have the following proposition.

Proposition 29. *The path homotopy relation between paths in a geometric space is an equivalence relation.*

Proposition 30. *Let $f: \mathcal{I}_m \longrightarrow S$ and $g: \mathcal{I}_n \longrightarrow S$ be paths in a geometric space $(S,\mathcal{B})$ from x to y. Let $(B_1,\ldots,B_m)$ and $(B_1,\ldots,B_i,A,B_{i+1},\ldots,B_m)$ for some $1 \le i < m$ be the reduced polygonals corresponded to paths f and g, respectively, such that $A \subseteq B_i \cup B_{i+1}$; then, f and g are path homotopic.*

Proof. Let $f': \mathcal{I}_m \longrightarrow S$ and $g': \mathcal{I}_{m+1} \longrightarrow S$ be paths from x to y corresponded to polygonals $(B_1,\ldots,B_m)$ and $(B_1,\ldots,B_i,A,B_{i+1},\ldots,B_m)$, respectively; hence, it follows that $f'(j) \in B_j \cap B_{j+1}$ for $1 \le j \le m-1$, that $g'(j) \in B_j \cap B_{j+1}$ for $1 \le j \le i-1$, that $g'(i) \in B_i \cap A$, that $g'(i+1) \in a \cap B_{i+1}$, and that $g'(j) \in B_{j-1} \cap B_j$ for $i+2 \le j \le m$. Define $f'': \mathcal{I}_{m+1} \longrightarrow S$ by

$$f''(j) = \begin{cases} f'(j), & \text{if } 0 \le j \le i, \\ g'(i+1), & \text{if } j = i+1, \\ f'(j-1), & \text{if } i+2 \le j \le m. \end{cases} \tag{9}$$

Since $A \subseteq B_i \cup B_{i+1}$, f'' is a well-defined good morphism, so f'' is an extension of f'. Define $F: \mathcal{I}_{m+1} \times \mathcal{I}_1 \longrightarrow S$ by $F(s,0) = f''(s)$ and $F(s,1) = g'(s)$. By the structure of f'' and g', $f''(j) \sim g'(j)$ for $0 \le j \le m+1$, so F is a good morphism, but $f''(0) = x = g'(0)$ and $f''(m+1) = y = g'(m+1)$; thus, $f'' \simeq_p g'$. Lemma 27 and Proposition 29 complete the proof. □

Lemma 31. *Let $(S,\mathcal{B})$ be a geometric space, and let $f: \mathcal{I}_1 \longrightarrow S$, $g: \mathcal{I}_2 \longrightarrow S$, and $h: \mathcal{I}_3 \longrightarrow S$ be paths from x to y; then, f, g, and h are path homotopic.*

Proof. By Proposition 29, it is enough to show that $f \simeq_p g$ and $f \simeq_p h$. Clearly, $f': \mathcal{I}_2 \longrightarrow S$ defined by $f'(0) = f'(1) = f(0) = x$ and $f'(2) = f(1) = y$ is an extension of f. Define $G: \mathcal{I}_2 \times \mathcal{I}_1 \longrightarrow S$ by $G(s,0) = f'(s)$ and $G(s,1) = g(s)$. Since g is a good morphism, $x \sim g(1) \sim y$, so G is a good morphism; thus, $f' \simeq_p g$, and hence $f \simeq_p g$.

Clearly, $f'': \mathcal{I}_3 \longrightarrow S$ defined by $f''(0) = f''(1) = f(0) = x$ and $f''(2) = f''(3) = f(1) = y$ is an extension of f. Define $H: \mathcal{I}_3 \times \mathcal{I}_1 \longrightarrow S$ by $H(s,0) = f''(s)$ and $H(s,1) = h(s)$. Since h is a good morphism, $x \sim h(1) \sim h(2) \sim y$, so H is a good morphism; thus, $f'' \simeq_p h$; therefore, $f \simeq_p h$. □

Proposition 32. *Let $(S,\mathcal{B})$ be a geometric space, and let $\mathcal{B}_1 \subseteq \mathcal{B}$. Let for each $x, y \in S$ with $x \sim y$, there exists a polygonal $(B_1,\ldots,B_n)$ in $\mathcal{B}_1$ such that $x \in B_1$, $y \in B_n$, and $n \le 3$; then, each path in $(S,\mathcal{B})$ is path homotopic to a path in $(S,\mathcal{B}_1)$ with the same initial and final points.*

Proof. Let $f: \mathcal{I}_m \longrightarrow (S,\mathcal{B})$ be a path from x to y; then, there exists a reduced polygonal $(B_1,\ldots,B_m)$ in $\mathcal{B}$ corresponded to f. Let $g: \mathcal{I}_m \longrightarrow (S,\mathcal{B})$ be a path from x to y corresponded to polygonal $(B_1,\ldots,B_m)$; then $g(i) \sim g(i+1)$ for $1 \le i \le m-1$. We can write the path $g = g_1 \star \ldots \star g_m$, where $g_i: \mathcal{I}_1 \longrightarrow (S,\mathcal{B})$ defined by $g_i(0) = g(i-1)$ and $g_i(1) = g(i)$ is a path for $1 \le i \le m$. For each $1 \le i \le m$, there are two cases: Case 1: If $g(i-1) \sim g(i)$ in $(S,\mathcal{B}_1)$, then define $h_i: \mathcal{I}_1 \longrightarrow (S,\mathcal{B}_1)$ by $h_i = g_i$. Case 2: If $g(i-1) \sim g(i)$ in $(S,\mathcal{B}_1)$, then, by the hypothesis, there exists a polygonal $(A_{i_1},\ldots,A_{i_{n_i}})$ in $\mathcal{B}_1$ such that $g(i-1) \in A_{i_1}$, $g(i) \in A_{n_i}$ and $n_i \le 3$, so $h: \mathcal{I}_{n_i} \longrightarrow (S,\mathcal{B}_1)$ is defined as a path from x to y corresponded to $(A_{i_1},\ldots,A_{i_{n_i}})$; by Lemma 31, $g_i \simeq_p h_i$ for all $1 \le i \le m$; thus, by Proposition 10, $g \simeq_p h$, where $h = h_1 \star \ldots \star h_m$ is a path in $(S,\mathcal{B}_1)$.

Immediately, the above proposition yields the following corollary. □

Corollary 33. *Let $(S,\mathcal{B})$ be a geometric space, and let $\mathcal{B}_1 \subseteq \mathcal{B}$ such that $\mathcal{B}_1$ preserves the connectedness of S (i.e., for each $x, y \in S$, if $x \sim y$ in $(S,\mathcal{B})$, then $x \approx y$ in $(S,\mathcal{B}_1)$). If $|\mathcal{B}_1| \le 3$, then each path in $(S,\mathcal{B})$ is homotopic to a path in $(S,\mathcal{B}_1)$.*

Definition 34. Let $f: \mathcal{I}_m \longrightarrow S$ and $g: \mathcal{I}_n \longrightarrow S$ be two paths in geometric space S, such that $f(m) = g(0)$. The operation $\star$ between two path homotopy classes $[f]_p$ and $[g]_p$ is defined as $[f]_p \star [g]_p = [f \star g]_p$, where $f \star g$ is the product of f and g.

Lemma 35. *Let $F: S_1 \times \mathcal{I}_m \longrightarrow S_2$ be a homotopy map between two good morphisms $f, g: S_1 \longrightarrow S_2$ and $n \in \mathbb{N}$; then $G: S_1 \times \mathcal{I}_{m+n} \longrightarrow S_2$ by*

$$G(x,t) = \begin{cases} F(x,t), & \text{if } t \in [0,m]_{\mathbb{Z}}, \\ F(x,m), & \text{if } t \in [m+1,m+n]_{\mathbb{Z}}, \end{cases} \tag{10}$$

for all $x \in S_1$ is a homotopy map between f and g. If F is a relative homotopy with respect to $K \subseteq S_1$, then G is too.

Proof. To show that G is a homotopy between f and g, first, we show that G is a well-defined good morphism. We can write G as

$$G(x,t)=\begin{cases} F(x,t), & \text{if } t\in[0,m]_{\mathbb{Z}},\\ F(x,m), & \text{if } t\in[m,m+n]_{\mathbb{Z}},\end{cases} \tag{11}$$

for all $x\in S_1$, so by Lemma 11, G is a well-defined good morphism. Indeed $G(x,0)=F(x,0)=f(x)$ and $G(x,m+n)=F(x,m)=g(x)$; thus, $G\colon f\simeq g$.

Let F be a relative homotopy with respect to $K\subseteq S_1$; then, for all $k\in K$,

$$G(k,t)=\begin{cases} F(k,t)=f(k)=g(k), & \text{if } t\in[0,m]_{\mathbb{Z}},\\ F(k,m)=g(k)=f(k), & \text{if } t\in[m+1,m+n]_{\mathbb{Z}},\end{cases} \tag{12}$$

which completes the proof. □

Proposition 36. *The operation $\star$ between path homotopy classes in a geometric space is well defined.*

Proof. Let $f_1,f_2\colon \mathcal{I}_m\longrightarrow S$ be path homotopic under $F\colon \mathcal{I}_m\times\mathcal{I}_{m'}\longrightarrow S$, and let $g_1,g_2\colon \mathcal{I}_n\longrightarrow S$ be path homotopic under $G\colon \mathcal{I}_n\times\mathcal{I}_{n'}\longrightarrow S$, such that $f_1(m)=f_2(m)=g_2(0)=g_1(0)$. Define $H\colon \mathcal{I}_{m+n}\times\mathcal{I}_l$ by

$$H(s,t)=\begin{cases} F'(s,t), & \text{if } s\in[0,m]_{\mathbb{Z}},\\ G'(s-m,t), & \text{if } s\in[m,m+n]_{\mathbb{Z}},\end{cases} \tag{13}$$

for all $t\in\mathcal{I}_l$, where $l=\max\{m',n'\}$, $F'\colon \mathcal{I}_m\times\mathcal{I}_l\longrightarrow S$, and $G'\colon \mathcal{I}_n\times\mathcal{I}_l\longrightarrow S$ are, respectively, defined by

$$\begin{aligned} F'(s,t)&=\begin{cases} F(s,t), & \text{if } s\in\left[0,m'\right]_{\mathbb{Z}},\\ F\left(s,m'\right), & \text{if } s\in\left[m',l\right]_{\mathbb{Z}},\end{cases}\\ G'(s,t)&=\begin{cases} G(s,t) & \text{if } s\in\left[0,n'\right]_{\mathbb{Z}},\\ F\left(s,n'\right) & \text{if } s\in\left[n',l\right]_{\mathbb{Z}}.\end{cases}\end{aligned} \tag{14}$$

By Lemma 35, F' and G' are path homotopy between f_1,f_2 and g_1,g_2, respectively. By Lemma 11, H is a well-defined good morphism. Indeed

$$\begin{aligned} H(s,0)&=\begin{cases} F'(s,0)=f_1(s), & \text{if } s\in[0,m]_{\mathbb{Z}},\\ G'(s-m,0)=g_1(s), & \text{if } s\in[m,m+n]_{\mathbb{Z}},\end{cases} = f_1\star g_1,\\ H(s,l)&=\begin{cases} F'(s,l)=f_2(s), & \text{if } s\in[0,m]_{\mathbb{Z}},\\ G'(s-m,l)=g_2(s), & \text{if } s\in[m,m+n]_{\mathbb{Z}},\end{cases} = f_2\star g_2,\end{aligned} \tag{15}$$

which completes the proof. □

Theorem 37. *Let S be a geometric space; then, the operation "$\star$" has the following properties.*

Associativity. Let f, g, and h be paths in the geometric space S. If $[f]_p\star([g]_p\star[h]_p)$ is defined, then $([f]_p\star[g]_p)\star[h]_p$ is defined, and they are equal.

Right and Left Identity. Let f be a path from x to y; then, $[e_x]_p\star[f]_p=[f]_p$ and $[f]_p\star[e_y]_p=[f]_p$, where e_x is a trivial loop at x.

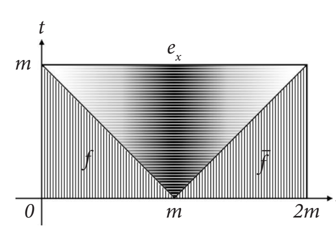
Inverse. Let $f\colon \mathcal{I}_m\longrightarrow S$ be a path from x to y. Let $\overline{f}\colon \mathcal{I}_m\longrightarrow S$ be the path defined by $\overline{f}(t)=f(m-t)$; then $[f]_p\star[\overline{f}]_p=[e_x]_p$ and $[\overline{f}]_p\star[f]_p=[e_y]_p$.

Proof. Associativity. Clearly, by the definition of operation "$\star$" and Proposition 22, $[f]_p\star([g]_p\star[h]_p)$ and $([f]_p\star[g]_p)\star[h]_p$ are equal to $[f\star g\star h]_p$.

Right and Left Identity. Since $[e_x]_p\star[f]_p=[e_x\star f]_p$ and $e_x\star f$ is an extension of f, $e_x\star f\simeq_p f$; thus, $[e_x]_p\star[f]_p=[f]_p$. Similarly, $[f]_p\star[e_y]_p=[f]_p$.

Inverse. Define $F\colon \mathcal{I}_{2m}\times\mathcal{I}_m\longrightarrow S$ by

$$F(s,t)=\begin{cases} f(s) & \text{if } s\in[0,m-t]_{\mathbb{Z}},\\ f(m-t)=e_{f(m-t)} & \text{if } s\in[m-t,m+t]_{\mathbb{Z}},\\ \bar{f}(s-m)=f(2m-s) & \text{if } s\in[m+t,2m]_{\mathbb{Z}};\end{cases}$$



By Lemma 11, F is a good morphism. Since $F(s,0) = f\star\overline{f}(s)$ and $F(s,m) = e_{f(0)}(s) = e_x(s)$, so $f\star\overline{f} \simeq e_x$. Indeed $F(0,t) = x = F(2m,t)$; hence, $f\star\overline{f} \simeq_p e_x$. Similarly, $[\overline{f}]_p \star [f]_p = [e_y]_p$. □

Definition 38. Let $(S_i, \mathscr{B}_i)$ be a geometric space, and let $x_i \in S_i$ for $i = 1,2$. Let $(K, \mathscr{C})$ be a(n) (induced) geometric subspace of S_1 such that $x_1 \in K$. The triple $(S_1, \mathscr{B}_1, x_1)$ is called a *pointed geometric space*. The triple $(K, \mathscr{C}, x_1)$ is called a(n) *(induced) pointed geometric subspace*. A map $f\colon (S_1, \mathscr{B}_1, x_1) \longrightarrow (S_2, \mathscr{B}_2, x_2)$ is called a *pointed good morphism* if $f\colon (S_1, \mathscr{B}_1) \longrightarrow (S_2, \mathscr{B}_2)$ is a good morphism and $f(x_1) = x_2$.

Immediately, by Theorem 37, we have the following corollary.

Corollary 39. *Let $(S, \mathscr{B}, x)$ be a pointed geometric space, and let $\pi_1(S, \mathscr{B}, x)$ (if there is no ambiguity $\pi_1(S, x)$) be the family of all path classes of all loops in S based at x; then, $\pi_1(S, \mathscr{B}, x)$ with operation $\star$ is a group, which is called the fundamental group or first homotopy group of the geometric space S.*

Example 2. Let (S^1, d) be a metric space, where $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ is the unit circle and $d(x,y)$ is the length of the shortest arc connecting x to y. We calculate the fundamental group of induced geometric space $(S^1, \mathscr{N}_{2r}(S^1))$ for multivalued radius $r > 0$. In general, there are two cases: Case I: the fundamental group is a trivial group and Case II: the fundamental group is isomorphic to $(\mathbb{Z}, +)$.

Case I. Let $\pi/3 < r < \pi/2$, let $\mathscr{B} = \mathscr{N}_{2r}(S^1)$, and let $\mathscr{B}_1 = \{A_1, A_2, A_3\}$, where A_i is the arc clockwisely from a_i to a_i' (which is presented in Figure 3 by colored close curve) for $i = 1,2,3$. By Corollary 33, it is enough to calculate the fundamental group of $(S^1, \mathscr{B}_1)$.

Define $f\colon \mathscr{I}_3 \longrightarrow (S^1, \mathscr{B}_1)$ by $f(0) = f(3) = N$ and $f(i) = x_i$, where $x_i \in A_i \cap A_{i+1}$ for $i = 1,2$, so f is a loop at N and (A_1, A_2, A_3) is the reduced polygonal corresponded to f. Define $F\colon \mathscr{I}_3 \times \mathscr{I}_2 \longrightarrow S^1$ by

$$F(s,t) = \begin{cases} N, & \text{if } t = 0, \\ f(s), & \text{if } t = 1, \end{cases} \tag{16}$$

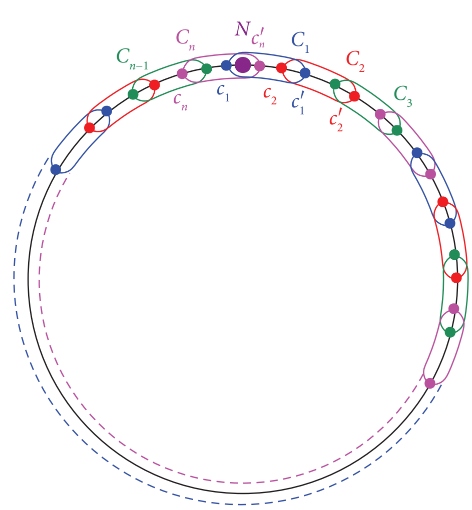
for all $s \in \mathscr{I}_3$. Since $N \sim x_1, N \sim x_2$, and $x_1 \sim x_2$, F is a good morphism, but $F(s,0) = e_N(s)$ and $F(s,t) = f(s)$ for all $s \in \mathscr{I}_3$; thus, $e_N \simeq_p f$. Since every nonconstant simple loop at N in $(S^1, \mathscr{B}_1)$ has the reduced polygonal (A_1, A_2, A_3) (or (A_3, A_2, A_1) resp.), then, by Proposition 28, every nonconstant simple loop at N in $(S^1, \mathscr{B}_1)$ is in the path homotopy class $[f]_p$ (or $[\overline{f}]_p$); hence, $\pi_1(S^1, N)$ is a trivial group.

By a similar argument, $\pi_1(S^1, N)$ is a trivial group when $r \geq \pi/3$.

Case II. Let $\pi/4 < r < \pi/3$, let $\mathscr{B}' = \mathscr{N}_{2r}(S^1)$, and let $\mathscr{B}_1' = \{B_1, B_2, B_3, B_4\}$, where B_i is the arc clockwisely from b_i to b_i' (which is presented in Figure 3 by colored close curve) for $i = 1,2,3,4$. For each $x, y \in S$, if $x \sim y$, then $d(x,y) \leq 2r$, so there are two cases. Case 1: there exists $1 \leq i \leq 4$ such that $\{x,y\} \subseteq B_i$. Case 2: there exists $1 \leq i \leq 4$ such that $\{x,y\} \subseteq B_i \cup B_j$, where $j = i + 1 \mod 4$; thus, by Proposition 32, it is enough to find the fundamental group of $(S^1, \mathscr{B}_1')$.

Define $g\colon \mathscr{I}_4 \longrightarrow (S^1, \mathscr{B}_1')$ by $g(0) = g(4) = N$ and $g(i) = x_i$, where $x_i \in B_i \cap B_{i+1}$ for $i = 1,2,3$, so g is a loop at N and (B_1, B_2, B_3, B_4) is the reduced polygonal corresponded to g. Since every nonconstant simple loop at N in $(S^1, \mathscr{B}_1')$ has the reduced polygonal (B_1, B_2, B_3, B_4) (or (B_4, B_3, B_2, B_1) resp.), by Proposition 28, every nonconstant simple loop at N in $(S^1, \mathscr{B}_1')$ is in the path homotopy class $[g]_p$ (or $[\overline{g}]_p$ resp.). Indeed there is not homotopy map between g and e_N, so $[g]_p \neq [e_N]_p$; thus, the fundamental group of $(S^1, \mathscr{B}_1')$ is generated by $[g]_p$; hence, $\pi_1(S^1, N) \cong (\mathbb{Z}, +)$ as groups.

By a similar argument, $\pi_1(S^1, N) \cong (\mathbb{Z}, +)$ as groups when $r = \pi/4$.



$r < \frac{\pi}{n-1}$

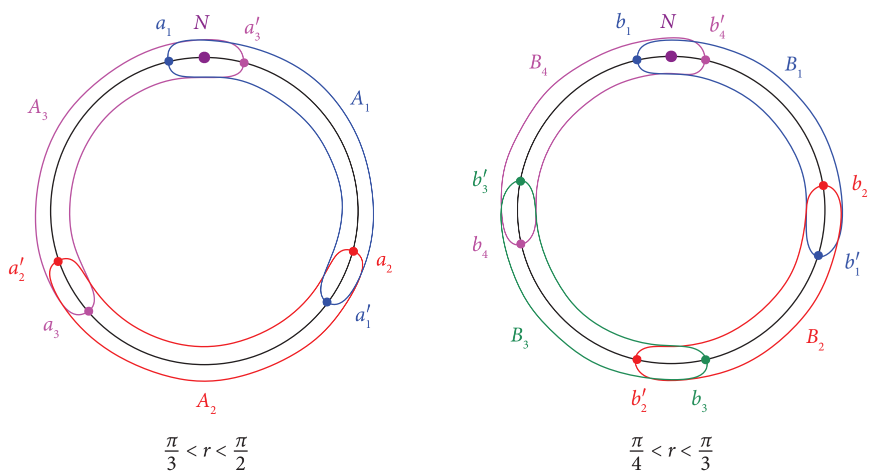


FIGURE 3: Example 2.

Let $r < \pi/n - 1$ where $n \geq 5$, let $\mathscr{B}'' = \mathscr{N}_{2r}(S^1)$, and let $\mathscr{B}_1'' = \{C_1, C_2, \ldots, C_n\}$, where C_i is the arc clockwisely from c_i to c_i' (which is presented in above figure by colored close curve) for $i = 1, 2, \ldots, n$. Now, we rewrite the argument of the case $\pi/4 < r < \pi/3$ for $r < \pi/n - 1$. Similarly, it is enough to calculate the fundamental group of $(S^1, \mathscr{B}_1'')$. Let $h\colon \mathscr{I}_n \longrightarrow (S^1, \mathscr{B}_1'')$ by $h(0) = h(n) = N$ and $h(i) = x_i$, where $x_i \in C_i \cap C_{i+1}$ for $i = 1, \ldots, n-1$, so h is a loop at N and $(C_1, C_2, \ldots, C_n)$ is the reduced polygonal corresponded to C. Since every nonconstant simple loop at N in $(S^1, \mathscr{B}_1'')$ has the reduced polygonal $(C_1, C_2, \ldots, C_n)$ (or $(C_n, C_{n-1}, \ldots, C_1)$ resp.), by Proposition 28, every nonconstant simple loop at N in $(S^1, \mathscr{B}_1'')$ is in the path homotopy class $[h]_p$ (or $[\overline{h}]_p$ resp.). Indeed there is not homotopy map between h and e_N, so $[h]_p \neq [e_N]_p$; thus, the fundamental group of $(S^1, \mathscr{B}_1'')$ is generated by $[h]_p$; hence $\pi_1(S^1, N) \cong (\mathbb{Z}, +)$ as groups.

Let $(S, \mathscr{B})$ be a geometric space, and let $x, y \in S$. If α is a path in S from x to y, then we identify $\widehat{\alpha}\colon \pi_1(S, x) \longrightarrow \pi_1(S, y)$ by $\widehat{\alpha}([f]_p) = [\overline{\alpha} \star f \star \alpha]_p$ for all $[f]_p \in \pi_1(S, x)$.

Theorem 40. *Let $(S, \mathscr{B})$ be a geometric space, and let $x, y \in S$ such that $x \approx y$; then $\pi_1(S, x) \cong \pi_1(S, y)$ as groups.*

Proof. Since $x \approx y$, by Proposition 10, there exists a path α in S from x to y. Define $\widehat{\alpha}\colon \pi_1(S, x) \longrightarrow \pi_1(S, y)$ by $\widehat{\alpha}([f]_p) = [\overline{\alpha} \star f \star \alpha]_p$ for all $[f]_p \in \pi_1(S, x)$. By Corollary 21, $\overline{\alpha} \star f \star \alpha$ is a path in S. Clearly, the initial and final points of it are y, so $[\overline{\alpha} \star f \star \alpha]_p \in \pi_1(S, y)$; thus, $\widehat{\alpha}$ is well defined.

To show that $\widehat{\alpha}$ is a group homomorphism, let $[f]_p, [g]_p \in \pi_1(S, x)$; then, by Theorem 37 and the concept of trivial extension of a path, we have

$$\begin{aligned} \widehat{\alpha}\left([f]_p\right) \star \widehat{\alpha}\left([g]_p\right) &= [\overline{\alpha} \star f \star \alpha]_p \star [\overline{\alpha} \star g \star \alpha]_p \\ &= [\overline{\alpha} \star f \star \alpha \star \overline{\alpha} \star g \star \alpha]_p \\ &= [\overline{\alpha} \star f \star g \star \alpha]_p = \widehat{\alpha}\left([f \star g]_p\right). \end{aligned} \tag{17}$$

To show that $\widehat{\alpha}$ is an isomorphism, let $\beta = \alpha$. Define $\widehat{\beta}\colon \pi_1(S, y) \longrightarrow \pi_1(S, x)$ by $\widehat{\beta}([g]_p) = [\overline{\beta} \star h \star \beta]_p = [\alpha \star g \star \overline{\alpha}]_p$ for all $[g]_p \in \pi_1(S, y)$; then, $\widehat{\beta}$ is a well-defined group homomorphism, by the similar argument of $\widehat{\alpha}$. By Theorem 37 and the concept of trivial extension of a path, we have

$$\widehat{\alpha}\left(\widehat{\beta}\left([g]_p\right)\right) = [\overline{\alpha} \star (\alpha \star g \star \overline{\alpha}) \star \alpha]_p = [g]_p, \tag{18}$$

for all $[g]_p \in \pi_1(S, y)$. Similarly, $\widehat{\beta}(\widehat{\alpha}([f]_p)) = [f]_p$, for all $[f]_p \in \pi_1(S, x)$; thus, $\widehat{\beta}$ is an inverse of $\widehat{\alpha}$, which completes the proof. □

Corollary 41. *If S is a connected geometric space, then $\pi_1(S, x) \cong \pi_1(S, y)$ as groups, for each $x, y \in S$.*

Proposition 42. *Let $(S, \mathscr{B})$ be a geometric space, and let $x, y \in S$. If α and β are two homotopic paths from x to y, then $\widehat{\alpha} = \widehat{\beta}$.*

Proof. We must show that $\widehat{\alpha}(f) = \widehat{\beta}(f)$ or $\overline{\alpha} \star f \star \alpha \simeq_p \overline{\beta} \star f \star \beta$ for all $[f]_p \in \pi_1(S, x)$. It is enough to show that if $\alpha \simeq_p \beta$, then $\overline{\alpha} \simeq_p \overline{\beta}$ and $f \star \alpha \simeq_p f \star \beta$.

Let $\alpha \simeq_p \beta$ under $F\colon \mathscr{I}_m \times \mathscr{I}_n \longrightarrow S$. Define $G\colon \mathscr{I}_m \times \mathscr{I}_n \longrightarrow S$ by $G(s, t) = F(m - s, t)$. Since F is a good morphism, G is too, but

$$\begin{aligned} G(s, 0) &= F(m - s, 0) = \alpha(m - s) = \overline{\alpha}(s), \\ G(s, n) &= F(m - s, n) = \beta(m - s) = \overline{\beta}(s), \end{aligned} \tag{19}$$

for all $s \in \mathscr{I}_m$ and

$$\begin{aligned} G(0, t) &= F(m, t) = \alpha(m) = \overline{\alpha}(0) = y = \overline{\beta}(0), \\ G(m, t) &= F(0, t) = \alpha(0) = \overline{\alpha}(m) = x = \overline{\beta}(m), \end{aligned} \tag{20}$$

for all $t \in \mathscr{I}_n$; thus, $G\colon \overline{\alpha} \simeq_p \overline{\beta}$.

Since $[\alpha]_p = [\beta]_p$, $[f \star \alpha]_p = [f]_p \star [\alpha]_p = [f]_p \star [\beta]_p = [f \star \beta]_p$. □

Theorem 43. *Let $(S, \mathscr{B})$ be a connected geometric space, and let $x, y \in S$. The group $\pi_1(S, x)$ is a commutative group if and only if $\widehat{\alpha} = \widehat{\beta}$ for all paths α and β in S from x to y.*

Proof. Let $\pi_1(S,x)$ be a commutative group, let $[f]_p \in \pi_1(S,x)$, and let α and β be paths in S from x to y. Since $[\alpha\star\overline{\beta}]_p \in \pi_1(S,x)$, the commutativity of $\pi_1(S,x)$ yields $[f]_p\star[\alpha\star\overline{\beta}]_p = [\alpha\star\overline{\beta}]_p\star[f]_p$, so $[f\star\alpha\star\overline{\beta}]_p = [\alpha\star\overline{\beta}\star f]_p$; thus, $[\overline{\alpha}\star f\star\alpha\star\overline{\beta}\star\beta]_p = [\overline{\alpha}\star\alpha\star\overline{\beta}\star f\star\beta]_p$; then, by Theorem 37, $[\overline{\alpha}\star f\star\alpha]_p = [\overline{\beta}\star f\star\beta]_p$; hence, $\widehat{\alpha}([f]_p) = \widehat{\beta}([f]_p)$.

Conversely, consider β in S from x to y, and let $[f]_p, [g]_p \in \pi_1(S,x)$; then, $f\star\alpha$ is a path in S from x to y, so by the hypothesis, $\widehat{f\star\alpha} = \widehat{\alpha}$; thus, $\widehat{f\star\alpha}([g]_p) = \widehat{\alpha}([g]_p)$; therefore, $[\overline{f\star\alpha}\star g\star f\star\alpha]_p = [\overline{\alpha}\star f\star\alpha]_p$. Since $\overline{f\star\alpha} = \overline{\alpha}\star\overline{f}$, $[\alpha\star\overline{\alpha}\star\overline{f}\star g\star f\star\alpha\star\overline{\alpha}]_p = [\alpha\star\overline{\alpha}\star f\star\alpha\star\overline{\alpha}]_p$, so by Theorem 37, $[\overline{f}\star g\star f]_p = [g]_p$; thus, $[f\star\overline{f}\star g\star f]_p = [f\star g]_p$; hence, by Theorem 37, $[g\star f]_p = [f\star g]_p$. □

Theorem 44. *Let $h: (S_1,\mathscr{B}_1,x_1) \longrightarrow (S_2,\mathscr{B}_2,x_2)$ be a pointed good morphism between pointed geometric spaces; then, $h_*: \pi_1(S_1,x_1) \longrightarrow \pi_1(S_2,x_2)$ defined by $h_*([f]_p) = [h\circ f]$ for all $[f]_p \in \pi_1(S_1,x_1)$ is a group homomorphism.*

Proof. Let $f: \mathscr{I}_m \longrightarrow S_1$ be a loop in S_1 at x_1. By Proposition 4, the map $h\circ f: \mathscr{I}_m \longrightarrow S_2$ is a good morphism, so it is a path in S_2, but $h\circ f(0) = h\circ f(m) = x_2$; then, $h\circ f$ is a loop in S_2 at x_2; thus, by Theorem 17, $[h\circ f]_p \in \pi_1(S_2,x_2)$; hence, h_* is well defined.

To show that h_* is a good homomorphism, let $[f]_p, [g]_p \in \pi_1(S_1,x_1)$; then,

$$\begin{aligned} h_*\big([f]_p\star_1[g]_p\big) &= h\circ\big([f\star_1 g]_p\big) = [h\circ f\star_2 h\circ g]_p \\ &= h_*\big([f]_p\big)\star h_*\big([g]_p\big), \end{aligned} \tag{21}$$

where $\star_i$ is the group operation of $\pi_1(S_i,x_i)$ for $i = 1,2$. □

Theorem 45. *Let $h: (S_1,\mathscr{B}_1,x_1) \longrightarrow (S_2,\mathscr{B}_2,x_2)$ and $k: (S_2,\mathscr{B}_2,x_2) \longrightarrow (S_3,\mathscr{B}_3,x_3)$ be pointed good morphisms between geometric spaces; then, $(k\circ h)_* = k_*\circ h_*$; moreover, if $id_S: (S,\mathscr{B},x) \longrightarrow (S,\mathscr{B},x)$ is the identity map in the geometric space S, then $(id_S)_* = id_{\pi_1(S,x)}$.*

Proof. Let $[f] \in \pi_1(S_1,x_1)$; then,

$$\begin{aligned} (k\circ h)_*\big([f]_p\big) &= [k\circ h\circ f]_p = [k\circ(h\circ f)]_p \\ &= k_*\big([h\circ f]_p\big) = k_*\circ h_*\big([f]_p\big). \end{aligned} \tag{22}$$

Let $h = \mathrm{id}_S: (S,\mathscr{B},x) \longrightarrow (S,\mathscr{B},x)$ be the identity map; then,

$$(\mathrm{id}_S)_*\big([f]_p\big) = [\mathrm{id}_S\circ f]_p = [f]_p, \tag{23}$$

for all $[f]_p \in \pi_1(S,x)$; thus, $(\mathrm{id}_S)_* = \mathrm{id}_{\pi_1(S,x)}$. □

Corollary 46. *Let $h: (S_1,\mathscr{B}_1,x_1) \longrightarrow (S_2,\mathscr{B}_2,x_2)$ be a bijective pointed good morphism between geometric spaces such that the inverse map $h^{-1}: (S_2,\mathscr{B}_2,x_2) \longrightarrow (S_1,\mathscr{B}_1,x_1)$ is a pointed good morphism; then, h_* is a group isomorphism.*

Proof. By Theorem 44, h_* is a group homomorphism. By Theorem 45, $h_*^{-1}\circ h_* = \mathrm{id}_{\pi_1(S_1,x_1)}$ and $h_*\circ h_*^{-1} = \mathrm{id}_{\pi_1(S_2,x_2)}$, so h_* is a group isomorphism. □

Proposition 47. *Let $(S,\mathscr{B})$ be a geometric space, and let $x \in S$; then, $\pi_1(\Gamma(x),x) \cong \pi_1(S,x)$ as groups.*

Proof. Let $i: (\Gamma(x),\mathscr{B}[x]) \longrightarrow (S,\mathscr{B})$ be the inclusion map; then, i is a good morphism; thus, by Theorem 44, $i_*: \pi_1(\Gamma(x),x) \longrightarrow \pi_1(S,x)$ is a group homomorphism. To show that i_* is one-to-one, let $[f]_p \in \pi_1(\Gamma(x),x)$ such that $i_*([f]_p) = i_*([e_x]_p)$; then, $[i\circ f]_p = [e_x]_p$, so there exists a homotopy map $F: \mathscr{I}_m\times\mathscr{I}_n \longrightarrow S$ in the geometric space S such that $F: i\circ f\simeq_p[e_x]_p$. Since F is a good morphism and $\mathscr{I}_m\times\mathscr{I}_n$ is a connected subset of $\mathscr{Z}\times\mathscr{Z}$, by Proposition 1, $F(\mathscr{I}_m\times\mathscr{I}_n)$ is a connected subset of S, but $F(0,0) = f(0) = x \in \Gamma(x)$; thus, $F(\mathscr{I}_m\times\mathscr{I}_n)\subseteq\Gamma(x)$; therefore, F is a homotopy map between $i\circ f$ and e_x in the subspace $\Gamma(x)$; hence, i_* is one-to-one.

To show that i_* is onto, let $[g]_p \in \pi_1(S,x)$; then, $g: \mathscr{I}_m \longrightarrow S$ is a loop in S at x. By Proposition 1, $g(\mathscr{I}_m)\subseteq\Gamma(x)$; thus, $[g]_p \in \pi_1(\Gamma(x),x)$; therefore, $i_*([g]_p) = [g]_p$. □

3.1. Deformation Retracts. In this section, we introduce the concept of "deformation retract," some of its properties, and some related concepts.

Definition 48. Let $(S,\mathscr{B})$ be a geometric space, let $K\leqslant S$, and let $i: K \longrightarrow S$ be the inclusion map. A *retraction* of S onto K is a good morphism $r: S \longrightarrow K$ such that $r|_K = \mathrm{id}_K$ (i.e., $r\circ i = \mathrm{id}_K$). In this case, r is called a *retract* of S. A retract r is called a *deformation retract* if $\mathrm{id}_S\simeq i\circ r$. In this case, K is called a *deformation retraction* of S. A retract r is called a *strong deformation retract* if $\mathrm{id}_S\simeq_K i\circ r$. In this case, K is called a *strong deformation retraction* of S.

Proposition 49. *Let $(S,\mathscr{B})$ be a geometric space, and let $K\leqslant S$ be a retraction of S with retract map $r: S \longrightarrow K$; then, the group homomorphism $r_*: \pi_1(S,x) \longrightarrow \pi_1(K,x)$ is onto and the group homomorphism $i_*: \pi_1(K,x) \longrightarrow \pi_1(S,x)$ is one-to-one, where $i: K \longrightarrow S$ is the inclusion map and $x \in K$.*

Proof. Let $r: S \longrightarrow K$ be a retract of S. Since $r\circ i = \underset{K}{\mathrm{id}}$, by Theorem 45, $r_*\circ i_* = (r\circ i)_* = (\mathrm{id}_K)_* = \mathrm{id}_{\pi_1(K,x)}$, but $\mathrm{id}_{\pi_1(K,x)}$ is a group isomorphism; thus, r_* is onto, and i_* is one-to-one. □

Theorem 50. *Let $h,k: (S_1,x_1) \longrightarrow (S_2,x_2)$ be two pointed good morphisms between pointed geometric spaces such that $h\simeq_{\{x_1\}}k$; then, the induced group homomorphisms $h_*,k_*: \pi_1(S_1,x_1) \longrightarrow \pi_1(S_2,x_2)$ are equal.*

Proof. Let $[f]_p \in \pi_1(S_1,x_1)$. Since $h\simeq_{\{x_1\}}k$, by Theorem 17, $h\circ f\simeq k\circ f$, but f is a loop at x_1 and $h\circ f(0) = h(x_1) = x_2 = k(x_1) = k\circ f(0)$; thus, $h\circ f\simeq_p k\circ f$, and hence $h_*([f]_p) = [h\circ f]_p = [k\circ f]_p = k_*([f]_p)$. □

Corollary 51. *Let $K \leqslant S$ be a strong deformation retract of geometric space $(S, \mathscr{B})$; then, $\pi_1(S, x) \cong \pi_1(K, x)$ as groups, for all $x \in K$.*

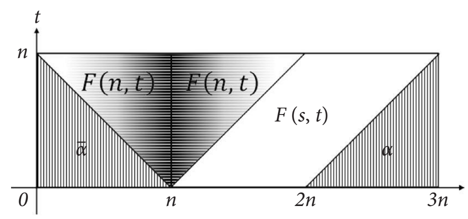
Proof. Let $i: K \longrightarrow s$ be the inclusion map, and let $r: S \longrightarrow K$ be a strong retract map of S; then, $r \circ i = \mathrm{id}_K$ and $i \circ r \simeq_K \mathrm{id}_X$, so by Theorem 45, $i_* \circ r_* = (\mathrm{id}_S)_* = \mathrm{id}_{\pi_1(S,x)}$ for all $x \in K$. Since $\mathrm{id}_{\pi_1(S,x)}$ is a group isomorphism, r_* is one-to-one; thus, by Proposition 49, $r_*: \pi_1(S, x) \longrightarrow \pi_1(K, x)$ is a group isomorphism.

In Corollary 51, if K is a singleton, then $\pi_1(S, x)$ is a trivial group. □

Theorem 52. *Let $(S_i, \mathscr{B}_i)$ be a geometric space for $i = 1, 2$, and let $h, k: (S_1, \mathscr{B}_1) \longrightarrow (S_2, \mathscr{B}_2)$ be two homotopic good morphisms under homotopy map $H: S_1 \times \mathscr{I}_n \longrightarrow S_2$. Let $x \in S_1$, and let $\alpha: \mathscr{I}_n \longrightarrow S_2$ defined by $\alpha(t) = H(x, t)$ be a path in S_2 from $h(x)$ to $k(x)$; then, the group homomorphisms $\widehat{\alpha} \circ h_*, k_*: \pi_1(S_1, x) \longrightarrow \pi_1(S_2, k(x))$ are equal.*

Proof. Let $[f]_p \in \pi_1(S_1, x)$. We must show that $\widehat{\alpha} \circ h_*([f]_p) = k_*([f]_p)$, or identically $\overline{\alpha} \star h \circ f \star \alpha \simeq_p k \circ f$. Define $F: \mathscr{I}_n \times \mathscr{I}_n \longrightarrow S_2$ by $F(s, t) = H(f(s), t)$ for all $(s, t) \in \mathscr{I}_n \times \mathscr{I}_n$. By Propositions 4 and 6, G is a good morphism, but $F(s, 0) = H(f(s), 0) = h \circ f(s)$ and $F(s, n) = H(f(s), n) = k \circ f(s)$, so $F: h \circ f \simeq k \circ f$. Define $F: \mathscr{I}_{3n} \times \mathscr{I}_n \longrightarrow S_2$ by

$$G(s,t) = \begin{cases} \bar{\alpha}(s) = \alpha(n-s) & \text{if } s \in [0, n-t]_{\mathbb{Z}}, \\ F(n,t) = H(x,t) & \text{if } s \in [n-t, n]_{\mathbb{Z}}, \\ F(n,t) = H(x,t) & \text{if } s \in [n, n+t]_{\mathbb{Z}}, \\ F(s-n-t, t) & \text{if } s \in [n+t, 2n+t]_{\mathbb{Z}}, \\ \alpha(s-2n) & \text{if } s \in [2n+t, 3n]_{\mathbb{Z}}; \end{cases}$$



Clearly, G is well defined, and by Lemma 11, G is a good morphism, but $G(s, 0) = (\overline{\alpha} \star h \circ f \star \alpha)(s)$ and $G(s, n) = (k \circ e_{f(n)} \star k \circ e_{f(n)} \star k \circ f)(s) = k \circ (e_{f(n)} \star e_{f(n)} \star f)(s)$. Since $e_{f(n)} \star e_{f(n)} \star f$ is a trivial extension of f, by Theorem 17, $k \circ (e_{f(n)} \star e_{f(n)} \star f) \simeq_p k \circ f$; thus, $\overline{\alpha} \star h \circ f \star \alpha \simeq k \circ f$, but $G(0, t) = \overline{\alpha}(0) = \alpha(n) = H(x, n) = k(x) = G(3n, t)$, which completes the proof.

In Theorem 52, if $h(x) = k(x)$, then $\alpha: \mathscr{I}_n \longrightarrow S_2$ is a loop in S_2 at $h(x)$ and $\widehat{\alpha}: \pi_1(S_2, h(x)) \longrightarrow \pi_1(S_2, h(x))$ by $\widehat{\alpha}([g]_p) = [\overline{\alpha} \star g \star \alpha]_p = [\overline{\alpha}]_p \star [g]_p \star [\alpha]_p$ for all $[g]_p \in \pi_1(S_2, h(x))$ is a group automorphism; moreover, if $h \simeq_{\{x\}} k$, then α is the trivial loop at x, so $\widehat{\alpha} = \mathrm{id}_{\pi_1(S_2, f(x))}$; thus, $h_* = k_*$, similar to Theorem 50.

Using Theorem 52, immediately, we have the following corollary. □

Corollary 53. *Let $h, k: (S_1, \mathscr{B}_1) \longrightarrow (S_2, \mathscr{B}_2)$ be homotopic good morphisms between geometric spaces, and let $x_1 \in S_1$. If $h_*: \pi_1(S_1, x_1) \longrightarrow \pi_1(S_2, h(x_1))$ is a one-to-one (onto, or trivial, resp.) group homomorphism, then $k_*: \pi_1(S_1, x_1) \longrightarrow \pi_1(S_2, k(x_1))$ is a one-to-one (onto, or trivial resp.) group homomorphism.*

Corollary 54. *Let $h: (S_1, \mathscr{B}_1, x_1) \longrightarrow (S_2, \mathscr{B}_2, x_2)$ be a null-homotopic pointed good morphism between geometric spaces; then, $h_*: \pi_1(S_1, x_1) \longrightarrow \pi_1(S_2, x_2)$ is a trivial group homomorphism.*

Proof. Since h is null-homotopic, $h \simeq c_{x_2}$ under homotopy map $H: S_1 \times \mathscr{I}_n \longrightarrow S_2$, where $c_{x_2}: (S_1, \mathscr{B}_1, x_1) \longrightarrow (S_2, \mathscr{B}_2, x_2)$ is the constant map at x_2. Define $\alpha: \mathscr{I}_n \longrightarrow S_2$ by $\alpha(t) = H(x_1, t)$, so α is a path from $h(x_1) = x_2$ to $c_{x_2}(x_1) = x_2$. By Theorem 52, $\widehat{\alpha} \circ h_* = (c_{x_2})_*$; then, for all $[f]_p \in \pi_1(S_1, x_1)$,

$$\begin{aligned}[\overline{\alpha} \star h \circ f \star \alpha]_p &= \widehat{\alpha} \circ h_*\left([f]_p\right) = \left(c_{x_2}\right)_*\left([f]_p\right) \\ &= \left[c_{x_2} \circ f\right]_p = \left[e_{x_2}\right]_p.\end{aligned} \tag{24}$$

Hence, by Theorem 37, $[h \circ f]_p = [\alpha \star e_{x_2} \star \overline{\alpha}]_p = [e_{x_2}]_p$, which completes the proof. □

3.2. Homotopy Type. In this section, we define the concept of "homotopy type" and present some of its properties.

Definition 55. A good morphism $f: S_1 \longrightarrow S_2$ between two geometric spaces is called a *homotopy equivalence* if there exists a good morphism $g: S_2 \longrightarrow S_1$ such that $f \circ g \simeq \mathrm{id}_{S_2}$ and $g \circ f \simeq \mathrm{id}_{S_1}$, where $\mathrm{id}_{S_i}: S_i \longrightarrow S_i$ is the identity map for $i = 1, 2$; moreover, g is called a *homotopy inverse* of f. If $f: S_1 \longrightarrow S_2$ is a homotopy equivalence, then S_1 and S_2 are called *homotopic equivalent* and denoted by $f: S_1 \simeq S_2$. Two spaces that are homotopy equivalent are called to have the same *homotopy type*.

Theorem 56. *The homotopy equivalence relation between geometric spaces is an equivalence relation.*

Proof. Reflexivity. Since $\mathrm{id}: (S, \mathscr{B}) \longrightarrow (S, \mathscr{B})$ is a good morphism for all geometric space S and $\mathrm{id} \circ \mathrm{id} \simeq \mathrm{id}$, $S \simeq S$. □

3.2.1. Symmetry. If $f\colon S_1 \longrightarrow S_2$ is a homotopy equivalence between two geometric spaces S_1 and S_2, then there exists a good morphism $g\colon S_2 \longrightarrow S_1$ such that $f\circ g \simeq \mathrm{id}_{S_2}$ and $g\circ f \simeq \mathrm{id}_{S_1}$, so g is a homotopy equivalence between S_2 and S_1.

3.2.2. Transitivity. If S_i is a geometric space for $i = 1, 2, 3$, $f_1\colon S_1 \simeq S_2$, and $f_2\colon S_2 \simeq S_3$, then there exist good morphisms $g_1\colon S_2 \longrightarrow S_1$ and $g_2\colon S_3 \longrightarrow S_2$ such that $g_i\circ f_i \simeq \mathrm{id}_{S_i}$ for $i = 1, 2$, $f_1\circ g_1 \simeq \mathrm{id}_{S_2}$, and $f_2\circ g_2 \simeq \underset{S_3}{\mathrm{id}}$. By Proposition 4, $f = f_2\circ f_1\colon S_1 \longrightarrow S_3$ and $g = g_1\circ g_2\colon S_3 \longrightarrow S_1$ are good morphisms, and

$$\begin{aligned} g\circ f &= (g_1\circ g_2)\circ(f_2\circ f_1) = g_1\circ(g_2\circ f_2)\circ f_1 \simeq g_1\circ \mathrm{id}_{S_2}\circ f_1 = g_1\circ f_1 \simeq \mathrm{id}_{S_1},\\ f\circ g &= (f_2\circ f_1)\circ(g_1\circ g_2) = f_2\circ(f_1\circ g_1)\circ g_2 \simeq f_2\circ \mathrm{id}_{S_2}\circ g_2 = f_2\circ g_2 \simeq \mathrm{id}_{S_3}, \end{aligned} \tag{25}$$

hence $f\colon S_1 \simeq S_3$.

Theorem 57. *Let $f\colon S_1 \longrightarrow S_2$ be a good morphism, and let $x \in S$. If f is a homotopy equivalence map, then $f_*\colon \pi_1(S_1, x) \longrightarrow \pi_1(S_2, f(x))$ is a group isomorphism.*

Proof. By Theorem 44, f_* is a group homomorphism. Since f is a homotopy equivalence map, there exists a good morphism $g\colon S_2 \longrightarrow S_1$ and homotopy maps $F\colon S_1 \times \mathcal{I}_m \longrightarrow S_2$ and $G\colon S_2 \times \mathcal{I}_n \longrightarrow S_1$ such that $F\colon g\circ f \simeq \mathrm{id}_{S_1}$ and $f\circ g \simeq \mathrm{id}_{S_2}$. Define $\alpha\colon \mathcal{I}_m \longrightarrow S_1$ by $\alpha(t) = F(x, t)$ for all $t \in \mathcal{I}_m$; then, α is a path in S_1 from x to $g\circ f(x)$, so by Theorems 45 and 52, $g_*\circ f_* = (g\circ f)_* = \widehat{\alpha}\circ(\mathrm{id}_{S_1})_* = \widehat{\alpha}$, but by the proof of Theorem 40, $\widehat{\alpha}$ is a group isomorphism; thus, f_* is one-to-one. By a similar argument, $f_*\circ g_*$ is a group isomorphism, so f_* is onto, which completes the proof.

According to Theorem 57, if two geometric spaces have the same homotopy type, then the fundamental groups of them are isomorphic in the group theory. □

3.3. Contractible Space. In this section, we define the concept of "Contractible" and present some of its properties.

Definition 58. A geometric space $(S, \mathcal{B})$ is called *contractible* if the identity map $\mathrm{id}_S\colon S \longrightarrow S$ is null-homotopic.

Theorem 59. *Geometric space S_2 is contractible if and only if each good morphism $f\colon (S_1, \mathcal{B}_1) \longrightarrow (S_2, \mathcal{B}_2)$ is null-homotopic for all geometric space $(S_1, \mathcal{B}_1)$.*

Proof. Let $(S_2, \mathcal{B}_2)$ be contractible; then, there exists $y \in S_2$ such that $\mathrm{id}_{S2} \simeq c_y$, so by Theorem 17, $\mathrm{id}_{S_2}\circ f \simeq e_y\circ \mathrm{f} = \mathrm{c_y}$; thus, f is a null-homotopic map. Conversely, let each good morphism $f\colon (S_1, \mathcal{B}_1) \longrightarrow (S_2, \mathcal{B}_2)$ be null-homotopic for all geometric space $(S_1, \mathcal{B}_1)$; then, the good morphism $\mathrm{id}_{S2}\colon S_2 \longrightarrow S_2$ is null-homotopic; thus, S_2 is contractible. □

Theorem 60. *Each contractible geometric space is connected.*

Proof. Let $(S, \mathcal{B})$ be a contractible geometric space; then, for some $x \in S$, $\mathrm{id}_S \simeq c_x$ under homotopy map $F\colon \mathcal{I}_m \longrightarrow S$. For each $y \in S$, clearly, $\alpha_y\colon \mathcal{I}_m \longrightarrow S$ defined by $\alpha_y(t) = F(y, t)$ is a path from y to x; thus, for each $y, z \in S$, $\alpha_y \circ \overline{\alpha}_z$ is a path from y to z; hence, S is connected. □

Theorem 61. *Geometric space $(S, \mathcal{B})$ is contractible if and only if S and $\{x\}$ have the same homotopy type for some $x \in S$.*

Proof. Let $(S, \mathcal{B})$ be contractible; then, there exists $y \in S$ such that $\mathrm{id}_S \simeq c_y$. Define $f\colon \{x\} \longrightarrow S$ by $f(x) = y$ and $g\colon S \longrightarrow \{x\}$ by $g(s) = x$ for all $s \in S$. Clearly, f and g are good morphisms, but $g\circ f(x) = g(y) = x = \mathrm{id}_{\{x\}}(x)$ and $f\circ g(s) = y$ for all $s \in S$; thus, $g\circ f \simeq \mathrm{id}_{\{x\}}$ and $f\circ g = c_y$, respectively, but by Theorem 56, $\mathrm{id}_{\{x\}} \simeq \mathrm{id}_{\{x\}}$, and since S is contractible, $e_y \simeq \mathrm{id}_S$; hence, S and $\{x\}$ have the same homotopy type.

Conversely, S and $\{x\}$ have the same homotopy type; then, there exist good morphisms $f\colon \{x\} \longrightarrow S$ and $g\colon S \longrightarrow \{x\}$ such that $g\circ f \simeq \mathrm{id}_{\{x\}}$ and $f\circ g \simeq \underset{S}{\mathrm{id}}$, but clearly, $g = c_x$ and $f\circ g(s) = f(x)$ for all $s \in S$; hence, $e_{\{f(x)\}} = f\circ g \simeq \mathrm{id}_S$; thus, S is contractible. □

Corollary 62. *Let $(S, \mathcal{B})$ be a contractible geometric space; then, $\pi_1(S, x)$ is the trivial group for all $x \in S$.*

Proof. By Theorem 61, S and $\{x\}$ have the same homotopy type for some $x \in S$; then, by Theorem 57, $\pi_1(S, x) = \pi_1(\{x\}, x) = \{[e_x]_p\}$. Theorems 40 and 60 and Proposition 10 complete the proof.

The converse of Theorem 60 necessarily is not true; see Example 3. □

Example 3. Consider the geometric space $(\{x, y, z, t\}, \{\{x, y\}, \{x, t\}, \{y, z\}, \{z, t\}\})$. Since $\Gamma(x) = \{x, y, z, t\}$, this geometric space is connected. Define $f\colon \mathcal{I}_4 \longrightarrow \{x, y, z, t\}$ by $f(0) = f(4) = x$, $f(1) = y$, $f(2) = z$, and $f(3) = t$; then, f is a loop in the geometric space $\{x, y, z, t\}$ at x. Clearly, the fundamental group $\pi_1(\{x, y, z, t\}, x)$ is generated by $[f]_p$; thus, $\pi_1(\{x, y, z, t\}, x) \cong (\mathbb{Z}, +)$ as groups; therefore, by Corollary 62, this geometric space is not contractible.

Definition 63. A connected geometric space $(S, \mathcal{B})$ is called *simply connected* if $\pi_1(S, x)$ is the trivial group for all $x \in S$.

By Corollary 62, each contractible geometric space is simply connected. To examine the simply connection of a geometric space, according to Theorem 40, it is enough to examine it for some $x \in S$.

Theorem 64. *A geometric space $(S, \mathscr{B})$ is simply connected if and only if any two paths having the same initial and final points are path homotopic.*

Proof. Let $x \in S$, let $\pi_1(S, x) = \{[e_x]_p\}$, and let $f\colon \mathscr{I}_m \longrightarrow S$ and $g\colon \mathscr{I}_n \longrightarrow S$ be two paths in S from x to y; then, $f \star \overline{g}$ is a loop in S at x; therefore, $[f \star \overline{g}]_p \in \pi_1(S, x) = \{[e_x]_p\}$; thus, $f \star \overline{g} \simeq_p e_x$; hence, by Theorem 37, $f \simeq_p g$.

Conversely, let $x \in S$ and $[f]_p \in \pi_1(S, x)$; then, by the hypothesis, $f \simeq_p e_x$, so $[f]_p = [e_x]_p$; thus, $\pi_1(S, x) = \{[e_x]_p\}$. □

4. The Relation between the Fundamental Group of Geometric Space with the Fundamental Group of Other Spaces

In this section, we try to find the relation between the discrete fundamental group, digital fundamental group, and graph homotopy group with the fundamental group of corresponding geometric space. First, we prove a helpful lemma in general.

Lemma 65. *Let A be a nonempty set with an operation $\circ$ on A. Let ρ_i be an equivalence relation on A. Assume that $[a]_i$ is an equivalence class of relation ρ_i for all $a \in A$ and $i = 1, 2$. Let $(A/\rho_i, \circ)$ be a group, for $i = 1, 2$, where $[a]_i \circ [b]_i = [a \circ b]_i$, for all $a, b \in A$ and $i = 1, 2$. If $[a]_1 \subseteq [a]_2$ for all $a \in A$, then there exists a surjective homomorphism from $(A/\rho_1, \circ)$ to $(A/\rho_2, \circ)$.*

Proof. Define $\varphi(A/\rho_1, \circ) \longrightarrow (A/\rho_2, \circ)$ by $\varphi([a]_1) = [a]_2$ for all $a \in A$. Since $[a]_1 \subseteq [a]_2$, for all $a \in A$, φ is well defined. To prove that φ is a group homomorphism, let $a, b \in A$, then $\varphi([a]_1 \circ [b]_1) = \varphi([a \circ b]_1) = [a \circ b]_2 = [a]_2 \circ [b]_2$. Since ρ_i is an equivalence relation on A, the set $\{[a]_i | a \in A\}$ is a partition for set A, for $i = 1, 2$. So, for each $[a]_2 \in A/\rho_2$, we have $[a_0]_1 \in A/\rho_1$ such that $\varphi([a_0]_1) = [a]_2$ because $\{[a]_1 | a \in A\}$ is a partition of A; thus, φ is surjective.

Clearly, the homomorphism φ in the above lemma is one-to-one if and only if $\rho_1 = \rho_2$. □

4.1. Graph Homotopy Groups. In this section, we discuss about the relation between graph homotopy groups with the fundamental group of geometric space. First, we recall the concept of graph homotopy groups and some related concepts and properties in graph. For more information, see [7].

Let $m \in \mathbb{N}$ and I_m be the graph $(\{0, \ldots, m\}, \{\{i-1, i\} | i = 1, \ldots, m\})$. The *distinguished base point* and the *boundary* of the 1-cube I_m of graph I_m, respectively, are $\varnothing = 0$ and $\partial I_m = \{0, m\}$. The extension of $f\colon (I_m, \{0, m\}) \longrightarrow (V, E, v_0)$ to a graph map $f'\colon (I_p, \{0, p\}) \longrightarrow (V, E, v_0)$ for $m \leq p$ is $f'(i) = f(i)$ for $i = 0, 1, \ldots, m$ and $f'(i) = v_0$ for $i > m$ [7]. Clearly, the extension of a graph map is a special case of trivial extension of a path in the induced geometric space.

The family of homotopy classes of graph maps $f\colon (I_m, \partial I_m) \longrightarrow (V, E, v_0)$ is denoted by $A_1^G(V, E, v_0)$ where $n \geq 1$. The multiplication $[f]_G * [g]_G$ on $A_1^G(V, E, v_0)$ is defined as the G-homotopy class of the map $h\colon (I_{2m}, \partial I_{2m}) \longrightarrow (V, E, v_0)$ by

$$h(i) = \begin{cases} f(i), & \text{if } i \leq m, \\ g(i-m), & \text{if } i > m. \end{cases} \tag{26}$$

Obviously, the multiplication on $A_1^G(V, E, v_0)$ coincides with the product of two homotopy classes of induced paths in the corresponding geometric space.

Proposition 66. *Let (V, E) be a graph with $v_0 \in V$; then, there is an onto group homomorphism from $\pi_1(V, E, v_0)$ to $A_1^G(V, E, v_0)$.*

Proof. Let $\tilde{f}\colon \mathscr{I}_m \longrightarrow (V, E)$ be a loop at v_0 in geometric space (V, E), so $f\colon (I_m, \{0, m\}) \longrightarrow (V, E, v_0)$ is a graph map by $f = \tilde{f}$. Let $[\tilde{f}]_p \in \pi_1(V, E, v_0)$, then, by Proposition 16, $[\tilde{f}]_p \subseteq [f]_G$, where $[f]_G \in A_1^G(V, E, v_0)$. Since the multiplication "$*$" on $A_n^G(V, E, v_0)$ coincides with the product "$\star$" on $\pi_1(V, E, v_0)$, by Lemma 65, $\varphi\colon \pi_1(V, E, v_0) \longrightarrow A_1^G(V, E, v_0)$ by $\varphi([\tilde{f}]_p) = [f]_G$, for all $[\tilde{f}] \in \pi_1(V, E, v_0)$, is an onto group homomorphism.

The following examples show that $\pi_1(V, E, v_0)$ and $A_1^G(V, E, v_0)$ sometimes are isomorphic and sometimes are not isomorphic. □

Example 4

(1) Let $(V, E) = (\{v_0, v_1, v_2, v_3\}, \{\{v_0, v_1\}, \{v_0, v_3\}, \{v_1, v_2\}, \{v_2, v_3\}\})$ be a graph; then, the induced geometric space is equal to (V, E) (Figure 4). With respect to Example (1) in page 120 and Proposition 5.10 in [7], $A_1^G(V, E, v_0)$ is a trivial group. By Example 3, $\pi_1(V, E, v_0) \cong (\mathbb{Z}, +)$ as groups.

(2) Let $(V', E') = (\{v'_0, v'_1, v'_2, v'_3, v'_4\}, \{\{v'_0, v'_1\}, \{v'_0, v'_4\}, \{v'_1, v'_2\}, \{v'_2, v'_3\}, \{v'_3, v'_4\}\})$ be a graph; then, the induced geometric space is equal to (V', E') (Figure 4). With respect to Example (3) in page 120 and Proposition 5.10 in [7], $A_1^G(V', E', v'_0) \cong (\mathbb{Z}, +)$.

By a similar argument about the fundamental group of geometric space (V, E, v_0) in Example 3, we have $\pi_1(V', E', v'_0) \cong (\mathbb{Z}, +)$ as groups.

Let (V, E) be a graph with $v_0 \in V$, and let $f, g\colon (I_m, \{0, m\}) \longrightarrow (V, E, v_0)$ be G-homotopic under G-homotopy map $F\colon I_m \times I_n \longrightarrow (V, E)$. If $\{F(t, s), F(t+1, s+1)\}, \{F(t+1, s), F(t, s+1)\} \in E$, for every $s \in I_{m-1}$ and every $t \in I_{n-1}$, then the induced map of F in corresponding geometric space is a homotopy map; thus, the homomorphism of Proposition 66 is an isomorphism.

4.2. Digital Fundamental Group. In this section, we consider the relation between the digital fundamental group and the fundamental group of geometric space. First, we recall the

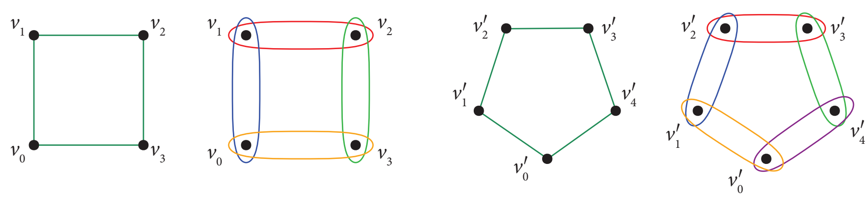


Figure 4: Example 4.

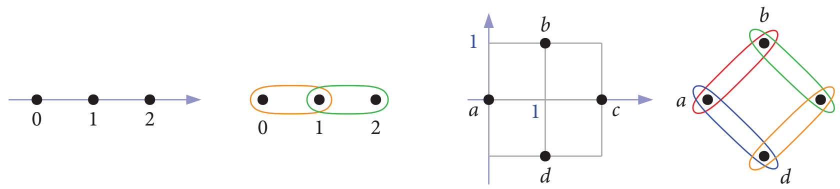


Figure 5: Example 5.

concept of digital fundamental group and some related concepts and properties in digital space. For more information, see [2, 10, 17, 18].

Recall from [2] that a *pointed digital image* is (X, κ, x_0) where (X, κ) is a digital image and $x \in X$. A pointed digital continuous function $f\colon (X, \kappa_x, x_0) \longrightarrow (y, \kappa_y, y_0)$ is a (κ_x, κ_y)-continuous function $f\colon (X, \kappa_x) \longrightarrow (y, \kappa_y)$ such that $f(x_0) = y_0$. Let $f, g\colon (X, \kappa_x, x_0) \longrightarrow (y, \kappa_y, y_0)$ be pointed (κ_x, κ_y)-continuous function. A (κ_x, κ_y)-homotopy function $H\colon (X, \kappa_x) \times [0, m]_{\mathbb{Z}} \longrightarrow (Y, \kappa_y)$, between f and g, is called a *pointed digital homotopy* if $H(x_0, t) = y_0$ for $t \in [0, m]_{\mathbb{Z}}$ [2].

A *digital κ-path* in a digital image (X, κ) is a $(2, \kappa)$-continuous function $f\colon [0, m]_{\mathbb{Z}} \longrightarrow X$. It is called a *digital κ-loop* if $f(0) = f(m)$; in this case, $f(0)$ is called *the base point of the loop* f. If f is a constant function, it is called a *trivial loop* [2].

Recall from [2, 18] that $f\colon [0, m_1]_{\mathbb{Z}} \longrightarrow X$ and $g\colon [0, m_2]_{\mathbb{Z}} \longrightarrow X$ be two paths such that $f(m-1) = g(0)$, then the *product* of f and g is the path $(f \cdot g)\colon [0, m_1 + m_2]_{\mathbb{Z}} \longrightarrow X$ with

$$(f \cdot g)(t) = \begin{cases} f(t), & \text{if } t \in [0, m_1]_{\mathbb{Z}}, \\ g(t - m_1), & \text{if } t \in [m_1, m_1 + m_2]_{\mathbb{Z}}. \end{cases} \tag{27}$$

Let f and f' be κ-loops in (X, κ, x_0). f' is called *trivial extension* of f if there exist sets of κ-paths $\{f_1, \ldots, f_k\}$ and $\{F_1, \ldots, F_l\}$ in X such that $k \leq l$, $f = f_1 \cdot \ldots \cdot f_k$, and $f' = F_1 \cdot \ldots \cdot F_l$ and there are indices $1 \leq i_1 \leq \cdots \leq i_k \leq l$ such that $F_{i_j} = f_j$ for $1 \leq j \leq k$ and F_i is a trivial loop if $i \notin \{i_1, \ldots, i_k\}$. Two κ-loops f and g are pointed digital homotopic if there exist trivial extensions f' and g' of f and g, respectively, with the same cardinality in their domains and a pointed digital homotopy between f' and g'. The homotopy class of κ-loop $f\colon [0, m]_{\mathbb{Z}} \longrightarrow (X, \kappa, x_0)$ is denoted by $[f]_X$, and the family of all homotopy classes of κ-loops in (X, κ, x_0) is denoted by $\Pi_1^{\kappa}(X, p)$. The set $\Pi_1^{\kappa}(X, p)$ is a group with product operation $[f]_X \cdot [g]_X = [f.g]_x$ is a group which is called the *digital fundamental group* [2].

Let (X, κ, x_0) be a pointed digital image, then, by Example 1, the corresponding geometric space of it is $(X, \mathscr{A}(X), x_0)$. By Proposition 3, the digital κ-path in (X, κ, x_0) coincides with a path in the corresponding geometric space $(X, \mathscr{A}(X))$. Clearly, the product of two digital κ-paths coincides with the product of two induced paths in corresponding geometric space. Obviously, the trivial extension of a digital κ-loop coincides with the trivial extension of a loop in geometric space. Now, we can identify the relation between the digital fundamental group and the fundamental group of corresponding geometric space.

Proposition 67. *Let (X, κ, x_0) be a pointed digital image; then, there is an onto group homomorphism from $\pi_1(X, \mathscr{A}(X), x_0)$ to $\Pi_1^{\kappa}(X, x_0)$.*

Proof. Let $\tilde{f}\colon \mathscr{I}_m \longrightarrow (X, \mathscr{A}(X))$ be a loop at x_0 in geometric space $(X, \mathscr{A}(X))$, so $f\colon [0, m]_{\mathbb{Z}} \longrightarrow X$ is a digital κ-loop by $f = \tilde{f}$. Let $[\tilde{f}]_p \in \pi_1(X, \mathscr{A}(X), x_0)$, then, by Proposition 14, $[\tilde{f}]_p \subseteq [f]_X$, where $[f]_X \in \Pi_1^{\kappa}(X, x_0)$. Since the multiplication "$\cdot$" on $\Pi_1^{\kappa}(X, x_0)$ coincides with the product "$\star$" on $\pi_1(X, \mathscr{A}(X), x_0)$, by Lemma 65, $\varphi\colon \pi_1(X, \mathscr{A}(X), x_0) \longrightarrow \Pi_1^{\kappa}(X, x_0)$ by $\varphi([\tilde{f}]_p) = [f]_X$, for all $[\tilde{f}]_p \in \pi_1(X, \mathscr{A}(X), x_0)$, is an onto group homomorphism.

The following examples show that $\pi_1(X, \mathscr{A}(X), x_0)$ and $\Pi_1^{\kappa}(X, x_0)$ sometimes are isomorphic and sometimes are not isomorphic. □

Example 5

(1) Consider the pointed digital image $(I_2, 0)$, then its corresponding geometric space is $(\mathscr{I}_2, 0)$ (Figure 5). By Example 5.4 in [10] and Theorem 4.16 in [2], $\Pi_1^2(I_2, 0)$ is a trivial group. Consider $F\colon \mathscr{I}_2 \times \mathscr{I}_1 \longrightarrow \mathscr{I}_2$ with $F(s, 0) = s = \mathrm{id}_{\mathscr{I}_2}(s)$ and $F(s, 1) = 1 = e_1(s)$, for $s \in \mathscr{I}_2$. Clearly, F is a good morphism, so F is a homotopy map between $\mathrm{id}_{\mathscr{I}_2}$ and e_1; thus, $\mathscr{I}_n$ is contractible; therefore, by Corollary 62, $\pi_1(\mathscr{I}_2, 0)$ is a trivial group.

(2) Let a, b, c, d be $(0,0), (1,1), (2,0), (1,-1)$, respectively. Let $(\{a,b,c,d\}, 8, (0,0))$ be a pointed digital image; then, the corresponding geometric space is $(\{a,b,c,d\}, \mathscr{B}, (0,0))$, where $\mathscr{B} = \{\{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}\}$ (Figure 5). With respect to Theorem 4.16 in page 6 in [2], $\Pi_1^8(\{a,b,c,d\}, (0,0))$ is a trivial group. By Example 3, $\pi_1(\{a,b,c,d\}, \mathscr{B}, (0,0)) \cong (\mathbb{Z}, +)$.

Let (X, κ, x_0) be a pointed digital image, and let $f, g\colon [0,m]_{\mathbb{Z}} \longrightarrow (X,\kappa)$ be digital homotopic κ-loops at x_0 under digital homotopy map $F\colon [0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}} \times I_n \longrightarrow (X,\kappa)$. If $F(t,s)$ and $F(t+1,s+1)$, and $F(t+1,s)$ and $F(t,s+1)$ are κ-adjacent, for each $s \in [0,m-1]_{\mathbb{Z}}$ and each $t \in [0,n-1]_{\mathbb{Z}}$, then the induced map of F in corresponding geometric space is a homotopy map; thus, the homomorphism of Proposition 67 is an isomorphism.

4.3. Discrete Fundamental Group. In this section, we discuss about the relation between the discrete fundamental group of metric space with the fundamental group of geometric space. First, we recall the concept of discrete fundamental group (fundamental group at scale r) and some related concepts and properties in metric space. For more information, see [1].

Let (X,d) be a metric space and $x, y, p \in X$, and let $r > 0$, recall from [1] that an *r-path* from x to y is a finite sequence of points $x_0 x_1 \cdots x_n x_{n+1}$ such that $x_0 = x$, $x_{n+1} = y$, and $d(x_i, x_{i+1}) \leq r$, for $i = 1, \ldots, n$. An r-path from x to y is called an r-loop based on p if $x = y = p$. The family of r-loops based on p is denoted by $\mathscr{C}_r(X,p)$. The *concatenation* of two r-paths $(x_0 x_1 \ldots x_n)$ and $(y_0 y_1 \ldots y_n)$ is defined as

$$(x_0 x_1 \ldots x_n)(y_0 y_1 \ldots y_n) = (x_0 x_1 \ldots x_n y_0 y_1 \ldots y_n), \tag{28}$$

$\mathscr{C}_r(X,p)$ is a monoid under concatenation of r-loops [1].

Recall from [1] that the *r-homotopy equivalence* on $\mathscr{C}_r(X,p)$ is an equivalence relation which is defined as follows:

(1) Each r-loop $(x_0 x_1 \cdots x_n)$ is r-homotopy equivalent to $(x_0 x_1 \cdots x_n x_{n+1})$, where $x_{n+1} = p$. It allows to increase the length of the sequence $(x_0 x_1 \cdots x_n)$ by adding p to the end of the sequence.

(2) Two r-loops $(x_0 x_1 \ldots x_n x_{n+1})$ and $(y_0 y_1 \ldots y_n y_{n+1})$ are r-homotopy equivalent if there is an r-homotopy grid:

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n & x_{n+1} \\ z_0^1 & z_1^1 & \cdots & z_n^1 & z_{n+1}^1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ z_0^t & z_1^t & \cdots & z_n^t & z_{n+1}^t \\ y_0 & y_1 & \cdots & y_n & y_{n+1} \end{pmatrix}, \tag{29}$$

where

(a) each row is an r-loop based on p

(b) each column is an r-path

Let $A_{1,r}(X,x)$ be a family of r-homotopy equivalence class of $\mathscr{C}_r(X,p)$; then, $A_{1,r}(X,p)$ with the operation of concatenation is a group which is called the *fundamental group at scale r*. A metric space (X,d) is called *connected at scale* $r > 0$ if for all $x, y \in X$, there are $x_0, x_1, \ldots, x_n, x_{n+1} \in X$ such that $x_0 = x$, $x_{n+1} = y$, and $d(x_i, x_{i+1}) \leq r$, for $i = 1, \ldots, n$. If (X,d) is connected at scale r, then the fundamental group $A_{1,r}(X,x)$ does not depend on x. So, it is denoted by $A_{1,r}(X)$ [1].

Proposition 68. *Let (X,d) be a metric space and $r > 0$. $(x_0 x_1 \cdots x_n x_{n+1})$ is an r-path if and only if $f\colon \mathscr{I}_{n+1} \longrightarrow X$ by $f(i) = x_i$, for $i \in \mathscr{I}_{n+1}$, is a path in the geometric space $(X, \mathscr{N}_r(X))$.*

Proof. With respect to the definition of the geometric space $(X, \mathscr{N}_r(X))$ in Example 1, $d(x,y) \leq r$ if and only if $x \sim y$ in geometric space $(X, \mathscr{N}_r(X))$, for all $x, y \in X$, which completes the proof.

Obviously, the concatenation of two r-paths in a metric space (X,d) coincides with the product of two induced paths in the corresponding geometric space $(X, \mathscr{N}_r(X))$. Increasing the length of an r-loop in $\mathscr{C}_r(X,p)$ by repeating the base point at the end of the sequence of the r-loop, in the definition of r-homotopy equivalence, is a particular case of trivial extension of the induced loop in the corresponding geometric space. □

Proposition 69. *Let (X,d) be a metric space and $p \in X$. Let $r > 0$ and $(x_0 x_1 \cdots x_n), (y_0 y_1 \cdots y_n) \in \mathscr{C}_r(X,p)$; these two r-loops are r-homotopy equivalent if and only if the induced loops in the corresponding geometric space $(X, \mathscr{N}_r(X))$ are path homotopic.*

Proof. Let $f, g\colon \mathscr{I}_n \longrightarrow (X, \mathscr{N}_r(X))$ by $f(i) = x_i$ and $g(i) = y_i$, for $i \in \mathscr{I}_n$, be the induced loops of r-loops $(x_0 x_1 \cdots x_n)$ and $(y_0 y_1 \cdots y_n)$, respectively.

Let r-loops $(x_0 x_1 \cdots x_n)$ and $(y_0 y_1 \cdots y_n)$ be r-homotopy equivalent, then, by definition of r-homotopy equivalence, there is a grid

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ z_0^1 & z_1^1 & \cdots & z_n^1 \\ \vdots & \vdots & \cdots & \vdots \\ z_0^t & z_1^t & \cdots & z_n^t \\ y_0 & y_1 & \cdots & y_n \end{pmatrix}, \tag{30}$$

such that each row is an r-loop based at p, and each column is an r-path; moreover, $x_n = y_n = p$ and $z_n^j = p$ for $j = 1, \ldots, t$. Define $F\colon \mathscr{I}_n \times \mathscr{I}_{t+1} \longrightarrow (X, \mathscr{N}_r(X))$ by $F(i,0) = x_i$, $F(i,t+1) = y_i$ and $F(i,j) = z_i^j$ for $i \in \mathscr{I}_n$ and $j = 1, \ldots, t$. By Proposition 68, each row and each column of the above grid are a path in geometric space $(X, \mathscr{N}_r(X))$, so F is a good morphism, and hence F is a path homotopy map between the loops f and g.

Conversely, let the induced loops f and g be path homotopic; then, there exists a homotopy map $F\colon \mathscr{I}_n \times \mathscr{I}_{t+1} \longrightarrow (X, \mathscr{N}_r(X))$ with $F(i,0) = f(i)$, $F(i,t+1) = g(y)$, for $i \in \mathscr{I}_n$ and $F(0,j) = F(n,j) = p$, for $j \in \mathscr{I}_{t+1}$. Define grid

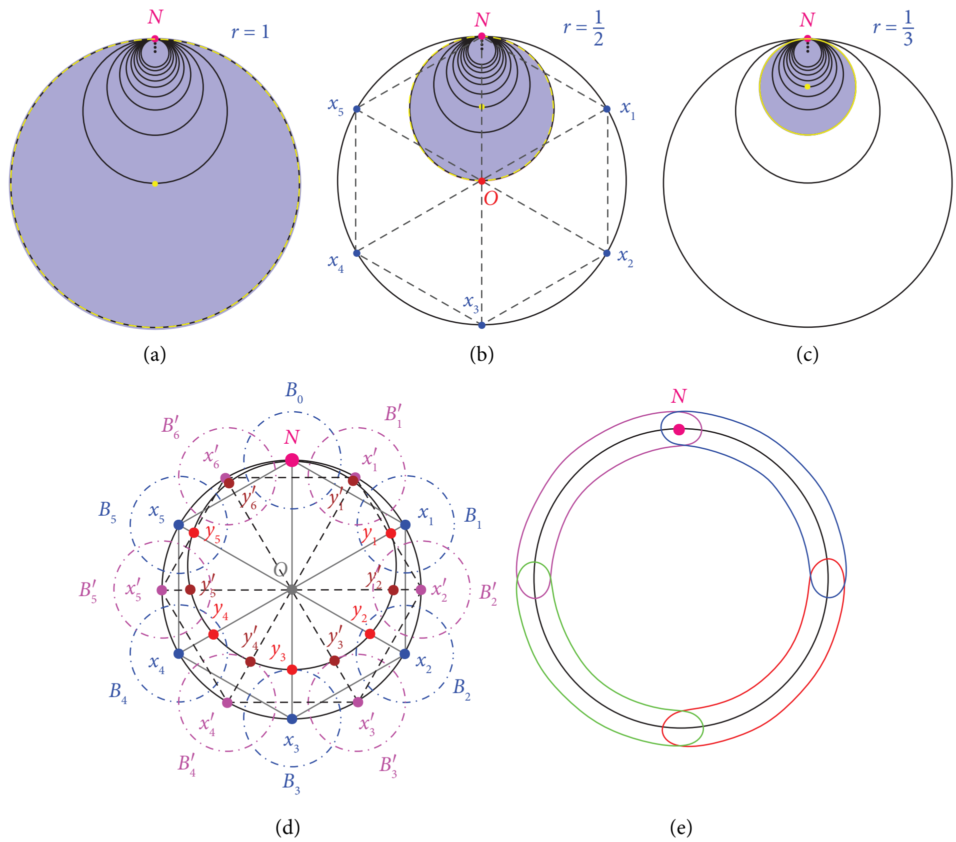


Figure 6: Example 6.

$$Z = \begin{pmatrix} F(0,0) & F(0,1) & \cdots & F(0,n) \\ \vdots & \vdots & \cdots & \vdots \\ F(t+1,0) & F(t+1,1) & \cdots & F(t+1,n) \end{pmatrix}. \quad (31)$$

By Proposition 5, $F_i(j) = F(i,j)$ and $F_j(i) = F(i,j)$ are good morphisms for $i \in \mathcal{I}_n$ and $j \in \mathcal{I}_{t+1}$, so by Proposition 68, each row of grid Z is an r-loop at p and each column of Z is an r-path, which completes the proof. □

Lemma 70. *Let (X,d) be a metric space and $r>0$; then, r-loops $(x_0x_1\cdots x_n)$ and $(x_0x_0x_1\cdots x_n)$ are r-homotopic.*

Proof. By definition, $(x_0x_1x_2\cdots x_{n-1}x_nx_n)$ is r-homotopy equivalent to $(x_0x_1x_2\cdots x_{n-1}x_n)$. Consider

$$\begin{pmatrix} x_0 & x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} & x_n \\ x_0 & x_1 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} & x_n \\ x_0 & x_1 & x_2 & x_2 & \cdots & x_{n-2} & x_{n-1} & x_n \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ x_0 & x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_{n-1} & x_n \\ x_0 & x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n & x_n \end{pmatrix}. \quad (32)$$

Clearly, each row is an r-loop, and each column is an r-path; thus, r-loops $(x_0x_0x_1x_2\cdots x_{n-1}x_n)$ and $(x_0x_1x_2\cdots x_{n-1}x_nx_n)$ are r-homotopy equivalent. Since r-homotopy is an equivalence relation, it is transitive, which completes the proof. □

Lemma 71. *Let (X,d) be a metric space and $r>0$; then, r-loop $(x_0x_1\cdots x_{n-1}x_n)$ is r-homotopy equivalent to r-loop $(x_0x_1\cdots x_i \underset{x_i}{e} x_{i+1}\cdots x_{n-1}x_n)$.*

Proof. By a similar argument of the proof of Lemma 70, the r-loop $(x_0x_1\cdots x_{n-1}x_n)$ is r-homotopy equivalent to r-loop $(x_0x_1\cdots x_ix_ix_{i+1}\cdots x_{n-1}x_n)$. Repeating this argument completes the proof.

In the following, we compare the r-homotopy equivalence class of an r-loop in metric space with the homotopy equivalence class of the induced loop in the corresponding geometric space. □

Proposition 72. *Let (X,d) be a metric space and $r>0$. Let $a=(x_0x_1\cdots x_n)$ and $b=(y_0y_1\cdots y_m)$ be r-loops, where $m\geq n$. Let f and g be induced loops of a and b, respectively, in geometric space $(X,\mathcal{N}_r(X))$. If g is a trivial extension of f, then a and b are r-homotopy equivalent.*

Proof. By definition of trivial extension, there exist sets $\{f_1,\ldots,f_p\}$ and $\{g_1,\ldots,g_q\}$ of paths in geometric space $(X,\mathcal{N}_r(X))$ and there exists a sequence $1\leq i_1<i_2<\cdots<i_p\leq q$ such that $g_{i_j}=f_j$ for $1\leq j\leq p$ and g_i is the trivial loop for all $i\in\{1,\ldots,q\}\backslash\{i_1,\ldots,i_p\}$. Since $f\colon \mathcal{I}_n \longrightarrow (X,\mathcal{N}_r(X))$ and $g\colon \mathcal{I}_m \longrightarrow (X,\mathcal{N}_r(X))$ are defined by $f(s)=x_s$ and $g(t)=y_t$ respectively, for $s\in\mathcal{I}_n$ and $t\in\mathcal{I}_m$, $y_{i_j}=x_{i_j}$ for $1\leq j\leq p$ and y_i is equal to y_{i-1} or

y_{i+1} for all $i \in \{1, \ldots, q\} \backslash \{i_1, \ldots, i_p\}$; thus, the proof is completed by Lemma 70 and 71.

By Proposition 72 and Proposition 69, we have the following corollary. □

Corollary 73. *Let (X, d) be a metric space and $p \in X$. Let $r > 0$ and $(x_0 x_1 \cdots x_n) \in \mathscr{C}_r(X, p)$; then, the r-homotopy equivalence class of $(x_0 x_1 \cdots x_n)$ is equal to the path homotopy class of the induced loop in geometric space $(X, \mathscr{N}_r(X))$.*

Now, we can identify the relation between the discrete fundamental group of metric space and the fundamental group of corresponding geometric space.

Proposition 74. *Let (X, d) be a metric space with $p \in X$ and $r > 0$; then, there is a group isomorphism from $A_{1,r}(X, p)$ to $\pi_1(X, \mathscr{N}_r(X), p)$.*

Proof. If $(x_0 x_1 \cdots x_n) \in \mathscr{C}_r(X, p)$, then $f: \mathscr{I}_n \longrightarrow (X, \mathscr{N}_r(X))$ by $f(i) = x_i$, for $i \in \mathscr{I}_n$, is the induced loop of r-loop $(x_0 x_1 \cdots x_n)$. Since the concatenation of two r-paths in a metric space (X, d) coincides with the product of two induced paths in the corresponding geometric space $(X, \mathscr{N}_r(X))$, by Corollary 73 and Lemma 65, $\varphi: A_{1,r}(X, p) \longrightarrow \pi_1(X, \mathscr{N}_r(X), p)$ by $\varphi([(x_0 x_1 \cdots x_n)]_r) = [f]_p$, for all $(x_0 x_1 \cdots x_n) \in \mathscr{C}_r(X, p)$, is an onto group homomorphism. However, $[(x_0 x_1 \cdots x_n)]_r = [f]_r$, for all $(x_0 x_1 \cdots x_n) \in \mathscr{C}_r(X, p)$, so the kernel of φ is empty; thus, φ is an isomorphism.

In the following examples, we compare the discrete fundamental group of Hawaiian earring and the fundamental group of geometric space of Hawaiian earring. □

Example 6. Conciser to the well-known Hawaiian earring $H := \cup_{n \in \mathbb{N}} C_n$ in Euclidean metric space $(\mathbb{R}^2, d)$, where

$$C_n := \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + \left(y + \frac{1}{n}\right)^2 = \left(\frac{1}{n}\right)^2\right\}, \quad \text{for } n \in \mathbb{N}. \tag{33}$$

By Proposition 74, the discrete fundamental group of H is isomorphic to fundamental group of corresponding geometric space of H. In the following, we calculate the fundamental group of induced geometric space $(H, \mathscr{N}_{2r}(H))$ at $N: = (0, 0)$ for multivalued radius $r > 0$ and the fundamental group of an arbitrary geometric space of H.

Case 1. Let $B_r = H \cap D_r \in \mathscr{N}_{2r}$ where $D_r := \{(x, y) \in \mathbb{R}^2 | x^2 + (y + r)^2 \leq r^2\}$; then $B_r \in \mathscr{N}_{2r}(H)$ and $\cup_{n \geq 1/r}^{\infty} C_n \subseteq B_r$.

(1) Let $r \geq 1$, then $H \subseteq B_r$, (a in Figure 6); thus, by Proposition 28, each loop at N on C_n is homotopic to e_N, where $n \geq r$; thus, $\pi_1(H, \mathscr{N}_{2r}(H), N)$ is a trivial group.

(2) Let $1/2 \leq r < 1$, then $\cup_{n=2}^{\infty} C_n \subseteq B_r$; thus, by a similar argument of case $r = 1$, each loop at N on C_n is homotopic to e_N, where $n \geq 2$ (b in Figure 6). Let $(N, x_1, x_2, \ldots, x_5)$ be a regular inscribed hexagonal in the circle C_1 (b 1). Since $d(x_i, O) = 1 \leq 2r$, $x_i \sim O$, for $i = 1, \ldots, 5$, where O is the center of C_1; thus, the loop $f: N \sim x_1 \sim \cdots \sim x_5 \sim N$ is homotopic to $g': N \sim O \sim \cdots \sim O \sim N$ which is a trivial extension of loop $g: N \sim O \sim N$. However, g is a loop on C_2; thus, g is homotopic to e_N; therefore, $\pi_1(H, \mathscr{N}_{2r}(H), N)$ is a trivial group.

(3) Let $r < 1/2$, then $\cup_{n \geq 1/r}^{\infty} C_n \subseteq B_r$; thus, by a similar argument of case $r = 1$, each loop at N on C_n is homotopic to e_N, where $n \geq 1/r$ (c in Figure 6).

Let $a_n = (0, 2/n)$ for $n \in \mathbb{N}$. If $d_n: = d(a_n, a_{n-1}) = 2/(n-1) - 2/n = 2/n(n-1) \leq r < 2r$ for some n, then $a_n \sim a_{n-1}$. In the following, we show that a loop at N on C_n is homotopic to a loop at N on C_{n-1} when $d_n \leq r$; then, by transitivity of homotopy relation, all the loops at N on C_n and on C_{n-1} are homotopic.

Let $(N, x_1, x_2, \ldots, x_5)$ be a regular inscribed hexagonal in the circle C_{n-1}, and let $y_i \in C_n$ be in the intersection of C_n and line $\overline{O_{n-1} x_i}$, for $i = 1, \ldots 5$, where O_{n-1} is the center of C_{n-1} (Figure 6). Clearly, $x_i \sim y_i$, for $i = 1, \ldots, 5$. Now, we define a path from x_i to x_{i+1}. Select $x'_{i1}, \ldots, x'_{ik} \in C_{n-1}$, for $i = 1, \ldots, 6$ and $k \geq n/2$, such that $d(x_{ij}, x_{ij+1}) = 1/k(n-1)$. Let $y'_{ij} \in C_n$ be in the intersection of C_n and line $\overline{O_{n-1} x'_i}$, for $i = 1, \ldots, 6$ and $j = 1, \ldots, k$ (d in Figure 6). Clearly, $x_i \sim y_i$ for $i = 1, \ldots, 5$ and $x_{ij} \sim y_{ij}$ for $i = 1, \ldots, 6$ and $j = 1, \ldots, k$. Let B_i be the disc with radius d_n and center x_i and B'_{ij} be the disc with radius d_n and center x'_{ij}, for $i = 1, \ldots, 6$ and $j = 1, \ldots, k$. Clearly, $x'_{ij} \approx y'_{ij}$, for $i = 1, \ldots, 6$ and $j = 1, \ldots, k$.

Since $2k/n(n-1) \geq 1/n - 1$, the discs $B_{i-1}, B'_{i1}, \ldots, B'_{ik}, B_i$ cover the arc of C_{n-1} from x_i to x_{i+1}, for $i = 1, \ldots, 4$. Since $d_n < r$, for each B_i, there exists a block $A_i \in \mathscr{N}_{2r}(H)$ such that $B_i \cap H \subseteq A_i$, for $i = 1, \ldots, 5$. Similarly, there exists a block $A'_i \in \mathscr{N}_{2r}(H)$ such that $B'_{ij} \cap H \subseteq A'_i$, for $i = 1, \ldots, 6$ and $j = 1, \ldots, k$; thus, $(A_{i-1}, A'_{i1}, \ldots, A'_{i,k}, A_i)$ is a polygonal corresponded to paths $f_i: x_{i-1} \sim x'_{i1} \sim \cdots \sim x'_{ik} \sim x_i$ and $g_i: y_{i-1} \sim y'_{i1} \sim \cdots \sim y'_{ik} \sim y_i$, for $i = 2, \ldots, 5$; therefore, by Proposition 28, the paths f_i and g_i are path homotopic, for $i = 2, \ldots, 5$. By a similar argument, paths $f_1: N \sim x'_{11} \sim \cdots \sim x'_{1k} \sim x_2$ and $g_1: N \sim y'_{11} \sim \cdots \sim y'_{1k} \sim y_1$ are homotopic and $f_6: x_5 \sim x'_{61} \sim \cdots \sim x'_{6k} \sim N$ and $g_6: y_5 \sim y'_{61} \sim \cdots \sim y'_{6k} \sim N$ are homotopic; hence, by transitivity of the homotopy equivalence relation, loops $f := f_1 \star \cdots \star f_6$ and $g := g_1 \star \cdots \star g_6$ are homotopic.

We can continue some way like above to find other homotopic loops at N on C_n and C_{n-1}, for $n \in \mathbb{N}$, but it is not our issue. For some enough little $r < 1/3$, there exists $m \in \mathbb{N}$ such that the loops at N on C_n and C_{n+1} are not homotopic, for $n = 1, \ldots, m$, and the loops at N on C_n are not homotopic to e_N, for $n > m$. By a similar argument of fundamental group of H in algebraic topology and Example 2 Case II, $\pi_1(H, \mathscr{N}_{2r}(H), N)$ is isomorphic to free product of m-copies of $(\mathbb{Z}, +)$.

Case 2. Now, we define a geometric space on H which is different from $(H, \mathscr{N}_{2r}(H))$. Let $(H, \mathscr{B})$ be a geometric space where $\mathscr{B} = \cup_{n \in N} \mathscr{B}_n$, and $\mathscr{B}_n$ contains four arcs of circle C_n like that e in Figure 6, for all $n \in N$. By Case II in Example 2 and a similar argument of fundamental group of H in algebraic topology, $\pi_1(H, \mathcal{N}_{2r}(H), N)$ is isomorphic to free product of infinite copies of $(\mathbb{Z}, +)$.

Data Availability

No data were used for the research described in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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