Research Article

A New Graphical Representation of the Old Algebraic Structure

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The most recent advancements in algebra and graph theory enable us to ask a straightforward question: what practical use does this graph connected with a mathematical system have in the real world? With the use of algebraic approaches, we may now tackle a wide range of graph theory-related problems. We compute loop-involutions and nonself-loop-involutions of flexible weak inverse property loops. Importantly, diameter, radius, and eccentricity of this loop structure’s inverse graph have all been calculated in a broad manner. Some topological indices, as an application of inverse graph, are also given at the end of the paper.

1. Introduction and Definitions

The Königsberg problem was one of mathematics’s most intriguing and illustrious puzzles in 1735. There was a well-known Pregel River in the Russian city of Königsberg that had seven bridges. Four districts of the city were connected by these bridges. The Russian president studied mathematics as well. He questioned whether one could explore every area of the community by crossing each bridge just once. He made several attempts to remedy this issue but was unsuccessful. Finally, he sent a letter to the Swiss mathematician Leonhard Euler asking for an explanation of the issue. Although Euler denied this letter and claimed it was not a work of mathematics, the query caught his interest [1]. He made a lot of effort.

In his study article from 1735, Euler explained the connection between this concept and others. His essay was 21 pages long. He thought that although this question has to do with geometry, this geometry is not the same as measurements and shapes [2]. This kind of geometry is novel. He created a sketch of this issue. He used dots for the terrain and lines for the bridges. Euler’s path concept was then presented. Odd-degree nodes should be two or zero in number, and if there are two nodes of odd degree, they serve as the starting and ending points of a graph path that moves each edge just once. As a result, the sketch must follow the Euler path to provide the correct response to the earlier query. However, there are four odd-degree vertices in the given design. As a result, it deviates from Euler’s path. As a result, there is no way to respond to the issue mentioned above, and the Königsberg problem provided the basis for the first graph theory theorem [3].

In the field of mathematics known as algebraic graph theory, graph-related problems are solved using algebraic approaches [4]. The area of algebraic graph theory known as group theory examines graphs in relation to the group theory. From 1975 to 2022, different researchers used algebraic structures, groups, rings, and loops, to overcome the problems connected with the graph theory. The author demonstrated, with reference to [5], that the vertex independence number of the intersecting graph connected to this Abelian group is a maximal prime order of the nontrivial subgroup of the finite Abelian group. He also provided this number for $p$ groups. However, eight years after this article’s publication, different mathematicians introduced a fresh concept for determining the vertex independence number of a finite simple graph in terms of the vertex degrees [6].
Cayley’s graphs for cyclic, symmetric, and dihedral groups, which are studied in the paper cited in [7], are another illustration of the link between algebra and graph theory. In 1990, the author used connected components of finite simple graphs, whose vertices were the noncentral conjugacy classes of the group explored by [8] and the class of finite groups whose Cayley’s graphs are planar to characterise well-known groups, quasi-Frobenius groups [9]. Inspired work has been carried out in [10] to connect the total graph and its related induced subgraphs, the zero-divisor graph, the nilpotent elements graph, and the regular elements graph, which correspond to the commutative ring and its subsets, the sets of zero-divisors, the nilpotent elements, and the regular elements, and the substructures and comaximal ideals, of the ring associated with finite graphs were introduced attractively in [11].

The unit graph is a finite simple graph in which any two different vertices have an edge if their total is the unit element of the ring and each node is an element of the finite ring with nonzero identity. In [12], planarity, girth, diameter, chromatic index, and connectedness of unit graph were thoroughly explored. Mathematicians established novel ideas of cototal graph and counit graph, examined structural features of rings, and characterised a variety of rings after the algebraic and graphical properties connected to total graphs, unit graphs, and comaximal graphs [13]. To study graph-theoretic properties with ring-theoretic properties, it is important to examine the set of the nonzero zero-divisors of the commutative ring with identity. The problems of when the zero-divisor graph of the commutative ring with identity is empty, finite, and connected are quite reasonable. The first authors to respond to these inquiries in a general way can be seen in [14].

In addition, it would be interesting to know what conditions exist on the ring for the unit graph to become Hamiltonian. The answer to this question was written by some authors with a citation to [15]. In-depth research was done on a very significant finding, “unit and unitary Cayley’s graph corresponding to a ring has genus 1 if and only if ring is the commutative Artinian” see [16]. The prime graphs connected to solvable finite groups provide as another illustration of algebraic graph theory.

The authors introduced a new class of graphs called prime graphs in [17], where any two vertices \( x_1 \) and \( x_2 \) are the prime divisors of the order of finite group and they are adjacent if the group has an element of order \( x_1 x_2 \). In addition, they invented the notion of prime graphs for infinitely many solvable groups. During his work on colouring the graphs, the first mathematician to propose the concept of zero-divisor graphs of the algebraic structure; commutative ring with nonzero identity was [18]. Different writers diligently investigated the fundamental characteristics of directed zero-divisor graphs connected to the ring of upper triangular matrices in 2013 by [19].

The reference paper [20] also gave an essential direction for the future studies and connectedness of simple graphs, square elements graphs of finite rings, was given in [21, 22] attractively in the manner of very important characteristics of Cayley graphs, commuting graphs, intersection graphs, prime graphs, noncommuting graphs, conjugacy class graphs, and inverse graphs of finite groups. Researchers in computer science and mathematics are currently interested in determining the energy of simple graphs, and the authors of [23] demonstrated some important findings involving inverse graphs of two well-known kinds of finite groups.

Upper bounds for the clique number and maximum degree associated with square graphs are given in [24, 25], where authors also computed the result regarding the number of edges of this graph. Some authors calculated some results of equal-square graphs, a subclass of simple graphs where all the elements of the finite group are the vertices and any two distinct vertices \( x_1, x_2 \) are adjacent iff \( x_1^2 = x_2^2 \) see [26]. With the aid of loop structures and a finite Boolean commutative ring, the authors of [27, 28] discovered properties of balanced bipartite graphs and zero-divisor graphs, while [29] gave a concept of directed inverse graphs of anti-automorphic inverse property loops and star graphs of substructures of these loops through edge labelings.

In order to clarify key characteristics of cycles, writers combined three well-known branches of mathematics: algebra, linear algebra, and graph theory. They also looked into a class of Koszul algebras using these cycles [30]. Interrelationship of complicated algebraic objects, burnside groups, and collectives of automata is nicely done in [31], and the study of chromatic numbers, clique numbers, cycles, and distance graphs presented comprehensively in the papers [32, 33].

Chemical graph theory is a fascinating area of mathematics that combines graph theory and chemistry. Through mathematical methods, the molecules from the Chemistry model produce a molecular graph. Atoms serve as the vertices and chemical bonds serve as the edges of a molecular graph. To get their different topological and structural characteristics, these molecular formations have been subjected to a variety of graph theory techniques [34]. For instance, the degree and distance between the vertices of a chemical compound can be used to determine the boiling point of the chemical compound. As a result, mathematicians might state that when describing a chemical problem when it comes of mathematical form, the molecular structure’s topology is important in deciding the usable characteristics of the related chemical compound [35].

In 1988, it was estimated that a few hundred professional researchers produced 500 study publications annually researching different chemical structural aspects, including meticulously researched contents see [36]. Commercial, pharmaceutical, and industrial chemistry make extensive use of a number of chemical compounds possesses distinct mathematical structures. Atomic arrangements inside chemical compounds follow clear structural laws that have practical hidden properties. Chemical graph theory is a predominant bough of graph theory in practical research that explores these aspects using combinatorics and topology as mathematical tools. It is important to note that chemical graph theory has benefited much from mathematical chemistry [37–39]. Many of the invariants found in chemical graph theory, like indices or descriptors, are used
in a variety of commercial areas, most notably in the chemical and pharmaceutical industries [40, 41].

More specifically, the development of associated fields is actively influenced by the study of degree-based indices. For further study in this direction, one may read the papers bounds on the partition dimension of convex polytopes, on the partition dimension of trihexagonal alpha-boron nanotube, the locating number of hexagonal Mobius ladder network, and lower bounds for Gaussian Estrada index of graphs [42, 43]. It is desirable to gather a lot of information in the way of numerical values connected to chemical structures using modern computer systems and to compare them [44]. In order to satisfy the demands of chemists, many topological descriptors were developed in the final ten years of the nineteenth century [45, 46].

With the help of two foremost mathematical objects \( * : Q_{FW}(Q_{FW}) \rightarrow Q_{FW}(Q_{FW}) \), a binary operation defined on \( Q_{FW}(Q_{FW}) \) and \( Q_{FW}(Q_{FW}), \) a nonempty set, the pair \((Q_{FW}(Q_{FW}), \ast)\) is said to be a groupoid and this algebraic structure is quasigroup if and only if \((Q_{FW}(Q_{FW}), \ast)\) is a groupoid with this algebraic structure is quasigroup if and only if \((Q_{FW}(Q_{FW}), \ast)\) is a groupoid, and lower bounds for Gaussian Estrada index of graphs [42, 43]. It is desirable to gather a lot of information in the way of numerical values connected to chemical structures using modern computer systems and to compare them [44]. In order to satisfy the demands of chemists, many topological descriptors were developed in the final ten years of the nineteenth century [45, 46].

An equation \( x \ast e = e \ast x = x \forall x \in Q_{FW}(Q_{FW}) \) can be taken as identity law and in the presence of neutral element \( e \) of \( Q_{FW}(Q_{FW}) \) any quasigroup \((Q_{FW}(Q_{FW}), \ast)\) or simply \( Q_{FW}(Q_{FW}) \) with this law is said to be a loop. The left and right inverse of each element in a group are always the same; however, this is not true for loops. Self bijectons \( J_1 : Q_{FW}(Q_{FW}) \rightarrow Q_{FW}(Q_{FW}) \) defined by \( J_1(x) = x^l \) and \( J_2 : Q_{FW}(Q_{FW}) \rightarrow Q_{FW}(Q_{FW}) \) by \( J_2(x) = x^r \forall x \in Q_{FW}(Q_{FW}) \) are called left inverse permutation and right inverse permutation, respectively. Moreover, in addition of the following identities, equations (3) and (4) are as follows:

\[
x_{1} \ast (x_{2} \ast x_{1}) = (x_{1} \ast x_{2}) \ast x_{1},
\]

\[
x_{1} \ast (x_{2} \ast x_{1})^l = x_{2}^l,
\]

\[
\forall x_{1}, x_{2} \in Q_{FW}(Q_{FW}),
\]

any loop can be taken as flexible weak inverse property loop. Left-non-self-loop-involution of loop \( Q_{FW}(Q_{FW}) \) is an element \( x \) in \( Q_{FW}(Q_{FW}) \) such that \( x^l \neq x \) and \( x^r \neq x \) has a unique left inverse \( x^l \). We denote the set of these elements of \( Q_{FW}(Q_{FW}) \) by \( I^l \) and it is straightforward to say that \( I^l \subset Q_{FW}(Q_{FW}) \). In the similar way, we can define \( I^r \). For the left inverse graph \( G_{Q_{FW}}(I^l) \), a finite simple graph, we assign all the elements of \( Q_{FW}(Q_{FW}) \) as vertices and any two distinct vertices \( x_{1}, x_{2} \) of \( G_{Q_{FW}}(I^l) \) are adjacent if and only if either \( x_{1} \ast x_{2} \in I^l \) or \( x_{2} \ast x_{1} \in I^l \) and analogously we can define \( G_{Q_{FW}}(I^r) \). Let \( G_{Q_{FW}}(I^l) \) be the connected graph then the distance \( d(x_{1}, x_{2}) \) between two different vertices \( x_{1}, x_{2} \) is the shortest path’s length from \( x_{1} \) to \( x_{2} \) in \( G_{Q_{FW}}(I^l) \) and the number of edges that a vertex \( x \) shares is referred to as its degree \( d_{x} \). Also, \( ecc(x_{1}) = Max \{ d(x_{1}, x_{2}) : \forall x_{2} \in V(G_{Q_{FW}}(I^l)) \} \) is called eccentricity of vertex \( x_{1} \), where minimum and maximum eccentricity are taken as radius and diameter of \( G_{Q_{FW}}(I^l) \) denoted by \( rad(G_{Q_{FW}}(I^l)) \) and \( diam(G_{Q_{FW}}(I^l)) \), respectively.

According to Gutman and Trinajstic [47, 48], the Zagreb indices are as follows:

\[
Z_{1}(G) = \sum_{x_{1}, x_{2} \in E(G)} (d_{x_{1}} + d_{x_{2}}),
\]

\[
Z_{2}(G) = \sum_{x_{1}, x_{2} \in E(G)} (d_{x_{1}} \times d_{x_{2}}).
\]

Following are Došlić’s definitions of the Zagreb indices [49]:

\[
Z_{1}^{m}(G) = \sum_{x_{1}, x_{2} \in E(G)} (d_{x_{1}} + d_{x_{2}}),
\]

\[
Z_{2}^{m}(G) = \sum_{x_{1}, x_{2} \in E(G)} (d_{x_{1}} \times d_{x_{2}}).
\]

In [50], Ghorbani and Azimi define the multiple Zagreb indices as follows:

\[
PZ_{1}(G) = \prod_{x_{1}, x_{2} \in E(G)} (d_{x_{1}} + d_{x_{2}}),
\]

\[
PZ_{2}(G) = \prod_{x_{1}, x_{2} \in E(G)} (d_{x_{1}} \times d_{x_{2}}).
\]

In [51], Albertson initiated the irregularity of \( G \) as follows:

\[
\text{irr}(G) = \sum_{e \in E(G)} \text{imb}(e) = \sum_{x_{1}, x_{2} \in E(G)} |d_{x_{1}} - d_{x_{2}}|.
\]

Two other most of the time utilized topological descriptors that give the measurement about irregularity of a graph are the Collatz–Sinogowitz index and the degree-based variance index see [52].

\[
\text{Var}(G) = \frac{1}{n} \sum_{i=1}^{n} d_{i}^{2} - \frac{1}{n^{2}} \left( \sum_{i=1}^{n} d_{i} \right)^{2},
\]

Also,

\[
\text{CS}(G) = \lambda_{1}(G) - \frac{2|E|}{|V|},
\]

where \( \lambda_{1} \) be the greatest eigenvalue of \( A = (a_{ij}) \), adjacency matrix, of the graph \( G \).

2. Main Results

Let \( C_{2} = \langle y; j^{2} = e_{C_{2}} \rangle \) be two cyclic groups under multiplication and addition, respectively, where \( \alpha \) is the positive integer and \( \{ f_{1}^{1}, f_{2}^{1}, f_{1}^{2}, f_{2}^{2} \} \) be the set of mappings \( f_{1}^{1}: C_{2} \rightarrow C_{2}, \; f_{2}^{1}: C_{2} \rightarrow C_{2}, \; f_{1}^{2}: C_{2} \rightarrow C_{2}, \; f_{2}^{2}: C_{2} \rightarrow C_{2} \), and
Let $\mathbb{I} = \{y, y^2 \in \mathbb{C}_2\}$ and $\mathbb{Z}_{2\alpha}$ be cyclic group of order 2 and group of residue classes of even modulo $2\alpha$. Then, $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$ is the flexible weak inverse property loop.

Remark 2. The sets of left-non-self-loop-involutions $I^1$ and right-non-self-loop-involutions $I^2$ of the flexible weak inverse property loop $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$ are always same. So, we can write $I^1 = I^2 = I_{(\lambda, \rho)}$.

Proof. Since $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$ is a flexible loop so left and right inverse of each element of this algebra is same. It completes the proof. \qed

Now, we move to write our first result associated with loop-involutions of this structure.

Theorem 3. Let $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$ be the flexible weak inverse property loop. Then, the set of loop-involutions of $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$ is $\{\{e_{\mathbb{C}_2}, 0\}, (y, 0)\}$.

Proof. Let $(y^p, z)$ be any element of $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$ with $p \in \{1, 2\}$ such that

\[
(y^p, z) \ast (y^p, z) = (e_{\mathbb{C}_2}, 0),
\]

\[
(y^p \mathbb{I}_z, y^p, z + 2z) = (e_{\mathbb{C}_2}, 0),
\]

\[
y^p \mathbb{I}_z \ast y^p = e_{\mathbb{C}_2},
\]

\[
y^p \mathbb{I}_z y^p = e_{\mathbb{C}_2},
\]

\[
y^p \mathbb{I}_z y^p = e_{\mathbb{C}_2},
\]

\[
y^p = y^2,
\]

\[
p = 1.
\]

We have only one nontrivial loop-involution $(y, 0)$. Thus, the desired set is $\{\{e_{\mathbb{C}_2}, 0\}, (y, 0)\}$. It completes the proof. \qed

Remark 4. Since $|\{\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast\}| = 4\alpha$, so we have $|I_{(\lambda, \rho)}| = 2(2\alpha - 1)$.

Proof. Because the order of the flexible loop is $4\alpha$ and it has only two loop-involutions by the previous theorem so the cardinality of $I_{(\lambda, \rho)}$ is $4\alpha - 2 = 2(2\alpha - 1)$.

It is natural to see that under what condition this inverse graph will be an empty graph. Following result gives the answer of this question.

Theorem 5. Let $G^{(\lambda, \rho)}_{(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ be the finite inverse graph of the flexible weak inverse property loop $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$. Then, it is an empty graph if and only if $I_{(\lambda, \rho)} = \emptyset$.

Proof. We suppose that the graph $G^{(\lambda, \rho)}_{(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ is empty. Then, by definition, for any $(y^p, z_1), (y^p, z_2) \in (\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$, with $(y^p, z_1) \neq (y^p, z_2)$ we have $(y^p, z_1) \ast (y^p, z_2) \notin I_{(\lambda, \rho)}$. Hence, $I_{(\lambda, \rho)}$ is vacuous set. The converse is obvious to do. It completes the proof. \qed

Theorem 6. For any finite loop $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$, the inverse graph $G^{(\lambda, \rho)}_{(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ is always not complete.

Proof. We suppose on contrary that there exists an inverse graph $G^{(\lambda, \rho)}_{(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ which is complete. Then, for each vertex $(y^p, z) \in V(G^{(\lambda, \rho)}_{(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)})$, we can write $deg((y^p, z)) = 4\alpha - 1$ where $|\{\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast\}| = 4\alpha$. Since $4\alpha$ is even and $deg((e_{\mathbb{C}_2}, 0)) = 4\alpha - 1$. Moreover, $|I_{(\lambda, \rho)}| = 4\alpha - 1$ is always even for any inverse graph. Thus, the process can get complete. \qed

Theorem 7. Let $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$ be the flexible weak inverse property loop. Then, $G^{(\lambda, \rho)}_{(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ has no any vertex of zero degree.

Proof. We suppose on contrary that $(y^p, z)$ is the vertex of zero degree of $G^{(\lambda, \rho)}_{(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)}$. Then, there will be following two cases:

Case 1: if $(y^p, z) \in I_{(\lambda, \rho)}$. But this is not possible for if $(y^p, z) \ast (e_{\mathbb{C}_2}, 0) \in I_{(\lambda, \rho)}$, where $(e_{\mathbb{C}_2}, 0)$ is the identity element of $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$ then $(y^p, z)$ is connected to $(e_{\mathbb{C}_2}, 0)$. This is the contradiction to our supposition that $(y^p, z)$ is isolated.

Case 2: if $(y^p, z) \notin I_{(\lambda, \rho)}$, then either $(y^p, z)$ is the identity or $(y^p, z)$ is the nontrivial self-loop-involution of $(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)$. It follows from our supposition that $(y^p, z)$ cannot be $(e_{\mathbb{C}_2}, 0)$. Hence, $(y^p, z)$ is the non-trivial self-loop-involution. Let $(y^p, z_1) \in I_{(\lambda, \rho)}$. We know that nontrivial self-loop-involution is only $(y, 0)$. So, $(y^p, z_1) \ast (y, 0) = (e_{\mathbb{C}_2}, 0)$. This implies $(y^p, z_1) = (y, 0)^2$ a contradiction. Therefore, $G^{(\lambda, \rho)}_{(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ has no isolated vertex. It completes the proof. \qed

Theorem 8. With the usual notations and symbols, the inverse graph $G^{(\lambda, \rho)}_{(\mathbb{C}_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ is connected.

Proof. As identity $(e_{\mathbb{C}_2}, 0)$ is adjacent to every element of $I_{(\lambda, \rho)}$. We are left to show that $(y, 0)$ is adjacent to each
element of $I_{(\lambda \rho)}$. For this, consider the product $(y^p, z) \ast (y, 0)$ where $(y^p, z) \in I_{(\lambda \rho)}$ and $\ast$ is the binary operation defined on $(C_2 \times \mathbb{Z}_{2\alpha}, \ast)$. Clearly, $(y^p, z) \ast (y, 0) \in I_{(\lambda \rho)}$. Thus, $(y, 0)$ is connected to every element of $I_{(\lambda \rho)}$. Therefore, each vertex of $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ is reachable and so $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ is connected. □

**Theorem 9.** With the usual notations and symbols, the diameter of inverse graph $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ is 2.

**Proof.** Due to the connectivity of $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$, associated with $(C_2 \times \mathbb{Z}_{2\alpha}, \ast)$, we consider the following vertex partition:

$$V(G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}) = \{e_{C_2}, 0\} \cup I_{(\lambda \rho)} \cup I_{(\lambda \rho)}^r,$$

where $I_{(\lambda \rho)}$ is the set of nonself-loop-involutions and $I_{(\lambda \rho)}^r = (y, 0)$ is the set of nontrivial self-loop-involution. Since the binary operation $\ast$ between any nonself-loop-involution of $I_{(\lambda \rho)}$ and $(e_{C_2}, 0)$ is also a nonself-loop-involution and the binary operation $\ast$ between $(e_{C_2}, 0)$ and $(y, 0)$ is not the nonself-loop-involution. So, we can write $ecc((e_{C_2}, 0)) = 2$.

For any arbitrary element $(y^p, z) \in I_{(\lambda \rho)}$, $ecc((y^p, z)) = 2$ because $(y^p, z) \ast (y^p, z)^1$ is the loop-involution and $(y^p, z) \ast (e_{C_2}, 0)$ and $(y^p, z) \ast (y, 0)$ are the nonself-loop-inventions. Also, $(y, 0)$ is not adjacent to $(e_{C_2}, 0)$ but adjacent to every vertex of $I_{(\lambda \rho)}$. Hence, $ecc((y, 0)) = 2$. Thus, $\text{di}am(G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}) = 2$. □

**Theorem 10.** If $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ is a nonempty inverse graph of order $4\alpha$ then the sum of the degrees is bounded above by $2(8\alpha^2 - 4\alpha + 1)$.

**Proof.** Let $V(G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}) = \{x_1, x_2, \ldots, x_{4\alpha}\}$. By the fundamental theorem of graph theory, we have

$$\sum_{j=1}^{4\alpha} \deg(x_j) = 2\left|\text{E}(G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)})\right|.$$  

Since $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ is not complete, so

$$\sum_{j=1}^{4\alpha} \deg(x_j) < 2 \left(\frac{4\alpha (4\alpha - 1)}{2}\right) = 4\alpha (4\alpha - 1).$$

Because $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ is a nonempty graph so $|I_{(\lambda \rho)}| \neq 0$. As any pair $(y^p, z_1), (y^p, z_2) \in I_{(\lambda \rho)}$ with $(y^p, z_1) = (y^p, z_2)$ has no edge in $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ such a pair provides 2 to the entire sum of the vertices’ degrees of $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$. Hence, a total of $|I_{(\lambda \rho)}|$ degrees will be shared by the elements of $I_{(\lambda \rho)}$ as a shortage in the sum of the degrees of $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$, so

$$\sum_{j=1}^{4\alpha} \deg(x_j) \leq 4\alpha (4\alpha - 1) - |I_{(\lambda \rho)}|,$$

which is equal to

$$= 4\alpha (4\alpha - 1) - (4\alpha - 2) \quad (18)$$

$$= 16\alpha^2 - 4\alpha - 4\alpha + 2$$

Finally,

$$\sum_{j=1}^{4\alpha} \deg(x_j) \leq 2 \left(8\alpha^2 - 4\alpha + 1\right). \quad (19)$$

As desired so it completes the proof. □

Now, we see the following result to get rid of the sign $<$ from the previous Theorem 10.

**Theorem 11.** Let $(C_2 \times \mathbb{Z}_{2\alpha}, \ast)$ be the flexible weak inverse property loop with inverse graph $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$. Then, $\sum_{j=1}^{4\alpha} \deg(x_j) = 16\alpha^2 - 12\alpha + 4$.

**Example 12.** If $C_2 = <i; i^2 = e_{C_2}>, \mathbb{Z}_4 = \{0, 1, 2, 3\}$ with even $\alpha = 2$ then the following Table 1 and Figure 1 indicate flexible weak inverse property loop of order 8 and its inverse graph, respectively. Moreover, blue and red vertices in the Figure 1 represent self-loop-inventions and nonself-loop-inventions of $(C_2 \times \mathbb{Z}_4, \ast)$, respectively.

**Example 13.** If $C_2 = <i; i^2 = e_{C_2}>, \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ with odd $\alpha = 3$, then Table 2 and Figure 2 indicate flexible weak inverse property loop of order 12 and its inverse graph, respectively. Moreover, blue and red vertices in the Figure 2 represent self-loop-inventions and nonself-loop-inventions of $(C_2 \times \mathbb{Z}_6, \ast)$, respectively.

Let $0 \neq \epsilon', \epsilon^2 \in \mathbb{Z}_{2\alpha}$, respectively, with any element $y^p \in C_2$, then Table 3 indicates the recapitulation of all the abovementioned results:

**3. Topological Indices of Inverse Graph**

In this section, we shall denote the inverse graph $G^{I_{(\lambda \rho)}}_{(C_2 \times \mathbb{Z}_{2\alpha}, \ast)}$ of flexible loop $(C_2 \times \mathbb{Z}_{2\alpha}, \ast)$ by $\Gamma$. Let $\{V_1, V_2\}$ and $\{E_1, E_2, E_3, E_4\}$ be the partitions of $V(\Gamma)$ and $E(\Gamma)$, respectively, where
Table 1: Flexible weak inverse property loop of order 8.

<table>
<thead>
<tr>
<th>*</th>
<th>(e,0)</th>
<th>(e,1)</th>
<th>(e,2)</th>
<th>(e,3)</th>
<th>(i,0)</th>
<th>(i,1)</th>
<th>(i,2)</th>
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Figure 1: Inverse graph of \((C_2 \times \mathbb{Z}_4, \ast)\).

Table 2: Flexible weak inverse property loop of order 12.

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Theorem 14. Let $\Gamma$ be the inverse graph of $(C_2 \times \mathbb{Z}_2, *)$ then first and second Zagreb indices of $\Gamma$ are $Z_{1}(\Gamma) = 64 \alpha \gamma^3 - 96 \alpha^2 + 67 \alpha^3 - 18 \text{ and } Z_{2}(\Gamma) = 128 \alpha^4 - 288 \alpha^3 + 308 \alpha^2 - 171 \alpha + 40.$
With the help of equation (5), the first Zagreb index can be obtained as follows:

\[ Z_1(\Gamma) = \sum_{x_i \in V(\Gamma)} (d_{x_i} + d_{x_i}), \]
\[ Z_1(\Gamma) = \sum_{x_i \in E(\Gamma)} (d_{x_i} + d_{x_i}) + \sum_{x_i \in V(\Gamma)} (d_{x_i} + d_{x_i}) + \sum_{x_i \in V(\Gamma)} (d_{x_i} + d_{x_i}) + \sum_{x_i \in V(\Gamma)} (d_{x_i} + d_{x_i}), \]
\[ Z_1(\Gamma) = 4(4\omega - 2 + 4\omega - 2) + 10(\omega - 1)(4\omega - 2 + 4\omega - 3) + (5\omega - 4), \]
\[ Z_1(\Gamma) = 4(8\omega - 4) + 10(\omega - 1)(8\omega - 5) + (5\omega - 4)(8\omega - 5) + (8\omega^2 - 21\omega + 12)(8\omega - 6), \]
\[ Z_1(\Gamma) = 64\omega^3 - 96\omega^2 + 67\omega - 18. \]

By equation (6), we can calculate second Zagreb index as follows:

\[ Z_2(\Gamma) = \sum_{x_i \in V(\Gamma)} (d_{x_i} \times d_{x_i}), \]
\[ Z_2(\Gamma) = \sum_{x_i \neq x_j \in E(\Gamma)} (d_{x_i} \times d_{x_j}) \times \sum_{x_i \in V(\Gamma)} (d_{x_i} + d_{x_i}) \times \sum_{x_i \in V(\Gamma)} (d_{x_i} + d_{x_i}) \times \sum_{x_i \in V(\Gamma)} (d_{x_i} + d_{x_i}), \]
\[ Z_2(\Gamma) = 4(4\omega - 2)^2 + 10(\omega - 1)(4\omega - 2)(4\omega - 3) + (5\omega - 4)(4\omega - 3)(4\omega - 2) + (8\omega^2 - 21\omega + 12)(4\omega - 3)^2, \]
\[ Z_2(\Gamma) = 128\omega^4 - 288\omega^3 + 308\omega^2 - 171\omega + 40. \]

It completes the proof. \(\square\)

**Theorem 15** (see [54]). Let \( G = (V, E) \) be a simple graph then first Zagreb coindex \( Z_1(G) \) is given by the following equation:

\[ Z_1(G) = 2|E(G)|(|V(G)| - 1) - Z_1(\Gamma). \] (23)

**Theorem 16** (see [54]). Let \( G = (V, E) \) be a simple graph then second Zagreb coindex \( Z_2(G) \) is given by the following equation:

\[ Z_2(G) = 2|E(G)|(|V(G)| - 1) - Z_1(\Gamma) \]
\[ = 2(8\omega^2 - 6\omega + 2)(4\omega - 1) - (64\omega^3 - 96\omega^2 + 67\omega - 18) \]
\[ = 32\omega^2 - 39\omega + 14. \] (25)

Also, with the help of Theorem 16 and equation (8) second Zagreb coindex can be calculated as follows:

\[ Z_2(G) = 2|E(G)|^2 - \frac{1}{2}Z_1(\Gamma) - Z_2(\Gamma) \]
\[ = 2(8\omega^2 - 6\omega + 2)^2 - \frac{1}{2}(64\omega^3 - 96\omega^2 + 67\omega - 18) - (240\omega^3 - 460\omega^2 + 306\omega - 68) \]
\[ = 128\omega^4 - 464\omega^3 + 644\omega^2 - \frac{727}{2}\omega + 85. \] (26)

**Theorem 17.** Let \( \Gamma \) be the inverse graph of \( (G_2 \times Z_{2\omega}, \ast) \) then first and second Zagreb coindices of \( \Gamma \) are \( Z_1(\Gamma) = 32\omega^2 - 39\omega + 14 \) and \( Z_2(\Gamma) = 128\omega^4 - 464\omega^3 + 644\omega^2 - (727/2)\omega + 85. \)

**Proof.** The first Zagreb coindex by Theorem 15 and equation (7) can be written as follows:

\[ Z_2(G) = 2|E(G)|^2 - \frac{1}{2}Z_1(\Gamma) - Z_2(\Gamma). \] (24)
Theorem 18. Let $\Gamma$ be the inverse graph of $(C_2 \times Z_{2\alpha}, \ast)$ then first and second multiple Zagreb indices of $\Gamma$ are

$$PZ_1(G_{(C_2 \times Z_{2\alpha}, \ast)}) = (8\alpha - 4)^4 (8\alpha - 5)^{15\alpha - 14} (8\alpha - 6)^{8\alpha - 21\alpha + 12}$$
and

$$PZ_2(G_{(C_2 \times Z_{2\alpha}, \ast)}) = (4\alpha - 2)^{15\alpha - 6} (4\alpha - 3)^{16\alpha - 27\alpha + 10}.$$ It completes the proof.

Proof. Equation (9) helps us to find first multiple Zagreb index as follows:

$$PZ_1(\Gamma) = \prod_{x_i, x_j \in E(\Gamma)} (d_{x_i} + d_{x_j}),$$

$$PZ_1(\Gamma) = \prod_{x_i, x_j \in E(\Gamma)} (d_{x_i} + d_{x_j}) \prod_{x_i, x_j \in E(\Gamma)} (d_{x_i} + d_{x_j}) \prod_{x_i, x_j \in E(\Gamma)} (d_{x_i} + d_{x_j}),$$

$$PZ_1(\Gamma) = (4\alpha - 2 + 4\alpha - 2)(4\alpha - 2 + 4\alpha - 3)^{10\alpha - 1}(4\alpha - 3 + 4\alpha - 2)^{8\alpha - 21\alpha + 12},$$
$$PZ_1(\Gamma) = (8\alpha - 4)^4 (8\alpha - 5)^{10\alpha - 1}(8\alpha - 5)^{8\alpha - 21\alpha + 12},$$
$$PZ_1(G_{(C_2 \times Z_{2\alpha}, \ast)}) = (4\alpha - 2)(8\alpha - 5)^{15\alpha - 14}(8\alpha - 6)^{8\alpha - 21\alpha + 12}.$$ By equation (10) the second multiple Zagreb index is given by the following equation:

$$PZ_2(\Gamma) = \prod_{x_i, x_j \in E(\Gamma)} (d_{x_i} \times d_{x_j}),$$

$$PZ_2(\Gamma) = \prod_{x_i, x_j \in E(\Gamma)} (d_{x_i} \times d_{x_j}) \prod_{x_i, x_j \in E(\Gamma)} (d_{x_i} \times d_{x_j}) \prod_{x_i, x_j \in E(\Gamma)} (d_{x_i} \times d_{x_j}) \prod_{x_i, x_j \in E(\Gamma)} (d_{x_i} \times d_{x_j}),$$

$$PZ_2(\Gamma) = (4\alpha - 2)^{15\alpha - 6}(4\alpha - 3)^{16\alpha - 27\alpha + 10}.$$ It completes the proof.

Proof. Equation (11) allows us to find irregularity index as follows:

Theorem 19. Let $\Gamma$ be the inverse graph of $(C_2 \times Z_{2\alpha}, \ast)$ then irregularity index of $\Gamma$ is $\text{irr}(\Gamma) = 5\alpha - 6.$

$$\text{irr}(\Gamma) = \sum_{x_i, x_j \in E(\Gamma)} |d_{x_i} - d_{x_j}|,$$

$$\text{irr}(\Gamma) = \sum_{x_i, x_j \in E(\Gamma)} |d_{x_i} - d_{x_j}| + \sum_{x_i, x_j \in E(\Gamma)} |d_{x_i} - d_{x_j}| + \sum_{x_i, x_j \in E(\Gamma)} |d_{x_i} - d_{x_j}| + \sum_{x_i, x_j \in E(\Gamma)} |d_{x_i} - d_{x_j}|,$$

$$\text{irr}(\Gamma) = (4\alpha - 2 - (4\alpha - 2))(4\alpha + (4\alpha - 2) - (4\alpha - 3)|10(4\alpha - 1) + |(4\alpha - 3)$$
$$- 4\alpha - 2)(5\alpha - 4) + (4\alpha - 3) - (4\alpha - 3)(8\alpha - 21\alpha + 12),$$
$$\text{irr}(\Gamma) = 10\alpha - 10 - 5\alpha + 4,$$
$$\text{irr}(\Gamma) = 5\alpha - 6.$$
It completes the proof.

**Theorem 20.** Let $\Gamma$ be the inverse graph of $(C_2 \times \mathbb{Z}_{2\alpha}, \ast)$ then $\var(\Gamma) = 1/\alpha^2 (\alpha - 1)$.

\[ \var(\Gamma) = \frac{1}{4\alpha} \sum_{x \in V(\Gamma)} d_x^2 - \frac{1}{16\alpha} \left[ \sum_{x \in V(\Gamma)} d_x^2 \right]^2, \]
\[ \var(\Gamma) = \frac{1}{4\alpha} \left[ \sum_{x \in V_1(\Gamma)} d_x^2 + \sum_{x \in V_2(\Gamma)} d_x^2 \right] - \frac{1}{16\alpha} \left\{ \left[ \sum_{\phi \in \varphi(\Gamma)} \xi(\phi) \right] + \left[ \sum_{\phi \in \varphi(\Gamma)} \xi(\phi) \right] \right\}^2, \]
\[ \var(\Gamma) = \frac{1}{4\alpha} \left[ 4(4\alpha - 2)^2 + 4(\alpha - 1)(4\alpha - 3)^2 \right] - \frac{1}{16\alpha} \left[ 4(4\alpha - 2) + 4(\alpha - 1)(4\alpha - 3) \right]^2, \]
\[ \var(\Gamma) = \frac{1}{4\alpha} \left[ 64\alpha^2 + 16 - 64\alpha + (4\alpha - 4)(16\alpha^2 + 9 - 24\alpha) \right] - \frac{1}{16\alpha} \left[ 16\alpha - 8 + 16\alpha^2 - 12\alpha - 16\alpha + 12 \right]^2, \]
\[ \var(\Gamma) = \frac{1}{4\alpha} \left[ 64\alpha^3 - 96\alpha^2 + 68\alpha - 20 \right] - \frac{1}{16\alpha} \left[ 16\alpha^2 - 12\alpha + 4 \right]^2, \]
\[ \var(\Gamma) = \frac{1}{4\alpha} \left( \alpha - 1 \right). \]

It completes the proof.

**Theorem 21.** Let $\Gamma$ be the inverse graph of $(C_2 \times \mathbb{Z}_{2\alpha}, \ast)$ then Collatz–Singowitz index of $\Gamma$ is $\text{CS}(\Gamma) = \lambda_1(\Gamma) - 4(\alpha^2 - 3\alpha + 1/\alpha)$.

**Proof.** By equation (13), the Collatz–Singowitz index can be calculated as follows:
\[ \text{CS}(\Gamma) = \lambda_1(\Gamma) - 2 \frac{|E(\Gamma)|}{|V(\Gamma)|} = \lambda_1(\Gamma) - 2 \frac{8\alpha^2 - 6\alpha + 2}{4\alpha}, \]
\[ \text{CS}(\Gamma) = \lambda_1(\Gamma) - \frac{4\alpha^2 - 3\alpha + 1}{\alpha}. \]

It completes the proof.

Table 4 indicates the values of all abovementioned indices.

As we can see the Figures 1 and 2, if $\alpha$ increases then the order of inverse graphs obviously increases. The important thing is only variance and CS indices decreases, they treat like decreasing functions but all the other indices move in the direction of $\alpha$. Moreover, in comparison, we have observed both first and second Zagreb multiple indices increases exponentially. It means, mathematically we can write for the very very large $\alpha$ these indices tend to infinity and on the other hand irregularity index shows a slow behavior but never overlap to zero because inverse graph is not regular.

### 4. Conclusion and Future Direction

In algebraic graph theory, we solve the problems related with graphs by taking into account some algebraic structures. This paper is actually a portrayal of this notion. Through flexible loops, a generalisation of commutative loops, and inverse graphs, we have uncovered different important numbers of simple graphs. In future, we can emphasize the notions, energy, chromatic number, clique number, vertex connectivity, edge connectivity, algebraic connectivity, adjacency spectrum, Laplacian spectrum of inverse graphs associated with this algebraic structure, or other structures of nonassociative binary operations. Moreover we can discuss optimization problems of this class of simple graphs related with spectral and polynomial methods. Other algebraic

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findings connected to adjacency matrices of inverse graphs of loop structures and their applications in data structure via adjacency list would be intriguing to study. In time to come, those molecular graphs that intersect with simple graphs discovered using mathematical structures will be engrossing and we shall be able to find Hosoya polynomial, Schultz polynomial, modified Schultz polynomial, and M-polynomial.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


