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Research Article

The Stability Analysis for a 2×2 Conservation Law with PI Controller and PDP Controller

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This article is concerned with the stability analysis of a 2×2 conservation law with PI and PDP boundary feedback control. In the PI controller case, the authors establish a sufficient and necessary stability condition for feedback parameters by Walton–Marshall criterion. Moreover, in PDP controller case, the characteristic equation is an exponential polynomial which includes two exponential arguments. Based on Ruan's results on stability for quasi-polynomials and Schur–Cohn criterion on the roots inside the unit circle for a real coefficient polynomial, a complete characterization for system parameters and the feedback delay is presented. Also, the delay-independent and delay-dependent stability intervals are obtained in this article.

1. Introduction and Problem Formulation

For the feedback control problem of hyperbolic systems, the PI (proportional-integral) controller may be one of the most widely used controller (see [1–3]). With PI controller, the system is directed to a designated equilibrium based on the present state and the accumulated past error. Moreover, the main approach to stability analysis is the Lyapunov function technique (see [4, 5]). In [6], the authors propose a strict Lyapunov functional for the hyperbolic systems with conservation laws. And then, in [7], the quadratic Lyapunov function is used to demonstrate the local exponential stability in H^2 norm. Moreover, the boundary dissipative conditions are presented explicitly. By using the Lyapunov direct method and spectral analysis, the local exponential stability was proved in [8]. Necessary and sufficient stability conditions are obtained in [9], which improves the result in [8].

However, there is little reference applying the characteristic method to investigate the stability of a hyperbolic system. The exponential stability of the quasilinear wave equation is established in [10] by studying the solution along the characteristic curves. The feedback stabilization problem of a scalar conservation law with PI controller

$$\begin{cases} \partial_t \xi(t,x) + \bar{\lambda}\partial_x \xi(t,x) = 0, t \in [0,\infty), x \in (0,L), \\ \xi(t,0) = k_p \xi(t,L) + k_i \int_0^t \xi(u,L)\mathrm{d}u, \end{cases} \tag{1}$$

is discussed in [11], where $\bar{\lambda}$ is a positive constant, k_p and k_i are feedback parameters, and the characteristic equation is a first-order exponential polynomial:

$$\mu - \left(k_p \mu + k_i\right) e^{-L/\bar{\lambda}\mu} = 0. \tag{2}$$

And then, the sufficient and necessary stability conditions for the parameters are deduced by the Walton–Marshall stability criterion. It is noted that it poses an open problem in dealing with hyperbolic systems of $n > 1$ conservation or balance laws in [11].

In this article, we are mainly concerned with the stability analysis of a 2×2 hyperbolic system with the form:

$$\begin{cases} \partial_t \xi_1(t,x) + \lambda_1 \partial_x \xi_1(t,x) = 0, \\ \partial_t \xi_2(t,x) - \lambda_2 \partial_x \xi_2(t,x) = 0, \end{cases} \tag{3}$$

with the following PI controller and initial conditions:

$$\xi_1(t,0) = k_0\xi_2(t,0) + k_1\int_0^t \xi_2(s,0)\mathrm{d}s, \xi_2(t,L) = k_2\xi_1(t,L) + k_3\int_0^t \xi_1(s,L)\mathrm{d}s, \quad (4)$$

$$\xi_1(0,x) = \xi_1^0(x), \xi_2(0,x) = \xi_2^0(x), \quad (5)$$

where $\xi_i(t,x)\,(i=1,2)$ are the system states over the space interval $(0,L)$ and time domain $[0,+\infty)$. Here, λ_1 and λ_2 denote the velocity of the system state, and we assume that λ_1 and λ_2 are two positive constants. The parameters k_0, k_1, k_2, and k_3 denote the proportional and integral control gain, respectively. Furthermore, we ignore the memory term $\int_0^t \xi_i(s,\cdot)\mathrm{d}s$ and substitute it by the delayed position $\xi_i(t-\sigma,\cdot)$; i.e., the PDP (position and delayed position) controller is as follows:

$$\begin{aligned}\xi_1(t,0) &= k_0\xi_2(t,0) + k_1\xi_2(t-\sigma,0), \xi_2(t,L)\\ &= k_2\xi_1(t,L) + k_3\xi_1(t-\sigma,L),\end{aligned} \quad (6)$$

where $\sigma > 0$ denotes the feedback time delay. Similarly, the PDP controller means that the system is designated to the equilibrium based upon the information of the present state and some past states. Our question is as follows: can system (3) be stabilized by PDP feedback (6)? In [12, 13], the PDP feedback control method was used to stabilize an inverted pendulum system, and it shows that the inverted equilibrium position is asymptotically stable for $\forall\tau > 0$. In this paper, the delay-independent stability and instability for systems (3) and (6) can also be analyzed.

The contribution of this research is as follows: (i) we construct the characteristic equation for systems (3) and (4) and (3) and (6): a kind of second-order exponential polynomial with single delay and multiple delays, respectively. (ii) We study the root distribution of the characteristic equation and give the sufficient and necessary conditions about the feedback parameters $k_i\,(i=0,1,2,3)$ such that both the closed-loop systems (3) and (4) and (3) and (6) are asymptotically stable. Additionally, the delay-independent and delay-dependent stability intervals are obtained in this article.

The structure of the article is as follows: we deduce the characteristic equation of (3) (4), and then, by Walton–Marshall stability criterion, we obtain the stability conditions for the system parameters $k_i\,(i=0,1,2,3)$ in Section 2. In Section 3, the sufficient and necessary stability conditions for (3) and (6) are given by Schur–Cohn criterion. Next, we present some numerical simulations in Section 4. Finally, several concluding remarks are presented in Section 5.

2. Stability of the Closed-Loop System (3) and (4)

2.1. The Characteristic Equation. Given the formal solution of (4) can be calculated as follows:

$$\xi_1(t,x) = y\left(t - \frac{1}{\lambda_1}x\right), \xi_2(t,x) = z\left(t + \frac{1}{\lambda_2}x\right). \quad (7)$$

Substitute (7) into the PI boundary (4), and assume $y(t) = ce^{\mu t}\,(c \neq 0)$ is a solution. By a series of calculation, we can get the characteristic equation of the closed-loop system (3) with PI control (4):

$$\Delta(\mu) = \mu^2 - (k_0\mu + k_1)(k_2\mu + k_3)e^{-\tau\mu} = 0, \quad (8)$$

where

$$\tau = a + b > 0, \frac{L}{\lambda_1} = a > 0, \frac{L}{\lambda_2} = b > 0. \quad (9)$$

Remark 1. We can also obtain the characteristic equation (8) *as follows. Consider the characteristic problem for system* (3)

$$\begin{aligned}\mu\xi_1 &= -\lambda_1\xi_1',\\ \mu\xi_2 &= \lambda_2\xi_2',\end{aligned} \quad (10)$$

whose fundamental *solution matrix under PI boundary* (4) *is*

$$\begin{pmatrix} \mu & -(k_0\mu + k_1) \\ (k_2\mu + k_3)e^{-\mu L/\lambda_1} & -\mu e^{\mu L/\lambda_2} \end{pmatrix}. \quad (11)$$

Hence, let the determinant of (11) equal to 0, we can have (8).

2.2. Walton–Marshall Stability Criterion. In this section, firstly, we state the Walton–Marshall stability criterion. It gives the sufficient and necessary conditions under which all the roots of the characteristic equation with an exponential argument

$$f(\mu) + g(\mu)e^{-\tau\mu} = 0, \quad (12)$$

have negative real parts, where $\tau \geq 0$ denotes time delay, $\mu \in C$, $f(\mu)$ is a p–order polynomial with real coefficients, $g(\mu)$ is a q–order polynomial with real coefficients, and $p \geq q$.

The Walton–Marshall stability criterion ([14, 15] (subsection 5.6)) can be summarized in three steps as follows:

Step 1: We show the stability if $\tau = 0$.

Step 2: We increase τ from zero to $q+$ and analyze how the eigenvalues move with the increasing τ. The process can be divided into two parts:

(i) If $p > q$, for an infinitely small τ, the new roots must appear at infinity; otherwise, $e^{-\tau\mu}$ would be near 1 and there cannot be any new roots. Hence, if $p > q$, (12) can be satisfied for large μ if and only if $e^{-\tau\mu}$ is large, i.e., $\mathrm{Re}(\mu) < 0$. So, in this situation, we can omit this step.

(ii) If $p = q$, we study the potential crossings of the imaginary axis, i.e., we analyze the purely imaginary roots $\mu = i\omega, \omega \in R$. Given to conjugate symmetry of (12), we get that if $\mu = i\omega$ is a solution, then $-\mu = -i\omega$ is also a solution. Substitute $\mu = \pm i\omega$ into (12), we get a real polynomial in ω^2, and for the sake of simplicity, we denote it by

$$P(\omega^2) = f(i\omega)f(-i\omega) - g(i\omega)g(-i\omega). \tag{13}$$

The location of new roots is determined by the sign of $P(\omega^2)$ for large ω. A necessary condition for exponential stability is

$$P(\omega^2) > 0, \text{for large } \omega \in R, \tag{14}$$

which can be found in [15] (subsection 5.6).

Step 3 We study the positive real roots of $P(\omega^2) = 0$. Moreover, according to $f(i\omega) + g(i\omega)e^{-i\omega\tau} = 0$, the corresponding critical values for time-delay τ can be calculated:

$$\begin{aligned}\cos(\tau\omega) &= \mathrm{Re}\left(-\frac{f(i\omega)}{g(i\omega)}\right),\\ \sin(\tau\omega) &= \mathrm{Im}\left(\frac{f(i\omega)}{g(i\omega)}\right).\end{aligned} \tag{15}$$

When we got a value of τ such that a pure imaginary root appears, we study the roots' behavior as τ changes. To be specific, we study whether, as τ is increasing, this root crosses the imaginary axis from the right half plane into the left plane (a stabilizing root), or vice versa, if it goes from the left half plane into the right half plane (a destabilizing root). And then, the stability switching would be analyzed. A nonnegative root y of P is stabilizing if the first derivative $P'(y) < 0$ and destabilizing if $P'(y) > 0$. If $P'(y) = 0$, one should consider the higher-order derivatives.

2.3. Stability Conditions for System Parameters. For characteristic equation (8), we assume

$$f(\mu) = \mu^2, g(\mu) = -(k_0\mu + k_1)(k_2\mu + k_3), \tag{16}$$

where $k_i \neq 0\,(i = 0, 1, 2, 3)$, and $p = q = 2$. Now, we analyze according to the three steps of Walton–Marshall stability criterion.

Step 1. If $\tau = 0$, (8) is equivalent to

$$(k_0k_2 - 1)\mu^2 + (k_0k_3 + k_1k_2)\mu + k_1k_3 = 0, \tag{17}$$

which is a quadratic equation in μ. The discriminant of (17) is

$$\Delta_1 = (k_0k_3 - k_1k_2)^2 + 4k_1k_3. \tag{18}$$

Hence, the system without delay is asymptotically stable if and only if one of the following cases holds.

Case 1 $\Delta_1 \geq 0$, $k_0k_2 < 1$, and $-(k_0k_3 + k_1k_2) - \sqrt{\Delta_1} > 0$

Case 2 $\Delta_1 < 0$, $k_0k_2 < 1$, and $k_0k_3 + k_1k_2 < 0$

Remark 2. Since in step 2, we have $|k_0k_2| < 1$ holds. Here, the case $k_0k_2 \geq 1$ can be omitted.

Step 2. Let $\mu = i\omega$ is a pure imaginary root of (8), by (13), we have that

$$\begin{aligned}P(\omega^2) &= \omega^4 - (k_1^2 + k_0^2\omega^2)(k_3^2 + k_2^2\omega^2)\\ &= (1 - k_0^2k_2^2)\omega^4 - (k_1^2k_2^2 + k_0^2k_3^2)\omega^2 - k_1^2k_3^2.\end{aligned} \tag{19}$$

It is obvious that $P(\omega^2) > 0$, for large $\omega \in R$ if and only if

$$|k_0k_2| < 1. \tag{20}$$

Step 3. Assume $y = \omega^2$, we can have

$$P(y) = (1 - k_0^2k_2^2)y^2 - (k_1^2k_2^2 + k_0^2k_3^2)y - k_1^2k_3^2, \tag{21}$$

and the discriminant is

$$\Delta_2 = (k_1^2k_2^2 + k_0^2k_3^2)^2 + 4(1 - k_0^2k_2^2)k_1^2k_3^2 > 0, \tag{22}$$

under the condition (20). Thus, we can have one positive root y_1 and one negative root y_2:

$$\begin{aligned}y_1 &= \frac{k_1^2k_2^2 + k_0^2k_3^2 + \sqrt{\Delta_2}}{2(1 - k_0^2k_2^2)} > 0,\\ y_2 &= \frac{k_1^2k_2^2 + k_0^2k_3^2 - \sqrt{\Delta_2}}{2(1 - k_0^2k_2^2)} < 0.\end{aligned} \tag{23}$$

Since $y = \omega^2 > 0$, so we only need to consider $y_1 := \omega_0^2$. Noticed that

$$P'(y) = 2(1 - k_0^2k_2^2)y - (k_1^2k_2^2 + k_0^2k_3^2), \tag{24}$$

and

$$P'(y_1) = \sqrt{\Delta_2} > 0, \tag{25}$$

the root is destabilizing. By (15), time delay $\tilde{\tau}$ satisfies

$$\begin{aligned}\cos\tilde{\tau}\omega &= \frac{-\omega^2(-k_0k_2\omega^2 + k_1k_3)}{(-k_0k_2\omega^2 + k_1k_3)^2 + (k_0k_3 + k_1k_2)^2\omega^2},\\ \sin\tilde{\tau}\omega &= \frac{-\omega^3(k_0k_3 + k_1k_2)}{(-k_0k_2\omega^2 + k_1k_3)^2 + (k_0k_3 + k_1k_2)^2\omega^2},\end{aligned} \tag{26}$$

and then,

$$\tan \tilde{\tau}\omega = \frac{\omega\left(k_0k_3 + k_1k_2\right)}{-k_0k_2\omega^2 + k_1k_3}. \tag{27}$$

Assume τ_0 is the smallest $\tilde{\tau}$, so we have

$$\tau_0 = \frac{1}{\omega_0}\left[\arctan\frac{\omega_0\left(k_0k_3 + k_1k_2\right)}{-k_0k_2\omega^2 + k_1k_3} + j\pi\right], \tag{28}$$

where

$$j = \begin{cases} 0, & \text{if } \dfrac{\omega_0\left(k_0k_3 + k_1k_2\right)}{-k_0k_2\omega^2 + k_1k_3} > 0, \\ \\ 1, & \text{if } \dfrac{\omega_0\left(k_0k_3 + k_1k_2\right)}{-k_0k_2\omega^2 + k_1k_3} < 0. \end{cases} \tag{29}$$

In conclusion, we can have the following results.

Theorem 1. *Suppose $k_i \neq 0$. The closed-loop system (3) and (4) are asymptotically stable for the L^2-norm if and only if the feedback parameters satisfy $|k_0k_2| < 1$ and one of the following conditions:*

(i) $(k_0k_3 - k_1k_2)^2 + 4k_1k_3 \geq 0$, $k_0k_3 + k_1k_2 < 0$, and $k_1k_3 < 0$

(ii) $(k_0k_3 - k_1k_2)^2 + 4k_1k_3 < 0$, and $k_0k_3 + k_1k_2 < 0$

Furthermore, τ satisfies $\tau \in (0, \tau_0)$, where τ_0 is given by (28).

Remark 3. If $k_0k_2 = 0, k_1k_3 \neq 0$, from the characteristic equation (8)*, we have*

$$f(\mu) = \mu^2, g(\mu) = -(k_0k_3 + k_1k_2)\mu - k_1k_3, \tag{30}$$

where $p = 2, q = 1$. Hence, $p > q$. By Walton–Marshall stability criterion, we can have results as follows.

Theorem 2. *Let us assume $k_0k_2 = 0, k_1k_3 \neq 0$, and*

$$\Delta_3 = \left(k_0k_3 + k_1k_2\right)^2 + 4k_1k_3. \tag{31}$$

The closed-loop system (3) and (4) is asymptotically stable for $\forall\tau > 0$ if and only if the feedback parameters satisfy one of the following conditions:

(i) $\Delta_3 < 0$, and $k_0k_3 + k_1k_2 < 0$

(ii) $\Delta_3 \geq 0$, and $k_1k_3 < 0, k_0k_3 + k_1k_2 < 0$

Remark 4. If $k_1k_3 = 0$, the characteristic equation (8) *changes to*

$$\Delta(\mu) = \mu^2 e^{\tau\mu} - \left(k_0k_2\mu^2 + \left(k_0k_3 + k_1k_2\right)\mu\right) = 0, \tag{32}$$

which means $\mu = 0$ must be an eigenvalue, and then, systems (3) *and* (4) *cannot be stable.*

3. Stability of System (3) with PDP Control (6)

In this part, we will analyze the stability for the hyperbolic system (3) under PDP (position and delayed position) feedback instead of PI controller:

$$\begin{cases} \partial_t\xi_1(t,x) + \lambda_1\partial_x\xi_1(t,x) = 0, \\ \partial_t\xi_2(t,x) - \lambda_2\partial_x\xi_2(t,x) = 0, \\ \xi_1(t,0) = k_0\xi_2(t,0) + k_1\xi_2(t-\sigma,0), \\ \xi_2(t,L) = k_2\xi_1(t,L) + k_3\xi_1(t-\sigma,L), \end{cases} \tag{33}$$

where $x \in (0, L), \sigma > 0$ denotes the feedback delay. By a simple calculation as in Section 2, we can get the characteristic equation of the closed-loop system (33) as

$$e^{\mu(\tau+2\sigma)} = k_0k_2e^{2\mu\sigma} + (k_1k_2 + k_0k_3)e^{\mu\sigma} + k_1k_3, \tag{34}$$

where $\tau = L/\lambda_1 + L/\lambda_2$.

Remark 5. Here, the Walton–Marshall stability criterion in Section 2.2 cannot be applied, since there are two delays: τ and σ.

3.1. Delay-Independent Stability and Instability Results.

According to Corollary 2.4 in [16], we can get the following delay-independent stability and instability results.

Let

$$\Delta = 4\left(k_1k_3 + k_0k_2\right)^2\left(k_1k_2 + k_0k_3\right)^2 - 16k_0k_1k_2k_3\left(k_0^2k_2^2 + k_1^2k_3^2 + k_1^2k_2^2 + k_0^2k_3^2 - 1\right), \tag{35}$$

and

$$y_{1,2} = \frac{-2\left(k_1k_3 + k_0k_2\right)\left(k_1k_2 + k_0k_3\right) \pm \sqrt{\Delta}}{8k_0k_1k_2k_3}. \tag{36}$$

Theorem 3. *Assume the feedback parameters satisfy the following conditions:*

$$\left|k_0k_2 + k_1k_3 + k_1k_2 + k_0k_3\right| < 1. \tag{37}$$

Then,

(i) If $\Delta < 0$, system (33) is always asymptotically stable for $\forall\tau, \sigma > 0$

(ii) If $\Delta \geq 0$, and $|y_{1,2}| > 1$, system (33) is always asymptotically stable for $\forall\tau, \sigma > 0$

(iii) If $\Delta \geq 0$, and $|y_1| \leq 1$ (or $|y_2| \leq 1$), system (33) is always unstable for $\forall \tau, \sigma > 0$

Proof. Firstly, we consider the limit case $\sigma = 0$, and then, the characteristic equation (34) turns to

$$e^{\mu\tau} = k_0k_2 + k_1k_3 + k_1k_2 + k_0k_3. \tag{38}$$

Hence, all roots of (38) have negative real parts if and only if the condition (37) holds.

As the delay increases $\sigma > 0$, let $\mu = i\omega$ is a pure imaginary root of (34). Hence, we have that

$$e^{i\omega(\tau+2\sigma)} = k_0k_2e^{2i\omega\sigma} + (k_1k_2 + k_0k_3)e^{i\omega\sigma} + k_1k_3. \tag{39}$$

Separating the real part and imaginary part, we get

$$\begin{aligned}\cos\,\omega(\tau + 2\sigma) &= k_0k_2\cos\,2\,\omega\sigma + (k_1k_2 + k_0k_3)\cos\,\omega\sigma + k_1k_3,\\ \sin\,\omega(\tau + 2\sigma) &= k_0k_2\sin\,2\,\omega\sigma + (k_1k_2 + k_0k_3)\sin\,\omega\sigma.\end{aligned} \tag{40}$$

By squaring both sides of the (40), we have

$$\begin{aligned}1 = {} & k_0^2k_2^2 + (k_1k_2 + k_0k_3)^2 + k_1^2k_3^2 + 2k_0k_1k_2k_3\cos\,2\,\omega\sigma\\ & + 2(k_1k_3 + k_0k_2)(k_1k_2 + k_0k_3)\cos\,\omega\sigma,\end{aligned} \tag{41}$$

and then,

$$4k_0k_1k_2k_3\cos^2\,\omega\sigma + 2(k_1k_3 + k_0k_2)(k_1k_2 + k_0k_3)\cos\,\omega\sigma + k_0^2k_2^2 + k_1^2k_3^2 + k_1^2k_2^2 + k_0^2k_3^2 - 1 = 0, \tag{42}$$

which is a quadratic equation of $\cos\,\omega\sigma$.

Let $y = \cos\,\omega\sigma$, and $y_{1,2}$ denotes the two roots of (42). If (40) has no real root (i.e., $\Delta < 0$), or the absolute values of both roots $y_{1,2}$ are bigger than 1 (i.e. $\Delta \geq 0$ and $|y_{1,2}| > 1$), it means that the characteristic equation has no pure imaginary roots. Hence, system (33) is always stable for any positive value of the delay σ, if and only if (37) holds. On the other hand, if $\Delta \geq 0$, and $|y_1| \leq 1$ (or $|y_2| \leq 1$), no matter what the time delay $\sigma > 0$ is, the characteristic equation (34) must have pure imaginary root. Hence, system (33) is always unstable for any positive value of the delay σ. □

3.2. Schur–Cohn Criterion and Stability Analysis. In this section, we discuss a special case: τ is an integral multiple of σ. Let

$$\frac{\tau}{\sigma} = k, \tag{43}$$

where $k > 0$ is an integral number. Then, the characteristic (34) can be written as follows:

$$1 - k_0k_2e^{-k\mu\sigma} - (k_1k_2 + k_0k_3)e^{-(k+1)\mu\sigma} - k_1k_3e^{-(k+2)\mu\sigma} = 0. \tag{44}$$

Remark 6. It should be noted that the Walton–Marshall stability criterion for systems with multiple commensurate delays

$$\sum_{j=0}^{n} A_j(\mu)e^{-j\mu\tau} = 0, \tag{45}$$

cannot be applied here, where $\tau > 0$ denotes the delay, and $A_k(\mu)\,(k = 0, 1, \ldots, n)$ are polynomials in μ with real coefficients. In fact, in (41), $A_j(\mu)$ are constant, and then, the recursive procedure for finding the pure imaginary roots in Walton–Marshall stability criterion cannot be used.

On the other hand, denote

$$e^{\mu\sigma} = z, \tag{46}$$

and then, the characteristic equation (34) can be written as follows:

$$F(z) = z^{2+k} - k_0k_2z^2 - (k_1k_2 + k_0k_3)z - k_1k_3 = 0, \tag{47}$$

which is a real $(2+k)-$ order polynomial in z. Hence, $\mathrm{Re}\,\mu < 0$ in (34) if and only if $|z| < 1$ in (47).

Next, we shall apply the Schur–Cohn criterion (see, e.g., [17] on pp. 34–36 or Proposition 5.3 of [18] on p. 27) to check that the roots of the polynomial $F(z)$ are inside the unit circle.

Theorem 4. *Suppose that all the roots of $F(z)$ in (43) lie inside the unit circle. Then,*

$$1 - k_0k_2 - (k_1k_2 + k_0k_3) - k_1k_3 > 0, \tag{48}$$

and

$$1 + (-1)^k(-k_0k_2 + k_1k_2 + k_0k_3 - k_1k_3) > 0. \tag{49}$$

More accurately, there are two cases:

Case 1: if k is an even number, then

$$1 - (k_0k_2 + k_1k_3) > |k_1k_2 + k_0k_3|. \tag{50}$$

Case 2: if k is an odd number, then

$$1 - (k_1k_2 + k_0k_3) > |k_0k_2 + k_1k_3|. \tag{51}$$

Proof. By Schur–Cohn Criterion, we have

$$F(1) > 0,\ (-1)^{2+k}F(-1) > 0. \tag{52}$$

Substitute this into (43), we can get (44) and (45), respectively.

Furthermore, when k is an even number, (45) becomes

$$1+\left(-k_0k_2+k_1k_2+k_0k_3-k_1k_3\right)>0. \tag{53}$$

Combining (44) with (48), we have (46) directly. Similarly, if k is an odd number, we get (47).

Next, for the given k, we show the necessary and sufficient conditions of the stability of the feedback parameters $k_i\,(i=0,1,2,3)$. □

Theorem 5

(i) For $k=1$, all the roots of $F(z)$ lie inside the unit circle if and only if

$$\begin{cases} 1-\left(k_1k_2+k_0k_3\right)>\left|k_0k_2+k_1k_3\right|,\\ 1-\left(k_1k_2+k_0k_3\right)-k_1k_3\left(k_0k_2+k_1k_3\right)>0,\\ 1+\left(k_1k_2+k_0k_3\right)-k_1k_3\left(-k_0k_2+k_1k_3\right)>0. \end{cases} \tag{54}$$

(ii) For $k=2$, all the roots of $F(z)$ lie inside the unit circle if and only if

$$\begin{cases} 1-\left(k_0k_2+k_1k_3\right)>\left|k_1k_2+k_0k_3\right|,\\ \left|k_1k_3\right|<1,\\ \left(1-k_1k_3\right)\left(1-k_0k_2\right)-\left(k_1k_2+k_0k_3\right)^2-k_1k_3\left(k_0k_2+k_1k_3\right)\left(1-k_1k_3\right)>0,\\ \left(1+k_1k_3\right)\left(1+k_0k_2\right)-\left(k_1k_2+k_0k_3\right)^2-k_1k_3\left(k_1k_3-k_0k_2\right)\left(1+k_1k_3\right)>0. \end{cases} \tag{55}$$

Proof. If $k=1$,

$$F(z)=z^3-k_0k_2z^2-\left(k_1k_2+k_0k_3\right)z-k_1k_3=0. \tag{56}$$

By Theorem 4, we have the first inequality of (54): $1-\left(k_1k_2+k_0k_3\right)>|k_0k_2+k_1k_3|$. Noting that

$$\Delta_2^{\pm}=\begin{pmatrix}1 & 0\\ -k_0k_2 & 1\end{pmatrix}\pm\begin{pmatrix}0 & -k_1k_3\\ -k_1k_3 & -\left(k_1k_2+k_0k_3\right)\end{pmatrix}=\begin{pmatrix}1 & \mp k_1k_3\\ -k_0k_2\mp k_1k_3 & 1\mp\left(k_1k_2+k_0k_3\right)\end{pmatrix}, \tag{57}$$

all the determinants of all inners are

$$\det\left(\Delta_2^{\pm}\right)=1\mp\left(k_1k_2+k_0k_3\right)-k_1k_3\left(\pm k_0k_2+k_1k_3\right). \tag{58}$$

So $\det\left(\Delta_2^{\pm}\right)>0$ yields the last two equalities of (54).

If $k=2$,

$$F(z)=z^4-k_0k_2z^2-\left(k_1k_2+k_0k_3\right)z-k_1k_3=0. \tag{59}$$

Since $F(1)>0$, and $F(-1)>0$, we have the first inequality of (55) by Theorem 4. Noting that

$$\begin{aligned}\Delta_3^{\pm}&=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -k_0k_2 & 0 & 1\end{pmatrix}\pm\begin{pmatrix}0 & 0 & -k_1k_3\\ 0 & -k_1k_3 & -\left(k_1k_2+k_0k_3\right)\\ -k_1k_3 & -\left(k_1k_2+k_0k_3\right) & -k_0k_2\end{pmatrix}\\ &=\begin{pmatrix}1 & 0 & \mp k_1k_3\\ 0 & 1\mp k_1k_3 & \mp\left(k_1k_2+k_0k_3\right)\\ -k_0k_2\mp k_1k_3 & \mp\left(k_1k_2+k_0k_3\right) & 1\mp k_0k_2\end{pmatrix}.\end{aligned} \tag{60}$$

So the determinants of all inners $\det\left(\Delta_1^{\pm}\right)>0$ and $\det\left(\Delta_3^{\pm}\right)>0$ yield the last three inequalities in (55). □

4. Simulation Results

In this part, we will give some simulation results by Matlab software.

Example 1. Let the feedback parameters $k_0=-2, k_1=-1, k_2=k_3=1/4$ in (4). Then, we have $|k_0k_2|=1/2<1$ and

$$\left(k_0k_3-k_1k_2\right)^2+4k_1k_3=-15/16<0, k_0k_3+k_1k_2=-3/4<0, \tag{61}$$

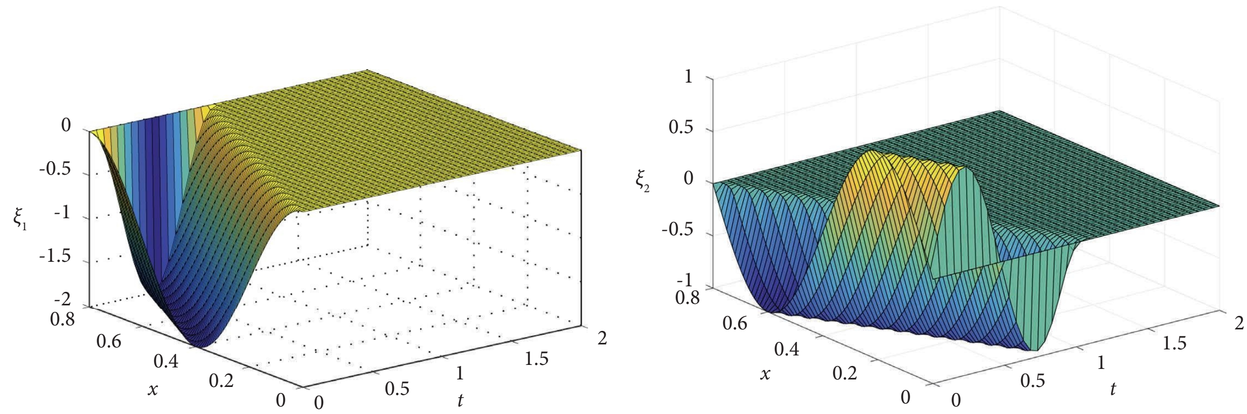

Figure 1: The convergence of the state $\xi_1(t,x)$ and $\xi_2(t,x)$ for system (3) with boundary conditions (4).

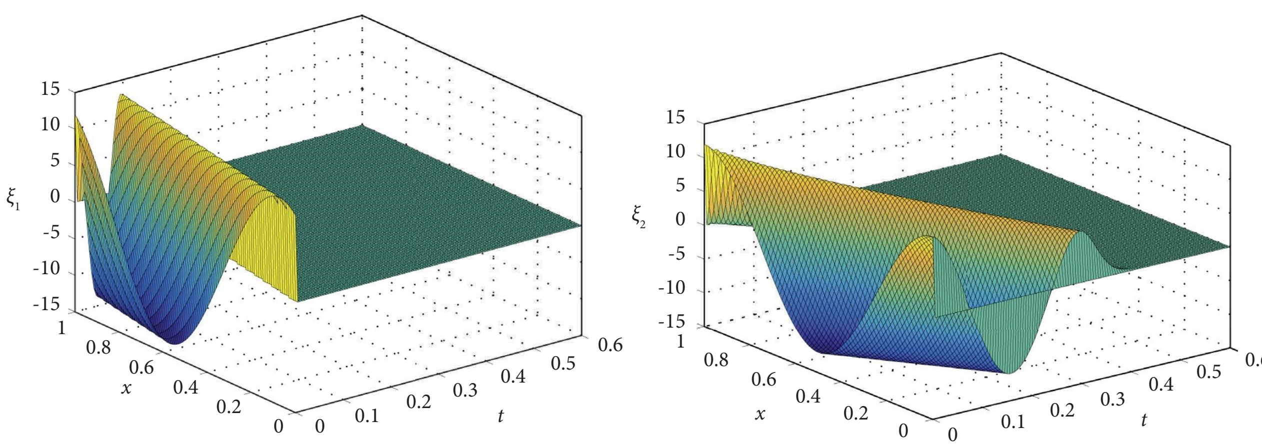

Figure 2: The convergence of the state $\xi_1(t,x)$ and $\xi_2(t,x)$ for system (33).

which satisfy the conditions of Theorem 1 *(ii). Furthermore, by a series of calculations, we can get the critical value* $\tau_0 \approx 2.1616$. *Therefore, systems* (3) *and* (4) *is asymptotically stable if* $\tau = L/\lambda_1 + L/\lambda_2 < \tau_0$. *Hence, we can choose* $\lambda_1 = 0.9, \lambda_2 = 0.8, L = 0.8$, *and then* $\tau \approx 1.8889 < \tau_0$. Figure 1 *shows the stability of* (3) *and* (4) *with the initial conditions* $\xi_1(0,x) = -\cos\ (2\pi x/L) + 1, \xi_2(0,x) = \sin(2\pi x/L)$.

Example 2. Let the feedback parameters $k_0 = 0.3, k_1 = 0.1, k_2 = 0.2, k_3 = 0.4$ *in* (33). *Then, we have* $|k_0k_2 + k_1k_3 + k_1k_2 + k_0k_3| = 0.24 < 1$ *and* $\Delta \approx -0.036848 < 0$*in* (35). *Hence, by Theorem* 3 *(i), system* (33) *is always stable for* $\forall \sigma > 0$. Figure 2 *shows the stability of* (33) *with the parameters* $\lambda_1 = 10, \lambda_2 = 3, L = 1, \sigma = 0.5$ *and the initial conditions* $\xi_1(0,x) = 12\ \cos(2\pi x) + 6\ \sin(2\pi x), \xi_2(0,x) = 12\ \cos(2\pi x)$.

5. Concluding Remarks

In this paper, the PI and PDP boundary feedback controllers are applied to a 2×2 hyperbolic system with conservation law. By using Walton–Marshall stability criterion and Schur–Cohn criterion, the sufficient and necessary stability and instability conditions for feedback parameters are established individually. And especially, the delay-dependent and delay-independent stabilities are analyzed.

It is worth noting that a 2×2 hyperbolic system with balance laws

$$\begin{cases} \partial_t \xi_1(t,x) + \lambda_1 \partial_x \xi_1(t,x) = \gamma \xi_1(t,x), \\ \partial_t \xi_2(t,x) - \lambda_2 \partial_x \xi_2(t,x) = \delta \xi_1(t,x), \end{cases} \tag{62}$$

which can depict the phenomenon of biological population affected by age structure in population ecology and describe the traffic dynamics in a freeway segment, which can also be discussed by the methods of the Walton–Marshall stability criterion and Schur–Cohn criterion, where the right-hand side of (62) can be regarded as a nonhomogeneous disturbance term.

In the future, our objective is to study the stability of a 2×2 coupled hyperbolic system

$$\begin{cases} \partial_t \xi_1(t,x) + \lambda_1 \partial_x \xi_1(t,x) + \gamma \xi_1(t,x) + \delta \xi_2(t,x) = 0, \\ \partial_t \xi_2(t,x) - \lambda_2 \partial_x \xi_2(t,x) + \gamma \xi_1(t,x) + \delta \xi_2(t,x) = 0, \end{cases} \tag{63}$$

which can describe the dynamics of a single open-channel system with nonzero bottom slope and bottom friction. The main difficulty is to find the characteristic equation under PI and PDP boundary feedback controller and choose a suitable method to analyze the stability and instability conditions for system parameters and the time delay.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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