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Research Article

On Partition Dimension of Generalized Convex Polytopes

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Let $\mathbb{G}$ be a graph having no loop or multiple edges, k–order vertex partition for $\mathbb{G}$ is represented by $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. The vector $r(\phi \mid \gamma) = (d(\phi, \gamma_1), d(\phi, \gamma_2), d(\phi, \gamma_3) \cdots, d(\phi, \gamma_k))$ is the representation of vertex ϕ with respect to γ. If the representation of all the vertices with respect to γ is different, then γ is said to be resolving partition for the graph $\mathbb{G}$. The minimum number k is resolving partition for $\mathbb{G}$ and is termed as partition dimension for $\mathbb{G}$, represented by $pd(\mathbb{G})$. There are numerous applications of partition dimension in different fields such as optimization, computer, mastermind games, and networking and also in modeling of numerical structure. The problem of finding constant value of partition dimension for a graph or network is very hard, so one can find bounds for the partition dimension. In this work, we consider convex polytopes in their generalized forms that are E_n, S_n, and G_n, and we compute upper bounds for the partition dimension of the desired polytopes.

1. Introduction

Let us consider a connected graph $\mathbb{G}$ having finite vertices and edges. Let θ and ϑ be the vertices in $\mathbb{G}$, then $d(\theta, \vartheta)$ is the distance between vertex θ and ϑ which is the shortest path between θ and ϑ. For the subset $\mho$ of $\mathbb{G}$ and vertex $\phi \in V(\mathbb{G})$. The distance of vertex ϕ and set $\mho$ is defined as $d(\phi, \mho) = \min\{d(\phi, \nu) | \nu \in \mho\}$. The order set $\mho = \{q_1, q_2, \ldots, q_l\}$ of $V(\mathbb{G})$ is referred to the $l-$ vector $r(\phi|\mho) = \{d(\phi, q_1), d(\phi, q_2), \ldots, d(\phi, q_l)\}$ as the representation of ϕ w.r.t $\mho$. The set $\mho$ is said to be resolving set if $\forall u \in \mathbb{G}$ has different representations w.r.t $\mho$. The minimum number of sets in resolving set is termed as metric dimension for $\mathbb{G}$ which is denoted by $\dim(\mathbb{G})$. Since 1975, the concept of metric dimension and metric bases was discussed in literature by different names. The name of locating set was given by Slater [1]. Melter and Harary introduce this concept by using term *metric bases* instead of locating set [2]. In [3], Chartrand introduced this concept by the name of minimum resolving set. For more about resolving set and metric dimension, we refer [4–9].

As the partition of a set is the collection of its subset such that no two subsets overlap and the union of all such sets form the original set. Similarly, partition dimension is also concerned about partitioning of vertex set $V(\mathbb{G})$ and resolvability. The partition dimension is actually the generalization of metric dimension. For given l–ordered partition of vertices of $\mathbb{G}$, where $\mathbb{G}$ is connected and simple is represented by $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$. The representation for vertex $\phi \in V(\mathbb{G})$ is the vector $r(\phi|\gamma) = (d(\phi, \gamma_1), d(\phi, \gamma_2), \ldots, d(\phi, \gamma_l))$. The partition γ is the resolving partition if for all vertices in $\mathbb{G}$ this representation is unique w.r.t γ. $pd(\mathbb{G})$ is the smallest number of sets in resolving set γ [10]. The problem of finding the resolving set for a graph is NP-hard [11]. As we know that, partition dimension for a graph is the generalization of metric dimension. Therefore, the problem of partition dimension is also NP-hard.

Graphs having $n-3$ as partition dimension are discussed in [7]. Graphs that are obtained by sum of path and cycle graph and its partition dimension are in [12, 13], and also the bounds for partition dimension are provided. In [14, 15], partition dimension of complete multipartite graphs is discussed, where strong partition dimension is discussed in [16, 17]. In [10], it is shown that the partition dimension of a graph G is bounded above by 1 more than its metric dimension. An upper bound for the partition dimension of a bipartite graph G is given in terms of the cardinalities of its partite sets, and it is shown that the bound is attained if and only if G is a complete bipartite graph. Graphs of order n having partition dimension 2, n, or $n-1$ are characterized. In [18], the authors consider relationships between metric dimension, partition dimension, diameter, and other graph parameters. They constructed universal examples of graphs with given partition dimension and used these to provide bounds on various graph parameters based on metric and partition dimensions. In [19], authors studied the partition dimension of Cartesian product graphs. More precisely, they showed that for all pairs of connected graphs G and H, $pd(G\times H)\leq pd(G)+pd(H)$ and $pd(G\times H)\leq pd(G)+\dim(H)$. The authors also showed that $pd(G\times H)\leq \dim(G)+\dim(H)+1$. In [15], the authors studied the partition dimension of circulant graphs, which are Cayley graphs of cyclic groups. In [20], the authors found bounds for the cardinality of vertices in some wheel-related graphs, namely, gear graph, helm, sun flower, and friendship graph with given partition dimension k. In [21], the authors calculated the partition dimension of two $(4,6)$−fullerene graphs. They also gave conjectures on the partition dimension of two $(3,6)$−fullerene graphs. In [22], the authors obtained several tight bounds on the partition dimension of trees. In [23], the authors studied partition dimension of some families of convex polytopes with pendant edge and proved that these graphs have bounded partition dimension. In [24], sharp bounds for the fault tolerant partition dimension of certain well-known families of convex polytopes are studied. Furthermore, it was studied that graphs having fault tolerant partition dimension are bounded below by 4. In [25], the authors considered the upper bound for the partition dimension of the generalized Petersen graph in terms of the cardinalities of its partite sets. In [17], the authors determined the partition dimension and strong metric dimension of a chain cycle constructed by even cycles and a chain cycle constructed by odd cycles [26] that mainly deal with metric dimension and partition dimension of tessellation of plane by boron nanosheets. It has been highlighted that there is a discrepancy between the mentioned parameters of the boron nanosheets. Moreover, some induced subgraphs of the stated sheets have been considered for the study of their metric dimension. For detail and brief review regarding partition dimension, we refer [13, 26–31] and the references therein.

There are various applications of resolving partition in various fields and can be found in robot navigation, network discovery, network verification, in representing chemical compounds, strategies for the master mind games, Djokovic–Winkler relation, image processing and pattern recognition, and hierarchical data structure; for more applications of the desired study, we refer [2, 5–8, 32–34].

In the study of the partition dimension for graph, the following theorems are very helpful.

Theorem 1 (see [13]). *Let γ be the resolving partition of $V(\mathbb{G})$ and $\theta,\vartheta\in V(\mathbb{G})$. If $d(\theta,u)=d(\vartheta,u)\,\forall\, u\in V(\mathbb{G})\setminus\{\theta,\vartheta\}$, then θ and ϑ be from different classes of γ.*

Theorem 2 (see [13]). *Let $\mathbb{G}$ be a simple and connected graph, then*

(i) $pd(\mathbb{G})$ is 2 iff $\mathbb{G}$ is a path graph

(ii) $pd(\mathbb{G})$ is n iff $\mathbb{G}$ is a complete graph

Let us consider K, which is family of connected graphs $\mathbb{G}_n$: $K=(\mathbb{G}_n)_{n\geq 1}$, where $V(\mathbb{G})=\tau(n)$ and $\lim_{n\longrightarrow\infty}\tau(n)=\infty$. If there is a constant $\beta\geq 1$ having the property that $pd(\mathbb{G})\leq\beta$, then partition dimension of K is bounded otherwise unbounded. Investigation of partition dimension of graphs is hard for some one, but one can easily compute bounds for the partition dimension in general family of graphs. From the research work given in [13], where the authors presented the graphs and the results in very organized way and found the upper bounds. In this work, we obtained upper bounds for various convex polytopes in their generalized form by adding prisms up to infinity. The generalized form of the polytopes is studied and denoted by E_n, G_n, and S_n. We found that partition dimension for the considered polytopes cannot be greater than 4. For lower bound of partition dimension, we present a consequence of Proposition 2.1 in the article [10], and we have that for a connected graph G, $pd(G)\geq 2$ and equality holds for path graph of order n.

2. New Results

In this section, we investigated E_n, G_n, and S_n in their generalized forms for partition dimension. We observed that partition dimension of these graphs is bounded by four, while generalization is made in terms of adding cycles that are extended into infinite numbers and can be seen in Figures 1–3.

2.1. Generalized Convex Polytope E_n. The convex polytope E_n is composed of two convex polytopes, antiprism A_n and T_n [13]. The generalized form is obtained by using the combination of prism with A_n and T_n, that is discussed and is given in Figure 1. E_n consists of n-, 5-, 4-, and 3-sided faces. The desired figure consists of various cycles induced by

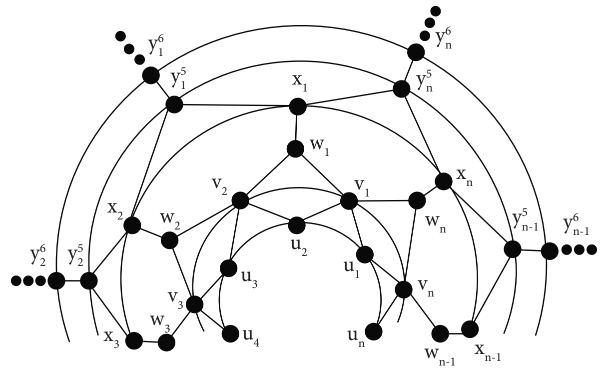

Figure 1: Convex polytope E_n.

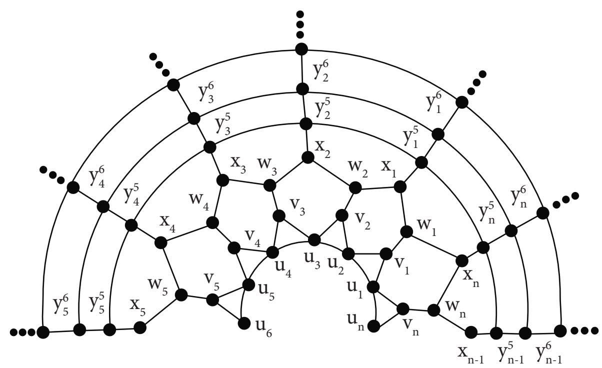

Figure 2: Convex polytope G_n.

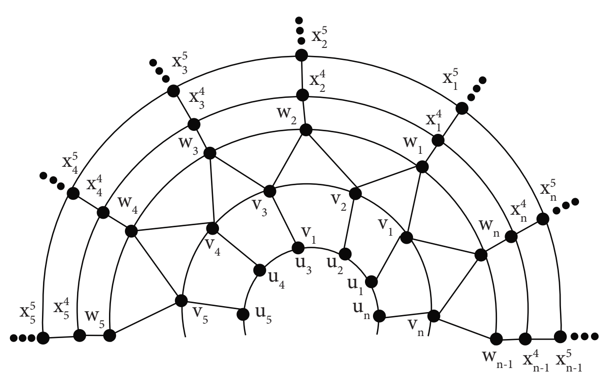

Figure 3: Convex polytope S_n.

vertices as first cycle is $\{u_\xi\colon 1 \leq \xi \leq n\}$, second cycle is $\{v_\xi\colon 1 \leq \xi \leq n\}$, and the generalized way is in Figure 1. The theorem given in the following is for the bound of $pd(E_n)$, where four sets of vertices are enough for $V(\mathbb{G})$.

Theorem 3. *Let E_n be the generalized convex polytope that has $n \geq 6$, then $pd(E_n) \leq 4$.*

Proof. The proof of the desired theorem is discussed in the following cases. □

Case 4. When $n = 2\varrho$, $\varrho \geq 3$, and $\varrho \in Z^+$, then the vertices of E_n are divided into four partition resolving sets that are $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$, where $\Gamma_1 = \{u_1\}$, $\Gamma_2 = \{u_3\}$, $\Gamma_3 = \{u_{\varrho+1}\}$, and $\Gamma_4 = \{\forall V E_n | \notin \{\Gamma_1, \Gamma_2, \Gamma_3\}\}$. For the desired proof, this will be enough to show that the representation of all the vertices of E_n is different w.r.t partition resolving set Γ and then $pd(E_n) \leq 4$. That is why, we give the representation of E_n w.r.t partition resolving set Γ.

The representation of vertices of the inner cycle or first cycle of E_n is given in the following equation:

$$r\left(u_\xi|\Gamma\right)=\begin{cases}(1,1,\varrho-1,0), & \text{if } \xi=2,\\ (\xi-1,\xi-3,\varrho-\xi+1,0), & \text{if } 4\leq\xi\leq\varrho+1,\\ (\varrho-1,\varrho-1,1,0), & \text{if } \xi=\varrho+2,\\ (2\varrho-\xi+1,2\varrho-\xi+3,\xi-\varrho-1,0), & \text{if } \varrho+3\leq\xi\leq 2\varrho.\end{cases} \quad (1)$$

The vertices of second cycle of E_n have the following representation:

$$r\left(v_\xi|\Gamma\right)=\begin{cases}(1,2,\varrho,0), & \text{if } \xi=1,\\ (2,1,\varrho-1,0), & \text{if } \xi=2,\\ (\xi,\xi-2,\varrho-\xi+1,0), & \text{if } 3\leq\xi=\varrho,\\ (\varrho,\varrho-1,1,0), & \text{if } \xi=\varrho+1,\\ (\varrho-1,\varrho,2,0), & \text{if } \xi=\varrho+2,\\ (2\varrho-\xi+1,2\varrho-\xi+3,\xi-\varrho,0), & \text{if } \varrho+3\leq\xi\leq 2\varrho.\end{cases} \quad (2)$$

The vertices of third cycle of E_n have the following representation:

$$r\left(w_\xi|\Gamma\right)=\begin{cases}(2,2,\varrho,0), & \text{if } \xi=1,\\ (3,2,\varrho-1,0), & \text{if } \xi=2,\\ (\xi+1,\xi-1,\varrho-\xi+1,0), & \text{if } 3\leq\xi=\varrho+1,\\ (\varrho+1,\varrho-1,2,0), & \text{if } \xi=\varrho,\\ (\varrho,\varrho,2,0), & \text{if } \xi=\varrho+1,\\ (\varrho-\xi+1,2\varrho-\xi+3,\xi-\varrho+1,0), & \text{if } \varrho+2\leq\xi\leq 2\varrho,\\ (2,3,\varrho+1,0), & \text{if } \xi=2\varrho.\end{cases} \quad (3)$$

The vertices of fourth cycle of E_n have the following representation:

$$r\left(x_\xi|\Gamma\right)=\begin{cases}(3,3,\varrho+1,0), & \text{if } \xi=1,\\ (4,3,\varrho,0), & \text{if } \xi=2,\\ (\xi+2,\xi,\varrho-\xi+2,0), & \text{if } 3\leq\xi=\varrho+1,\\ (\varrho+2,\varrho,3,0), & \text{if } \xi=\varrho,\\ (\varrho+1,\varrho+1,3,0), & \text{if } \xi=\varrho+1,\\ (\varrho-\xi+2,2\varrho-\xi+4,\xi-\varrho+2,0), & \text{if } \varrho+2\leq\xi\leq 2\varrho,\\ (3,3,\varrho+2,0), & \text{if } \xi=2\varrho.\end{cases} \quad (4)$$

The vertices of fifth cycle of E_n have the following representation:

$$r\left(y_{\xi}^{5}|\Gamma\right)=\begin{cases}(4,4,\varrho+1,0), & \text{if } \xi=1,\\ (5,4,\varrho,0), & \text{if } \xi=2,\\ (\xi+3,\xi+1,\varrho-\xi+2,0), & \text{if } 3\leq\xi\leq\varrho-2,\\ (\varrho+2,\varrho,4,0), & \text{if } \xi=\varrho-1,\\ (\varrho+2,\varrho+1,4,0), & \text{if } \xi=\varrho,\\ (\varrho+1,\varrho+2,4,0), & \text{if } \xi=\varrho+1,\\ (2\varrho-\xi+2,2\varrho-\xi+3,\xi-\varrho+3,0), & \text{if } \varrho+2\leq\xi\leq 2\varrho-1,\\ (4,4,\varrho+2,0), & \text{if } \xi=2\varrho.\end{cases}\tag{5}$$

The vertices of sixth and onward cycles of E_n have the following representation, where $k\in N$ and $k\geq 6$, and show the position of the cycles:

$$r\left(y_{\xi}^{k}|\Gamma\right)=\begin{cases}(1,1,1,0)+r\left(y_{\xi}^{k-1}|\Gamma\right), & \text{if } \xi=1,\\ (1,1,1,0)+r\left(y_{\xi}^{k-1}|\Gamma\right), & \text{if } \xi=2,\\ (1,1,1,0)+r\left(y_{\xi}^{k-1}|\Gamma\right), & \text{if } 3\leq\xi\leq\varrho-2,\\ (1,1,1,0)+r\left(y_{\xi}^{k-1}|\Gamma\right), & \text{if } \xi=\varrho-1,\\ (1,1,1,0)+r\left(y_{\xi}^{k-1}|\Gamma\right), & \text{if } \xi=\varrho,\\ (1,1,1,0)+r\left(y_{\xi}^{k-1}|\Gamma\right), & \text{if } \xi=\varrho+1,\\ (1,1,1,0)+r\left(y_{\xi}^{k-1}|\Gamma\right), & \text{if } \varrho+2\leq\xi\leq 2\varrho-1,\\ (1,1,1,0)+r\left(y_{\xi}^{k-1}|\Gamma\right), & \text{if } \xi=2\varrho.\end{cases}\tag{6}$$

Case 5. When $n=2\varrho+1$, $\varrho\geq 3$, similarly as in Case 4, the vertices of E_n are resolved into four partitions that are $\Gamma=\{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\}$, $\varrho\in Z^{+}$ where $\Gamma_1=\{u_1\}$, $\Gamma_2=\{u_3\}$, $\Gamma_3=\{u_{\varrho+1}\}$, and $\Gamma_4=\{\forall VE_n|\notin\{\Gamma_1,\Gamma_2,\Gamma_3\}\}$. Our aim is to show that the vertices of E_n have unique representation w.r.t Γ and then $pd(E_n)\leq 4$.

The following are the representations of vertices of E_n w.r.t Γ.

The vertices of inner cycle of E_n have the following representation:

$$r\left(u_{\xi}|\Gamma\right)=\begin{cases}(1,1,\varrho-1,0), & \text{if } \xi=2,\\ (\xi-1,\xi-3,\varrho-\xi+1,0), & \text{if } 4\leq\xi\leq\varrho+1,\\ (\varrho,\varrho-1,1,0), & \text{if } \xi=\varrho+2,\\ (\varrho-1,\varrho,2,0), & \text{if } \xi=\varrho+3,\\ (2\varrho-\xi+2,2\varrho-\xi+4,\xi\varrho-1,0), & \text{if } \varrho+4\leq\xi\leq 2\varrho+1.\end{cases}\tag{7}$$

The vertices of second cycle of E_n have the following representation:

$$r\left(v_{\xi}|\Gamma\right)=\begin{cases}(1,2,\varrho,0), & \text{if } \xi=1,\\ (2,1,\varrho-1,0), & \text{if } \xi=2,\\ (\xi,\xi-2,\varrho-\xi+1,0), & \text{if } 3\leq\xi\leq\varrho,\\ (\varrho+1,\varrho-1,1,0), & \text{if } \xi=\varrho+1,\\ (2\varrho-\xi+2,2\varrho-\xi+2,2,0), & \text{if } \xi=\varrho+2,\\ (2\varrho-\xi+2,2\varrho-\xi+4,\xi-\varrho,0), & \text{if } \varrho+3\leq\xi\leq 2\varrho+1.\end{cases}\tag{8}$$

The vertices of third cycle of E_n have the following representation:

$$r\left(w_\xi|\Gamma\right) = \begin{cases} (2, 2, \varrho, 0), & \text{if } \xi = 1, \\ (3, 2, \varrho - 1, 0), & \text{if } \xi = 2, \\ (\xi + 1, \xi - 1, \varrho - \xi + 1, 0), & \text{if } 3 \leq \xi \leq \varrho - 1, \\ (\varrho + 1, \varrho - 1, 2, 0), & \text{if } \xi = \varrho, \\ (\varrho + 1, \varrho, 2, 0), & \text{if } \xi = \varrho + 1, \\ (\varrho, \varrho + 1, 3, 0), & \text{if } \xi = \varrho + 2, \\ (2\varrho - \xi + 2, 2\varrho - \xi + 4, \xi - \varrho + 1, 0), & \text{if } \varrho + 3 \leq 2\varrho, \\ (2, 3, \varrho + 1, 0), & \text{if } \xi = 2\varrho + 1. \end{cases} \tag{9}$$

The vertices of fourth cycle of E_n have the following representation:

$$r\left(x_\xi|\Gamma\right) = \begin{cases} (3, 3, \varrho + 1, 0), & \text{if } \xi = 1, \\ (4, 3, \varrho, 0), & \text{if } \xi = 2, \\ (\xi + 2, \xi, \varrho - \xi + 2, 0), & \text{if } 3 \leq \xi \leq \varrho - 1, \\ (\varrho + 2, \varrho, 3, 0), & \text{if } \xi = \varrho, \\ (\varrho + 2, \varrho + 1, 3, 0), & \text{if } \xi = \varrho + 1, \\ (\varrho + 1, \varrho + 2, 4, 0), & \text{if } \xi = \varrho + 2, \\ (2\varrho - \xi + 3, 2\varrho - \xi + 5, \xi - \varrho + 2, 0), & \text{if } \varrho + 3 \leq 2\varrho, \\ (3, 4, \varrho + 2, 0), & \text{if } \xi = 2\varrho + 1. \end{cases} \tag{10}$$

The vertices of fifth cycle of E_n have the following representation:

$$r\left(y_\xi^5|\Gamma\right) = \begin{cases} (4, 4, \varrho + 1, 0), & \text{if } \xi = 1, \\ (5, 4, \varrho, 0), & \text{if } \xi = 2, \\ (\xi + 3, \xi + 1, \varrho - \xi + 2, 0), & \text{if } 3 \leq \xi \leq \varrho - 1, \\ (\varrho + 3, \varrho + 1, 4, 0), & \text{if } \xi = \varrho, \\ (\varrho + 2, \varrho + 2, 4, 0), & \text{if } \xi = \varrho + 1, \\ (2\varrho - \xi + 3, 2\varrho - \xi + 5, \xi - \varrho + 3, 0), & \text{if } \varrho + 2 \leq \xi \leq 2\varrho - 1, \\ (4, 5, \varrho + 3, 0), & \text{if } \xi = 2\varrho, \\ (4, 4, \varrho + 2, 0), & \text{if } \xi = 2\varrho + 1. \end{cases} \tag{11}$$

The vertices of sixth and onward cycles of E_n have the representation given in the following equation, where $k \in N$ and $k \geq 6$:

$$r\left(y_\xi^k|\Gamma\right)=\begin{cases}(1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \xi=1,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \xi=2,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } 3\le\xi\le\varrho-1,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \xi=\varrho,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \xi=\varrho+1,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \varrho+2\le\xi\le2\varrho-1,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \xi=2\varrho,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \xi=2\varrho+1.\end{cases} \tag{12}$$

2.2. Generalized Convex Polytope G_n. The graph G_n in generalized form has n-, 6-, 5-, 4-, and 3-sided faces [13]; such graph is shown in Figure 2. The first cycle consists of vertices $\{u_\xi\colon 1\le\xi\le n\}$, second cycle consists of vertices $\{v_\xi\colon 1\le\xi\le n\}$, similarly the other cycles are in the desired figure. For the bound of $pd(G_n)$, we represent the following theorem. The theorem shows that, for the desired purpose, only four sets of vertices are enough for partition of $V(G_n)$.

Theorem 6. *Let G_n be the generalized convex polytope with $n\ge6$, then $pd(G_n)\le4$.*

Proof. For the proof, the following cases are discussed. ☐

Case 7. For $n=2\varrho$, with $\varrho\ge3$ and $\varrho\in Z^+$. For the desired purpose, the vertices of G_n are divided into four sets. The sets are $\Gamma=\{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\}$, where $\Gamma_1=\{u_1\}$, $\Gamma_2=\{u_2\}$, $\Gamma_3=\{u_{\varrho+1}\}$, and $\Gamma_4=\{\forall V(G_n)|\notin\{\Gamma_1,\Gamma_2,\Gamma_3\}\}$. This will be enough to show that the vertices of G_n have unique representation of vertices w.r.t Γ and then $pd(G_n)\le4$. For this, the following is the representation w.r.t Γ.

The vertices of first cycle of G_n have the following representation:

$$r\left(u_\xi|\Gamma\right)=\begin{cases}(\xi-1,\xi-2,\varrho-\xi+1,0), & \text{if } 3\le\xi\le\varrho,\\ (\varrho-1,\varrho,1,0), & \text{if } \xi=\varrho+2,\\ (2\varrho-\xi+1,2\varrho-\xi+2,\xi-\varrho-1,0), & \text{if } \varrho+3\le\xi\le2\varrho.\end{cases} \tag{13}$$

The vertices of second cycle of G_n have the following representation:

$$r\left(v_\xi|\Gamma\right)=\begin{cases}(1,1,\varrho,0), & \text{if } \xi=1,\\ (\xi,\xi-1,\varrho-\xi+1,0), & \text{if } 2\le\xi\le\varrho,\\ (\varrho,\varrho,1,0), & \text{if } \xi=\varrho+1,\\ (2\varrho-\xi+1,2\varrho-\xi+2,\xi-\varrho,0), & \text{if } \varrho+2\le\xi\le2\varrho.\end{cases} \tag{14}$$

The vertices of third cycle of G_n have the following representation:

$$r\left(w_\xi|\Gamma\right)=\begin{cases}(2,2,\varrho+1,0), & \text{if } \xi=1,\\ (\xi+1,\xi,\varrho-\xi+2,0), & \text{if } 2\le\xi\le\varrho,\\ (\varrho+1,\varrho+1,2,0), & \text{if } \xi=\varrho+1,\\ (2\varrho-\xi+2,2\varrho-\xi+3,\xi-\varrho+1,0), & \text{if } \varrho+2\le\xi\le2\varrho.\end{cases} \tag{15}$$

The vertices of fourth cycle of G_n have the following representation:

$$r\left(x_\xi|\Gamma\right)=\begin{cases}(3,3,\varrho+2,0), & \text{if } \xi=1,\\ (\xi+2,\xi+1,\varrho-\xi+2,0), & \text{if } 2\le\xi\le\varrho-1,\\ (\varrho+2,\varrho+1,3,0), & \text{if } \xi=\varrho,\\ (2\varrho-\xi+2,2\varrho-\xi+3,\xi-\varrho+2,0), & \text{if } \varrho+1\le\xi\le 2\varrho-1,\\ (3,3,\varrho+2,0), & \text{if } \xi=2\varrho.\end{cases} \tag{16}$$

The vertices of fifth cycle of G_n have the following representation:

$$r\left(y_\xi^5|\Gamma\right)=\begin{cases}(4,4,\varrho+2,0), & \text{if } \xi=1,\\ (\xi+3,\xi+2,\varrho-\xi+2,0), & \text{if } 2\le\xi\le\varrho-1,\\ (\varrho+3,\varrho+2,4,0), & \text{if } \xi=\varrho,\\ (2\varrho-\xi+3,2\varrho-\xi+4,\xi-\varrho+3,0), & \text{if } \varrho+1\le\xi\le 2\varrho-1,\\ (4,4,\varrho+3,0), & \text{if } \xi=2\varrho.\end{cases} \tag{17}$$

The vertices of remaining cycles of G_n have the representation given in the following equation with $k\in N$ and $k\ge 6$:

$$r\left(y_\xi^k|\Gamma\right)=\begin{cases}(1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \xi=1,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } 2\le\xi\le\varrho-1,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \xi=\varrho,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \varrho+1\le\xi\le 2\varrho-1,\\ (1,1,1,0)+r\left(y_\xi^{k-1}|\Gamma\right), & \text{if } \xi=2\varrho.\end{cases} \tag{18}$$

Case 8. When $n=2\varrho+1$, $\varrho\ge 3, \varrho\in Z^+$, such as Case 7, where the vertex set of G_n is divided into four sets that are $\Gamma=\left\{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\right\}$ such that $\Gamma_1=\{u_1\}$, $\Gamma_2=\{u_3\}$, $\Gamma_3=\left\{u_{\varrho+1}\right\}$, and $\Gamma_4=\{\forall V(E_n)|\notin\{\Gamma_1,\Gamma_2,\Gamma_3\}\}$. For the desired purpose, we have to show that the vertices of G_n have unique representation w.r.t Γ and then $pd(G_n)\le 4$.

The representations w.r.t Γ are given in the following.

The vertices of first cycle of G_n have the following representation:

$$r\left(u_\xi|\Gamma\right)=\begin{cases}(\xi-1,\xi-2,\varrho-\xi+1,0,\varrho-1,0), & \text{if } 3\le\xi\le\varrho,\\ (\varrho,\varrho,1,0), & \text{if } \xi=\varrho+2,\\ (2\varrho-\xi+2,2\varrho-\xi+3,\xi-\varrho,0), & \text{if } \varrho+2\le\xi\le 2\varrho+1.\end{cases} \tag{19}$$

The vertices of second cycle of G_n have the following representation:

$$r\left(v_\xi|\Gamma\right)=\begin{cases}(1,1,\varrho,0,\varrho-1,0), & \text{if } \xi=1,\\ (\xi,\xi-1,\varrho-\xi+1,0), & \text{if } 2\le\xi\le\varrho,\\ (\varrho+1,\varrho,1,0), & \text{if } \xi=\varrho+1,\\ (2\varrho-\xi+2,2\varrho-\xi+3,\xi-\varrho,0), & \text{if } \varrho+2\le\xi\le 2\varrho+1.\end{cases} \tag{20}$$

The vertices of third cycle of G_n have the following representation:

$$r\left(w_{\xi}|\Gamma\right)=\begin{cases}(2,2,\varrho+1,0,\varrho-1,0), & \text{if } \xi=1,\\ (\xi+1,\xi,\varrho-\xi+2,0), & \text{if } 2\le\xi\le\varrho,\\ (\varrho+2,\varrho+1,2,0), & \text{if } \xi=\varrho+1,\\ (2\varrho-\xi+3,2\varrho-\xi+4,\xi-\varrho+1,0), & \text{if } \varrho+2\le\xi\le 2\varrho+1.\end{cases} \tag{21}$$

The vertices of fourth cycle of G_n have the following representation:

$$r\left(x_{\xi}|\Gamma\right)=\begin{cases}(3,3,\varrho+1,0), & \text{if } \xi=1,\\ (\xi+2,\xi+1,\varrho-\xi+2,0), & \text{if } 2\le\xi\le\varrho-1,\\ (\varrho+2,\varrho+1,3,0), & \text{if } \xi=\varrho,\\ (\varrho+2,\varrho+2,3,0), & \text{if } \xi=\varrho+1,\\ (2\varrho-\xi+3,2\varrho-\xi+4,\xi-\varrho+2,0), & \text{if } \varrho+2\le\xi\le 2\varrho,\\ (3,3,\varrho+2,0), & \text{if } \xi=2\varrho+1.\end{cases} \tag{22}$$

The vertices of fifth cycle of G_n have the following representation:

$$r\left(y_{\xi}^{5}|\Gamma\right)=\begin{cases}(4,4,\varrho+2,0), & \text{if } \xi=1,\\ (\xi+3,\xi+2,\varrho-\xi+3,0), & \text{if } 2\le\xi\le\varrho-1,\\ (\varrho+3,\varrho+2,4,0), & \text{if } \xi=\varrho,\\ (\varrho+3,\varrho+3,4,0), & \text{if } \xi=\varrho+1,\\ (2\varrho-\xi+4,2\varrho-\xi+5,\xi-\varrho+3,0), & \text{if } \varrho+2\le\xi\le 2\varrho,\\ (4,4,\varrho+3,0), & \text{if } \xi=2\varrho+1.\end{cases} \tag{23}$$

The remaining cycles of G_n have the following representation with $k\ge 6$ and $k\in N$:

$$r\left(y_{\xi}^{k}|\Gamma\right)=\begin{cases}(1,1,1,0)+r\left(y^{k-1}|\Gamma\right), & \text{if } \xi=1,\\ (1,1,1,0)+r\left(y^{k-1}|\Gamma\right), & \text{if } 2\le\xi\le\varrho-1,\\ (1,1,1,0)+r\left(y^{k-1}|\Gamma\right), & \text{if } \xi=\varrho,\\ (1,1,1,0)+r\left(y^{k-1}|\Gamma\right), & \text{if } \xi=\varrho+1,\\ (1,1,1,0)+r\left(y^{k-1}|\Gamma\right), & \text{if } \varrho+2\le\xi\le 2\varrho,\\ (1,1,1,0)+r\left(y^{k-1}|\Gamma\right), & \text{if } \xi=2\varrho+1.\end{cases} \tag{24}$$

2.3. Generalized Convex Polytope S_n. The formation of convex polytopes is in [13]. S_n consists of 3-, 4-, 5-, and n–sided faces. The arrangement of cycles in S_n is like the first cycle is composed of vertices $\{u_{\xi}\colon 1\le\xi\le n\}$ and the second cycle is composed of vertices $\{v_{\xi}\colon 1\le\xi\le n\}$. The general way of arrangement of cycles is shown in Figure 3. The following theorem is for $pd(S_n)$, which shows that only four sets of vertices are required for the desired purpose.

Theorem 9. *Let S_n be the generalized convex polytope with $n\ge 6$, then $pd(S_n)\le 4$.*

Proof. We represent the proof in two cases given as follows. □

Case 10. For $n = 2\varrho$, with $\varrho \geq 3$ and $\varrho \in Z^{+}$, the vertices of S_n are divided into four sets, as shown in Figure 3. The sets are $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$, where $\Gamma_1 = \{u_1\}$, $\Gamma_2 = \{u_2\}$, $\Gamma_3 = \{u_{\varrho+1}\}$, and $\Gamma_4 = \{\forall V(S_n) | \notin \{\Gamma_1, \Gamma_2, \Gamma_3\}\}$. For the desired proof, this will be enough to show that the vertices of S_n have unique representation w.r.t Γ and then $pd(S_n) \leq 4$. The representation is given in the following equation.

The vertices of first cycle in S_n have the following representation:

$$r(u_\xi|\Gamma) = \begin{cases} (\xi - 1, \xi - 2, \varrho - \xi + 1, 0), & \text{if } 3 \leq \xi \leq \varrho, \\ (2\varrho - \xi + 1, 2\varrho - \xi + 2, \xi - \varrho - 1, 0), & \text{if } \varrho + 2 \leq \xi \leq 2\varrho. \end{cases} \tag{25}$$

The vertices of second cycle of S_n have the following representation:

$$r(v_\xi|\Gamma) = \begin{cases} (1, 2, \varrho + 1, 0), & \text{if } \xi = 1, \\ (\xi, \xi - 1, \varrho - \xi + 2, 0), & \text{if } 2 \leq \xi \leq \varrho + 1, \\ (2\varrho - \xi + 2, 2\varrho - \xi + 3, \xi - \varrho, 0), & \text{if } \varrho + 2 \leq \xi \leq 2\varrho. \end{cases} \tag{26}$$

The vertices of third cycle of S_n have the following representation:

$$r(w_\xi|\Gamma) = \begin{cases} (2, 2, \varrho + 1, 0), & \text{if } \xi = 1, \\ (\xi + 1, \xi, \varrho - \xi + 2, 0), & \text{if } 2 \leq \xi \leq \varrho, \\ (\varrho + 1, \varrho + 1, 2, 0), & \text{if } \xi = \varrho + 1, \\ (2\varrho - \xi + 2, 2\varrho - \xi + 3, \xi - \varrho + 1, 0), & \text{if } \varrho + 2 \leq \xi \leq 2\varrho. \end{cases} \tag{27}$$

The vertices of fourth cycle of S_n have the following representation:

$$r(w_\xi^4|\Gamma) = \begin{cases} (3, 3, \varrho + 2, 0), & \text{if } \xi = 1, \\ (\xi + 2, \xi + 1, \varrho - \xi + 3, 0), & \text{if } 2 \leq \xi \leq \varrho, \\ (\varrho + 2, \varrho + 2, 3, 0), & \text{if } \xi = \varrho + 1, \\ (2\varrho - \xi + 3, 2\varrho - \xi + 4, \xi - \varrho + 2, 0), & \text{if } \varrho + 2 \leq \xi \leq 2\varrho. \end{cases} \tag{28}$$

The representation for the fifth and onward cycles is given in the following equation, where $k \in N$ and $k \geq 5$:

$$r(w_\xi^k|\Gamma) = \begin{cases} (1, 1, 1, 0) + r(x_\xi^{k-1}|\Gamma), & \text{if } \xi = 1, \\ (1, 1, 1, 0) + r(x_\xi^{k-1}|\Gamma), & \text{if } 2 \leq \xi \leq \varrho, \\ (1, 1, 1, 0) + r(x_\xi^{k-1}|\Gamma), & \text{if } \xi = \varrho + 1, \\ (1, 1, 1, 0) + r(x_\xi^{k-1}|\Gamma), & \text{if } \varrho + 2 \leq \xi \leq 2\varrho. \end{cases} \tag{29}$$

Case 11. For $n = 2\varrho + 1$, with $\varrho \geq 3$ and $\varrho \in Z^{+}$. Here, vertices of S_n are divided into four sets. The sets are $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ such that $\Gamma_1 = \{u_1\}$, $\Gamma_2 = \{u_2\}$, $\Gamma_3 = \{u_{\varrho+1}\}$, and $\Gamma_4 = \{\forall V(S_n) | \notin \{\Gamma_1, \Gamma_2, \Gamma_3\}\}$. For our purpose, we show that all the vertices of S_n have unique representation w.r.t Γ and then $pd(S_n) \leq 4$. The desired representations are given in the following equation.

The vertices of first cycle of S_n have the following representation:

$$r\left(u_{\xi}|\Gamma\right) = \begin{cases} (\xi + 1, \xi - 2, \varrho - \xi + 1, 0), & \text{if } 3 \leq \xi \leq \varrho, \\ (\varrho, \varrho, 1, 0), & \text{if } \xi = \varrho + 2, \\ (2\varrho - \xi + 2, 2\varrho - \xi + 3, \xi - \varrho - 1, 0), & \text{if } \varrho + 3 \leq \xi \leq \varrho + 1, \\ (2\varrho - \xi + 2, 2\varrho - \xi + 3, \xi - \varrho + 1, 0), & \text{if } \varrho + 2 \leq \xi \leq 2\varrho + 1. \end{cases} \tag{30}$$

The vertices of second cycle of S_n have the following representation:

$$r\left(v_{\xi}|\Gamma\right) = \begin{cases} (1, 2, \varrho + 1, 0), & \text{if } \xi = 1, \\ (\xi, \xi - 1, \varrho - \xi + 2, 0), & \text{if } 2 \leq \xi \leq \varrho + 1, \\ (\varrho + 1, \varrho + 1, 2, 0), & \text{if } \xi = \varrho + 2, \\ (2\varrho - \xi + 3, 2\varrho - \xi + 4, \xi - \varrho, 0), & \text{if } \varrho + 3 \leq \xi \leq 2\varrho + 1. \end{cases} \tag{31}$$

The vertices of third cycle of S_n haves the following representation:

$$r\left(w_{\xi}|\Gamma\right) = \begin{cases} (2, 2, \varrho + 1, 0), & \text{if } \xi = 1, \\ (\xi + 1, \xi, \varrho - \xi + 2, 0), & \text{if } 2 \leq \xi \leq \varrho, \\ (\varrho + 2, \varrho + 1, 2, 0), & \text{if } \xi = \varrho + 1, \\ (2\varrho - \xi + 3, 2\varrho - \xi + 4, \xi - \varrho + 1, 0), & \text{if } \varrho + 2 \leq \xi \leq 2\varrho + 1. \end{cases} \tag{32}$$

The vertices of fourth cycle of S_n have the following representation:

$$r\left(x_{\xi}^{4}|\Gamma\right) = \begin{cases} (3, 3, \varrho + 2, 0), & \text{if } \xi = 1, \\ (\xi + 2, \xi + 1, \varrho - \xi + 3, 0), & \text{if } 2 \leq \xi \leq \varrho, \\ (\varrho + 3, \varrho + 2, 3, 0), & \text{if } \xi = \varrho + 1, \\ (2\varrho - \xi + 4, 2\varrho - \xi + 5, \xi - \varrho + 2, 0), & \text{if } \varrho + 2 \leq \xi \leq 2\varrho + 1. \end{cases} \tag{33}$$

The representation for the vertices of fifth cycle and onward is in the following equation, where $k \in N$ and $k \geq 5$:

$$r\left(x_{\xi}^{k}|\Gamma\right)=\begin{cases}(1,1,1,0)+r\left(x_{\xi}^{k-1}|\Gamma\right), & \text{if } \xi=1,\\(1,1,1,0)+r\left(x_{\xi}^{k-1}|\Gamma\right), & \text{if } 2\leq\xi\leq\varrho,\\(1,1,1,0)+r\left(x_{\xi}^{k-1}|\Gamma\right), & \text{if } \xi=\varrho+1,\\(1,1,1,0)+r\left(x_{\xi}^{k-1}|\Gamma\right), & \text{if } \varrho+2\leq\xi\leq 2\varrho+1.\end{cases}\tag{34}$$

3. Conclusion

In this work, different types of convex polytopes are considered, and these polytopes are generalized by the addition of some cycles that are discussed in the main work. All the new cycles are generated up to some number; then, general representations were given for representing further cycles. The polytopes that are discussed for the partition dimension in generalized form are E_n, G_n, and S_n. Also, we obtained the bounds for the partition dimension of the desired polytopes, and the bound for the partition dimension of the considered polytopes is found to be 4 or less [35].

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest and all the authors agree to publish this paper under academic ethics.

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