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Research Article

On Novel Fractional Integral and Differential Operators and Their Properties

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The main goal of this paper is to describe the new version of extended Bessel–Maitland function and discuss its special cases. Then, using the aforementioned function as their kernels, we develop the generalized fractional integral and differential operators. The convergence and boundedness of the newly operators and compare them with the existing operators such as the Saigo and Riemann–Liouville fractional operators are explored. The integral transforms of newly defined and generalized fractional operators in terms of the generalized Fox–Wright function are presented. Additionally, we discuss a few exceptional cases of the main result.

1. Introduction

Bessel–Maitland function has great importance in the advanced study of fractional operators. This function has gained more recognition due to its numerous applications in different areas of fractional calculus. The development of double integral involving the Bessel–Maitland function and established some results in term of Wright functions by Khan and Nisar [1]. Abouzaid et al. [2] established the integral representations of Bessel–Maitland function and also discussed some properties. Khan et al. developed the Mellin transform of Wright–Bessel function and some other useful properties regarding integral operators and transformations [3, 4]. The generalization of Bessel–Maitland functions [5–7] has modified the theory of fractional operators and gave many significant improved results in the area of fractional operator theory. Its generalizations and extensions have great contribution in the mathematical analysis and field of fractional theory. The rapid developments of fractional calculus have increased the demand of generalized fractional operators and various types of transformations. Due to this, many researchers have been worked on the multi-index spacial functions and established generalized version of fractional operators, that deals many real fractional problem and improved the theory of fractional inequalities. Huang et al. [8] developed the Hermite–Hadamard type inequalities by k-fractional integrals, which have immense applications in the fields of mathematical analysis. Nisae et al. [9] discussed the some refinements of Chebyshev type inequalities in aspects of generalized conformable integrals. Yang and Vivas-Cortez et al. modified some useful inequalities like Hermite–Hadamard and Fej\'er-Hadamard type inequalities and discussed their immense applications [10, 11]. The integral relations have been developed of generalized Mittag-Leffler type functions [12] by the researchers, and investigated many applications. Haubold and Diethelm have great contribution in advance theory of special functions and discuss many applications in fractional theory [13, 14], which gave to resolved many problems of fractional area in better way. Many extension of Pochhammer-s symbols, and extend the theory of fractional operators have been solved various fractional models [15–17].

Definition 1 (see [18]). The Bessel–Maitland function is defined for $\Re(\beta_1) > -1, \Re(\alpha_1) \geq 0$ $\alpha_1, \beta_1 \in \mathbb{C}$, as follows:

$$J_{\beta_1}^{\alpha_1}(s) = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!\Gamma(\alpha_1 n + \beta_1 + 1)}. \quad (1)$$

Definition 2 (see [19]). The generalized Bessel–Maitland function is defined as

$$J_{\beta_1,q}^{\alpha_1,\gamma}(s) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-s)^n}{n!\Gamma(\alpha_1 n + \beta_1 + 1)}, \quad (2)$$

where $\alpha_1, \gamma, \beta_1 \in \mathbb{C}$, $\Re(\beta_1) > -1, \Re(\alpha_1) \geq 0, \Re(\gamma) \geq 0$, $q \in (0,1) \cup \mathbb{N}$, and

$$(\gamma)_0 = 1, (\gamma)_{qn} = \frac{\Gamma(\gamma + qn)}{\Gamma(\gamma)}. \quad (3)$$

Definition 3 (see [20]). The extended Bessel–Maitland function is given as follows:

$$J_{\beta_1,q,\delta_1}^{\alpha_1,\gamma,p}(s) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-s)^n}{\Gamma(\alpha_1 n + \beta_1 + 1)(\delta_1)_{pn}}, \quad (4)$$

where $\alpha_1, \gamma, \beta_1, \delta_1 \in \mathbb{C}$, $\Re(\beta_1) > -1, \Re(\gamma) > 0, \Re(\delta_1) > 0$, $\Re(\alpha_1) \geq 0, q, p \geq 0$, and $q < \Re(\alpha_1) + p$.

Definition 4 (see [21]). The generalization of generalized Bessel–Maitland function is given as follows:

$$J_{\beta_1,q,\delta_1\nu_1}^{\alpha_1,\gamma,p,\mu_1}(s) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\gamma)_{qn}(-s)^n}{\Gamma(\alpha_1 n + \beta_1 + 1)(\delta_1)_{pn}}, \quad (5)$$

where $\alpha_1, \gamma, \beta_1, \delta_1, \mu_1 \in \mathbb{C}$, $\Re(\beta_1) > -1, \Re(\alpha_1) \geq 0$, $\Re(\gamma) > 0$, $\Re(\delta_1) > 0$, $\Re(\nu_1) > 0$, and $p, q, \nu_1 \geq 0$.

Definition 5 (see [22]). The Bessel–Maitland function of ten parameters is given as follows:

$$J_{\beta_1,q,\delta_1,\nu_1,m}^{\alpha_1,\gamma,p,\mu_1,\rho_1}(s) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\gamma)_{qn}(-s)^n}{\Gamma(\alpha_1 n + \beta_1 + 1)(\rho_1)_{mn}(\delta_1)_{pn}}, \quad (6)$$

where $\alpha_1, \gamma, \beta_1, \delta_1, \mu_1, \rho_1 \in \mathbb{C}$, $\Re(\beta_1) > -1$, $\Re(\alpha_1) \geq 0$, $\Re(\gamma) > 0$, $\Re(\delta_1) > 0, \Re(\mu_1) > 0$, and $p, q, \nu_1, m \geq 0$.

Definition 6 (see [23]). The Gauss hypergeometric function defined for all $a, b, c, d \in \mathbb{C}, a \neq 0, -1, -2, -3, \ldots$, and $|s| < 1$ is given below:

$${}_2F_1(b, -d; a; s) = \sum_{n=0}^{\infty} \frac{(b)_n(-d)_n s^n}{(a)_n n!}, \quad (7)$$

where $(a)_n$, $(b)_n$ and $(-d)_n$ are Pochhammer's symbols.

Definition 7 (see [24]). Pochhammer's symbol is defined by

$$(\nu)_n = \begin{cases} \nu(\nu+1)(\nu+2)\cdots(\nu+n-1), & \text{for } n \geq 1, \\ 1 & \text{for } n = 0, \nu \neq 0, \end{cases} \quad (8)$$

where $\nu \in \mathbb{C}$ and $n \in \mathbb{N}$, and using gamma function, it can be written as

$$(\nu)_n = \frac{\Gamma(\nu + n)}{\Gamma(\nu)}. \quad (9)$$

Definition 8 (see [23]). The Saigo fractional integral operators are defined for $s > 0, a, c, d \in \mathbb{C}$ and $\Re(a) > 0$, as follows:

$$\begin{aligned} \left(\mathfrak{F}_{0+}^{a,c,d} g\right)(s) &= \frac{s^{-a-c}}{\Gamma(a)} \int_0^s (s-\tau)^{a-1} {}_2R_1\left(a+c, -d; a; \left(1 - \frac{\tau}{s}\right)\right) g(\tau) d\tau, \\ \left(\mathfrak{F}_{0-}^{a,c,d} g\right)(s) &= \frac{1}{\Gamma(a)} \int_s^\infty (\tau - s)^{a-1} \tau^{-a-c} {}_2R_1\left(a+c, -d; a; \left(1 - \frac{s}{\tau}\right)\right) g(\tau) d\tau. \end{aligned} \quad (10)$$

Definition 9 (see [24]). The integral representation of gamma function is defined for $\Re(u) > 0$, as follows:

$$\Gamma(u) = \int_0^\infty v^{u-1} e^{-v} dv. \quad (11)$$

Definition 10 (see [25]). The Dirichlet formula is defined by the following relation:

$$\int_x^c du \int_u^c z(u,v) dv = \int_x^c dv \int_x^v z(u,v) du. \quad (12)$$

Definition 11 (see [25]). The Riemann–Liouville fractional integral and differential operators $J_{u^+}^{\eta_1}\Upsilon$ and $J_{v^-}^{\eta_1}\Upsilon$ of order $\eta_1 > 0$ are defined for $u, v \geq 0$, as follows:

$$J_{u^+}^{\eta_1}\Upsilon(x) = \frac{1}{\Gamma(\eta_1)}\int_u^x (x-t)^{\eta_1-1}\Upsilon(t)\mathrm{d}t, x > u,$$
$$J_{v^-}^{\eta_1}\Upsilon(x) = \frac{1}{\Gamma(\eta_1)}\int_x^v (t-x)^{\eta_1-1}\Upsilon(t)\mathrm{d}t, x < v, \tag{13}$$

and differential operator is defined by

$$\left(D_{0^+}^{a}\Upsilon\right)(s) = \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^n \left(\mathfrak{F}_{0^+}^{n-a}\Upsilon\right)(s). \tag{14}$$

Definition 12 (see [26]). The Fox–Wright function is defined as

$$_{\mu_1}\psi_{\nu_1}(s) = {}_{\mu_1}\psi_{\nu_1}\left[\begin{matrix}(z_j, y_j)_{1,\mu_1} \\ (x_i, w_i)_{1,\nu_1}\end{matrix} \mid s\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\mu_1}\Gamma(z_j + y_j n)}{\prod_{i=1}^{\nu_1}\Gamma(x_i + w_i n)} \frac{s^n}{n!}, \tag{15}$$

where $z_j, x_i \in \mathbb{C}$ and $y_j, w_i \in \mathfrak{R}$ and $(y_j, w_i, j = 1, 2, \ldots, \mu_1; i = 1, 2, \ldots, \nu_1) \neq 0$.

Motivated by interesting researches in field of fractional calculus and the videly use of Bessel–Maitland function by the scholars for new extensions, this paper is organized as follows. In Section 2, extended version of generalized Bessel–Maitland function (EvBMF) and will discuss its special cases. In Section 3, we will discuss the convergence and boundedness of fractional integral operator which utilized EvBMF as its kernel in the form of theorem. Moreover, we will discuss the behavior of generalized fractional integral operators with the EvBMF. In Section 4, we will prove some results of generalized fractional operators (Saigo and Riemann–Liouville) with the EvBMF and will obtain results in terms of generalized Fox–Wright function. In Section 5, we will discuss the Laplace transform and the behavior of Riemann–Liouville fractional operator with new fractional integral operator. In Section 6, we will present some interesting applications of the inverse fractional operator. We will derive some results of inverse fractional operator with Mittag-Leffler function and Bessel–Maitland function and will obtain results in terms of Fox–Wright function. Conclusions and future research are given in Section 7.

2. Extended Version of Generalized Bessel–Maitland Function (EvBMF)

In this section, we define the following extended version of generalized Bessel–Maitland function and discuss its special cases.

Definition 13. The extended version of Bessel–Maitland function (EvBMF) is defined for $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \in \mathbb{C}$ and $\mathfrak{R}(\alpha_1) \geq 0$, $\mathfrak{R}(\varphi_1) \geq -1$, $\eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \mathfrak{R}(\alpha_1)$, $\mathfrak{R}(\eta_1) > 0, \mathfrak{R}(\xi_1) > 0, \mathfrak{R}(\nu_1) > 0, \mathfrak{R}(\rho_1) > 0, \mathfrak{R}(\delta_1) > 0$, and $\gamma, q, \zeta_1, m, p \geq 0$, as follows:

$$J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(s) = \sum_{n=0}^{\infty} \frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-s)^n}{\Gamma(\alpha_1 n + \varphi_1 + 1)(\rho_1)_{mn}(\delta_1)_{pn}}. \tag{16}$$

If we replace $s = 1$ in equation (16), then we have

$$J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(1) = \sum_{n=0}^{\infty} \frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-1)^n}{\Gamma(\alpha_1 n + \varphi_1 + 1)(\rho_1)_{mn}(\delta_1)_{pn}}. \tag{17}$$

Special cases:

(i) If we put $q = 0$ in equation (16), then we have generalized Bessel–Maitland function (GBMF) of ten parameters defined by Khan et al. [22]:

$$J_{\varphi_1,\gamma,0,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(s) = J_{\varphi_1,\gamma,\zeta_1,m,p}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}(s). \tag{18}$$

(ii) If we replace $q = m = 0$ and $\rho_1 = 1$ in equation (16), then we have the Bessel–Maitland function of eight parameters defined by Ali et al. in [21]:

$$J_{\varphi_1,\gamma,0,\zeta_1,0,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,1,\delta_1}(s) = J_{\varphi_1,\gamma,\zeta_1,p}^{\alpha_1,\eta_1,\nu_1,\delta_1}(s). \tag{19}$$

(iii) If $q = m = \zeta_1 = 0$ in (16), then we have the Bessel–Maitland function of six parameters defined by Ghayasuddin and Khan in [20]:

$$J_{\varphi_1,\gamma,0,0,0,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(s) = J_{\varphi_1,\gamma,p}^{\alpha_1,\eta_1,\delta_1}(s). \tag{20}$$

(iv) If $q = m = \zeta_1 = 0$ and $p = \delta_1 = 1$ in equation (16), then we have Bessel–Maitland function defined in [19]:

$$J_{\varphi_1,\gamma,0,0,0,1}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,1}(s) = J_{\varphi_1,\gamma}^{\alpha_1,\eta_1}(s). \tag{21}$$

(v) If $q = m = \zeta_1 = \gamma = 0$ and $p = \eta_1 = \delta_1 = 1$ in equation (16), then we have Bessel–Maitland function in [18]:

$$J_{\varphi_1,0,0,0,0,1}^{\alpha_1,1,\xi_1,\nu_1,\rho_1,1}(s) = J_{\varphi_1}^{\alpha_1}(s). \tag{22}$$

(vi) If $q = 0$ and we replace φ_1 by $\varphi_1 - 1$ in equation (16), then we obtain

$$J_{\varphi_1-1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(-s) = E_{\varphi_1,\gamma,\zeta_1,m,p}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}(s), \tag{23}$$

where $E^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p}(s)$ is Mittag-Leffler function, which is defined by Khan and Ahmad [27].

(vii) If = 0, $\rho_1 = m = 1$ and we replace φ_1 by $\varphi_1 - 1$ in equation (16), then we have Mittag-Leffler function:

$$J^{\alpha_1,\eta_1,\xi_1,\nu_1,1,\delta_1}_{\varphi_1-1,\gamma,0,\zeta_1,1,p}(-s) = E^{\alpha_1,\eta_1,\nu_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,p}(s). \tag{24}$$

(viii) If $q = 0$, $\xi_1 = \rho_1 = m = \nu_1 = 1$ and we replace φ_1 by $\varphi_1 - 1$ in equation (16), then we get

$$J^{\alpha_1,\eta_1,1,1,1,\delta_1}_{\varphi_1-1,\gamma,0,\zeta_1,1,p}(-s) = E^{\alpha_1,\eta_1,\delta_1}_{\varphi_1,\gamma,p}(s), \tag{25}$$

where $E^{\alpha_1,\eta_1,\delta_1}_{\varphi_1,\gamma,p}(s)$ is defined in [28].

(ix) If $q = 0$, $\zeta_1 = \rho_1 = m = p = 1$ and we replace φ_1 by $\varphi_1 - 1$ in equation (16), then we have Mittag-Leffler function $E^{\alpha_1,\eta_1}_{\varphi_1,\gamma}(s)$ defined by Shukla and Prajapati in [29]:

$$J^{\alpha_1,\eta_1,\xi_1,\nu_1,1,\delta_1}_{\varphi_1,\gamma,0,1,1,1}(-s) = E^{\alpha_1,\eta_1}_{\varphi_1,\gamma}(s). \tag{26}$$

(x) If $q = 0$, $\zeta_1 = \rho_1 = m = \eta_1 = p = \delta_1 = 1$ and we replace φ_1 by $\varphi_1 - 1$ in equation (16), then we get Mittag-Leffler function defined by Wiman [30]:

$$J^{\alpha_1,1,\xi_1,\nu_1,1,1}_{\varphi_1,\gamma,0,1,1,1}(-s) = E^{\alpha_1}_{\varphi_1}(s). \tag{27}$$

(xi) If $\varphi_1 = q = 0$, $q = 0$, and $\zeta_1 = \rho_1 = m = \eta_1 = p = \delta_1 = 1$ in equation (16), then we have

$$J^{\alpha_1,1,\xi_1,\nu_1,1,1}_{0,\gamma,0,1,1,1}(-s) = E^{\alpha_1}(s), \tag{28}$$

where $E^{\alpha_1}(s)$ is the Mittag-Leffler function [31].

Definition 14. The fractional integral operator with EvBMF as its kernel is defined as follows:

$$\left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon\right)(s) = \int_a^s (s-t)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(s-t)^{\alpha_1}\right)\Upsilon(t)\mathrm{d}t, \tag{29}$$

where $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \in \mathbb{C}$, $\mathfrak{R}(\alpha_1) \geq 0$, $\mathfrak{R}(\varphi_1) \geq -1$ $\mathfrak{R}(\eta_1) > 0, \mathfrak{R}(\xi_1) > 0, \mathfrak{R}(\nu_1) > 0, \mathfrak{R}(\rho_1) > 0, \mathfrak{R}(\delta_1) > 0$, $\eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \mathfrak{R}(\alpha_1)$, and $\gamma, q, \zeta_1, m, p \geq 0$.

Special cases:

(i) By applying (29) for $q = 0$, we have the following well-known results of [32]:

$$\left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon\right)(s) = \int_a^s (s-t)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(s-t)^{\alpha_1}\right)\Upsilon(t)\mathrm{d}t. \tag{30}$$

(ii) If we consider $\omega = 0$ and replace φ_1 by $\varphi_1 - 1$, then we get the left-sided Riemann–Liouville fractional operator.

(iii) If we apply (29) for $q = m = 0$ and $\rho_1 = 1$, then we obtain the fractional integral operator defined by Ali et al. in [21].

Definition 15. The left inverse integral operator having EvBMF as its kernel is defined as follows:

$$\begin{aligned}\left(D^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon\right)(s) &= \left(\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1-n,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon\right)\right)(s)\\ &= \frac{\mathrm{d}^n}{\mathrm{d}s^n}\int_a^s (s-t)^{n-\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(s-t)^{\alpha_1}\right)\Upsilon(t)\mathrm{d}t,\end{aligned} \tag{31}$$

where $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \in \mathbb{C}$, $\mathfrak{R}(\alpha_1) \geq 0$, $\mathfrak{R}(\varphi_1) \geq -1$ $\mathfrak{R}(\eta_1) > 0, \mathfrak{R}(\xi_1) > 0, \mathfrak{R}(\nu_1) > 0, \mathfrak{R}(\rho_1) > 0$, $\mathfrak{R}(\delta_1) > 0, \eta_1$, $\xi_1, \nu_1 < \rho_1 + \delta_1 + \mathfrak{R}(\alpha_1)$, and $\gamma, q, \zeta_1, m, p \geq 0$.

Remark 1. If we consider $\omega = 0$ and replace φ_1 by $\varphi_1 - 1$, then equation (31) becomes Riemann–Liouville fractional differential operator.

Remark 2. If we consider p = q = $\zeta_1 = 0$, $\eta_1 = -\eta_1$, and $\gamma = \rho_1$ = m = 1 and replace φ_1 by $\varphi_1 - 1$ in equation (31), we have

$$\left(D^{\alpha_1,1,-\eta_1,1,0,0}_{\varphi_1-1,\nu_1,1,1,\delta_1,0,\omega;a^+}\Upsilon\right)(s) = \left(D^{\gamma}_{\alpha_1,\varphi_1,\omega;a^+}\Upsilon\right)(s), \tag{32}$$

where the inverse operator $(\mathrm{D}^{\gamma}_{\alpha_1,\varphi_1,\omega;\mathrm{a}^+}\Upsilon)(s)$ is discussed and described by Polito and Tomovski in [33].

3. Convergence and Boundedness of Generalized Fractional Integral Operator

In this section, we discuss the convergence and boundedness of fractional integral operator which utilized extended version of Bessel–Maitland function (EvBMF) as its kernel in the form of theorem. Moreover, we discuss the behavior of generalized fractional integral operators with the EvBMF.

Theorem 1. *Let* $(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon)(s)$ *be bounded on* $L(a,b)$ *with* $\eta_1,\xi_1,\nu_1,\alpha_1,\varphi_1,\rho_1,\delta_1,\in\mathbb{C}$, $\Re(\alpha_1)\geq 0$, $\Re(\varphi_1)\geq -1$, $\Re(\eta_1)>0,\Re(\xi_1)>0,\Re(\nu_1)>0$, $\Re(\rho_1)>0$, $\Re(\delta_1)>0$, *and* $\gamma,q,\zeta_1,m,p\geq 0,\eta_1,\xi_1,\nu_1<\rho_1+\delta_1+\Re(\alpha_1)$; *then, the following relation holds:*

$$\left\|\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon\right\|\|b\leq A\|\Upsilon\|_b, \tag{33}$$

where

$$A:==(b-a)^{\Re(\varphi_1)}\sum_{n=0}^{\infty}\frac{\left|(\eta_1)_{\gamma n}\right\|(\xi_1)_{qn}\left\|(\nu_1)_{\zeta_1 n}\right|}{\left|(\rho_1)_{mn}\right\|(\delta_1)_{pn}\right|}\times\frac{\left(-\omega(b-a)^{\Re(\alpha_1)}\right)^n}{\left|\Gamma(\alpha_1 n+\varphi_1+1)\right\|\Re(\alpha_1)n+\Re(\varphi_1)+1\right|}. \tag{34}$$

Proof. Let K_n denote the n^{th} term of (34); then, we have

$$\begin{aligned}\left|\frac{K_{n+1}}{K_n}\right| &= \left|\frac{(\eta_1)_{\gamma n+\gamma}}{(\eta_1)_{\gamma n}}\right\|\frac{(\xi_1)_{qn+q}}{(\xi_1)_{qn}}\left\|\frac{(\nu_1)_{\zeta_1 n+\zeta_1}}{(\nu_1)_{\zeta_1 n}}\right\|\frac{(\rho_1)_{mn}}{(\rho_1)_{mn+m}}\right|,\\ &\left|\frac{(\delta_1)_{pn}}{(\delta_1)_{pn+p}}\right\|\frac{\Gamma(\alpha_1 n+\varphi_1+1)}{\Gamma(\alpha_1 n+\alpha_1+\varphi_1+1)}\right|,\\ &\left|\frac{\Gamma(\Re(\alpha_1)_n+\Re(\varphi_1)+1)}{\Gamma(\Re(\alpha_1)(n+1)+\Re(\varphi_1)+1)}\right\|\frac{(-1)^{n+1}}{(-1)^n}\left\|\omega(b-a)^{\Re(\alpha_1)}\right|,\\ &\approx\frac{(\gamma n)^{\gamma}(qn)^q(\zeta_1 n)_1^{\zeta}\left|\left(\omega(b-a)^{\Re(\alpha_1)}\right)\right|}{(\rho_1 n)_1^{\rho}(\delta_1 n)_1^{\delta}\left|n+1\right\|\left(|\alpha_1|n\right)^{\Re(\alpha_1)}\right|}\ \text{as}\, n\longrightarrow\infty.\end{aligned} \tag{35}$$

Hence, $|K_{n+1}/K_n|\longrightarrow 0$ as $n\longrightarrow\infty$ and $\eta_1,\xi_1,\nu_1<\rho_1+\delta_1+\Re(\alpha_1)$, which means that the right side of equation (34) is convergent and finite under the condition. The condition of boundedness of the integral operator $(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon)(s)$ is discussed in the space of Lebesgue measure $L(a,b)$ of a continuous function on (a,b), where $b>a$:

$$L(a,b):==\left\{g(x)\Big\|\|g\|_b:==\int_a^b|g(x)\Big|\mathrm{d}x<\infty\right\}. \tag{36}$$

According to equations (16) and (29), we have

$$\begin{aligned}\left\|\left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon\right)\right\|_b &= \int_a^b\left|\int_a^s(s-t)^{\varphi_1}J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(s-t)^{\alpha_1}\right)\Upsilon(t)\mathrm{d}t\right|\mathrm{d}s,\\ &\leq\int_a^b\int_t^b(s-t)^{\varphi_1}J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(s-t)^{\alpha_1}\right)\mathrm{d}s|\Upsilon(t)|\mathrm{d}t.\end{aligned} \tag{37}$$

By putting these values $s-t=y\Rightarrow ds=dy$ for $s=t$, $y=0$, $s=b$, $y=b-t$ in equation (37), we obtain

$$\begin{aligned}\left\|\left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon\right)\right\|_b &\leq \int_a^b\left[\int_0^{b-t} y^{\Re(\varphi_1)} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(y)^{\alpha_1}\right)dy\right]|\Upsilon(t)|dt,\\ &\leq \int_a^b\left[\int_0^{b-a} y^{\Re(\varphi_1)}\left|J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(y)^{\alpha_1}\right)\right|dy\right]|\Upsilon(t)|dt,\end{aligned} \tag{38}$$

Now, let

$$\begin{aligned}A &= \int_0^{b-a} y^{\Re(\varphi_1)}\left|J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(y)^{\alpha_1}\right)\right|dy,\\ &= \sum_{n=0}^{\infty}\frac{\left|(\eta_1)_{\gamma n}\right|\left|(\xi_1)_{qn}\right|\left|(\nu_1)_{\zeta_1 n}\right|\left|(-\omega)^n\right|}{\left|\Gamma(\alpha_1 n+\varphi_1+1)\right|\left|(\rho_1)_{mn}\right|\left|(\delta_1)_{pn}\right|}\int_0^{b-a}(y)^{\Re(\alpha_1)n+\Re(\varphi_1)}dy,\\ &= \sum_{n=0}^{\infty}\frac{(b-a)^{\Re(\varphi_1)+1}\left|(\eta_1)_{\gamma n}\right|\left|(\xi_1)_{qn}\right|\left|(\nu_1)_{\zeta_1 n}\right|\left|\left(-\omega(b-a)^{\Re(\alpha_1)}\right)^n\right|}{\left|\Gamma(\alpha_1 n+\varphi_1+1)\right|\left|(\Re(\alpha_1)n+\Re(\varphi_1)+1)\right|\left|(\rho_1)_{mn}\right|\left|(\delta_1)_{pn}\right|}.\end{aligned} \tag{39}$$

So,

$$\left\|\left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}\Upsilon\right)\right\|_b \leq \int_a^b A|\Upsilon(t)|dt \leq A\|\Upsilon\|_b, \tag{40}$$

which completes the proof. □

4. Behavior of Generalized Fractional Operators with EvBMF

In this section, we prove some results of generalized fractional operators (Saigo and Riemann–Liouville) with the EvBMF and obtain results in terms of generalized Fox–Wright function. First, we consider the following two lemmas.

Lemma 1 (see [34]). *For $a,b,c,\ \rho_1\in\mathbb{C}$ with*

$$\begin{aligned}&\Re(a)>0,\\ &\Re(\rho_1+c-b)>0,\end{aligned} \tag{41}$$

the following result holds:

$$\left(I^{a,b,c}_{0,u}t^{\rho_1-1}\right)(u) = u^{\rho_1-b-1}\frac{\Gamma(\rho_1)\Gamma(\rho_1+c-b)}{\Gamma(\rho_1-b)\Gamma(\rho_1+a+c)}. \tag{42}$$

Lemma 2 (see [34]). *For $a,b,c\in\mathbb{C}$ with*

$$\Re(a)>0, \Re(\rho_1)<1+\min[\Re(b),\Re(c)], \tag{43}$$

the following relation holds:

$$\left(I^{a,b,c}_{u,\infty}t^{\rho_1-1}\right)(u) = u^{\rho_1-b-1}\frac{\Gamma(b-\rho_1+1)\Gamma(c-\rho_1+1)}{\Gamma(1-\rho_1)\Gamma(a+b+c-\rho_1+1)}. \tag{44}$$

Theorem 2. *Let $a,c,d,\eta_1,\xi_1,\nu_1,\alpha_1,\varphi_1,\rho_1,\delta_1,\in\mathbb{C}$ with $\Re(\alpha_1)>0$,
$\rho_1>\max[0,\Re(c-d)], \Re(\xi_1)>0, \Re(\alpha_1)\geq-1, \Re(\eta_1)$,
$\eta_1,\xi_1,\nu_1<\rho_1+\delta_1+\Re(\alpha_1)$,
$\Re(\eta_1)>0, \Re(\xi_1)>0, \Re(\nu_1)>0, \Re(\rho_1)>0, \Re(\delta_1)>0$, and $\gamma,q,\zeta_1,m,p\geq 0$; then, the following result holds:*

$$\begin{aligned}\mathfrak{F}^{a,c,d}_{0^+}\left[(\tau^{\rho_1})J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\tau^{\delta_1}\right)\right](s) &= \frac{s^{\rho_1-c}\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\\ &\times{}_6\psi_5\left[\begin{matrix}(\eta_1,\gamma)(\xi_1,q)(\nu_1,\zeta_1)(\rho_1+1,\delta_1)(\rho_1+1+d+c,\delta_1)(1,1)\\ (\varphi_1+1,\alpha_1)(\rho_1,m)(\delta_1,p)(\rho_1+1-c,\delta_1)(\rho_1+1+a+d,\delta_1)-s^{\delta_1}\end{matrix}\right].\end{aligned} \tag{45}$$

Proof. Consider the left-sided Saigo fractional integral operator with (EvBMF), and we have

$$\mathfrak{F}_{0^+}^{a,c,d}\left[\left(\tau^{\rho_1}\right)J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\tau_1^{\delta}\right)\right](s) = \mathfrak{F}_{0^+}^{a,c,d}\left[\left(\tau^{\rho_1}\right)\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}\left(-\tau^{\delta_1}\right)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\right](s)$$
$$= \mathfrak{F}_{0^+}^{a,c,d}\left[\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-1)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\right]\left(\tau^{\rho_1+\delta_1 n}(s)\right)$$
$$= \left[\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-1)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\right]\mathfrak{F}_{0^+}^{a,c,d}\left(\tau^{\rho_1+\delta_1 n}(s)\right). \tag{46}$$

By applying Lemma 1, we get

$$\mathfrak{F}_{0^+}^{a,c,d}\left[\left(\tau^{\rho_1}\right)J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\tau^{\delta_1}\right)\right](s) = \sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-1)^n(s)^{\rho_1+\delta_1 n-c}}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}$$
$$\frac{\Gamma(\rho_1+\delta_1 n+1)\Gamma(\rho_1+\delta_1 n+1+d-c)}{\Gamma(\rho_1+\delta_1 n+1-c)\Gamma(\rho_1+\delta_1 n+1+a+d)}$$
$$= \sum_{n=0}^{\infty}\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)\Gamma(\rho_1)\Gamma(\delta_1)\left(-s_1^{\delta}\right)^n(s)^{\rho_1-c}}{\Gamma(\alpha_1 n+\varphi_1+1)\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)}$$
$$\times\frac{\Gamma(\rho_1+\delta_1 n+1)\Gamma(\rho_1+\delta_1 n+1+d-c)}{\Gamma(\rho_1+\delta_1 n+1-c)\Gamma(\rho_1+\delta_1 n+1+a+d)}. \tag{47}$$

Hence, we obtain the following result:

$$\mathfrak{F}_{0^+}^{a,c,d}\left[\left(\tau^{\rho_1}\right)J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\tau^{\delta_1}\right)(s)\right] = \frac{s^{\rho_1-c}\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}$$
$$\times {}_6\psi_5\left[\begin{matrix}(\eta_1,\gamma)(\xi_1,q)(\nu_1,\zeta_1)(\rho_1+1,\delta_1)(\rho_1+1+d-c,\delta_1)(1,1)\\ (\varphi_1+1,\alpha_1)(\rho_1,m)(\delta_1,p)(\rho_1+1-c,\delta_1)(\rho_1+1+a+d,\delta_1)-s^{\delta_1}\end{matrix}\right]. \tag{48}$$

□

Corollary 1. *If we take $q=0$ in equation (48), we have the following result as in [32]:*

$$\mathfrak{F}_{0^+}^{a,c,d}\left[\left(\tau^{\rho_1}\right)J_{\varphi_1,\gamma,\zeta_1,m,p}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}\left(\tau^{\delta_1}\right)(s)\right] = \frac{s^{\rho_1-c}\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\nu_1)}$$
$$\times {}_5\psi_5\left[\begin{matrix}(\eta_1,\gamma)(\nu_1,\zeta_1)(\rho_1+1,\delta_1)(\rho_1+1+d-c,\delta_1)(1,1)\\ (\varphi_1+1,\alpha_1)(\rho_1,m)(\delta_1,p)(\rho_1+1-c,\delta_1)(\rho_1+1+a+d,\delta_1)-s^{\delta_1}\end{matrix}\right]. \tag{49}$$

Theorem 3. *Let* $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \in \mathbb{C}$, $\Re(\alpha_1) \geq 0$, $\Re(\varphi_1) \geq -1$, $\Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0$, $\Re(\rho_1) > 0, \Re(\delta_1) > 0$, *and* $\gamma, q, \zeta_1, m, p \geq 0$; *then, the following result holds:*

$$\mathfrak{F}_{a^+}^{\lambda}\left[(\tau-a)^{\varphi_1} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(\tau-a)^{\alpha_1}\right)\right](s-a) = (s-a)^{\lambda+\varphi_1+1} J_{\varphi_1+\lambda,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(\tau-a)^{\alpha_1}\right). \tag{50}$$

Proof. Consider the left-sided Riemann–Liouville fractional integral operator, in which using the power function with (EvBMF), we get

$$\begin{aligned}
&\mathfrak{F}_{a^+}^{\lambda}\left[(\tau-a)^{\varphi_1} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(\tau-a)^{\alpha_1}\right)\right](s-a) = \frac{1}{\Gamma(\lambda)}\int_a^s (s-\tau)^{\lambda-1}(\tau-a)^{\varphi-1} \\
&\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}\left(-w(\tau-a)^{\alpha_1}\right)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}(s-a)\mathrm{d}\tau \\
&= \frac{(\omega)^n}{\Gamma(\lambda)}\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-1)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\int_a^s (s-\tau)^{\lambda-1}(\tau-a)^{\alpha_1 n+\varphi_1}(s-a)\mathrm{d}\tau \\
&= \frac{(\omega)^n}{\Gamma(\lambda)} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(1)\int_a^s (s-\tau)^{\lambda-1}(\tau-a)^{\alpha_1 n+\varphi_1}(s-a)\mathrm{d}\tau.
\end{aligned} \tag{51}$$

Putting the values $\tau - a/s - a = u \Rightarrow \mathrm{d}\tau = (s-a)\mathrm{d}u$, $\tau = u(s-a)+a$, for $\tau = a, u = 0$ and for $\tau = s, u = 1$ in equation (51), we have

$$\begin{aligned}
\mathfrak{F}_{a^+}^{\lambda}\left[(\tau-a)^{\varphi_1} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(\tau-a)^{\alpha_1}\right)\right](s-a) &= \frac{(\omega)^n}{\Gamma(\lambda)} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(1) \\
&\int_0^1 (s-(s-a)u-a)^{\lambda-1}(u(s-a))^{\alpha_1 n+\varphi_1}(s-a)(s-a)\mathrm{d}u \\
&= \frac{(s-a)^{\varphi_1+1+1}(\omega)^n}{\Gamma(\lambda)} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(1)(s-a)^{\alpha_1 n}\int_0^1 (s-(s-a)u-a)^{\lambda-1}u^{\alpha_1 n+\varphi_1}\mathrm{d}u \\
&= \frac{(s-a)^{\varphi_1+\lambda+1}(\omega)^n}{\Gamma(\lambda)} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(1)(s-a)^{\alpha_1 n}\int_0^1 (1-u)^{\lambda-1}u^{\alpha_1 n+\varphi_1}\mathrm{d}u \\
&= \frac{(s-a)^{\varphi_1+\lambda+1}(\omega)^n}{\Gamma(\lambda)} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(1)(s-a)^{\alpha_1 n}\frac{\Gamma(\lambda)\Gamma(\alpha_1 n+\varphi_1+1)}{\Gamma(\alpha_1 n+\varphi_1+\lambda+1)} \\
&= (s-a)^{\varphi_1+\lambda+1}\frac{\Gamma(\alpha_1 n+\varphi_1+1)}{\Gamma(\alpha_1 n+\varphi_1+\lambda+1)}\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}\left(\omega(\tau-a)^{\alpha_1}\right)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}} \\
&= (s-a)^{\varphi_1+\lambda+1} J_{\varphi_1+\lambda,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(\tau-a)^{\alpha_1}\right),
\end{aligned} \tag{52}$$

which is the required result. □

Corollary 2. *If we take* $q = 0$ *in equation (50), we get the result as described in [32]:*

$$\mathfrak{F}_{a^+}^{\lambda}\left[(\tau-a)^{\varphi_1} J_{\varphi_1,\gamma,\zeta_1,m,p}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}\left(\omega(\tau-a)^{\alpha_1}\right)\right](s-a) = (s-a)^{\lambda+\varphi_1+1} J_{\varphi_1+\lambda,\gamma,\zeta_1,m,p}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}\left(\omega(\tau-a)^{\alpha_1}\right). \tag{53}$$

5. Laplace Transform of EvBMF and Behavior of Riemann–Liouville Fractional Operator

In this section, we discuss the Laplace transform and the behavior of Riemann–Liouville fractional operator with EvBMF.

Theorem 4. *Let* $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \in \mathbb{C}$, $\mathfrak{R}(\alpha_1) \geq 0$, $\mathfrak{R}(\varphi_1) \geq -1\, \mathfrak{R}(\eta_1) > 0, \mathfrak{R}(\xi_1) > 0, \mathfrak{R}(\nu_1) > 0$, $\mathfrak{R}(\rho_1) > 0, \mathfrak{R}(\delta_1) > 0$, $\gamma, q, \zeta_1, m, p \geq 0$, *and* $s > 0$; *then, the following relation holds:*

$$\mathfrak{L}\left[\daleth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right] = \frac{s^{-\varphi_1-1}\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)} \times {}_5\psi_3\left[\begin{matrix}(\eta_1,\gamma), (\xi_1,q), (\nu_1,\zeta_1), (1,1)\\ (\rho_1,m), (\delta_1,p), (\varphi_1+1,\alpha_1)\end{matrix} - \left(\frac{\omega}{s}\right)^{\alpha_1}\right]. \tag{54}$$

Proof. Consider the well-known Laplace transform operator, and we have

$$\begin{aligned}\mathfrak{L}\left[\daleth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right] &= \int_0^\infty e^{-st}\left[\int_0^t (t-u)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(\tau-a)^{\alpha_1}\right)^n \Upsilon(u)\mathrm{d}u\right]\mathrm{d}t\\ &= \int_0^\infty e^{-st}\left[\int_0^t (t-u)^{\varphi_1}\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}\left(\omega(\tau-a)^{\alpha_1}\right)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\Upsilon(u)\mathrm{d}u\right]\mathrm{d}t.\end{aligned} \tag{55}$$

Changing order of integration and applying Dirichlet formula, we get

$$\begin{aligned}&= \int_0^\infty e^{-st}\int_u^\infty (t-u)^{\varphi_1}\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-w)^n\left((\tau-u)^{\alpha_1 n}\right)}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\mathrm{d}t\,\Upsilon(u)\mathrm{d}u\\ &= \sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-w)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\int_0^\infty\int_u^\infty\frac{(t-u)^{\alpha_1 n+\varphi_1}}{e^{st}}\mathrm{d}t\,\Upsilon(u)\mathrm{d}u.\end{aligned} \tag{56}$$

Putting $t-u=\tau$ implies $\mathrm{d}t=\mathrm{d}\tau$. Now for limits of integration $t=u, \tau=0$ and if $t=\infty, \tau=\infty$ in equation (56), we obtain

$$\begin{aligned}\mathfrak{L}\left[\daleth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right] &= \sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-w)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\int_0^\infty\Upsilon(u)\mathrm{d}u\int_0^\infty\tau^{\alpha_1 n+\varphi_1}e^{-s(\tau+u)}\mathrm{d}\tau\,\mathrm{d}u\\ &= \sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-w)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\int_0^\infty\frac{\Upsilon(u)}{e^{su}}\mathrm{d}u\int_0^\infty\tau^{\alpha_1 n+\varphi_1}e^{-s\tau}\mathrm{d}\tau\,\mathrm{d}u\\ &= \sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-w)^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\frac{\Gamma(\alpha_1 n+\varphi_1)}{s^{\alpha_1 n+\varphi_1+1}}\\ \Upsilon(s) &= \frac{s^{-\varphi_1-1}\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\sum_{n=0}^{\infty}\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)\left(-ws^{-\varphi_1}\right)^n\Gamma(\alpha_1 n+\varphi_1)}{\Gamma(\alpha_1 n+\varphi_1+1)\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)}\\ &= \frac{s^{-\varphi_1-1}\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\times{}_5\psi_3\left[\begin{matrix}(\eta_1,\gamma), (\xi_1,q), (\nu_1,\zeta_1), (\alpha_1,\varphi_1), (1,1)\\ (\rho_1,m), (\delta_1,p), (\varphi_1+1,\alpha_1)\end{matrix} - \left(\frac{\omega}{s}\right)^{\alpha_1}\right],\end{aligned} \tag{57}$$

which is the required result. □

Corollary 3. *If we put $q = 0$ in equation (41), we have the following result of [32]:*

$$\mathfrak{L}\left[\beth^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;0^+}\Upsilon\right] = \frac{s^{-\varphi_1-1}\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\nu_1)} \times {}_4\psi_3\left[\begin{matrix}(\eta_1,\gamma),(\nu_1,\zeta_1),(1,1)\\ \\ (\rho_1,m),(\delta_1,p),(\varphi_1+1,\alpha_1)\end{matrix} -\left(\frac{\omega}{s}\right)^{\alpha_1}\right]. \tag{58}$$

Theorem 5. *Let $\eta_1,\xi_1,\nu_1,\alpha_1,\varphi_1,\rho_1,\delta_1,\lambda\in\mathbb{C}$, $\mathfrak{R}(\alpha_1)\geq 0$, $\mathfrak{R}(\varphi_1)\geq -1\,\mathfrak{R}(\eta_1)>0,\mathfrak{R}(\xi_1)>0,\mathfrak{R}(\nu_1)>0,\mathfrak{R}(\rho_1)>0$, $\mathfrak{R}(\delta_1)>0$, and $\gamma,q,\zeta_1,m,p\geq 0$; then, the following relation holds:*

$$\left(\mathfrak{F}^{\lambda}_{0^+}\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right)(s) = \left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1+\lambda,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right)(s). \tag{59}$$

Proof. Consider the left-sided Riemann–Liouville integral operator involving new fractional integral operator (29), and we have

$$\begin{aligned}\left(\mathfrak{F}^{\lambda}_{0^+}\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right)(s) &= \frac{1}{\Gamma(\lambda)}\int_0^s (s-u)^{\lambda-1}\int_0^u (u-\tau)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(u-\tau)^{\alpha_1}\right)\Upsilon(\tau)\mathrm{d}\tau\,\mathrm{d}u\\ &= \frac{1}{\Gamma(\lambda)}\int_0^s\int_\tau^s (s-u)^{\lambda-1}(u-\tau)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(u-\tau)^{\alpha_1}\right)\mathrm{d}u\,\Upsilon(\tau)\mathrm{d}\tau.\end{aligned} \tag{60}$$

Putting $u-\tau=t$ implies $u=t+\tau,\mathrm{d}u=\mathrm{d}t$. Now for limits of integration, if $u=s$, then $t=s-\tau$ and $u=\tau$; then, $t=0$, so we get

$$\begin{aligned}\left(\mathfrak{F}^{\lambda}_{0^+}\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right)(s) &= \frac{1}{\Gamma(\lambda)}\int_0^s\int_0^{s-\tau}(s-t-\tau)^{\lambda-1}t^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(t)^{\alpha_1}\right)\mathrm{d}t\,\Upsilon(\tau)\mathrm{d}\tau\\ &= \int_0^s\frac{1}{\Gamma(\lambda)}\int_0^{s-\tau}(s-t-\tau)^{\lambda-1}t^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(t)^{\alpha_1}\right)\mathrm{d}t\,\Upsilon(\tau)\mathrm{d}\tau.\end{aligned} \tag{61}$$

By using (14) in equation (61), we have

$$\left(\mathfrak{F}^{\lambda}_{0^+}\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right)(s) = \int_0^s\left[\mathfrak{F}^{\lambda}_{0^+}(t)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(t)^{\alpha_1}\right)\right](s-\tau)\Upsilon(\tau)\mathrm{d}\tau. \tag{62}$$

Applying Theorem 3 in (62), we obtain

$$\begin{aligned}\left(\mathfrak{F}^{\lambda}_{0^+}\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right)(s) &= \int_0^s\frac{J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(\omega(s-\tau)^{\alpha_1}\right)}{(s-\tau)^{-\alpha_1-\varphi_1-1}}\Upsilon(\tau)\mathrm{d}\tau\\ &= \left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1+\lambda,\gamma,q,\zeta_1,m,p,\omega;0^+}\Upsilon\right)(s).\end{aligned} \tag{63}$$

□

Corollary 4. *If we take $q = 0$ in equation (63), we have the result as in [32]:*

$$\left(\mathfrak{F}_{0^+}^{\lambda} \beth_{\varphi_1,\gamma,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1} \Upsilon\right)(s) = \left(\beth_{\varphi_1+\lambda,\gamma,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1} \Upsilon\right)(s). \tag{64}$$

Theorem 6. *Let $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \in \mathbb{C}$, $\mathfrak{R}(\alpha_1) \geq 0$, $\mathfrak{R}(\varphi_1) \geq -1$, $\mathfrak{R}(\eta_1) > 0, \mathfrak{R}(\xi_1) > 0, \mathfrak{R}(\nu_1) > 0$, $\mathfrak{R}(\rho_1) > 0, \mathfrak{R}(\delta_1) > 0$, and $\gamma, q, \zeta_1, m, p \geq 0$; then, the following relation holds:*

$$\mathfrak{D}_{0^+}^{\lambda}\left(\beth_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1} \Upsilon\right)(s) = \left(\beth_{\varphi_1-\lambda,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1} \Upsilon\right)(s). \tag{65}$$

Proof. Consider the left-sided Riemann–Liouville differential operator and new fractional integral operator (29), and we have

$$\begin{aligned}\mathfrak{D}_{0^+}^{\lambda}\left(\beth_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1} \Upsilon\right)(s) &= \frac{1}{\Gamma(m-\lambda)}\left(\frac{d}{ds}\right)^m \int_0^s (s-u)^{m-\lambda-1} \\ &\times \int_0^u (u-\tau)^{\varphi_1} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(u-\tau)^{\alpha_1}\right)\Upsilon(\tau)\mathrm{d}\tau\,\mathrm{d}u.\end{aligned} \tag{66}$$

By applying Dirichlet formula in equation (66), we get

$$\begin{aligned}\mathfrak{D}_{0^+}^{\lambda}\left(\beth_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1} \Upsilon\right)(s) &= \frac{1}{\Gamma(m-\lambda)}\left(\frac{d}{ds}\right)^m \int_0^s \int_\tau^s (\tau-u)^{\varphi_1}(s-u)^{m-\lambda-1} \\ &J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(u-\tau)^{\alpha_1}\right)\mathrm{d}u\,\Upsilon(\tau)\mathrm{d}\tau.\end{aligned} \tag{67}$$

Substituting $u - \tau = t$ implies $\mathrm{d}t = \mathrm{d}u, u = \tau, t = 0, u = s$, and $t = s - \tau$; then, we obtain

$$\begin{aligned}\mathfrak{D}_{0^+}^{\lambda}\left(\beth_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1} \Upsilon\right)(s) &= \frac{1}{\Gamma(m-\lambda)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^m \int_0^s \int_0^{s-\tau} \frac{(-\tau)^{\varphi_1} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(t)^{\alpha_1}\right)}{(s-t-\tau)^{-m+\lambda+1}}\mathrm{d}t\,\Upsilon(\tau)\mathrm{d}\tau \\ &= \int_0^s \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^m \frac{1}{\Gamma(m-\lambda)} \int_0^{s-\tau} \frac{(-\tau)^{\varphi_1} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(t)^{\alpha_1}\right)}{(s-t-\tau)^{-m+\lambda+1}}\mathrm{d}t\,\Upsilon(\tau)\mathrm{d}\tau.\end{aligned} \tag{68}$$

By using equation (14) in equation (68), we have

$$\mathfrak{D}_{0^+}^{\lambda}\left(\beth_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1} \Upsilon\right)(s) = \int_0^s \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^m \mathfrak{F}_{0^+}^{m-\lambda}\left[\begin{matrix}(-\tau)^{\varphi_1} \\ J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(t)^{\alpha_1} - w(\tau-a)^{\alpha_1}\right)(s-\tau)\end{matrix}\right]\Upsilon(\tau)\mathrm{d}\tau. \tag{69}$$

By using equation (16) in (69), we get

$$\left(\mathfrak{D}_{0^+}^{\lambda}\beth_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\Upsilon\right)(s)=\int_0^s\left(\frac{d}{ds}\right)^m\frac{J_{m-\lambda+\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(s-\tau)^{\alpha_1}\right)}{(s-\tau)^{\lambda-m-\varphi_1}}\Upsilon(\tau)d\tau. \tag{70}$$

Using (16) in (70) and then taking one time derivative, we obtain

$$\begin{aligned}
&\left(\mathfrak{D}_{0^+}^{\lambda}\beth_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\Upsilon\right)(s)\\
&=\int_0^s\left(\frac{d}{ds}\right)^m(s-\tau)^{m-\lambda+\varphi_1}\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}\left(-w(s-\tau)^{\alpha_1}\right)^n}{\Gamma(\alpha_1 n+\varphi_1+m-\lambda+1)(\rho_1)_{mn}(\delta_1)_{pn}}\Upsilon(\tau)d\tau\\
&=\sum_{n=0}^{\infty}\frac{(-\omega)^n(\alpha_1 n+\varphi_1+m-\lambda)}{\Gamma(\alpha_1 n+\varphi_1+m-\lambda+1)}\times\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}\left(-\tau^{-\delta_1}\right)^n}{(\rho_1)_{mn}(\delta_1)_{pn}}\\
&\times\left(\frac{d}{ds}\right)^{m-1}\int_0^s(s-\tau)^{\alpha_1 n+\varphi_1+m-\lambda-1}\Upsilon(\tau)d\tau\\
&=\left(\frac{d}{ds}\right)^{m-1}\int_0^s\frac{J_{m-\lambda+\varphi_1-1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\omega(s-\tau)^{\alpha_1}\right)}{(s-\tau)^{-m-\varphi_1+\lambda+1}}\Upsilon(\tau)d\tau.
\end{aligned}\tag{71}$$

Now, taking the $(m-1)$ derivative of equation (71), we have

$$\begin{aligned}
\left(\mathfrak{D}_{0^+}^{\lambda}\beth_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\Upsilon\right)(s)&=\int_0^s\frac{J_{\varphi_1-\lambda,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(-\omega(s-\tau)^{\alpha_1}\right)}{(s-\tau)^{\lambda-\varphi_1}}\Upsilon(\tau)d\tau\\
&=\left(\beth_{\varphi_1-\lambda,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(-w(s-\tau)^{\alpha_1}\right)\Upsilon\right)(s),
\end{aligned}\tag{72}$$

which completes the proof. □

Corollary 5. *If we put $q=0$ in equation (65), we get the result discussed in [32].*

$$\mathfrak{D}_{0^+}^{\lambda}\left(\beth_{\varphi_1,\gamma,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}\Upsilon\right)(s)=\left(\beth_{\varphi_1-\lambda,\gamma,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}\Upsilon\right)(s). \tag{73}$$

Theorem 7. *Let $\eta_1,\lambda,\chi,\xi_1,\nu_1,\alpha_1,\varphi_1,\rho_1,\delta_1,\in\mathbb{C}$, $\Re(\alpha_1)\geq 0$, $\Re(\varphi_1)\geq -1$, $\Re(\eta_1)>0,\Re(\xi_1)>0,\Re(\nu_1)>0$, $\Re(\rho_1)>0,\Re(\delta_1)>0$, and $\gamma,q,\zeta_1,m,p\geq 0$; then, the following relation holds:*

$$\begin{aligned}
\left(\beth_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}t^{\frac{(\lambda+\delta_1)}{\chi}-1}\right)(s)&=s^{\varphi_1+\frac{\lambda+\delta_1}{\chi}}\frac{\Gamma(\rho_1)\Gamma(\delta_1)\Gamma(\lambda+\delta_1/\chi)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\\
&\times{}_4\Psi_3\left[\begin{matrix}(\eta_1,\gamma)(\xi_1,q)(\nu_1,\zeta_1)(1,1)\\ \left(\varphi_1+\frac{\lambda+\delta_1}{\chi}+1,\alpha_1\right)(\rho_1,m)(p,\delta_1)-\omega s^{\alpha_1}\end{matrix}\right].
\end{aligned}\tag{74}$$

Proof. Consider the new fractional integral operator (29), and we have

$$\left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+} t^{\frac{(\lambda+\delta_1)}{\chi}-1}\right)(s) = \int_0^s (s-\tau)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(w(s-\tau)^{\alpha_1}\right)\tau^{\frac{\lambda+\delta_1}{\chi}-1} d\tau$$

$$= \int_0^s (s-\tau)^{\varphi_1} \sum_{n=0}^{\infty} \frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}} \left(-w(s-\tau)^{\alpha_1}\right)^n \tau^{\frac{\lambda+\delta_1}{\chi}-1} d\tau \tag{75}$$

$$= \sum_{n=0}^{\infty} \frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}} (-\omega)^n s^{\alpha_1 n+\varphi_1} \int_0^s \left(1-\frac{\tau}{s}\right)^{\alpha_1 n+\varphi_1} \tau^{\frac{\lambda+\delta_1}{\chi}-1} d\tau.$$

By putting the values of $\tau/s = u$, $d\tau = \text{sdu}$ and if $\tau = 0$, $u = 0$, $\tau = s$, $u = 1$, we get

$$\left(\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+} t^{\frac{(\lambda+\delta_1)}{\chi}-1}\right)(s) = \sum_{n=0}^{\infty} \frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}} (-\omega)^n s^{\alpha_1 n+\varphi_1}$$

$$\int_0^s (1-u)^{\alpha_1 n+\varphi_1} (su)^{\frac{\lambda+\delta_1}{\chi}-1} \text{sdu}$$

$$= \sum_{n=0}^{\infty} \frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}} (-\omega)^n s^{\alpha_1 n+\varphi_1+\frac{\lambda+\delta_1}{\chi}} \int_0^s (1-u)^{\alpha_1 n+\varphi_1} u^{\frac{\lambda+\delta_1}{\chi}-1} du$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)\Gamma(\alpha_1 n+\varphi_1+1)} (-\omega)^n \tag{76}$$

$$\times s^{\alpha_1 n} s^{\varphi_1+\frac{\lambda+\delta_1}{\chi}} \frac{\Gamma(\alpha_1 n+\varphi_1+1)\Gamma(\lambda+\delta_1/\chi)}{\Gamma(\alpha_1 n+\varphi_1+\lambda+\delta_1/\chi-1)}$$

$$= \frac{s^{\varphi_1+\lambda+\delta_1/\chi}\Gamma(\rho_1)\Gamma(\delta_1)\Gamma(\lambda+\delta_1/\chi)(-\omega)^n(s)^{\alpha_1 n}}{\Gamma(\alpha_1 n+\varphi_1+1)\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)} \sum_{n=0}^{\infty} \frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)\Gamma(\alpha_1 n+\varphi_1+1)}{\Gamma(\alpha_1 n+\varphi_1+\lambda+\delta_1/\chi+1)\Gamma(\rho_1+mn)\Gamma(p+\delta_1 n)}$$

$$= s^{\varphi_1+\frac{\lambda+\delta_1}{\chi}} \frac{\Gamma(\rho_1)\Gamma(\delta_1)\Gamma(\lambda+\delta_1/\chi)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)} \sum_{n=0}^{\infty} \frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)(-\omega s^{\alpha_1})^n}{\Gamma(\rho_1+mn)\Gamma(p+\delta_1 n)\Gamma(\alpha_1 n+\lambda+\delta_1/\chi+1+\varphi_1)}$$

$$= s^{\varphi_1+\frac{\lambda+\delta_1}{\chi}} \frac{\Gamma(\rho_1)\Gamma(\delta_1)\Gamma(\lambda+\delta_1/\chi)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)} \times {}_4\psi_3\left[\begin{array}{c}(\eta_1,\gamma)(\xi_1,q)(\nu_1,\zeta_1)(1,1)\\ (\varphi_1+\lambda+\delta_1/\chi+1,\alpha_1)(\rho_1,m)(p,\delta_1)-\omega s^{\alpha_1}\end{array}\right],$$

which is the required result. □

Corollary 6. *If we take $q = 0$ in equation (74), we have the following result as in [32]:*

$$\left(\beth^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;0^+}t^{\lambda+\delta_1/\chi-1}\right)(s)=s^{\varphi_1+\frac{\lambda+\delta_1}{\chi}}\frac{\Gamma(\rho_1)\Gamma(\delta_1)\Gamma(\lambda+\delta_1/\chi)}{\Gamma(\eta_1)\Gamma(\nu_1)}\times{}_3\psi_3\left[\begin{array}{c}(\eta_1,\gamma)(\nu_1,\zeta_1)(1,1)\\ \left(\varphi_1+\frac{\lambda+\delta_1}{\chi}+1,\alpha_1\right)(\rho_1,m)(p,\delta_1)-\omega s^{\alpha_1}\end{array}\right]. \tag{77}$$

6. Inverse Operators with Some Special Functions

Here, we discuss some applications of the inverse fractional operator. We derive some results of inverse fractional operator with Mittag-Leffler function and Bessel–Maitland function and obtain results in terms of Fox–Wright function.

Theorem 8. *Let* $\eta_1,\vartheta,\varepsilon,\varpi_1,\varrho,\xi_1,\nu_1,\alpha_1,\varphi_1,\rho_1,\delta_1,\in\mathbb{C}$ *with* $\min\{\Re(\vartheta),\Re(\varepsilon),\Re(\varpi_1),\Re(\varrho)\}>0$ *and* $\Re(\alpha_1)\geq 0$, $\Re(\varphi_1)\geq -1$, $\Re(\eta_1)>0,\Re(\xi_1)>0\Re(\rho_1),\Re(\nu_1)>0$, $\Re(\mu_1)>0,\Re(\delta_1)>0$, *and* $\upsilon,\varsigma,\kappa,\pi>0$ *and* $\gamma,q,\zeta_1,m,p\geq 0$; *then, the following result holds:*

$$\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}(\tau-a)^{\rho_1-1}E^{\mu_1,\nu_1,\upsilon,\varsigma,\varpi_1}_{\vartheta,\varrho,\kappa,\varepsilon,\pi}(\tau-a)^{\lambda}\right](s)=\frac{\Gamma(\varrho)\Gamma(\varepsilon)\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)\Gamma(\upsilon)\Gamma(\mu_1)}\sum_{m=0}^{\infty}\frac{\Gamma(\rho_1+\lambda m)\Gamma(\nu_1+\varsigma m)\Gamma(\mu_1+\nu_1 m)\Gamma(\upsilon+\varsigma m)\left((s-a)^{-\varphi_1+\rho_1+\lambda m}\right)}{\Gamma(\vartheta m+\varpi_1)\Gamma(\varrho+\kappa m)\Gamma(\varepsilon+\pi m)}\times{}_4\psi_3\left[\begin{array}{c}(\eta_1,\gamma)(\xi_1,q)(\nu_1,\zeta_1)(1,1)\\ (\rho_1,m)(\delta_1,p)(h-\varphi_1+1+\rho_1+\lambda m,\alpha_1)\end{array}\bigg|-\omega(s-a)^{\alpha_1}\right]. \tag{78}$$

Proof. Consider inverse fractional integral operator (31) with generalized Mittag-Leffler function defined in (23); then, the following result holds:

$$\begin{aligned}
&\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}(\tau-a)^{\rho_1-1}E^{\mu_1,\nu_1,\upsilon,\varsigma,\varpi_1}_{\vartheta,\varrho,\kappa,\varepsilon,\pi}(\tau-a)^{\lambda}\right](s)=\left(\frac{d}{ds}\right)^{h}\\
&\int_a^s(s-\tau)^{h-\varphi_1}J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}\left(w(s-\tau)^{\alpha_1}\right)(\tau-a)^{\rho_1-1}\sum_{m=0}^{\infty}\frac{(\mu_1)_{\upsilon m}(\nu_1)_{\varsigma m}(\tau-a)^{\lambda m}}{\Gamma(\vartheta m+\varpi_1)(\varrho)_{\kappa m}(\varepsilon)_{\pi m}}d\tau\\
&=\left(\frac{d}{ds}\right)^{h}\int_0^s(s-\tau)^{h-\varphi_1}\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-w(s-\tau)^{\alpha_1})}{\Gamma(\alpha_1 n+h-\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}(\tau-a)^{\rho_1-1}\\
&\times\sum_{m=0}^{\infty}\frac{(\mu_1)_{\upsilon m}(\nu_1)_{\varsigma m}(\tau-a)^{\lambda m}}{\Gamma(\vartheta m+\varpi_1)(\varrho)_{\kappa m}(\varepsilon)_{\pi m}}d\tau\\
&=\left(\frac{d^h}{ds^h}\right)J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)(\omega)^n\sum_{m=0}^{\infty}\frac{(\mu_1)_{\upsilon m}(\nu_1)_{\varsigma m}}{\Gamma(\vartheta m+\varpi_1)(\varrho)_{\kappa m}(\varepsilon)_{\pi m}}\\
&\int_a^s(s-\tau)^{h-\varphi_1+\alpha_1 n}(\tau-a)^{\rho_1-1+\lambda m}d\tau.
\end{aligned} \tag{79}$$

Substituting $u = s - \tau/s - a$ implies $u(s-a) = s - \tau$, $-d\tau = (s-a)du$, and $\tau = a, u = 1$, $\tau = s, u = 0$ in (79), and we have

$$\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}(\tau-a)^{\rho_1-1}E^{\mu_1,\nu_1,\upsilon,\varsigma,\varpi_1}_{\vartheta,\varrho,\kappa,\varepsilon,\pi}(\tau-a)^{\lambda}\right](s) = \left(\frac{\mathrm{d}^h}{\mathrm{d}s^h}\right)J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)(\omega)^n \sum_{m=0}^{\infty}\frac{(\mu_1)_{\upsilon m}(\nu_1)_{\varsigma m}}{\Gamma(\vartheta m+\varpi_1)(\varrho)_{\kappa m}(\varepsilon)_{\pi m}}(s-a)^{h-\varphi_1+\alpha_1 n+\rho_1+\lambda m}\int_0^1 u^{h+\alpha_1 n-\varphi_1}(1-u)^{\rho_1+\lambda m-1}\mathrm{d}u. \tag{80}$$

From (80), we get

$$\begin{aligned}&\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}(\tau-a)^{\rho_1-1}E^{\mu_1,\nu_1,\upsilon,\varsigma,\varpi_1}_{\vartheta,\varrho,\kappa,\varepsilon,\pi}(\tau-a)^{\lambda}\right](s)\\ &= \left(\frac{\mathrm{d}^h}{\mathrm{d}s^h}\right)J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)(\omega)^n \sum_{m=0}^{\infty}\frac{(\mu_1)_{\upsilon m}(\nu_1)_{\varsigma m}}{\Gamma(\vartheta m+\varpi_1)(\varrho)_{\kappa m}(\varepsilon)_{\pi m}}\\ &\times (s-a)^{h-\varphi_1+\alpha_1 n+\rho_1+\lambda m}\frac{\Gamma(h+\alpha_1 n-\varphi_1+1)\Gamma(\rho_1+\lambda m)}{\Gamma(h+\alpha_1 n-\varphi_1+1+\rho_1+\lambda m)}.\end{aligned} \tag{81}$$

Now on back substitution, we obtain

$$\begin{aligned}&= \left(\frac{\mathrm{d}^h}{\mathrm{d}s^h}\right)\sum_{m,n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-\omega)^n(s-a)^{h-\varphi_1+\alpha_1 n+\rho_1+\lambda m}}{(\rho_1)_{mn}(\delta_1)_{pn}\Gamma(h+\alpha_1 n-\varphi_1+1+\rho_1+\lambda m)}\,\frac{(\mu_1)_{\upsilon m}(\nu_1)_{\varsigma m}\Gamma(\rho_1+\lambda m)}{\Gamma(\vartheta m+\varpi_1)(\varrho)_{\kappa m}(\varepsilon)_{\pi m}}\\
&= \sum_{m,n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-\omega)^n\left(\mathrm{d}^h/\mathrm{d}s^h\right)(s-a)^{h-\varphi_1+\alpha_1 n+\rho_1+\lambda m}}{(\rho_1)_{mn}(\delta_1)_{pn}\Gamma(h+\alpha_1 n-\varphi_1+1+\rho_1+\lambda m)}\,\frac{(\mu_1)_{\upsilon m}(\nu_1)_{\varsigma m}\Gamma(\rho_1+\lambda m)}{\Gamma(\vartheta m+\varpi_1)(\varrho)_{\kappa m}(\varepsilon)_{\pi m}}\\
&= \sum_{m,n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-\omega)^n(s-a)^{-\varphi_1+\alpha_1 n+\rho_1+\lambda m}}{(\rho_1)_{mn}(\delta_1)_{pn}\Gamma(h+\alpha_1 n-\varphi_1+1+\rho_1+\lambda m)}\,\frac{(\mu_1)_{\upsilon m}(\nu_1)_{\varsigma m}\Gamma(\rho_1+\lambda m)}{\Gamma(\vartheta m+\varpi_1)(\varrho)_{\kappa m}(\varepsilon)_{\pi m}}\\
&= \frac{\Gamma(\varrho)\Gamma(\varepsilon)\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)\Gamma(\upsilon)\Gamma(\mu_1)}\\
&\sum_{m,n=0}^{\infty}\frac{\Gamma(\mu_1+\upsilon m)\Gamma(\nu_1+\varsigma m)\Gamma(\rho_1+\lambda m)\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)(-\omega)^n(s-a)^{\alpha_1 n-\varphi_1+\rho_1+\lambda m}}{\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)\Gamma(h+\alpha_1 n-\varphi_1+1+\rho_1+\lambda m)\Gamma(\vartheta m+\varpi_1)\Gamma(\varrho+\kappa m)\Gamma(\varepsilon+\pi m)}\\
&= \frac{\Gamma(\varrho)\Gamma(\varepsilon)\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)\Gamma(\upsilon)\Gamma(\mu_1)}\sum_{m=0}^{\infty}\frac{\Gamma(\rho_1+\lambda m)\Gamma(\nu_1+\varsigma m)\Gamma(\mu_1+\nu_1 m)\Gamma(\upsilon+\varsigma m)\left(\left(s-a^{-\varphi_1+\rho_1+\lambda m}\right)\right.}{\Gamma(\vartheta m+\varpi_1)\Gamma(\varrho+\kappa m)\Gamma(\varepsilon+\pi m)}\\
&\times {}_4\psi_3\left[\begin{matrix}(\eta_1,\gamma)(\xi_1,q)(\nu_1,\zeta_1)(1,1)\\ (\rho_1,m)(\delta_1,p)(h-\varphi_1+1+\rho_1+\lambda m,\alpha_1)\end{matrix}\,\middle|\, -\omega(s-a)^{\alpha_1}\right],\end{aligned} \tag{82}$$

which completes the proof. □

Corollary 7. *If we replace φ_1 with $-\varphi_1$ in Theorem 8, we get*

$$\left[\beth^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}(\tau-a)^{\rho_1-1}E^{\mu_1,\nu_1,\upsilon,\varsigma,\varpi_1}_{\vartheta,\varrho,\kappa,\varepsilon,\pi}(\tau-a)^{\lambda}\right](s)=\frac{\Gamma(\varrho)\Gamma(\varepsilon)\Gamma(\rho_1)\Gamma(\delta_1)((s-a)^{\varphi_1+\rho_1+\lambda m})}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)\Gamma(\upsilon)\Gamma(\mu_1)}\times\sum_{m=0}^{\infty}\frac{\Gamma(\rho_1+\lambda m)\Gamma(\nu_1+\varsigma m)\Gamma(\mu_1+\nu_1 m)\Gamma(\upsilon+\varsigma m)}{\Gamma(\vartheta m+\varpi_1)\Gamma(\varrho+\kappa m)\Gamma(\varepsilon+\pi m)}\times{}_4\Psi_3\left[\begin{matrix}(\eta_1,\gamma)(\xi_1,q)(\nu_1,\zeta_1)(1,1)\\(\rho_1,m)(\delta_1,p)(h+\varphi_1+1+\rho_1+\lambda m,\alpha_1)\end{matrix}\bigg|-\omega(s-a)^{\alpha_1}\right]. \tag{83}$$

Theorem 9. *Let $\eta_1,\xi_1,\nu_1,\alpha_1,\varphi_1,\rho_1,\delta_1,\in\mathbb{C}$, $\Re(\alpha_1)\geq 0$, $\Re(\varphi_1)\geq -1$, $\Re(\eta_1)>0,\Re(\xi_1)>0,\Re(\nu_1)>0,\Re(\rho_1)>0,\Re(\delta_1)>0$, and $\gamma,q,\zeta_1,m,p\geq 0$; then, the following result holds:*

$$\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}(\tau)^{-1}J^{\alpha_1}_{\beta_1}(\tau)J^{\alpha_1,\gamma}_{\beta_1,q}\left(\tau^{-\lambda}\right)\right](s)=\frac{\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\times\sum_{m,n=0}^{\infty}\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)}{\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)}\,\frac{(-s)^m\left(-\omega s^{\alpha_1}\right)^n s^{-\varphi_1}}{m!\Gamma(\alpha_1 m+\beta_1+1)}\times{}_2\Psi_2\left[\begin{matrix}(\gamma,q)(m,-\lambda)\\(\varphi_1+1,\alpha_1)(\alpha_1 n+h-\varphi_1+m+1,-\lambda)-s^{-\lambda}\end{matrix}\right]. \tag{84}$$

Proof. Consider inverse fractional integral operator with the product of two Bessel–Maitland functions; then, the following result holds:

$$\begin{aligned}&\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}(\tau)^{-1}J^{\alpha_1}_{\beta_1}(\tau)J^{\alpha_1,\gamma}_{\beta_1,q}\left(\tau^{-\lambda}\right)\right](s)\\&=\left(\frac{d^h}{ds^h}\right)\int_0^s(s-\tau)^{h-\varphi_1}\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}\left(-\omega(s-\tau)^{\alpha_1}\right)^n}{\Gamma(\alpha_1 n+h-\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\\&\times\sum_{m=0}^{\infty}\frac{\tau^{-1}(-\tau)^m}{m!\Gamma(\alpha_1 m+\beta_1+1)}\times\sum_{\omega=0}^{\infty}\frac{(\eta_1)_{q\omega}(-1)^{\omega}\tau^{-\lambda\omega}}{\omega!\Gamma(\alpha_1\omega+\beta_1+1)}d\tau\\&=\left(\frac{d^h}{ds^h}\right)\quad J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)\int_0^s\sum_{m,\omega=0}^{\infty}(-1)^{m+\omega}s^{h-\varphi_1+\alpha_1 n}\left(1-\frac{\tau}{s}\right)^{h-\varphi_1+\alpha_1 n}\\&\times\frac{(\omega)^n(\eta_1)_{q\omega}}{\Gamma(\alpha_1\omega+\beta_1+1)\Gamma(\alpha_1 m+\beta_1+1)m!\omega!}\tau^{-1+m-\lambda\omega}d\tau\\&=\left(\frac{d^h}{ds^h}\right)J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)\sum_{m,\omega=0}^{\infty}\frac{(\omega)^n(\eta_1)_{q\omega}(-1)^{m+\omega}s^{h-\varphi_1+\alpha_1 n}}{\Gamma(\alpha_1\omega+\beta_1+1)\Gamma(\alpha_1 m+\beta_1+1)m!\omega!}\int_0^s\tau^{-1+m-\lambda\omega}\left(1-\frac{\tau}{s}\right)^{h-\varphi_1+\alpha_1 n}d\tau.\end{aligned} \tag{85}$$

Putting the values $u=\tau/s, d\tau = s du$, $\tau = 0, u = 0$, and $\tau = s, u = 1$, we have

$$
\begin{aligned}
&= \left(\frac{d^h}{ds^h}\right) J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)\sum_{m,\omega}^{\infty}\frac{(\omega)^n(\eta_1)_{q\omega}(-1)^{m+\omega}s^{h-\varphi_1+\alpha_1 n}}{\Gamma(\alpha_1\omega+\beta_1+1)\Gamma(\alpha_1 m+\beta_1+1)m!\omega!}\int_0^1(1-u)^{h-\varphi_1+\alpha_1 n}(us)^{m-\lambda\omega-1}s du\\
&= \left(\frac{d^h}{ds^h}\right) J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)\sum_{m,\omega}^{\infty}\frac{(\omega)^n(\eta_1)_{q\omega}(-1)^{m+\omega}s^{h-\varphi_1+\alpha_1 n+m-\lambda\omega}}{\Gamma(\alpha_1\omega+\beta_1+1)\Gamma(\alpha_1 m+\beta_1+1)m!\omega!}\int_0^1(1-u)^{h-\varphi_1+\alpha_1 n}u^{m-\lambda\omega-1}du\\
&= \left(\frac{d^h}{ds^h}\right) J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)\sum_{m,\omega}^{\infty}\frac{(\omega)^n(\eta_1)_{q\omega}s^{\alpha_1 n}(-1)^{m+\omega}s^{h-\varphi_1+m-\lambda\omega}}{\Gamma(\alpha_1\omega+\beta_1+1)\Gamma(\alpha_1 m+\beta_1+1)m!\omega!}\frac{\Gamma(h-\varphi_1+\alpha_1 n+1)\Gamma(m-\lambda\omega)}{\Gamma(h-\varphi_1+\alpha_1 n+1+m-\lambda\omega)}\\
&= \left(\frac{d^h}{ds^h}\right) J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)\sum_{m,\omega}^{\infty}\frac{(\omega)^n(\eta_1)_{q\omega}s^{\alpha_1 n}(-1)^{m+\omega}s^{h-\varphi_1+m-\lambda\omega}}{\Gamma(\alpha_1\omega+\beta_1+1)\Gamma(\alpha_1 m+\beta_1+1)m!\omega!}\frac{\Gamma(h-\varphi_1+\alpha_1 n+1)\Gamma(m-\lambda\omega)}{\Gamma(h-\varphi_1+\alpha_1 n+1+m-\lambda\omega)}\\
&= \sum_{m,n,\omega=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}}{\Gamma(\alpha_1 n+h-\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\\
&\quad\times\frac{(-\omega)^n(\eta_1)_{q\omega}s^{\alpha_1 n}(-1)^{m+\omega}(s)^{-\varphi_1+m-\lambda\omega}}{\Gamma(\alpha_1\omega+\beta_1+1)\Gamma(\alpha_1 m+\beta_1+1)m!\omega!}\frac{\Gamma(h-\varphi_1+\alpha_1 n+1)\Gamma(m-\lambda\omega)}{\Gamma(h-\varphi_1+\alpha_1 n+1+m-\lambda\omega)}=\frac{\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\\
&\quad\cdot\sum_{m,n=0}^{\infty}\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)}{\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)}\frac{(-1)^m s^{\alpha_1 n}s^{-\varphi_1+m}(-1)^{\omega}s^{-\lambda\omega}(-\omega)^n}{m!\Gamma(\alpha_1 m+\beta_1+1)}\\
&\quad\times\frac{\Gamma(m-\lambda\omega)\Gamma(\nu_1+q\omega)}{\omega!\Gamma(\alpha_1\omega+\beta_1+1)\Gamma(\alpha_1 n+h-\varphi_1+m-\lambda\omega+1)}\\
&= \frac{\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\sum_{m,n=0}^{\infty}\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)}{\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)}\frac{(-s)^m(-\omega s^{\alpha_1})^n s^{-\varphi_1}}{m!\Gamma(\alpha_1 m+\beta_1+1)}\\
&\quad\times {}_2\psi_2\left[\begin{matrix}(\gamma,q)(m,-\lambda)\\(\beta_1+1,\alpha_1)(\alpha_1 n+h-\varphi_1+m+1,-\lambda)-s^{-\lambda}\end{matrix}\right],
\end{aligned}
\tag{86}
$$

which is the required result. □

Corollary 8. *If we replace φ_1 by $-\varphi_1$ in Theorem 9, we have the result obtained by left inverse operator:*

$$
\begin{aligned}
&\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}(\tau)^{-1}J^{\alpha_1}_{\beta_1}(\tau)J^{\alpha_1,\gamma}_{\beta_1,q}\left(\tau^{-\lambda}\right)(s)\right]=\frac{\Gamma(\rho_1)\Gamma(\delta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\\
&\quad\times\sum_{m,n=0}^{\infty}\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)}{\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)}\times\frac{(-s)^m(-\omega s^{\alpha_1})^n s^{\varphi_1}}{m!\Gamma(\alpha_1 m+\beta_1+1)}\\
&\quad\times {}_2\psi_2\left[\begin{matrix}(\gamma,q)(m,-\lambda)\\(\beta_1+1,\alpha_1)(\alpha_1 n+h+\varphi_1+m+1,-\lambda)-s^{-\lambda}\end{matrix}\right].
\end{aligned}
\tag{87}
$$

Theorem 10. *Let* $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \in \mathbb{C}$, $\Re(\alpha_1) \geq 0$, $\Re(\varphi_1) \geq -1$, $\Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0$, *and* $\gamma, q, \zeta_1, m, p \geq 0$; *then, the following relation holds:*

$$\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\frac{\tau^{\alpha_1/\beta_1}}{\tau}\ {}_2R_1\left(\frac{\alpha_1}{\beta_1}+\xi_1,-\eta_1;\frac{\alpha_1}{\beta_1};\tau\right)\right](s)=\sum_{m,n=0}^{\infty}\frac{(\alpha_1/\beta_1+\xi_1)_m(-\eta_1)_m}{m!(\alpha_1 n-\varphi_1+1+\alpha_1/\beta_1)_m}\times\frac{(-\omega s^{\alpha_1})^n\Gamma(\rho_1)\Gamma(\delta_1)\Gamma(\alpha_1/\beta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)}{\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)\Gamma(\alpha_1 n-\varphi_1+\alpha_1/\beta_1+m)}s^{-\varphi_1+\frac{\alpha_1}{\beta_1}+m}. \tag{88}$$

Proof. Consider fractional integral operator with Gauss hypergeometric function; then, the following result holds:

$$\begin{aligned}
&\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\frac{\tau^{\alpha_1/\beta_1}}{\tau}\ {}_2R_1\left(\frac{\alpha_1}{\beta_1}+\xi_1,-\eta_1;\frac{\alpha_1}{\beta_1};\tau\right)\right](s)\\
&=\left(\frac{d^h}{ds^h}\right)\int_0^s(s-\tau)^{h-\varphi_1}\sum_{n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-\omega(s-\tau)^{\alpha_1})^n}{\Gamma(\alpha_1 n+\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\tau^{\frac{\alpha_1}{\beta_1}-1}\sum_{m=0}^{\infty}\frac{(\alpha_1/\beta_1+\xi_1)_m(-\eta_1)_m}{(\alpha_1/\beta_1)_m m!}\tau^m d\tau\\
&=\left(\frac{d^h}{ds^h}\right)J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)(\omega)^n\sum_{m=0}^{\infty}\frac{(\alpha_1/\beta_1+\xi_1)_m(-\eta_1)_m}{(\alpha_1/\beta_1)_m m!}s^{h-\varphi_1+\alpha_1 n}\times\int_0^s\left(1-\frac{\tau}{s}\right)^{h-\varphi_1+\alpha_1 n}\tau^{\frac{\alpha_1}{\beta_1}-1+m}d\tau.
\end{aligned} \tag{89}$$

Substituting $u=\tau/s$ implies $d\tau = \mathrm{sdu}$, $\tau = 0, u = 0,$ and $\tau = s, u = 1$ in equation (89), and we have

$$\begin{aligned}
&\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\frac{\tau^{\alpha_1/\beta_1}}{\tau}\ {}_2R_1\left(\frac{\alpha_1}{\beta_1}+\xi_1,-\eta_1;\frac{\alpha_1}{\beta_1};\tau\right)\right](s)\\
&=\left(\frac{d^h}{ds^h}\right)J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)(\omega)^n\sum_{m=0}^{\infty}\frac{(\alpha_1/\beta_1+\xi_1)_m(-\eta_1)_m}{(\alpha_1/\beta_1)_m m!}s^{h-\varphi_1+\alpha_1 n}\int_0^1(1-u)^{h-\varphi_1+\alpha_1 n}(us)^{\frac{\alpha_1}{\beta_1}+m-1}\mathrm{sdu}\\
&=\left(\frac{d^h}{ds^h}\right)J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{h-\varphi_1,\gamma,q,\zeta_1,m,p}(1)(\omega)^n\sum_{m=0}^{\infty}\frac{(\alpha_1/\beta_1+\xi_1)_m(-\eta_1)_m}{(\alpha_1/\beta_1\alpha_1/\beta_1)_m m!}s^{h-\varphi_1+\alpha_1 n+\frac{\alpha_1}{\beta_1}+m}\int_0^1(1-u)^{h-\varphi_1+\alpha_1 n}(u)^{\frac{\alpha_1}{\beta_1}+m-1}du\\
&=\left(\frac{d^h}{ds^h}\right)\sum_{m,n=0}^{\infty}\frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-\omega)^n}{\Gamma(\alpha_1 n+h-\varphi_1+1)(\rho_1)_{mn}(\delta_1)_{pn}}\frac{(\alpha_1/\beta_1+\xi_1)_m(-\eta_1)_m}{(\alpha_1/\beta_1)_m m!}s^{h-\varphi_1+\alpha_1 n+\frac{\alpha_1}{\beta_1}+m}\\
&\times\frac{\Gamma(h-\varphi_1+\alpha_1 n+1)\Gamma(\alpha_1/\beta_1+m)}{\Gamma(h-\varphi_1+\alpha_1 n+1+\alpha_1/\beta_1+m)}\\
&=\sum_{m,n=0}^{\infty}\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)\Gamma(\rho_1)\Gamma(\delta_1)(-\omega)^n}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)}\frac{(\alpha_1/\beta_1+\xi_1)_m(-\eta_1)_m}{(\alpha_1/\beta_1)_m m!}\left(\frac{d^h}{ds^h}\right)s^{h-\varphi_1+\alpha_1 n+\frac{\alpha_1}{\beta_1}+m}\\
&\times\frac{\Gamma(\alpha_1/\beta_1+m)}{\Gamma(\alpha_1 n+h-\varphi_1+1+\alpha_1/\beta_1+m)}\\
&=\sum_{m,n=0}^{\infty}\frac{(\alpha_1/\beta_1+\xi_1)_m(-\eta_1)_m}{m!(\alpha_1 n-\varphi_1+1+\alpha_1/\beta_1)_m}\frac{(-\omega s^{\alpha_1})^n\Gamma(\rho_1)\Gamma(\delta_1)\Gamma(\alpha_1/\beta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\\
&\times\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)}{\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)\Gamma(\alpha_1 n-\varphi_1+\alpha_1/\beta_1+m)}s^{-\varphi_1+\frac{\alpha_1}{\beta_1}+m}.
\end{aligned} \tag{90}$$

□

Corollary 9. *If we replace φ_1 by $-\varphi_1$ in Theorem 10, we have the following new result:*

$$\left[\mathfrak{D}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;0^+}\frac{\tau^{\alpha_1/\beta_1}}{\tau}\ {}_2R_1\left(\frac{\alpha_1}{\beta_1}+\xi_1,-\eta_1;\frac{\alpha_1}{\beta_1};\tau\right)\right](s)=\sum_{m,n=0}^{\infty}\frac{(\alpha_1/\beta_1+\xi_1)_m(-\eta_1)_m}{m!\,(\alpha_1 n+\varphi_1+1+\alpha_1/\beta_1)_m}$$
$$\times\frac{(-\omega s^{\alpha_1})^n\Gamma(\rho_1)\Gamma(\delta_1)\Gamma(\alpha_1/\beta_1)}{\Gamma(\eta_1)\Gamma(\xi_1)\Gamma(\nu_1)}\frac{\Gamma(\eta_1+\gamma n)\Gamma(\xi_1+qn)\Gamma(\nu_1+\zeta_1 n)}{\Gamma(\rho_1+mn)\Gamma(\delta_1+pn)\Gamma(\alpha_1 n+\varphi_1+\alpha_1/\beta_1+m)}s^{-\varphi_1+\frac{\alpha_1}{\beta_1}+m}. \tag{91}$$

7. Conclusions

In our present investigation, the extended version of Bessel–Maitland function (EvBMF), generalized fractional integral, and differential operators having EvBMF as their kernel have been established. Further, we discussed the convergent and bounded behavior of generalized fractional operator and also established its relation with other fractional operators. Moreover, we discussed the integral transforms of generalized fractional operator and also developed some applications. These extensions and generalizations have great contribution in the field of fractional operators and series type special functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors have read and approved the final manuscript.

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