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*Research Article*

# On Powers and Roots of Split Octonions

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In this article, we obtain the polar forms for two types of split octonions. We calculate De Moivre's formulas for all polar forms of split octonions. Thus, we give the $n^{th}-$ powers and roots of split octonions and the matrix representation of split octonions. In addition, we present an illustrative example with Matlab codes.

## 1. Introduction

De Moivre's formula,

$$(\cos\,\alpha + i\,\sin\,\alpha)^n = (\cos\,n\alpha + i\,\sin\,n\alpha), \tag{1}$$

is used to calculate the $n^{\text{th}}-$ powers and $n^{\text{th}}-$ roots of a complex number in polar form. Therefore, it has been one of the important subjects studied after some fundamental operations in mathematics. Many studies have been carried out on the adaptation of the formula to the lots of number systems, which are the expansion of the complex number system (references [1–7] and therein).

Split octonions have been worked on physics and mathematics [8–20]. Split octonions represent Minkowski space with $(4+4)$-signature. However, split octonions have also been studied with different orders of base elements in the articles in different fields. So split octonions have been studied as split octonions, countercomplex octonions, hyperbolic octonions, or split type octonions. But, it should be emphasized that there are direct relationships between all octonions in the literature. The geometrical fundamental properties for split octonions are located in the article [8–12].

In this study, we first give polar forms for two types of split octonions whose polar forms have not been given before. After that, we obtain De Moivre's formulas of the split octonions and the $8\times 8$ matrices related with the split octonions in all polar forms. Finally, we present the $n^{\text{th}}-$ roots of the split octonions and the matrix representation of split octonions in all polar forms.

## 2. Preliminaries

A split octonion $s$ [8, 9] is introduced as

$$s = w + V, \tag{2}$$

where the number $w$ is called scalar part of the split octonion and

$$V = \lambda_1 J_1 + \lambda_2 J_2 + \lambda_3 J_3 + x_1 j_1 + x_2 j_2 + x_3 j_3 + ctI, \tag{3}$$

is called vector parts of split octonion.

In (2), $c$, $t$, and $x_m$ denote the speed of the light, time, and space coordinates; $w$ and $\lambda_m$ are considered as the phase (classical action) and the wavelengths associated with octonionic signals in geometric application. The set of all split octonions is denoted by $\mathfrak{S}$. The fundamental algebraic properties for split octonions are located in Table 1. The conjugate of a split octonion, denoted by $s^c$, is introduced by the following equation:

$$s^c = w - \lambda_1 J_1 - \lambda_2 J_2 - \lambda_3 J_3 - x_1 j_1 - x_2 j_2 - x_3 j_3 - ctI. \tag{4}$$

The inner product over split octonions is defined by the following equation [12]:

$$\mathbb{I}(s) = ss^c = s^c s = w^2 - V^2, \tag{5}$$

Table 1: Split octonion multiplications.

| $\times$ | $1$ | $J_1$ | $J_2$ | $J_3$ | $j_1$ | $j_2$ | $j_3$ | $I$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $J_1$ | $J_2$ | $J_3$ | $j_1$ | $j_2$ | $j_3$ | $I$ |
| $J_1$ | $J_1$ | $1$ | $j_3$ | $-j_2$ | $I$ | $-J_3$ | $J_2$ | $j_1$ |
| $J_2$ | $J_2$ | $-j_3$ | $1$ | $j_1$ | $J_3$ | $I$ | $-J_1$ | $j_2$ |
| $J_3$ | $J_3$ | $j_2$ | $-j_1$ | $1$ | $-J_2$ | $J_1$ | $I$ | $j_3$ |
| $j_1$ | $j_1$ | $-I$ | $-J_3$ | $J_2$ | $-1$ | $j_3$ | $-j_2$ | $J_1$ |
| $j_2$ | $j_2$ | $J_3$ | $-I$ | $-J_1$ | $-j_3$ | $-1$ | $j_1$ | $J_2$ |
| $j_3$ | $j_3$ | $-J_2$ | $J_1$ | $-I$ | $j_2$ | $-j_1$ | $-1$ | $J_3$ |
| $I$ | $I$ | $-j_1$ | $-j_2$ | $-j_3$ | $-J_1$ | $-J_2$ | $-J_3$ | $1$ |

where $V^2 = \lambda^2 - x^2 + c^2t^2$ (where $\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^3$ and $x^2 = x_1^2 + x_2^2 + x_3^2$). So split octonions are divided into three classes having positive, negative, or zero value with respect to the inner product. In this case, similar to the literature of the split quaternions, if $\mathbb{I}(s) < 0$, $\mathbb{I}(s) > 0$, and $\mathbb{I}(s) = 0$, $s$ can be also called a spacelike split octonion, a timelike split octonion, and a lightlike split octonion, respectively. Also, the split octonions can be divided into three classes with respect to the vector part: if $V^2 < 0$, $V^2 > 0$, and $V^2 = 0$, then it is called a split octonion with spacelike, timelike, and lightlike vector parts, respectively. The norm of the split octonion, denoted by $N(s)$ or for brevity $N$, is defined by $N = \sqrt{|ss^c|}$ [12].

The polar representation of a split octonion depends on its inner product and its vector part. So, it can be summarized in 5-case as follows:

(i) Every split octonion with the inner product having a negative value is

$$s = N(\sinh \Upsilon + \mathfrak{U} \cosh \Upsilon), \tag{6}$$

where

$$\sinh \Upsilon = \frac{w}{N}, \quad \cosh \Upsilon = \frac{|V|}{N}, \quad \mathfrak{U} = \frac{\sum_{m=1}^{3}(\lambda_m J_m + x_m j_m) + ctI}{|V|}, \tag{7}$$

and $\mathfrak{U}$ is a unit timelike 7− vector [12].

(ii) Every split octonion with the inner product having a positive value and timelike vector part is

$$s = N(\cosh \Upsilon + \mathfrak{U} \sinh \Upsilon), \tag{8}$$

where

$$\cosh \Upsilon = \frac{w}{N}, \quad \sinh \Upsilon = \frac{|V|}{N}, \quad \mathfrak{U} = \frac{\sum_{m=1}^{3}(\lambda_m J_m + x_m j_m) + ctI}{|V|}, \tag{9}$$

and $\mathfrak{U}$ again is a unit timelike 7− vector [12].

(iii) Every split octonion with the inner product having a positive value and spacelike vector part is

$$s = N(\cos \Upsilon + \mathfrak{U} \sin \Upsilon), \tag{10}$$

where

$$\cos \Upsilon = \frac{w}{N}, \quad \sin \Upsilon = \frac{|V|}{N}, \quad \mathfrak{U} = \frac{\sum_{m=1}^{3}(\lambda_m J_m + x_m j_m) + ctI}{|V|}, \tag{11}$$

and $\mathfrak{U}$ now is a unit spacelike 7− vector [12].

Now, we will define the polar forms for two types of split octonions which is not given before as follows.

(iv) Every split octonion with the inner product having zero value, $w^2 = V^2$, is

$$s = w(1 + \mathfrak{U}), \tag{12}$$

where

$$\mathfrak{U} = \frac{\sum_{m=1}^{3}(\lambda_m J_m + x_m j_m) + ctI}{w}, \tag{13}$$

and $\mathfrak{U}$ now is a unit 7− vector and $\mathfrak{U}^2 = +1$ . Indeed, the following decomposition for the split octonion with the inner product having zero value, $w^2 = V^2$, can be written as follows:

$$s = w + V, \quad = w\left(1 + \frac{V}{w}\right), \tag{14}$$

and since $w^2 = V^2$,

$$\mathfrak{U}^2 = \left(\frac{V}{w}\right)^2 = +1, \tag{15}$$

is satisfied.

(v) Every split octonion with the lightlike vector part, $V^2 = 0$, is written as

$$s = N(\operatorname{sgn}(w) + \mathfrak{U}), \tag{16}$$

where

$$\mathfrak{U} = \frac{\sum_{m=1}^{3}(\lambda_m J_m + x_m j_m) + ctI}{N}, \tag{17}$$

and $\mathfrak{U}^2 = 0$. Because the split octonion $s$ with the lightlike vector part, $V^2 = 0$, can be determined as

$$\begin{aligned} s &= w + \lambda_1 J_1 + \lambda_2 J_2 + \lambda_3 J_3 + x_1 j_1 + x_2 j_2 + x_3 j_3 + ctI, \\ &= \sqrt{|w^2|}\left(\operatorname{sgn}(w) + \frac{\sum_{m=1}^{3}(\lambda_m J_m + x_m j_m) + ctI}{\sqrt{|w^2|}}\right), \end{aligned} \tag{18}$$

and since $V^2 = 0$,

$$\mathfrak{U}^2 = \left(\frac{V}{\sqrt{|w^2|}}\right)^2 = 0, \tag{19}$$

is provided.

## 3. De Moivre's Formulas of Split Octonions

In this section, we express De Moivre's formulas when split octonions are given in polar forms.

**Theorem 1.** *Given the polar form $s = \sinh\Upsilon + \mathfrak{U}\cosh\Upsilon$ for a unit split octonion with the inner product having a negative value. Then, we have the following equation:*

$$s^n = \sinh n\Upsilon + \mathfrak{U}\cosh n\Upsilon. \tag{20}$$

If $n$ is an odd integer, then the equation (20) is provided.

$$s^n = \cosh n\Upsilon + \mathfrak{U}\sinh n\Upsilon. \tag{21}$$

If $n$ is an even integer, then the equation (21) is provided.

*Proof.* For $n = 1$, it is trivial and for $n = 2$ and $n = 3$, with the hyperbolic angle sum formulas and considering that $\mathfrak{U}$ is timelike, we have the following equation:

$$\begin{aligned} s^2 &= \cosh 2\Upsilon + \mathfrak{U}\sinh 2\Upsilon, \\ s^3 &= \sinh 3\Upsilon + \mathfrak{U}\cosh 3\Upsilon. \end{aligned} \tag{22}$$

Now, let us assume that the formula is valid for $n = 2l, l \in \mathbb{Z}^+$, that is,

$$s^n = \cosh n\Upsilon + \mathfrak{U}\sinh n\Upsilon. \tag{23}$$

Then,

$$s^{2l}s = (\cosh 2l\Upsilon + \mathfrak{U}\sinh 2l\Upsilon)(\sinh\Upsilon + \mathfrak{U}\cosh\Upsilon), \tag{24}$$

or by using the hyperbolic angle sum formulas and $\mathfrak{U}^2 = 1$, we obtain the following equation:

$$s^{2l+1} = \sinh(2l+1)Y + \mathfrak{U}\cosh(2l+1)Y. \tag{25}$$

Similarly, suppose that the formula is provided for $n = 2l + 1, l \in \mathbb{Z}^+$, that is,

$$s^n = \sinh n\Upsilon + \mathfrak{U}\cosh n\Upsilon. \tag{26}$$

Then,

$$s^{2l+1}s = (\sinh(2l+1)\Upsilon + \mathfrak{U}\cosh(2l+1)\Upsilon)(\sinh\Upsilon + \mathfrak{U}\cosh\Upsilon), \tag{27}$$

or by using the hyperbolic angle sum formulas and $\mathfrak{U}^2 = 1$, we have the following equation:

$$s^{2l+2} = \cosh(2l+2)\Upsilon + \mathfrak{U}\sinh(2l+2)\Upsilon. \tag{28}$$

Also, we can calculate for negative integer power by a similar method. For $n = -1$, by the properties of the multiplicative inverse of the split octonion $s$, we have $s^{-1} = -\sinh\Upsilon + \mathfrak{U}\cosh\Upsilon$ and $s^{-2} = \cosh -2\Upsilon + \mathfrak{U}\cosh -2\Upsilon$. If the calculation process is repeated as mentioned above, the formula can be obtained similarly. □

**Theorem 2.** *Let $s = \cosh\Upsilon + \mathfrak{U}\sinh\Upsilon$ be a unit split octonion with the inner product having positive value and time-like vector part. Then, we have the following equation:*

$$s^n = \cosh n\Upsilon + \mathfrak{U}\sinh n\Upsilon, \tag{29}$$

*for all $n \in \mathbb{Z}$.*

*Proof.* Like the proof of Theorem 1, by using the induction and the hyperbolic angle sum formulas, the proof can be seen clearly. □

**Theorem 3.** *Given the polar form $s = \cos\Upsilon + \mathfrak{U}\sin\Upsilon$ for a unit split octonion with the inner product having a positive value and spacelike vector part. Then, we have the following equation:*

$$s^n = \cos n\Upsilon + \mathfrak{U}\sin n\Upsilon, \tag{30}$$

*for all $n \in \mathbb{Z}$.*

*Proof.* Like the steps of the proof of Theorem 1, by applying the induction, the poof can be obtained. □

**Theorem 4.** *Let us consider the polar form $s = w(1 + \mathfrak{U})$ for a split octonion with the inner product having zero. Then, we have the following equation:*

$$s^n = w^n 2^{n-1}(1 + \mathfrak{U}), \tag{31}$$

*for all $n \in \mathbb{Z}$.*

*Proof.* Let us apply the induction method. Suppose that $s$ is a split octonion with the inner product having zero and $s^n = w^n 2^{n-1}(1 + \mathfrak{U})$ holds. Then, $\mathfrak{U}^2 = 1$ and

$$
\begin{aligned}
s^{n+1} &= w^n 2^{n-1}(1+\mathfrak{U})w(1+\mathfrak{U}),\\
&= w^{n+1}2^{n-1}(1+\mathfrak{U})^2,\\
&= w^{n+1}2^{n}(1+\mathfrak{U}).
\end{aligned} \tag{32}
$$

□

**Theorem 5.** *Assume that $s=\operatorname{sgn}(w)+\mathfrak{U}$ is a unit split octonion with the light-like vector part. Then, we have the following equation:*

$$s^n = \operatorname{sgn}(w)^n + nw^{n-1}\mathfrak{U}, \tag{33}$$

*for all $n \in \mathbb{Z}$.*

*Proof.* By applying the induction method and considering $\mathfrak{U}^2=0$, the proof can be seen directly. □

## 4. Roots of Split Octonions

In this section, the roots of a split octonion or in other words the solutions of the equation $x^n = s$ will be examined.

**Theorem 6.** *Suppose that $s=\sinh\Upsilon+\mathfrak{U}\cosh\Upsilon$ is a unit split octonion with the inner product having a negative value. Let us consider the equation $x^n=s$.*

(i) If $n$ is an even number, then there is no root

(ii) If $n$ is an odd number, then there exist only 1− root as follows:

$$x_0 = \sinh\vartheta + \mathfrak{U}\cosh\vartheta, \tag{34}$$

where $\vartheta = \Upsilon/n$.

*Proof.* Let $s=\sinh\Upsilon+\mathfrak{U}\cosh\Upsilon$ be a unit split octonion with the inner product having a negative value and $n$ is an odd number. Then, $\mathfrak{U}^2=1$ and using Theorem 1, we can write

$$x_0^n = \sinh n\vartheta + \mathfrak{U}\cosh n\vartheta = s. \tag{35}$$

The other case is clear from Theorem 1.

**Theorem 7.** *Let $s=\cosh\Upsilon+\mathfrak{U}\sinh\Upsilon$ be a unit split octonion with the inner product having a positive value and timelike vector part. Let us consider the equation of $x^n=s$.*

(i) If $n$ is an even number, then the 4-distinct roots are given in the following forms:

$$
\begin{aligned}
x_0 &= \sinh\vartheta + \mathfrak{U}\cosh\vartheta,\\
x_1 &= -(\sinh\vartheta + \mathfrak{U}\cosh\vartheta),\\
x_2 &= \cosh\vartheta + \mathfrak{U}\sinh\vartheta,\\
x_3 &= -(\cosh\vartheta + \mathfrak{U}\sinh\vartheta),
\end{aligned} \tag{36}
$$

where $\vartheta = \Upsilon/n$.

(ii) If $n$ is an odd number, then the only 1− root is

$$x_0 = \sinh\vartheta + \mathfrak{U}\cosh\vartheta, \tag{37}$$

where $\vartheta = \Upsilon/n$.

*Proof.* Let $s=\sinh\Upsilon+\mathfrak{U}\cosh\Upsilon$ be a unit split octonion with the inner product having a positive value and timelike vector part:

(i) Assume that $n$ is an even number. And, let

$$
\begin{aligned}
x_0 &= \sinh\vartheta + \mathfrak{U}\cosh\vartheta,\\
x_1 &= -(\sinh\vartheta + \mathfrak{U}\cosh\vartheta),
\end{aligned} \tag{38}
$$

be roots of the equation $x^n=s$, where $\vartheta=\Upsilon/n$. Then, using Theorem 1, we can obtain the following equation:

$$
\begin{aligned}
x_0^n &= \sinh n\vartheta + \mathfrak{U}\cosh n\vartheta = s,\\
x_1^n &= +(\sinh n\vartheta + \mathfrak{U}\cosh n\vartheta) = s.
\end{aligned} \tag{39}
$$

Thus, $x_0$ and $x_1$ hold the equation $x^n=s$. The other cases can be obtained by using Theorems 1 and 2. □

**Theorem 8.** *Let $s=\cosh\Upsilon+\mathfrak{U}\sinh\Upsilon$ be a unit split octonion with the inner product having a positive value and spacelike vector part. Then, there exist $n$− roots*

$$x_p = \cos\vartheta + \mathfrak{U}\sin\vartheta, \tag{40}$$

*where $\vartheta = \Upsilon + 2p\pi/n$, $p=0,1,\ldots,n-1$.*

*Proof.* One can directly verify this result with a similar way of the proof of Theorem 6.

**Theorem 9.** *Let $s=w(1+\mathfrak{U})$ be a light-like split octonion with the inner product having zero. Then,*

*(i) If $n$ is an even number, then 2− roots of the equation $x^n=s$ are in the following forms:*

$$
\begin{aligned}
x_0 &= \Omega(1+\mathfrak{U}),\\
x_1 &= -\Omega(1+\mathfrak{U}),
\end{aligned} \tag{41}
$$

*where $\Omega = \sqrt[n]{2\omega}/2$.*

*(ii) If $n$ is an odd number, then only 1− root of the equation $x^n=s$ is*

$$x_0 = \Omega(1+\mathfrak{U}), \tag{42}$$

*where $\Omega = \sqrt[n]{2\omega}/2$.*

*Proof.* The proof can be easily obtained by using the same way of Theorem 6.

**Theorem 10.** *Let $s=\operatorname{sgn}(w)+\mathfrak{U}$ be a unit split octonion with the light-like vector part.*

(i) If $n$ is an even number, there are 2− roots of the equation $x^n=s$ for $\operatorname{sgn}(w)>0$. The roots are in the following forms:

$$x_0 = \sqrt[n]{w}\left(1 + \frac{\mathfrak{U}}{n}\right),$$
$$x_1 = -\sqrt[n]{w}\left(1 + \frac{\mathfrak{U}}{n}\right). \tag{43}$$

But there exist no roots of the equation $x^n = s$ for $\operatorname{sgn}(w) < 0$.

(ii) If $n$ is an odd number, the only 1− root of the equation of $x^n = s$ is in the form as follows:

$$x_0 = \sqrt[n]{w}\left(1 + \frac{\mathfrak{U}}{n}\right). \tag{44}$$

*Proof.* One can prove the formula by using the same way of Theorem 6. □

## 5. De Moivre's Formulas of the Matrices of Split Octonions

Let $s$ and $x$ be any two split octonions in $\mathfrak{S}$. Given the following two linear transformations in $\mathfrak{S}$,

$$\mathfrak{s}_L, \mathfrak{s}_R \colon \mathfrak{S} \longrightarrow \mathfrak{S},$$
$$\mathfrak{s}_L(x) = sx, \tag{45}$$
$$\mathfrak{s}_R(x) = xs.$$

Then, using the transformation $\mathfrak{s}_L$ and basis vectors $\{1, J_m, j_m, I\}$, for $m = 1, 2, 3$, the left matrix representation of the transformation $\mathfrak{s}_L$ is determined by

$$\mathfrak{s}_L = \begin{bmatrix} w & \lambda_1 & \lambda_2 & \lambda_3 & -x_1 & -x_2 & -x_3 & ct \\ \lambda_1 & w & x_3 & -x_2 & -ct & \lambda_3 & -\lambda_2 & x_1 \\ \lambda_2 & -x_3 & w & x_1 & -\lambda_3 & -ct & \lambda_1 & x_2 \\ \lambda_3 & x_2 & -x_1 & w & \lambda_2 & -\lambda_1 & -ct & x_3 \\ x_1 & -ct & -\lambda_3 & \lambda_2 & w & -x_3 & x_2 & \lambda_1 \\ x_2 & \lambda_3 & -ct & -\lambda_1 & x_3 & w & -x_1 & \lambda_2 \\ x_3 & -\lambda_2 & \lambda_1 & -ct & -x_2 & x_1 & w & \lambda_3 \\ ct & -x_1 & -x_2 & -x_3 & \lambda_1 & \lambda_2 & \lambda_3 & w \end{bmatrix}. \tag{46}$$

Similarly, using the transformation $\mathfrak{s}_R$ and basis vectors $\{1, J_m, j_m, I\}$, for $m = 1, 2, 3$, the right matrix representation of the transformation $\mathfrak{s}_R$ is given by

$$\mathfrak{s}_R = \begin{bmatrix} w & \lambda_1 & \lambda_2 & \lambda_3 & -x_1 & -x_2 & -x_3 & ct \\ \lambda_1 & w & -x_3 & x_2 & ct & -\lambda_3 & \lambda_2 & -x_1 \\ \lambda_2 & x_3 & w & -x_1 & \lambda_3 & ct & -\lambda_1 & -x_2 \\ \lambda_3 & -x_2 & x_1 & w & -\lambda_2 & \lambda_1 & ct & -x_3 \\ x_1 & ct & \lambda_3 & -\lambda_2 & w & x_3 & -x_2 & -\lambda_1 \\ x_2 & -\lambda_3 & ct & \lambda_1 & -x_3 & w & x_1 & -\lambda_2 \\ x_3 & \lambda_2 & -\lambda_1 & ct & x_2 & -x_1 & w & -\lambda_3 \\ ct & x_1 & x_2 & x_3 & -\lambda_1 & -\lambda_2 & -\lambda_3 & w \end{bmatrix}. \tag{47}$$

In addition to this, we can define an isomorphism $\phi \colon \mathfrak{S} \longrightarrow M_L$, where

$$M_L = \left\{ \begin{bmatrix} w & \lambda_1 & \lambda_2 & \lambda_3 & -x_1 & -x_2 & -x_3 & ct \\ \lambda_1 & w & x_3 & -x_2 & -ct & \lambda_3 & -\lambda_2 & x_1 \\ \lambda_2 & -x_3 & w & x_1 & -\lambda_3 & -ct & \lambda_1 & x_2 \\ \lambda_3 & x_2 & -x_1 & w & \lambda_2 & -\lambda_1 & -ct & x_3 \\ x_1 & -ct & -\lambda_3 & \lambda_2 & w & -x_3 & x_2 & \lambda_1 \\ x_2 & \lambda_3 & -ct & -\lambda_1 & x_3 & w & -x_1 & \lambda_2 \\ x_3 & -\lambda_2 & \lambda_1 & -ct & -x_2 & x_1 & w & \lambda_3 \\ ct & -x_1 & -x_2 & -x_3 & \lambda_1 & \lambda_2 & \lambda_3 & w \end{bmatrix} \middle| w, \lambda_m, x_m \in \mathbb{R}, \quad m = 1, 2, 3 \right\}. \tag{48}$$

Similarly, we can define a bijective and surjective transformation $\psi \colon \mathfrak{S} \longrightarrow M_R$, where

$$M_R = \left\{ \begin{bmatrix} w & \lambda_1 & \lambda_2 & \lambda_3 & -x_1 & -x_2 & -x_3 & ct \\ \lambda_1 & w & -x_3 & x_2 & ct & -\lambda_3 & \lambda_2 & -x_1 \\ \lambda_2 & x_3 & w & -x_1 & \lambda_3 & ct & -\lambda_1 & -x_2 \\ \lambda_3 & -x_2 & x_1 & w & -\lambda_2 & \lambda_1 & ct & -x_3 \\ x_1 & ct & \lambda_3 & -\lambda_2 & w & x_3 & -x_2 & -\lambda_1 \\ x_2 & -\lambda_3 & ct & \lambda_1 & -x_3 & w & x_1 & -\lambda_2 \\ x_3 & \lambda_2 & -\lambda_1 & ct & x_2 & -x_1 & w & -\lambda_3 \\ ct & x_1 & x_2 & x_3 & -\lambda_1 & -\lambda_2 & -\lambda_3 & w \end{bmatrix} \middle| w, \lambda_m, x_m \in \mathbb{R}, \quad m = 1, 2, 3 \right\}. \tag{49}$$

But note that, $\psi(s_1+s_2)=\psi(s_1)+\psi(s_2)$ and $\psi(s_1s_2)=\psi(s_2)\psi(s_1)$, for $\forall s_1,s_2\in\mathfrak{S}$ are hold.

**Lemma 1.** *Let s be a split octonion. Then,*

$$\det\mathfrak{s}_L=\det\mathfrak{s}_R=N^4, \tag{50}$$

*where N is the norm of s.*

*Proof.* One can easily prove with some algebraic operations.

From now on, we consider only the right matrix representation $\mathfrak{s}_R$ of any split octonion $s$ through all our calculations. But all results also can be easily obtained for the left matrix representations of any split octonion by the same methods:

(i) If $s=\sinh\Upsilon+\mathfrak{U}\cosh\Upsilon$ is a unit split octonion with the inner product having a negative value and a unit timelike 7− vector $\mathfrak{U}$, the right representation of the split octonion $\mathfrak{s}_R$ can be written as follows:

$$\begin{bmatrix}
\sinh\Upsilon & \frac{\lambda_1}{|V|}\cosh\Upsilon & \frac{\lambda_2}{|V|}\cosh\Upsilon & \frac{\lambda_3}{|V|}\cosh\Upsilon & -\frac{x_1}{|V|}\cosh\Upsilon & -\frac{x_2}{|V|}\cosh\Upsilon & -\frac{x_3}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon \\
\frac{\lambda_1}{|V|}\cosh\Upsilon & \sinh\Upsilon & -\frac{x_3}{|V|}\cosh\Upsilon & \frac{x_2}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & -\frac{\lambda_3}{|V|}\cosh\Upsilon & \frac{\lambda_2}{|V|}\cosh\Upsilon & -\frac{x_1}{|V|}\cosh\Upsilon \\
\frac{\lambda_2}{|V|}\cosh\Upsilon & \frac{x_3}{|V|}\cosh\Upsilon & \sinh\Upsilon & -\frac{x_1}{|V|}\cosh\Upsilon & \frac{\lambda_3}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & -\frac{\lambda_1}{|V|}\cosh\Upsilon & -\frac{x_2}{|V|}\cosh\Upsilon \\
\frac{\lambda_3}{|V|}\cosh\Upsilon & -\frac{x_2}{|V|}\cosh\Upsilon & \frac{x_1}{|V|}\cosh\Upsilon & \sinh\Upsilon & -\frac{\lambda_2}{|V|}\cosh\Upsilon & \frac{\lambda_1}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & -\frac{x_3}{|V|}\cosh\Upsilon \\
\frac{x_1}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & \frac{\lambda_3}{|V|}\cosh\Upsilon & -\frac{\lambda_2}{|V|}\cosh\Upsilon & \sinh\Upsilon & \frac{x_3}{|V|}\cosh\Upsilon & -\frac{x_2}{|V|}\cosh\Upsilon & -\frac{\lambda_1}{|V|}\cosh\Upsilon \\
\frac{x_2}{|V|}\cosh\Upsilon & -\frac{\lambda_3}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & \frac{\lambda_1}{|V|}\cosh\Upsilon & -\frac{x_3}{|V|}\cosh\Upsilon & \sinh\Upsilon & \frac{x_1}{|V|}\cosh\Upsilon & -\frac{\lambda_2}{|V|}\cosh\Upsilon \\
\frac{x_3}{|V|}\cosh\Upsilon & \frac{\lambda_2}{|V|}\cosh\Upsilon & -\frac{\lambda_1}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & \frac{x_2}{|V|}\cosh\Upsilon & -\frac{x_1}{|V|}\cosh\Upsilon & \sinh\Upsilon & -\frac{\lambda_3}{|V|}\cosh\Upsilon \\
\frac{ct}{|V|}\cosh\Upsilon & \frac{x_1}{|V|}\cosh\Upsilon & \frac{x_2}{|V|}\cosh\Upsilon & \frac{x_3}{|V|}\cosh\Upsilon & -\frac{\lambda_1}{|V|}\cosh\Upsilon & -\frac{\lambda_2}{|V|}\cosh\Upsilon & -\frac{\lambda_3}{|V|}\cosh\Upsilon & \sinh\Upsilon
\end{bmatrix}. \tag{51}$$

(ii) If $s=\cosh\Upsilon+\mathfrak{U}\sinh\Upsilon$ is a split octonion with the inner product having a positive value and a unit timelike vector part $\mathfrak{U}$, then the right representation of the split octonion $\mathfrak{s}_R$ can be written as follows:

$$\begin{bmatrix}
\cosh\Upsilon & \frac{\lambda_1}{|V|}\sinh\Upsilon & \frac{\lambda_2}{|V|}\sinh\Upsilon & \frac{\lambda_3}{|V|}\sinh\Upsilon & -\frac{x_1}{|V|}\sinh\Upsilon & -\frac{x_2}{|V|}\sinh\Upsilon & -\frac{x_3}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon \\
\frac{\lambda_1}{|V|}\sinh\Upsilon & \cosh\Upsilon & -\frac{x_3}{|V|}\sinh\Upsilon & \frac{x_2}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & -\frac{\lambda_3}{|V|}\sinh\Upsilon & \frac{\lambda_2}{|V|}\sinh\Upsilon & -\frac{x_1}{|V|}\sinh\Upsilon \\
\frac{\lambda_2}{|V|}\sinh\Upsilon & \frac{x_3}{|V|}\sinh\Upsilon & \cosh\Upsilon & -\frac{x_1}{|V|}\sinh\Upsilon & \frac{\lambda_3}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & -\frac{\lambda_1}{|V|}\sinh\Upsilon & -\frac{x_2}{|V|}\sinh\Upsilon \\
\frac{\lambda_3}{|V|}\sinh\Upsilon & -\frac{x_2}{|V|}\sinh\Upsilon & \frac{x_1}{|V|}\sinh\Upsilon & \cosh\Upsilon & -\frac{\lambda_2}{|V|}\sinh\Upsilon & \frac{\lambda_1}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & -\frac{x_3}{|V|}\sinh\Upsilon \\
\frac{x_1}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & \frac{\lambda_3}{|V|}\sinh\Upsilon & -\frac{\lambda_2}{|V|}\sinh\Upsilon & \cosh\Upsilon & \frac{x_3}{|V|}\sinh\Upsilon & -\frac{x_2}{|V|}\sinh\Upsilon & -\frac{\lambda_1}{|V|}\sinh\Upsilon \\
\frac{x_2}{|V|}\sinh\Upsilon & -\frac{\lambda_3}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & \frac{\lambda_1}{|V|}\sinh\Upsilon & -\frac{x_3}{|V|}\sinh\Upsilon & \cosh\Upsilon & \frac{x_1}{|V|}\sinh\Upsilon & -\frac{\lambda_2}{|V|}\sinh\Upsilon \\
\frac{x_3}{|V|}\sinh\Upsilon & \frac{\lambda_2}{|V|}\sinh\Upsilon & -\frac{\lambda_1}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & \frac{x_2}{|V|}\sinh\Upsilon & -\frac{x_1}{|V|}\sinh\Upsilon & \cosh\Upsilon & -\frac{\lambda_3}{|V|}\sinh\Upsilon \\
\frac{ct}{|V|}\sinh\Upsilon & \frac{x_1}{|V|}\sinh\Upsilon & \frac{x_2}{|V|}\sinh\Upsilon & \frac{x_3}{|V|}\sinh\Upsilon & -\frac{\lambda_1}{|V|}\sinh\Upsilon & -\frac{\lambda_2}{|V|}\sinh\Upsilon & -\frac{\lambda_3}{|V|}\sinh\Upsilon & \cosh\Upsilon
\end{bmatrix}. \tag{52}$$

(iii) If $s = \cos\Upsilon + \mathfrak{U}\sin\Upsilon$ is a split octonion with the inner product having a positive value and spacelike vector part, then the right representation of the split octonion $\mathfrak{s}_R$ can be obtained as follows:

$$
\begin{bmatrix}
\cos\Upsilon & \frac{\lambda_1}{|V|}\sin\Upsilon & \frac{\lambda_2}{|V|}\sin\Upsilon & \frac{\lambda_3}{|V|}\sin\Upsilon & -\frac{x_1}{|V|}\sin\Upsilon & -\frac{x_2}{|V|}\sin\Upsilon & -\frac{x_3}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon \\
\frac{\lambda_1}{|V|}\sin\Upsilon & \cos\Upsilon & -\frac{x_3}{|V|}\sin\Upsilon & \frac{x_2}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & -\frac{\lambda_3}{|V|}\sin\Upsilon & \frac{\lambda_2}{|V|}\sin\Upsilon & -\frac{x_1}{|V|}\sin\Upsilon \\
\frac{\lambda_2}{|V|}\sin\Upsilon & \frac{x_3}{|V|}\sin\Upsilon & \cos\Upsilon & -\frac{x_1}{|V|}\sin\Upsilon & \frac{\lambda_3}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & -\frac{\lambda_1}{|V|}\sin\Upsilon & -\frac{x_2}{|V|}\sin\Upsilon \\
\frac{\lambda_3}{|V|}\sin\Upsilon & -\frac{x_2}{|V|}\sin\Upsilon & \frac{x_1}{|V|}\sin\Upsilon & \cos\Upsilon & -\frac{\lambda_2}{|V|}\sin\Upsilon & \frac{\lambda_1}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & -\frac{x_3}{|V|}\sin\Upsilon \\
\frac{x_1}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & \frac{\lambda_3}{|V|}\sin\Upsilon & -\frac{\lambda_2}{|V|}\sin\Upsilon & \cos\Upsilon & \frac{x_3}{|V|}\sin\Upsilon & -\frac{x_2}{|V|}\sin\Upsilon & -\frac{\lambda_1}{|V|}\sin\Upsilon \\
\frac{x_2}{|V|}\sin\Upsilon & -\frac{\lambda_3}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & \frac{\lambda_1}{|V|}\sin\Upsilon & -\frac{x_3}{|V|}\sin\Upsilon & \cos\Upsilon & \frac{x_1}{|V|}\sin\Upsilon & -\frac{\lambda_2}{|V|}\sin\Upsilon \\
\frac{x_3}{|V|}\sin\Upsilon & \frac{\lambda_2}{|V|}\sin\Upsilon & -\frac{\lambda_1}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & \frac{x_2}{|V|}\sin\Upsilon & -\frac{x_1}{|V|}\sin\Upsilon & \cos\Upsilon & -\frac{\lambda_3}{|V|}\sin\Upsilon \\
\frac{ct}{|V|}\sin\Upsilon & \frac{x_1}{|V|}\sin\Upsilon & \frac{x_2}{|V|}\sin\Upsilon & \frac{x_3}{|V|}\sin\Upsilon & -\frac{\lambda_1}{|V|}\sin\Upsilon & -\frac{\lambda_2}{|V|}\sin\Upsilon & -\frac{\lambda_3}{|V|}\sin\Upsilon & \cos\Upsilon
\end{bmatrix}. \tag{53}
$$

(iv) If $s = w(1 + \mathfrak{U})$ is a split octonion with the inner product having zero. Then, the right representation of the split octonion $\mathfrak{s}_R$ can be given as follows:

$$
\mathfrak{s}_R = \begin{bmatrix}
w & \lambda_1 & \lambda_2 & \lambda_3 & -x_1 & -x_2 & -x_3 & ct \\
\lambda_1 & w & -x_3 & x_2 & ct & -\lambda_3 & \lambda_2 & -x_1 \\
\lambda_2 & x_3 & w & -x_1 & \lambda_3 & ct & -\lambda_1 & -x_2 \\
\lambda_3 & -x_2 & x_1 & w & -\lambda_2 & \lambda_1 & ct & -x_3 \\
x_1 & ct & \lambda_3 & -\lambda_2 & w & x_3 & -x_2 & -\lambda_1 \\
x_2 & -\lambda_3 & ct & \lambda_1 & -x_3 & w & x_1 & -\lambda_2 \\
x_3 & \lambda_2 & -\lambda_1 & ct & x_2 & -x_1 & w & -\lambda_3 \\
ct & x_1 & x_2 & x_3 & -\lambda_1 & -\lambda_2 & -\lambda_3 & w
\end{bmatrix}. \tag{54}
$$

(v) If $s = \mathrm{sgn}(w) + \mathfrak{U}$ is a unit split octonion with the lightlike vector part $s = 0$. Then, the right representation of the split octonion $\mathfrak{s}_R$ is found as

$$\mathfrak{s}_R = \begin{bmatrix} \operatorname{sgn}(w) & \lambda_1 & \lambda_2 & \lambda_3 & -x_1 & -x_2 & -x_3 & ct \\ \lambda_1 & \operatorname{sgn}(w) & -x_3 & x_2 & ct & -\lambda_3 & \lambda_2 & -x_1 \\ \lambda_2 & x_3 & \operatorname{sgn}(w) & -x_1 & \lambda_3 & ct & -\lambda_1 & -x_2 \\ \lambda_3 & -x_2 & x_1 & \operatorname{sgn}(w) & -\lambda_2 & \lambda_1 & ct & -x_3 \\ x_1 & ct & \lambda_3 & -\lambda_2 & \operatorname{sgn}(w) & x_3 & -x_2 & -\lambda_1 \\ x_2 & -\lambda_3 & ct & \lambda_1 & -x_3 & \operatorname{sgn}(w) & x_1 & -\lambda_2 \\ x_3 & \lambda_2 & -\lambda_1 & ct & x_2 & -x_1 & \operatorname{sgn}(w) & -\lambda_3 \\ ct & x_1 & x_2 & x_3 & -\lambda_1 & -\lambda_2 & -\lambda_3 & \operatorname{sgn}(w) \end{bmatrix}. \quad (55)$$

□

**Theorem 11.** *Let us consider the matrix* $\mathfrak{s}_R$ *defined by*

$$\begin{bmatrix} \sinh\Upsilon & \frac{\lambda_1}{|V|}\cosh\Upsilon & \frac{\lambda_2}{|V|}\cosh\Upsilon & \frac{\lambda_3}{|V|}\cosh\Upsilon & -\frac{x_1}{|V|}\cosh\Upsilon & -\frac{x_2}{|V|}\cosh\Upsilon & -\frac{x_3}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon \\ \frac{\lambda_1}{|V|}\cosh\Upsilon & \sinh\Upsilon & -\frac{x_3}{|V|}\cosh\Upsilon & \frac{x_2}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & -\frac{\lambda_3}{|V|}\cosh\Upsilon & \frac{\lambda_2}{|V|}\cosh\Upsilon & -\frac{x_1}{|V|}\cosh\Upsilon \\ \frac{\lambda_2}{|V|}\cosh\Upsilon & \frac{x_3}{|V|}\cosh\Upsilon & \sinh\Upsilon & -\frac{x_1}{|V|}\cosh\Upsilon & \frac{\lambda_3}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & -\frac{\lambda_1}{|V|}\cosh\Upsilon & -\frac{x_2}{|V|}\cosh\Upsilon \\ \frac{\lambda_3}{|V|}\cosh\Upsilon & -\frac{x_2}{|V|}\cosh\Upsilon & \frac{x_1}{|V|}\cosh\Upsilon & \sinh\Upsilon & -\frac{\lambda_2}{|V|}\cosh\Upsilon & \frac{\lambda_1}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & -\frac{x_3}{|V|}\cosh\Upsilon \\ \frac{x_1}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & \frac{\lambda_3}{|V|}\cosh\Upsilon & -\frac{\lambda_2}{|V|}\cosh\Upsilon & \sinh\Upsilon & \frac{x_3}{|V|}\cosh\Upsilon & -\frac{x_2}{|V|}\cosh\Upsilon & -\frac{\lambda_1}{|V|}\cosh\Upsilon \\ \frac{x_2}{|V|}\cosh\Upsilon & -\frac{\lambda_3}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & \frac{\lambda_1}{|V|}\cosh\Upsilon & -\frac{x_3}{|V|}\cosh\Upsilon & \sinh\Upsilon & \frac{x_1}{|V|}\cosh\Upsilon & -\frac{\lambda_2}{|V|}\cosh\Upsilon \\ \frac{x_3}{|V|}\cosh\Upsilon & \frac{\lambda_2}{|V|}\cosh\Upsilon & -\frac{\lambda_1}{|V|}\cosh\Upsilon & \frac{ct}{|V|}\cosh\Upsilon & \frac{x_2}{|V|}\cosh\Upsilon & -\frac{x_1}{|V|}\cosh\Upsilon & \sinh\Upsilon & -\frac{\lambda_3}{|V|}\cosh\Upsilon \\ \frac{ct}{|V|}\cosh\Upsilon & \frac{x_1}{|V|}\cosh\Upsilon & \frac{x_2}{|V|}\cosh\Upsilon & \frac{x_3}{|V|}\cosh\Upsilon & -\frac{\lambda_1}{|V|}\cosh\Upsilon & -\frac{\lambda_2}{|V|}\cosh\Upsilon & -\frac{\lambda_3}{|V|}\cosh\Upsilon & \sinh\Upsilon \end{bmatrix}, \quad (56)$$

for a unit split octonion with the inner product having a negative value and a unit timelike 7-vector $\mathfrak{U}$. Then, if $n \in \mathbb{Z}$ is an odd number, the matrix $\mathfrak{s}_R^n$ is

$$\begin{bmatrix}
\sinh n\Upsilon & \frac{\lambda_1}{|V|}\cosh n\Upsilon & \frac{\lambda_2}{|V|}\cosh n\Upsilon & \frac{\lambda_3}{|V|}\cosh n\Upsilon & -\frac{x_1}{|V|}\cosh n\Upsilon & -\frac{x_2}{|V|}\cosh n\Upsilon & -\frac{x_3}{|V|}\cosh n\Upsilon & \frac{ct}{|V|}\cosh n\Upsilon \\
\frac{\lambda_1}{|V|}\cosh n\Upsilon & \sinh n\Upsilon & -\frac{x_3}{|V|}\cosh n\Upsilon & \frac{x_2}{|V|}\cosh n\Upsilon & \frac{ct}{|V|}\cosh n\Upsilon & -\frac{\lambda_3}{|V|}\cosh n\Upsilon & \frac{\lambda_2}{|V|}\cosh n\Upsilon & -\frac{x_1}{|V|}\cosh n\Upsilon \\
\frac{\lambda_2}{|V|}\cosh n\Upsilon & \frac{x_3}{|V|}\cosh n\Upsilon & \sinh n\Upsilon & -\frac{x_1}{|V|}\cosh n\Upsilon & \frac{\lambda_3}{|V|}\cosh n\Upsilon & \frac{ct}{|V|}\cosh n\Upsilon & -\frac{\lambda_1}{|V|}\cosh n\Upsilon & -\frac{x_2}{|V|}\cosh n\Upsilon \\
\frac{\lambda_3}{|V|}\cosh n\Upsilon & -\frac{x_2}{|V|}\cosh n\Upsilon & \frac{x_1}{|V|}\cosh n\Upsilon & \sinh n\Upsilon & -\frac{\lambda_2}{|V|}\cosh n\Upsilon & \frac{\lambda_1}{|V|}\cosh n\Upsilon & \frac{ct}{|V|}\cosh n\Upsilon & -\frac{x_3}{|V|}\cosh n\Upsilon \\
\frac{x_1}{|V|}\cosh n\Upsilon & \frac{ct}{|V|}\cosh n\Upsilon & \frac{\lambda_3}{|V|}\cosh n\Upsilon & -\frac{\lambda_2}{|V|}\cosh n\Upsilon & \sinh n\Upsilon & \frac{x_3}{|V|}\cosh n\Upsilon & -\frac{x_2}{|V|}\cosh n\Upsilon & -\frac{\lambda_1}{|V|}\cosh n\Upsilon \\
\frac{x_2}{|V|}\cosh n\Upsilon & -\frac{\lambda_3}{|V|}\cosh n\Upsilon & \frac{ct}{|V|}\cosh n\Upsilon & \frac{\lambda_1}{|V|}\cosh n\Upsilon & -\frac{x_3}{|V|}\cosh \Upsilon & \sinh n\Upsilon & \frac{x_1}{|V|}\cosh \Upsilon & -\frac{\lambda_2}{|V|}\cosh \Upsilon \\
\frac{x_3}{|V|}\cosh n\Upsilon & \frac{\lambda_2}{|V|}\cosh n\Upsilon & -\frac{\lambda_1}{|V|}\cosh n\Upsilon & \frac{ct}{|V|}\cosh n\Upsilon & \frac{x_2}{|V|}\cosh n\Upsilon & -\frac{x_1}{|V|}\cosh n\Upsilon & \sinh n\Upsilon & -\frac{\lambda_3}{|V|}\cosh n\Upsilon \\
\frac{ct}{|V|}\cosh n\Upsilon & \frac{x_1}{|V|}\cosh n\Upsilon & \frac{x_2}{|V|}\cosh n\Upsilon & \frac{x_3}{|V|}\cosh n\Upsilon & -\frac{\lambda_1}{|V|}\cosh n\Upsilon & -\frac{\lambda_2}{|V|}\cosh n\Upsilon & -\frac{\lambda_3}{|V|}\cosh n\Upsilon & \sinh n\Upsilon
\end{bmatrix}, \quad (57)$$

*and if $n \in \mathbb{Z}$ is an even number, the matrix $\mathfrak{S}_R^n$ is*

$$\begin{bmatrix}
\cosh n\Upsilon & \frac{\lambda_1}{|V|}\sinh n\Upsilon & \frac{\lambda_2}{|V|}\sinh n\Upsilon & \frac{\lambda_3}{|V|}\sinh n\Upsilon & -\frac{x_1}{|V|}\sinh n\Upsilon & -\frac{x_2}{|V|}\sinh n\Upsilon & -\frac{x_3}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon \\
\frac{\lambda_1}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & -\frac{x_3}{|V|}\sinh n\Upsilon & \frac{x_2}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & -\frac{\lambda_3}{|V|}\sinh n\Upsilon & \frac{\lambda_2}{|V|}\sinh n\Upsilon & -\frac{x_1}{|V|}\sinh n\Upsilon \\
\frac{\lambda_2}{|V|}\sinh n\Upsilon & \frac{x_3}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & -\frac{x_1}{|V|}\sinh n\Upsilon & \frac{\lambda_3}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & -\frac{\lambda_1}{|V|}\sinh n\Upsilon & -\frac{x_2}{|V|}\sinh n\Upsilon \\
\frac{\lambda_3}{|V|}\sinh n\Upsilon & -\frac{x_2}{|V|}\sinh n\Upsilon & \frac{x_1}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & -\frac{\lambda_2}{|V|}\sinh n\Upsilon & \frac{\lambda_1}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & -\frac{x_3}{|V|}\sinh n\Upsilon \\
\frac{x_1}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & \frac{\lambda_3}{|V|}\sinh n\Upsilon & -\frac{\lambda_2}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & \frac{x_3}{|V|}\sinh n\Upsilon & -\frac{x_2}{|V|}\sinh n\Upsilon & -\frac{\lambda_1}{|V|}\sinh n\Upsilon \\
\frac{x_2}{|V|}\sinh n\Upsilon & -\frac{\lambda_3}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & \frac{\lambda_1}{|V|}\sinh n\Upsilon & -\frac{x_3}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & \frac{x_1}{|V|}\sinh n\Upsilon & -\frac{\lambda_2}{|V|}\sinh n\Upsilon \\
\frac{x_3}{|V|}\sinh n\Upsilon & \frac{\lambda_2}{|V|}\sinh n\Upsilon & -\frac{\lambda_1}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & \frac{x_2}{|V|}\sinh n\Upsilon & -\frac{x_1}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & -\frac{\lambda_3}{|V|}\sinh n\Upsilon \\
\frac{ct}{|V|}\sinh n\Upsilon & \frac{x_1}{|V|}\sinh n\Upsilon & \frac{x_2}{|V|}\sinh n\Upsilon & \frac{x_3}{|V|}\sinh n\Upsilon & -\frac{\lambda_1}{|V|}\sinh n\Upsilon & -\frac{\lambda_2}{|V|}\sinh n\Upsilon & -\frac{\lambda_3}{|V|}\sinh n\Upsilon & \cosh n\Upsilon
\end{bmatrix}. \quad (58)$$

*Proof.* By applying matrix multiplication and using the hyperbolic angle sum formulas and the fact

$$\frac{\lambda_1^2}{|V|^2}+\frac{\lambda_2^2}{|V|^2}+\frac{\lambda_3^2}{|V|^2}-\frac{x_1^2}{|V|^2}-\frac{x_2^2}{|V|^2}-\frac{x_3^2}{|V|^2}+\frac{(ct)^2}{|V|^2}=1, \tag{59}$$

The $2^{nd}$– and $3^{rd}$– power of the given matrix is calculated as

$$\begin{bmatrix}
\cosh 2\Upsilon & \frac{\lambda_1}{|V|}\sinh 2\Upsilon & \frac{\lambda_2}{|V|}\sinh 2\Upsilon & \frac{\lambda_3}{|V|}\sinh 2\Upsilon & -\frac{x_1}{|V|}\sinh 2\Upsilon & -\frac{x_2}{|V|}\sinh 2\Upsilon & -\frac{x_3}{|V|}\sinh 2\Upsilon & \frac{ct}{|V|}\sinh 2\Upsilon \\
\frac{\lambda_1}{|V|}\sinh 2\Upsilon & \cosh 2\Upsilon & -\frac{x_3}{|V|}\sinh 2\Upsilon & \frac{x_2}{|V|}\sinh 2\Upsilon & \frac{ct}{|V|}\sinh 2\Upsilon & -\frac{\lambda_3}{|V|}\sinh 2\Upsilon & \frac{\lambda_2}{|V|}\sinh 2\Upsilon & -\frac{x_1}{|V|}\sinh 2\Upsilon \\
\frac{\lambda_2}{|V|}\sinh 2\Upsilon & \frac{x_3}{|V|}\sinh 2\Upsilon & \cosh 2\Upsilon & -\frac{x_1}{|V|}\sinh 2\Upsilon & \frac{\lambda_3}{|V|}\sinh 2\Upsilon & \frac{ct}{|V|}\sinh 2\Upsilon & -\frac{\lambda_1}{|V|}\sinh 2\Upsilon & -\frac{x_2}{|V|}\sinh 2\Upsilon \\
\frac{\lambda_3}{|V|}\sinh 2\Upsilon & -\frac{x_2}{|V|}\sinh 2\Upsilon & \frac{x_1}{|V|}\sinh 2\Upsilon & \cosh 2\Upsilon & -\frac{\lambda_2}{|V|}\sinh 2\Upsilon & \frac{\lambda_1}{|V|}\sinh 2\Upsilon & \frac{ct}{|V|}\sinh 2\Upsilon & -\frac{x_3}{|V|}\sinh 2\Upsilon \\
\frac{x_1}{|V|}\sinh 2\Upsilon & \frac{ct}{|V|}\sinh 2\Upsilon & \frac{\lambda_3}{|V|}\sinh 2\Upsilon & -\frac{\lambda_2}{|V|}\sinh 2\Upsilon & \cosh 2\Upsilon & \frac{x_3}{|V|}\sinh 2\Upsilon & -\frac{x_2}{|V|}\sinh 2\Upsilon & -\frac{\lambda_1}{|V|}\sinh 2\Upsilon \\
\frac{x_2}{|V|}\sinh 2\Upsilon & -\frac{\lambda_3}{|V|}\sinh 2\Upsilon & \frac{ct}{|V|}\sinh 2\Upsilon & \frac{\lambda_1}{|V|}\sinh 2\Upsilon & -\frac{x_3}{|V|}\sinh 2\Upsilon & \cosh 2\Upsilon & \frac{x_1}{|V|}\sinh 2\Upsilon & -\frac{\lambda_2}{|V|}\sinh 2\Upsilon \\
\frac{x_3}{|V|}\sinh 2\Upsilon & \frac{\lambda_2}{|V|}\sinh 2\Upsilon & -\frac{\lambda_1}{|V|}\sinh 2\Upsilon & \frac{ct}{|V|}\sinh 2\Upsilon & \frac{x_2}{|V|}\sinh 2\Upsilon & -\frac{x_1}{|V|}\sinh 2\Upsilon & \cosh 2\Upsilon & -\frac{\lambda_3}{|V|}\sinh 2\Upsilon \\
\frac{ct}{|V|}\sinh 2\Upsilon & \frac{x_1}{|V|}\sinh 2\Upsilon & \frac{x_2}{|V|}\sinh 2\Upsilon & \frac{x_3}{|V|}\sinh 2\Upsilon & -\frac{\lambda_1}{|V|}\sinh 2\Upsilon & -\frac{\lambda_2}{|V|}\sinh 2\Upsilon & -\frac{\lambda_3}{|V|}\sinh 2\Upsilon & \cosh 2\Upsilon
\end{bmatrix},$$

$$\begin{bmatrix}
\sinh 3\Upsilon & \frac{\lambda_1}{|V|}\cosh 3\Upsilon & \frac{\lambda_2}{|V|}\cosh 3\Upsilon & \frac{\lambda_3}{|V|}\cosh 3\Upsilon & -\frac{x_1}{|V|}\cosh 3\Upsilon & -\frac{x_2}{|V|}\cosh 3\Upsilon & -\frac{x_3}{|V|}\cosh 3\Upsilon & \frac{ct}{|V|}\cosh 3\Upsilon \\
\frac{\lambda_1}{|V|}\cosh 3\Upsilon & \sinh 3\Upsilon & -\frac{x_3}{|V|}\cosh 3\Upsilon & \frac{x_2}{|V|}\cosh 3\Upsilon & \frac{ct}{|V|}\cosh 3\Upsilon & -\frac{\lambda_3}{|V|}\cosh 3\Upsilon & \frac{\lambda_2}{|V|}\cosh 3\Upsilon & -\frac{x_1}{|V|}\cosh 3\Upsilon \\
\frac{\lambda_2}{|V|}\cosh 3\Upsilon & \frac{x_3}{|V|}\cosh 3\Upsilon & \sinh 3\Upsilon & -\frac{x_1}{|V|}\cosh 3\Upsilon & \frac{\lambda_3}{|V|}\cosh 3\Upsilon & \frac{ct}{|V|}\cosh 3\Upsilon & -\frac{\lambda_1}{|V|}\cosh 3\Upsilon & -\frac{x_2}{|V|}\cosh 3\Upsilon \\
\frac{\lambda_3}{|V|}\cosh 3\Upsilon & -\frac{x_2}{|V|}\cosh 3\Upsilon & \frac{x_1}{|V|}\cosh 3\Upsilon & \sinh 3\Upsilon & -\frac{\lambda_2}{|V|}\cosh 3\Upsilon & \frac{\lambda_1}{|V|}\cosh 3\Upsilon & \frac{ct}{|V|}\cosh 3\Upsilon & -\frac{x_3}{|V|}\cosh 3\Upsilon \\
\frac{x_1}{|V|}\cosh 3\Upsilon & \frac{ct}{|V|}\cosh 3\Upsilon & \frac{\lambda_3}{|V|}\cosh 3\Upsilon & -\frac{\lambda_2}{|V|}\cosh 3\Upsilon & \sinh 3\Upsilon & \frac{x_3}{|V|}\cosh 3\Upsilon & -\frac{x_2}{|V|}\cosh 3\Upsilon & -\frac{\lambda_1}{|V|}\cosh 3\Upsilon \\
\frac{x_2}{|V|}\cosh 3\Upsilon & -\frac{\lambda_3}{|V|}\cosh 3\Upsilon & \frac{ct}{|V|}\cosh 3\Upsilon & \frac{\lambda_1}{|V|}\cosh 3\Upsilon & -\frac{x_3}{|V|}\cosh\Upsilon & \sinh 3\Upsilon & \frac{x_1}{|V|}\cosh\Upsilon & -\frac{\lambda_2}{|V|}\cosh\Upsilon \\
\frac{x_3}{|V|}\cosh 3\Upsilon & \frac{\lambda_2}{|V|}\cosh 3\Upsilon & -\frac{\lambda_1}{|V|}\cosh 3\Upsilon & \frac{ct}{|V|}\cosh 3\Upsilon & \frac{x_2}{|V|}\cosh 3\Upsilon & -\frac{x_1}{|V|}\cosh 3\Upsilon & \sinh 3\Upsilon & -\frac{\lambda_3}{|V|}\cosh 3\Upsilon \\
\frac{ct}{|V|}\cosh 3\Upsilon & \frac{x_1}{|V|}\cosh 3\Upsilon & \frac{x_2}{|V|}\cosh 3\Upsilon & \frac{x_3}{|V|}\cosh 3\Upsilon & -\frac{\lambda_1}{|V|}\cosh 3\Upsilon & -\frac{\lambda_2}{|V|}\cosh 3\Upsilon & -\frac{\lambda_3}{|V|}\cosh 3\Upsilon & \sinh 3\Upsilon
\end{bmatrix}, \tag{60}$$

respectively. It is assumed that the formula is valid for $n = 2l, l \in \mathbb{Z}^+$. In this case, using the hyperbolic sum formulas and matrix multiplications $\mathfrak{s}_R^n \mathfrak{s}_R$ and $\mathfrak{s}_R^{(n+1)} \mathfrak{s}_R$, the proof is completed easily for $n = 2l + 1$ and $= 2l + 2$.

On the other hand, for negative integer powers, the formula can be calculated as follows. For $n = -1$, the inverse of the given matrix can be found:

$$\begin{bmatrix} \sinh\widetilde{\Upsilon} & \frac{\lambda_1}{|V|}\cosh\widetilde{\Upsilon} & \frac{\lambda_2}{|V|}\cosh\widetilde{\Upsilon} & \frac{\lambda_3}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_1}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_2}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_3}{|V|}\cosh\widetilde{\Upsilon} & \frac{ct}{|V|}\cosh\widetilde{\Upsilon} \\ \frac{\lambda_1}{|V|}\cosh\widetilde{\Upsilon} & \sinh\widetilde{\Upsilon} & -\frac{x_3}{|V|}\cosh\widetilde{\Upsilon} & \frac{x_2}{|V|}\cosh\widetilde{\Upsilon} & \frac{ct}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_3}{|V|}\cosh\widetilde{\Upsilon} & \frac{\lambda_2}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_1}{|V|}\cosh\widetilde{\Upsilon} \\ \frac{\lambda_2}{|V|}\cosh\widetilde{\Upsilon} & \frac{x_3}{|V|}\cosh\widetilde{\Upsilon} & \sinh\widetilde{\Upsilon} & -\frac{x_1}{|V|}\cosh\widetilde{\Upsilon} & \frac{\lambda_3}{|V|}\cosh\widetilde{\Upsilon} & \frac{ct}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_1}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_2}{|V|}\cosh\widetilde{\Upsilon} \\ \frac{\lambda_3}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_2}{|V|}\cosh\widetilde{\Upsilon} & \frac{x_1}{|V|}\cosh\widetilde{\Upsilon} & \sinh\widetilde{\Upsilon} & -\frac{\lambda_2}{|V|}\cosh\widetilde{\Upsilon} & \frac{\lambda_1}{|V|}\cosh\widetilde{\Upsilon} & \frac{ct}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_3}{|V|}\cosh\widetilde{\Upsilon} \\ \frac{x_1}{|V|}\cosh\widetilde{\Upsilon} & \frac{ct}{|V|}\cosh\widetilde{\Upsilon} & \frac{\lambda_3}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_2}{|V|}\cosh\widetilde{\Upsilon} & \sinh\widetilde{\Upsilon} & \frac{x_3}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_2}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_1}{|V|}\cosh\widetilde{\Upsilon} \\ \frac{x_2}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_3}{|V|}\cosh\widetilde{\Upsilon} & \frac{ct}{|V|}\cosh\widetilde{\Upsilon} & \frac{\lambda_1}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_3}{|V|}\cosh\widetilde{\Upsilon} & \sinh\widetilde{\Upsilon} & \frac{x_1}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_2}{|V|}\cosh\widetilde{\Upsilon} \\ \frac{x_3}{|V|}\cosh\widetilde{\Upsilon} & \frac{\lambda_2}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_1}{|V|}\cosh\widetilde{\Upsilon} & \frac{ct}{|V|}\cosh\widetilde{\Upsilon} & \frac{x_2}{|V|}\cosh\widetilde{\Upsilon} & -\frac{x_1}{|V|}\cosh\widetilde{\Upsilon} & \sinh\widetilde{\Upsilon} & -\frac{\lambda_3}{|V|}\cosh\widetilde{\Upsilon} \\ \frac{ct}{|V|}\cosh\widetilde{\Upsilon} & \frac{x_1}{|V|}\cosh\widetilde{\Upsilon} & \frac{x_2}{|V|}\cosh\widetilde{\Upsilon} & \frac{x_3}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_1}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_2}{|V|}\cosh\widetilde{\Upsilon} & -\frac{\lambda_3}{|V|}\cosh\widetilde{\Upsilon} & \sinh\widetilde{\Upsilon} \end{bmatrix}, \tag{61}$$

where $\widetilde{\Upsilon} = -\Upsilon$. Also, the $(-2)^{th}-$ and $(-3)^{th}-$ power of the matrix can be obtained similarly. Finally, by using similar steps above, the proof for the negative integer is completed. □

**Theorem 12.** *Assumed that the matrix $\mathfrak{s}_R$ defined by*

$$\begin{bmatrix} \cosh\Upsilon & \frac{\lambda_1}{|V|}\sinh\Upsilon & \frac{\lambda_2}{|V|}\sinh\Upsilon & \frac{\lambda_3}{|V|}\sinh\Upsilon & -\frac{x_1}{|V|}\sinh\Upsilon & -\frac{x_2}{|V|}\sinh\Upsilon & -\frac{x_3}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon \\ \frac{\lambda_1}{|V|}\sinh\Upsilon & \cosh\Upsilon & -\frac{x_3}{|V|}\sinh\Upsilon & \frac{x_2}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & -\frac{\lambda_3}{|V|}\sinh\Upsilon & \frac{\lambda_2}{|V|}\sinh\Upsilon & -\frac{x_1}{|V|}\sinh\Upsilon \\ \frac{\lambda_2}{|V|}\sinh\Upsilon & \frac{x_3}{|V|}\sinh\Upsilon & \cosh\Upsilon & -\frac{x_1}{|V|}\sinh\Upsilon & \frac{\lambda_3}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & -\frac{\lambda_1}{|V|}\sinh\Upsilon & -\frac{x_2}{|V|}\sinh\Upsilon \\ \frac{\lambda_3}{|V|}\sinh\Upsilon & -\frac{x_2}{|V|}\sinh\Upsilon & \frac{x_1}{|V|}\sinh\Upsilon & \cosh\Upsilon & -\frac{\lambda_2}{|V|}\sinh\Upsilon & \frac{\lambda_1}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & -\frac{x_3}{|V|}\sinh\Upsilon \\ \frac{x_1}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & \frac{\lambda_3}{|V|}\sinh\Upsilon & -\frac{\lambda_2}{|V|}\sinh\Upsilon & \cosh\Upsilon & \frac{x_3}{|V|}\sinh\Upsilon & -\frac{x_2}{|V|}\sinh\Upsilon & -\frac{\lambda_1}{|V|}\sinh\Upsilon \\ \frac{x_2}{|V|}\sinh\Upsilon & -\frac{\lambda_3}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & \frac{\lambda_1}{|V|}\sinh\Upsilon & -\frac{x_3}{|V|}\sinh\Upsilon & \cosh\Upsilon & \frac{x_1}{|V|}\sinh\Upsilon & -\frac{\lambda_2}{|V|}\sinh\Upsilon \\ \frac{x_3}{|V|}\sinh\Upsilon & \frac{\lambda_2}{|V|}\sinh\Upsilon & -\frac{\lambda_1}{|V|}\sinh\Upsilon & \frac{ct}{|V|}\sinh\Upsilon & \frac{x_2}{|V|}\sinh\Upsilon & -\frac{x_1}{|V|}\sinh\Upsilon & \cosh\Upsilon & -\frac{\lambda_3}{|V|}\sinh\Upsilon \\ \frac{ct}{|V|}\sinh\Upsilon & \frac{x_1}{|V|}\sinh\Upsilon & \frac{x_2}{|V|}\sinh\Upsilon & \frac{x_3}{|V|}\sinh\Upsilon & -\frac{\lambda_1}{|V|}\sinh\Upsilon & -\frac{\lambda_2}{|V|}\sinh\Upsilon & -\frac{\lambda_3}{|V|}\sinh\Upsilon & \cosh\Upsilon \end{bmatrix}, \tag{62}$$

*is a right matrix representation for a unit split octonion with the inner product having a positive value and timelike vector part. Then, the matrix* $\mathfrak{s}_R^n$ *is*

$$\begin{bmatrix}
\cosh n\Upsilon & \frac{\lambda_1}{|V|}\sinh n\Upsilon & \frac{\lambda_2}{|V|}\sinh n\Upsilon & \frac{\lambda_3}{|V|}\sinh n\Upsilon & -\frac{x_1}{|V|}\sinh n\Upsilon & -\frac{x_2}{|V|}\sinh n\Upsilon & -\frac{x_3}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon \\
\frac{\lambda_1}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & -\frac{x_3}{|V|}\sinh n\Upsilon & \frac{x_2}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & -\frac{\lambda_3}{|V|}\sinh n\Upsilon & \frac{\lambda_2}{|V|}\sinh n\Upsilon & -\frac{x_1}{|V|}\sinh n\Upsilon \\
\frac{\lambda_2}{|V|}\sinh n\Upsilon & \frac{x_3}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & -\frac{x_1}{|V|}\sinh n\Upsilon & \frac{\lambda_3}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & -\frac{\lambda_1}{|V|}\sinh n\Upsilon & -\frac{x_2}{|V|}\sinh n\Upsilon \\
\frac{\lambda_3}{|V|}\sinh n\Upsilon & -\frac{x_2}{|V|}\sinh n\Upsilon & \frac{x_1}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & -\frac{\lambda_2}{|V|}\sinh n\Upsilon & \frac{\lambda_1}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & -\frac{x_3}{|V|}\sinh n\Upsilon \\
\frac{x_1}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & \frac{\lambda_3}{|V|}\sinh n\Upsilon & -\frac{\lambda_2}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & \frac{x_3}{|V|}\sinh n\Upsilon & -\frac{x_2}{|V|}\sinh n\Upsilon & -\frac{\lambda_1}{|V|}\sinh n\Upsilon \\
\frac{x_2}{|V|}\sinh n\Upsilon & -\frac{\lambda_3}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & \frac{\lambda_1}{|V|}\sinh n\Upsilon & -\frac{x_3}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & \frac{x_1}{|V|}\sinh n\Upsilon & -\frac{\lambda_2}{|V|}\sinh n\Upsilon \\
\frac{x_3}{|V|}\sinh n\Upsilon & \frac{\lambda_2}{|V|}\sinh n\Upsilon & -\frac{\lambda_1}{|V|}\sinh n\Upsilon & \frac{ct}{|V|}\sinh n\Upsilon & \frac{x_2}{|V|}\sinh n\Upsilon & -\frac{x_1}{|V|}\sinh n\Upsilon & \cosh n\Upsilon & -\frac{\lambda_3}{|V|}\sinh n\Upsilon \\
\frac{ct}{|V|}\sinh n\Upsilon & \frac{x_1}{|V|}\sinh n\Upsilon & \frac{x_2}{|V|}\sinh n\Upsilon & \frac{x_3}{|V|}\sinh n\Upsilon & -\frac{\lambda_1}{|V|}\sinh n\Upsilon & -\frac{\lambda_2}{|V|}\sinh n\Upsilon & -\frac{\lambda_3}{|V|}\sinh n\Upsilon & \cosh n\Upsilon
\end{bmatrix}. \quad (63)$$

*Proof.* The formula can be shown by considering the same steps at the proof of Theorem 11 and by using the hyperbolic angle sum formulas. □

**Theorem 13.** *Let us consider the matrix* $\mathfrak{s}_R$ *defined by*

$$\begin{bmatrix}
\cos\Upsilon & \frac{\lambda_1}{|V|}\sin\Upsilon & \frac{\lambda_2}{|V|}\sin\Upsilon & \frac{\lambda_3}{|V|}\sin\Upsilon & -\frac{x_1}{|V|}\sin\Upsilon & -\frac{x_2}{|V|}\sin\Upsilon & -\frac{x_3}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon \\
\frac{\lambda_1}{|V|}\sin\Upsilon & \cos\Upsilon & -\frac{x_3}{|V|}\sin\Upsilon & \frac{x_2}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & -\frac{\lambda_3}{|V|}\sin\Upsilon & \frac{\lambda_2}{|V|}\sin\Upsilon & -\frac{x_1}{|V|}\sin\Upsilon \\
\frac{\lambda_2}{|V|}\sin\Upsilon & \frac{x_3}{|V|}\sin\Upsilon & \cos\Upsilon & -\frac{x_1}{|V|}\sin\Upsilon & \frac{\lambda_3}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & -\frac{\lambda_1}{|V|}\sin\Upsilon & -\frac{x_2}{|V|}\sin\Upsilon \\
\frac{\lambda_3}{|V|}\sin\Upsilon & -\frac{x_2}{|V|}\sin\Upsilon & \frac{x_1}{|V|}\sin\Upsilon & \cos\Upsilon & -\frac{\lambda_2}{|V|}\sin\Upsilon & \frac{\lambda_1}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & -\frac{x_3}{|V|}\sin\Upsilon \\
\frac{x_1}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & \frac{\lambda_3}{|V|}\sin\Upsilon & -\frac{\lambda_2}{|V|}\sin\Upsilon & \cos\Upsilon & \frac{x_3}{|V|}\sin\Upsilon & -\frac{x_2}{|V|}\sin\Upsilon & -\frac{\lambda_1}{|V|}\sin\Upsilon \\
\frac{x_2}{|V|}\sin\Upsilon & -\frac{\lambda_3}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & \frac{\lambda_1}{|V|}\sin\Upsilon & -\frac{x_3}{|V|}\sin\Upsilon & \cos\Upsilon & \frac{x_1}{|V|}\sin\Upsilon & -\frac{\lambda_2}{|V|}\sin\Upsilon \\
\frac{x_3}{|V|}\sin\Upsilon & \frac{\lambda_2}{|V|}\sin\Upsilon & -\frac{\lambda_1}{|V|}\sin\Upsilon & \frac{ct}{|V|}\sin\Upsilon & \frac{x_2}{|V|}\sin\Upsilon & -\frac{x_1}{|V|}\sin\Upsilon & \cos\Upsilon & -\frac{\lambda_3}{|V|}\sin\Upsilon \\
\frac{ct}{|V|}\sin\Upsilon & \frac{x_1}{|V|}\sin\Upsilon & \frac{x_2}{|V|}\sin\Upsilon & \frac{x_3}{|V|}\sin\Upsilon & -\frac{\lambda_1}{|V|}\sin\Upsilon & -\frac{\lambda_2}{|V|}\sin\Upsilon & -\frac{\lambda_3}{|V|}\sin\Upsilon & \cos\Upsilon
\end{bmatrix}, \quad (64)$$

for a split octonion with the inner product having a positive value and spacelike vector part. Then, the matrix $\mathfrak{s}_R^n$ is

$$\begin{bmatrix} \cos n\Upsilon & \frac{\lambda_1}{|V|}\sin n\Upsilon & \frac{\lambda_2}{|V|}\sin n\Upsilon & \frac{\lambda_3}{|V|}\sin n\Upsilon & -\frac{x_1}{|V|}\sin n\Upsilon & -\frac{x_2}{|V|}\sin n\Upsilon & -\frac{x_3}{|V|}\sin n\Upsilon & \frac{ct}{|V|}\sin n\Upsilon \\ \frac{\lambda_1}{|V|}\sinh n\Upsilon & \cos n\Upsilon & -\frac{x_3}{|V|}\sin n\Upsilon & \frac{x_2}{|V|}\sin n\Upsilon & \frac{ct}{|V|}\sin n\Upsilon & -\frac{\lambda_3}{|V|}\sin n\Upsilon & \frac{\lambda_2}{|V|}\sin n\Upsilon & -\frac{x_1}{|V|}\sin n\Upsilon \\ \frac{\lambda_2}{|V|}\sin n\Upsilon & \frac{x_3}{|V|}\sin n\Upsilon & \cos n\Upsilon & -\frac{x_1}{|V|}\sin n\Upsilon & \frac{\lambda_3}{|V|}\sin n\Upsilon & \frac{ct}{|V|}\sin n\Upsilon & -\frac{\lambda_1}{|V|}\sin n\Upsilon & -\frac{x_2}{|V|}\sin n\Upsilon \\ \frac{\lambda_3}{|V|}\sin n\Upsilon & -\frac{x_2}{|V|}\sin n\Upsilon & \frac{x_1}{|V|}\sin n\Upsilon & \cos n\Upsilon & -\frac{\lambda_2}{|V|}\sin n\Upsilon & \frac{\lambda_1}{|V|}\sin n\Upsilon & \frac{ct}{|V|}\sin n\Upsilon & -\frac{x_3}{|V|}\sin n\Upsilon \\ \frac{x_1}{|V|}\sin n\Upsilon & \frac{ct}{|V|}\sin n\Upsilon & \frac{\lambda_3}{|V|}\sin n\Upsilon & -\frac{\lambda_2}{|V|}\sin n\Upsilon & \cos n\Upsilon & \frac{x_3}{|V|}\sin n\Upsilon & -\frac{x_2}{|V|}\sin n\Upsilon & -\frac{\lambda_1}{|V|}\sin n\Upsilon \\ \frac{x_2}{|V|}\sin n\Upsilon & -\frac{\lambda_3}{|V|}\sin n\Upsilon & \frac{ct}{|V|}\sin n\Upsilon & \frac{\lambda_1}{|V|}\sin n\Upsilon & -\frac{x_3}{|V|}\sin n\Upsilon & \cos n\Upsilon & \frac{x_1}{|V|}\sin n\Upsilon & -\frac{\lambda_2}{|V|}\sin n\Upsilon \\ \frac{x_3}{|V|}\sin n\Upsilon & \frac{\lambda_2}{|V|}\sin n\Upsilon & -\frac{\lambda_1}{|V|}\sin n\Upsilon & \frac{ct}{|V|}\sin n\Upsilon & \frac{x_2}{|V|}\sin n\Upsilon & -\frac{x_1}{|V|}\sin n\Upsilon & \cos n\Upsilon & -\frac{\lambda_3}{|V|}\sin n\Upsilon \\ \frac{ct}{|V|}\sin n\Upsilon & \frac{x_1}{|V|}\sin n\Upsilon & \frac{x_2}{|V|}\sin n\Upsilon & \frac{x_3}{|V|}\sin n\Upsilon & -\frac{\lambda_1}{|V|}\sin n\Upsilon & -\frac{\lambda_2}{|V|}\sin n\Upsilon & -\frac{\lambda_3}{|V|}\sin n\Upsilon & \cos n\Upsilon \end{bmatrix}. \quad (65)$$

*Proof.* One can show the proof like the proof of Theorem 11. □

**Theorem 14.** *Let us consider the following matrix:*

$$\mathfrak{s}_R = w\begin{bmatrix} 1 & \frac{\lambda_1}{w} & \frac{\lambda_2}{w} & \frac{\lambda_3}{w} & -\frac{x_1}{w} & -\frac{x_2}{w} & -\frac{x_3}{w} & \frac{ct}{w} \\ \frac{\lambda_1}{w} & 1 & -\frac{x_3}{w} & \frac{x_2}{w} & \frac{ct}{w} & -\frac{\lambda_3}{w} & \frac{\lambda_2}{w} & -\frac{x_1}{w} \\ \frac{\lambda_2}{w} & \frac{x_3}{w} & 1 & -\frac{x_1}{w} & \frac{\lambda_3}{w} & \frac{ct}{w} & -\frac{\lambda_1}{w} & -\frac{x_2}{w} \\ \frac{\lambda_3}{w} & -\frac{x_2}{w} & \frac{x_1}{w} & 1 & -\frac{\lambda_2}{w} & \frac{\lambda_1}{w} & \frac{ct}{w} & -\frac{x_3}{w} \\ \frac{x_1}{w} & \frac{ct}{w} & \frac{\lambda_3}{w} & -\frac{\lambda_2}{w} & 1 & \frac{x_3}{w} & -\frac{x_2}{w} & -\frac{\lambda_1}{w} \\ \frac{x_2}{w} & -\frac{\lambda_3}{w} & \frac{ct}{w} & \frac{\lambda_1}{w} & -x_3 & 1 & \frac{x_1}{w} & -\frac{\lambda_2}{w} \\ \frac{x_3}{w} & \frac{\lambda_2}{w} & -\frac{\lambda_1}{w} & \frac{ct}{w} & \frac{x_2}{w} & -\frac{x_1}{w} & 1 & -\frac{\lambda_3}{w} \\ \frac{ct}{w} & \frac{x_1}{w} & \frac{x_2}{w} & \frac{x_3}{w} & -\frac{\lambda_1}{w} & -\frac{\lambda_2}{w} & -\frac{\lambda_3}{w} & 1 \end{bmatrix}, \quad (66)$$

for a split octonion with the inner product having zero. In this case, the matrix $\mathfrak{s}_R^n$ is

$$w^n 2^{n-1}\begin{bmatrix} 1 & \frac{\lambda_1}{w} & \frac{\lambda_2}{w} & \frac{\lambda_3}{w} & -\frac{x_1}{w} & -\frac{x_2}{w} & -\frac{x_3}{w} & \frac{ct}{w} \\ \frac{\lambda_1}{w} & 1 & -\frac{x_3}{w} & \frac{x_2}{w} & \frac{ct}{w} & -\frac{\lambda_3}{w} & \frac{\lambda_2}{w} & -\frac{x_1}{w} \\ \frac{\lambda_2}{w} & \frac{x_3}{w} & 1 & -\frac{x_1}{w} & \frac{\lambda_3}{w} & \frac{ct}{w} & -\frac{\lambda_1}{w} & -\frac{x_2}{w} \\ \frac{\lambda_3}{w} & -\frac{x_2}{w} & \frac{x_1}{w} & 1 & -\frac{\lambda_2}{w} & \frac{\lambda_1}{w} & \frac{ct}{w} & -\frac{x_3}{w} \\ \frac{x_1}{w} & \frac{ct}{w} & \frac{\lambda_3}{w} & -\frac{\lambda_2}{w} & 1 & \frac{x_3}{w} & -\frac{x_2}{w} & -\frac{\lambda_1}{w} \\ \frac{x_2}{w} & -\frac{\lambda_3}{w} & \frac{ct}{w} & \frac{\lambda_1}{w} & -x_3 & 1 & \frac{x_1}{w} & -\frac{\lambda_2}{w} \\ \frac{x_3}{w} & \frac{\lambda_2}{w} & -\frac{\lambda_1}{w} & \frac{ct}{w} & \frac{x_2}{w} & -\frac{x_1}{w} & 1 & -\frac{\lambda_3}{w} \\ \frac{ct}{w} & \frac{x_1}{w} & \frac{x_2}{w} & \frac{x_3}{w} & -\frac{\lambda_1}{w} & -\frac{\lambda_2}{w} & -\frac{\lambda_3}{w} & 1 \end{bmatrix}. \quad (67)$$

*Proof.* Suppose that $s$ is a split octonion with the inner product having zero and $\mathfrak{U}^2 = 1$, that is,

$$\frac{\lambda_1^2}{w^2} + \frac{\lambda_2^2}{w^2} + \frac{\lambda_3^2}{w^2} - \frac{x_1^2}{w^2} - \frac{x_2^2}{w^2} - \frac{x_3^2}{w^2} + \frac{(ct)^2}{w^2} = 1. \tag{68}$$

One can verify the proof as similar to Theorem 4. □

**Theorem 15.** *Let us consider the following matrix:*

$$\mathfrak{s}_R = \begin{bmatrix} \operatorname{sgn}(w) & \lambda_1 & \lambda_2 & \lambda_3 & -x_1 & -x_2 & -x_3 & ct \\ \lambda_1 & \operatorname{sgn}(w) & -x_3 & x_2 & ct & -\lambda_3 & \lambda_2 & -x_1 \\ \lambda_2 & x_3 & \operatorname{sgn}(w) & -x_1 & \lambda_3 & ct & -\lambda_1 & -x_2 \\ \lambda_3 & -x_2 & x_1 & \operatorname{sgn}(w) & -\lambda_2 & \lambda_1 & ct & -x_3 \\ x_1 & ct & \lambda_3 & -\lambda_2 & \operatorname{sgn}(w) & x_3 & -x_2 & -\lambda_1 \\ x_2 & -\lambda_3 & ct & \lambda_1 & -x_3 & \operatorname{sgn}(w) & x_1 & -\lambda_2 \\ x_3 & \lambda_2 & -\lambda_1 & ct & x_2 & -x_1 & \operatorname{sgn}(w) & -\lambda_3 \\ ct & x_1 & x_2 & x_3 & -\lambda_1 & -\lambda_2 & -\lambda_3 & \operatorname{sgn}(w) \end{bmatrix}, \tag{69}$$

for a unit split octonion with the lightlike vector part $\mathfrak{U}, \mathfrak{U}^2 = 0$. Then, the matrix $\mathfrak{s}_R^n$ is

$$\begin{bmatrix} \operatorname{sgn}(w)^n & nw^{n-1}\lambda_1 & nw^{n-1}\lambda_2 & nw^{n-1}\lambda_3 & -nw^{n-1}x_1 & -nw^{n-1}x_2 & -nw^{n-1}x_3 & nw^{n-1}ct \\ nw^{n-1}\lambda_1 & \operatorname{sgn}(w)^n & -nw^{n-1}x_3 & nw^{n-1}x_2 & nw^{n-1}ct & -nw^{n-1}\lambda_3 & nw^{n-1}\lambda_2 & -nw^{n-1}x_1 \\ nw^{n-1}\lambda_2 & nw^{n-1}x_3 & \operatorname{sgn}(w)^n & -nw^{n-1}x_1 & nw^{n-1}\lambda_3 & nw^{n-1}ct & -nw^{n-1}\lambda_1 & -nw^{n-1}x_2 \\ nw^{n-1}\lambda_3 & -nw^{n-1}x_2 & nw^{n-1}x_1 & \operatorname{sgn}(w)^n & -nw^{n-1}\lambda_2 & nw^{n-1}\lambda_1 & nw^{n-1}ct & -nw^{n-1}x_3 \\ nw^{n-1}x_1 & nw^{n-1}ct & nw^{n-1}\lambda_3 & -nw^{n-1}\lambda_2 & \operatorname{sgn}(w)^n & nw^{n-1}x_3 & -nw^{n-1}x_2 & -nw^{n-1}\lambda_1 \\ nw^{n-1}x_2 & -nw^{n-1}\lambda_3 & nw^{n-1}ct & nw^{n-1}\lambda_1 & -nw^{n-1}x_3 & \operatorname{sgn}(w)^n & nw^{n-1}x_1 & -nw^{n-1}\lambda_2 \\ nw^{n-1}x_3 & nw^{n-1}\lambda_2 & -nw^{n-1}\lambda_1 & nw^{n-1}ct & nw^{n-1}x_2 & -nw^{n-1}x_1 & \operatorname{sgn}(w)^n & -nw^{n-1}\lambda_3 \\ nw^{n-1}ct & nw^{n-1}x_1 & nw^{n-1}x_2 & nw^{n-1}x_3 & -nw^{n-1}\lambda_1 & -nw^{n-1}\lambda_2 & -nw^{n-1}\lambda_3 & \operatorname{sgn}(w)^n \end{bmatrix}. \tag{70}$$

*Proof.* The formula can be found similar to Theorem 5. □

## 6. Roots of the Matrices of Split Octonions

In this section, we investigate the solutions of the equation $\mathfrak{x}^n = \mathfrak{s}_R$. The equation has some different solutions with respect to the type of split octonion matrix representations as follows:

**Theorem 16.** *Assume that $\mathfrak{s}_R$ is given as a right matrix representation for a unit split octonion with an inner product having a negative value and a unit timelike 7− vector $\mathfrak{U}$. Then, the equation $\mathfrak{x}^n = \mathfrak{s}_R$ has the following solutions:*

*(i) If $n$ is an even number, then there is no solution*

*(ii) If $n$ is an odd number, then there is only 1− solution $\mathfrak{x}_0$ given by*

$$\begin{bmatrix} \sinh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta \\ \frac{\lambda_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta \\ \frac{\lambda_2}{|V|}\cosh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta \\ \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta \\ \frac{x_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & \sinh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta \\ \frac{x_2}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \sinh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta \\ \frac{x_3}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta \\ \frac{ct}{|V|}\cosh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \sinh\vartheta \end{bmatrix}, \quad (71)$$

*where $\vartheta = \Upsilon/n$.*

*Proof.* Using Theorem 11, we can write

$$\begin{bmatrix} \sinh n\vartheta & \frac{\lambda_1}{|V|}\cosh n\vartheta & \frac{\lambda_2}{|V|}\cosh n\vartheta & \frac{\lambda_3}{|V|}\cosh n\vartheta & -\frac{x_1}{|V|}\cosh n\vartheta & -\frac{x_2}{|V|}\cosh n\vartheta & -\frac{x_3}{|V|}\cosh n\vartheta & \frac{ct}{|V|}\cosh n\vartheta \\ \frac{\lambda_1}{|V|}\cosh n\vartheta & \sinh n\vartheta & -\frac{x_3}{|V|}\cosh n\vartheta & \frac{x_2}{|V|}\cosh n\vartheta & \frac{ct}{|V|}\cosh n\vartheta & -\frac{\lambda_3}{|V|}\cosh n\vartheta & \frac{\lambda_2}{|V|}\cosh n\vartheta & -\frac{x_1}{|V|}\cosh n\vartheta \\ \frac{\lambda_2}{|V|}\cosh n\vartheta & \frac{x_3}{|V|}\cosh n\vartheta & \sinh n\vartheta & -\frac{x_1}{|V|}\cosh n\vartheta & \frac{\lambda_3}{|V|}\cosh n\vartheta & \frac{ct}{|V|}\cosh n\vartheta & -\frac{\lambda_1}{|V|}\cosh n\vartheta & -\frac{x_2}{|V|}\cosh n\vartheta \\ \frac{\lambda_3}{|V|}\cosh n\vartheta & -\frac{x_2}{|V|}\cosh n\vartheta & \frac{x_1}{|V|}\cosh \vartheta & \sinh n\vartheta & -\frac{\lambda_2}{|V|}\cosh n\vartheta & \frac{\lambda_1}{|V|}\cosh n\vartheta & \frac{ct}{|V|}\cosh n\vartheta & -\frac{x_3}{|V|}\cosh n\vartheta \\ \frac{x_1}{|V|}\cosh n\vartheta & \frac{ct}{|V|}\cosh n\vartheta & \frac{\lambda_3}{|V|}\cosh n\vartheta & -\frac{\lambda_2}{|V|}\cosh n\vartheta & \sinh n\vartheta & \frac{x_3}{|V|}\cosh n\vartheta & -\frac{x_2}{|V|}\cosh n\vartheta & -\frac{\lambda_1}{|V|}\cosh n\vartheta \\ \frac{x_2}{|V|}\cosh n\vartheta & -\frac{\lambda_3}{|V|}\cosh n\vartheta & \frac{ct}{|V|}\cosh n\vartheta & \frac{\lambda_1}{|V|}\cosh n\vartheta & -\frac{x_3}{|V|}\cosh \vartheta & \sinh n\vartheta & \frac{x_1}{|V|}\cosh n\vartheta & -\frac{\lambda_2}{|V|}\cosh n\vartheta \\ \frac{x_3}{|V|}\cosh n\vartheta & \frac{\lambda_2}{|V|}\cosh n\vartheta & -\frac{\lambda_1}{|V|}\cosh n\vartheta & \frac{ct}{|V|}\cosh n\vartheta & \frac{x_2}{|V|}\cosh n\vartheta & -\frac{x_1}{|V|}\cosh n\vartheta & \sinh n\vartheta & -\frac{\lambda_3}{|V|}\cosh n\vartheta \\ \frac{ct}{|V|}\cosh n\vartheta & \frac{x_1}{|V|}\cosh n\vartheta & \frac{x_2}{|V|}\cosh n\vartheta & \frac{x_3}{|V|}\cosh n\vartheta & -\frac{\lambda_1}{|V|}\cosh n\vartheta & -\frac{\lambda_2}{|V|}\cosh n\vartheta & -\frac{\lambda_3}{|V|}\cosh n\vartheta & \sinh n\vartheta \end{bmatrix}, \quad (72)$$

for the $n^{th}-$ power of $\mathfrak{x}_0$. Hence, $\mathfrak{x}_0^n = \mathfrak{S}_R$ is obtained. The other case is clear from Theorem 11.

**Theorem 17.** *Assume that $\mathfrak{S}_R$ is given as a right matrix representation for a unit split octonion with an inner product having a positive value and a unit timelike 7− vector $\mathfrak{U}$. Then, the equation $\mathfrak{x}^n = \mathfrak{S}_R$ has the following solutions:*

*(i) If $n$ is an even number, then the 4− distinct solutions are*

$$
\mathfrak{x}_0 = \begin{bmatrix}
\sinh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta \\
\frac{\lambda_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta \\
\frac{\lambda_2}{|V|}\cosh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta \\
\frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta \\
\frac{x_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & \sinh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta \\
\frac{x_2}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \sinh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta \\
\frac{x_3}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta \\
\frac{ct}{|V|}\cosh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \sinh\vartheta
\end{bmatrix},
$$

$$
\mathfrak{x}_1 = -\begin{bmatrix}
\sinh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta \\
\frac{\lambda_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta \\
\frac{\lambda_2}{|V|}\cosh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta \\
\frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta \\
\frac{x_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & \sinh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta \\
\frac{x_2}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \sinh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta \\
\frac{x_3}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta \\
\frac{ct}{|V|}\cosh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \sinh\vartheta
\end{bmatrix},
$$

$$
\mathfrak{x}_2 = \begin{bmatrix}
\cosh\vartheta & \frac{\lambda_1}{|V|}\sinh\vartheta & \frac{\lambda_2}{|V|}\sinh\vartheta & \frac{\lambda_3}{|V|}\sinh\vartheta & -\frac{x_1}{|V|}\sinh\vartheta & -\frac{x_2}{|V|}\sinh\vartheta & -\frac{x_3}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta \\
\frac{\lambda_1}{|V|}\sinh\vartheta & \cosh\vartheta & -\frac{x_3}{|V|}\sinh\vartheta & \frac{x_2}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & -\frac{\lambda_3}{|V|}\sinh\vartheta & \frac{\lambda_2}{|V|}\sinh\vartheta & -\frac{x_1}{|V|}\sinh\vartheta \\
\frac{\lambda_2}{|V|}\sinh\vartheta & \frac{x_3}{|V|}\sinh\vartheta & \cosh\vartheta & -\frac{x_1}{|V|}\sinh\vartheta & \frac{\lambda_3}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & -\frac{\lambda_1}{|V|}\sinh\vartheta & -\frac{x_2}{|V|}\sinh\vartheta \\
\frac{\lambda_3}{|V|}\sinh\vartheta & -\frac{x_2}{|V|}\sinh\vartheta & \frac{x_1}{|V|}\sinh\vartheta & \cosh\vartheta & -\frac{\lambda_2}{|V|}\sinh\vartheta & \frac{\lambda_1}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & -\frac{x_3}{|V|}\sinh\vartheta \\
\frac{x_1}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & \frac{\lambda_3}{|V|}\sinh\vartheta & -\frac{\lambda_2}{|V|}\sinh\vartheta & \cosh\vartheta & \frac{x_3}{|V|}\sinh\vartheta & -\frac{x_2}{|V|}\sinh\vartheta & -\frac{\lambda_1}{|V|}\sinh\vartheta \\
\frac{x_2}{|V|}\sinh\vartheta & -\frac{\lambda_3}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & \frac{\lambda_1}{|V|}\sinh\vartheta & -\frac{x_3}{|V|}\sinh\vartheta & \cosh\vartheta & \frac{x_1}{|V|}\sinh\vartheta & -\frac{\lambda_2}{|V|}\sinh\vartheta \\
\frac{x_3}{|V|}\sinh\vartheta & \frac{\lambda_2}{|V|}\sinh\vartheta & -\frac{\lambda_1}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & \frac{x_2}{|V|}\sinh\vartheta & -\frac{x_1}{|V|}\sinh\vartheta & \cosh\vartheta & -\frac{\lambda_3}{|V|}\sinh\vartheta \\
\frac{ct}{|V|}\sinh\vartheta & \frac{x_1}{|V|}\sinh\vartheta & \frac{x_2}{|V|}\sinh\vartheta & \frac{x_3}{|V|}\sinh\vartheta & -\frac{\lambda_1}{|V|}\sinh\vartheta & -\frac{\lambda_2}{|V|}\sinh\vartheta & -\frac{\lambda_3}{|V|}\sinh\vartheta & \cosh\vartheta
\end{bmatrix},
\tag{73}
$$

$$
\mathfrak{x}_3 = -\begin{bmatrix}
\cosh\vartheta & \frac{\lambda_1}{|V|}\sinh\vartheta & \frac{\lambda_2}{|V|}\sinh\vartheta & \frac{\lambda_3}{|V|}\sinh\vartheta & -\frac{x_1}{|V|}\sinh\vartheta & -\frac{x_2}{|V|}\sinh\vartheta & -\frac{x_3}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta \\
\frac{\lambda_1}{|V|}\sinh\vartheta & \cosh\vartheta & -\frac{x_3}{|V|}\sinh\vartheta & \frac{x_2}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & -\frac{\lambda_3}{|V|}\sinh\vartheta & \frac{\lambda_2}{|V|}\sinh\vartheta & -\frac{x_1}{|V|}\sinh\vartheta \\
\frac{\lambda_2}{|V|}\sinh\vartheta & \frac{x_3}{|V|}\sinh\vartheta & \cosh\vartheta & -\frac{x_1}{|V|}\sinh\vartheta & \frac{\lambda_3}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & -\frac{\lambda_1}{|V|}\sinh\vartheta & -\frac{x_2}{|V|}\sinh\vartheta \\
\frac{\lambda_3}{|V|}\sinh\vartheta & -\frac{x_2}{|V|}\sinh\vartheta & \frac{x_1}{|V|}\sinh\vartheta & \cosh\vartheta & -\frac{\lambda_2}{|V|}\sinh\vartheta & \frac{\lambda_1}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & -\frac{x_3}{|V|}\sinh\vartheta \\
\frac{x_1}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & \frac{\lambda_3}{|V|}\sinh\vartheta & -\frac{\lambda_2}{|V|}\sinh\vartheta & \cosh\vartheta & \frac{x_3}{|V|}\sinh\vartheta & -\frac{x_2}{|V|}\sinh\vartheta & -\frac{\lambda_1}{|V|}\sinh\vartheta \\
\frac{x_2}{|V|}\sinh\vartheta & -\frac{\lambda_3}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & \frac{\lambda_1}{|V|}\sinh\vartheta & -\frac{x_3}{|V|}\sinh\vartheta & \cosh\vartheta & \frac{x_1}{|V|}\sinh\vartheta & -\frac{\lambda_2}{|V|}\sinh\vartheta \\
\frac{x_3}{|V|}\sinh\vartheta & \frac{\lambda_2}{|V|}\sinh\vartheta & -\frac{\lambda_1}{|V|}\sinh\vartheta & \frac{ct}{|V|}\sinh\vartheta & \frac{x_2}{|V|}\sinh\vartheta & -\frac{x_1}{|V|}\sinh\vartheta & \cosh\vartheta & -\frac{\lambda_3}{|V|}\sinh\vartheta \\
\frac{ct}{|V|}\sinh\vartheta & \frac{x_1}{|V|}\sinh\vartheta & \frac{x_2}{|V|}\sinh\vartheta & \frac{x_3}{|V|}\sinh\vartheta & -\frac{\lambda_1}{|V|}\sinh\vartheta & -\frac{\lambda_2}{|V|}\sinh\vartheta & -\frac{\lambda_3}{|V|}\sinh\vartheta & \cosh\vartheta
\end{bmatrix},
$$

*where $\vartheta = \Upsilon/n$.*

*(ii) If $n$ is an odd number, the only $1-$ solution is*

$$\mathfrak{x}_0 = \begin{bmatrix} \sinh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta \\ \frac{\lambda_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta \\ \frac{\lambda_2}{|V|}\cosh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta \\ \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta \\ \frac{x_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{\lambda_3}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & \sinh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & -\frac{x_2}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta \\ \frac{x_2}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{x_3}{|V|}\cosh\vartheta & \sinh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta \\ \frac{x_3}{|V|}\cosh\vartheta & \frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & \frac{ct}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & -\frac{x_1}{|V|}\cosh\vartheta & \sinh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta \\ \frac{ct}{|V|}\cosh\vartheta & \frac{x_1}{|V|}\cosh\vartheta & \frac{x_2}{|V|}\cosh\vartheta & \frac{x_3}{|V|}\cosh\vartheta & -\frac{\lambda_1}{|V|}\cosh\vartheta & -\frac{\lambda_2}{|V|}\cosh\vartheta & -\frac{\lambda_3}{|V|}\cosh\vartheta & \sinh\vartheta \end{bmatrix}. \tag{74}$$

*Proof.* The proof can be obtained like Theorem 16. □

**Theorem 18.** *Given the matrix $\mathfrak{s}_R$ for a unit split octonion with an inner product having a positive value and a unit spacelike $7-$ vector $\mathfrak{U}$. Then, the equation $\mathfrak{x}^n = \mathfrak{s}_R$ has $n$ -solutions in the following form:*

$$\mathfrak{x}_p = \begin{bmatrix} \cos\vartheta & \frac{\lambda_1}{|V|}\sin\vartheta & \frac{\lambda_2}{|V|}\sin\vartheta & \frac{\lambda_3}{|V|}\sin\vartheta & -\frac{x_1}{|V|}\sin\vartheta & -\frac{x_2}{|V|}\sin\vartheta & -\frac{x_3}{|V|}\sin\vartheta & \frac{ct}{|V|}\sin\vartheta \\ \frac{\lambda_1}{|V|}\sin\vartheta & \cos\vartheta & -\frac{x_3}{|V|}\sin\vartheta & \frac{x_2}{|V|}\sin\vartheta & \frac{ct}{|V|}\sin\vartheta & -\frac{\lambda_3}{|V|}\sin\vartheta & \frac{\lambda_2}{|V|}\sin\vartheta & -\frac{x_1}{|V|}\sin\vartheta \\ \frac{\lambda_2}{|V|}\sin\vartheta & \frac{x_3}{|V|}\sin\vartheta & \cos\vartheta & -\frac{x_1}{|V|}\sin\vartheta & \frac{\lambda_3}{|V|}\sin\vartheta & \frac{ct}{|V|}\sin\vartheta & -\frac{\lambda_1}{|V|}\sin\vartheta & -\frac{x_2}{|V|}\sin\vartheta \\ \frac{\lambda_3}{|V|}\sin\vartheta & -\frac{x_2}{|V|}\sin\vartheta & \frac{x_1}{|V|}\sin\vartheta & \cos\vartheta & -\frac{\lambda_2}{|V|}\sin\vartheta & \frac{\lambda_1}{|V|}\sin\vartheta & \frac{ct}{|V|}\sin\vartheta & -\frac{x_3}{|V|}\sin\vartheta \\ \frac{x_1}{|V|}\sin\vartheta & \frac{ct}{|V|}\sin\vartheta & \frac{\lambda_3}{|V|}\sin\vartheta & -\frac{\lambda_2}{|V|}\sin\vartheta & \cos\vartheta & \frac{x_3}{|V|}\sin\vartheta & -\frac{x_2}{|V|}\sin\vartheta & -\frac{\lambda_1}{|V|}\sin\vartheta \\ \frac{x_2}{|V|}\sin\vartheta & -\frac{\lambda_3}{|V|}\sin\vartheta & \frac{ct}{|V|}\sin\vartheta & \frac{\lambda_1}{|V|}\sin\vartheta & -\frac{x_3}{|V|}\sin\vartheta & \cos\vartheta & \frac{x_1}{|V|}\sin\vartheta & -\frac{\lambda_2}{|V|}\sin\vartheta \\ \frac{x_3}{|V|}\sin\vartheta & \frac{\lambda_2}{|V|}\sin\vartheta & -\frac{\lambda_1}{|V|}\sin\vartheta & \frac{ct}{|V|}\sin\vartheta & \frac{x_2}{|V|}\sin\vartheta & -\frac{x_1}{|V|}\sin\vartheta & \cos\vartheta & -\frac{\lambda_3}{|V|}\sin\vartheta \\ \frac{ct}{|V|}\sin\vartheta & \frac{x_1}{|V|}\sin\vartheta & \frac{x_2}{|V|}\sin\vartheta & \frac{x_3}{|V|}\sin\vartheta & -\frac{\lambda_1}{|V|}\sin\vartheta & -\frac{\lambda_2}{|V|}\sin\vartheta & -\frac{\lambda_3}{|V|}\sin\vartheta & \cos\vartheta \end{bmatrix}, \tag{75}$$

where $\vartheta = \Upsilon + 2p\pi/n$, $p = 0, 1, 2, \ldots, n-1$.

*Proof.* The proof is similar to Theorem 16. ☐

**Theorem 19.** *Given the matrix $\mathfrak{s}_R$ for a split octonion with an inner product having zero value and a unit 7− vector $\mathfrak{U}$. Then, the equation $\mathfrak{x}^n = \mathfrak{s}_R$ has the following solutions as follows:*

*(i) If n is an even number, there exist 2− solutions in the following forms:*

$$\mathfrak{x}_0 = \Omega \begin{bmatrix} 1 & \frac{\lambda_1}{w} & \frac{\lambda_2}{w} & \frac{\lambda_3}{w} & -\frac{x_1}{w} & -\frac{x_2}{w} & -\frac{x_3}{w} & \frac{ct}{w} \\ \frac{\lambda_1}{w} & 1 & -\frac{x_3}{w} & \frac{x_2}{w} & \frac{ct}{w} & -\frac{\lambda_3}{w} & \frac{\lambda_2}{w} & -\frac{x_1}{w} \\ \frac{\lambda_2}{w} & \frac{x_3}{w} & 1 & -\frac{x_1}{w} & \frac{\lambda_3}{w} & \frac{ct}{w} & -\frac{\lambda_1}{w} & -\frac{x_2}{w} \\ \frac{\lambda_3}{w} & -\frac{x_2}{w} & \frac{x_1}{w} & 1 & -\frac{\lambda_2}{w} & \frac{\lambda_1}{w} & \frac{ct}{w} & -\frac{x_3}{w} \\ \frac{x_1}{w} & \frac{ct}{w} & \frac{\lambda_3}{w} & -\frac{\lambda_2}{w} & 1 & \frac{x_3}{w} & -\frac{x_2}{w} & -\frac{\lambda_1}{w} \\ \frac{x_2}{w} & -\frac{\lambda_3}{w} & \frac{ct}{w} & \frac{\lambda_1}{w} & -x_3 & 1 & \frac{x_1}{w} & -\frac{\lambda_2}{w} \\ \frac{x_3}{w} & \frac{\lambda_2}{w} & -\frac{\lambda_1}{w} & \frac{ct}{w} & \frac{x_2}{w} & -\frac{x_1}{w} & 1 & -\frac{\lambda_3}{w} \\ \frac{ct}{w} & \frac{x_1}{w} & \frac{x_2}{w} & \frac{x_3}{w} & -\frac{\lambda_1}{w} & -\frac{\lambda_2}{w} & -\frac{\lambda_3}{w} & 1 \end{bmatrix},$$

$$\mathfrak{x}_1 = -\Omega \begin{bmatrix} 1 & \frac{\lambda_1}{w} & \frac{\lambda_2}{w} & \frac{\lambda_3}{w} & -\frac{x_1}{w} & -\frac{x_2}{w} & -\frac{x_3}{w} & \frac{ct}{w} \\ \frac{\lambda_1}{w} & 1 & -\frac{x_3}{w} & \frac{x_2}{w} & \frac{ct}{w} & -\frac{\lambda_3}{w} & \frac{\lambda_2}{w} & -\frac{x_1}{w} \\ \frac{\lambda_2}{w} & \frac{x_3}{w} & 1 & -\frac{x_1}{w} & \frac{\lambda_3}{w} & \frac{ct}{w} & -\frac{\lambda_1}{w} & -\frac{x_2}{w} \\ \frac{\lambda_3}{w} & -\frac{x_2}{w} & \frac{x_1}{w} & 1 & -\frac{\lambda_2}{w} & \frac{\lambda_1}{w} & \frac{ct}{w} & -\frac{x_3}{w} \\ \frac{x_1}{w} & \frac{ct}{w} & \frac{\lambda_3}{w} & -\frac{\lambda_2}{w} & 1 & \frac{x_3}{w} & -\frac{x_2}{w} & -\frac{\lambda_1}{w} \\ \frac{x_2}{w} & -\frac{\lambda_3}{w} & \frac{ct}{w} & \frac{\lambda_1}{w} & -x_3 & 1 & \frac{x_1}{w} & -\frac{\lambda_2}{w} \\ \frac{x_3}{w} & \frac{\lambda_2}{w} & -\frac{\lambda_1}{w} & \frac{ct}{w} & \frac{x_2}{w} & -\frac{x_1}{w} & 1 & -\frac{\lambda_3}{w} \\ \frac{ct}{w} & \frac{x_1}{w} & \frac{x_2}{w} & \frac{x_3}{w} & -\frac{\lambda_1}{w} & -\frac{\lambda_2}{w} & -\frac{\lambda_3}{w} & 1 \end{bmatrix}, \tag{76}$$

*where* $\Omega = \sqrt[n]{2w}/2$.

*(ii) If $n$ is an odd number, there exists only* $1-$ *solution*

$$\mathfrak{x}_0 = \Omega \begin{bmatrix} 1 & \frac{\lambda_1}{w} & \frac{\lambda_2}{w} & \frac{\lambda_3}{w} & -\frac{x_1}{w} & -\frac{x_2}{w} & -\frac{x_3}{w} & \frac{ct}{w} \\ \frac{\lambda_1}{w} & 1 & -\frac{x_3}{w} & \frac{x_2}{w} & \frac{ct}{w} & -\frac{\lambda_3}{w} & \frac{\lambda_2}{w} & -\frac{x_1}{w} \\ \frac{\lambda_2}{w} & \frac{x_3}{w} & 1 & -\frac{x_1}{w} & \frac{\lambda_3}{w} & \frac{ct}{w} & -\frac{\lambda_1}{w} & -\frac{x_2}{w} \\ \frac{\lambda_3}{w} & -\frac{x_2}{w} & \frac{x_1}{w} & 1 & -\frac{\lambda_2}{w} & \frac{\lambda_1}{w} & \frac{ct}{w} & -\frac{x_3}{w} \\ \frac{x_1}{w} & \frac{ct}{w} & \frac{\lambda_3}{w} & -\frac{\lambda_2}{w} & 1 & \frac{x_3}{w} & -\frac{x_2}{w} & -\frac{\lambda_1}{w} \\ \frac{x_2}{w} & -\frac{\lambda_3}{w} & \frac{ct}{w} & \frac{\lambda_1}{w} & -x_3 & 1 & \frac{x_1}{w} & -\frac{\lambda_2}{w} \\ \frac{x_3}{w} & \frac{\lambda_2}{w} & -\frac{\lambda_1}{w} & \frac{ct}{w} & \frac{x_2}{w} & -\frac{x_1}{w} & 1 & -\frac{\lambda_3}{w} \\ \frac{ct}{w} & \frac{x_1}{w} & \frac{x_2}{w} & \frac{x_3}{w} & -\frac{\lambda_1}{w} & -\frac{\lambda_2}{w} & -\frac{\lambda_3}{w} & 1 \end{bmatrix}, \quad (77)$$

*where* $\Omega = \sqrt[n]{2w}/2$.

*Proof.* The proof can be presented by the similar way to Theorem 16. □

**Theorem 20.** *Let* $\mathfrak{s}_R$ *be a right matrix representation for a split octonion with an inner product having zero value and a unit* $7-$ *vector* $\mathfrak{U}$. *Then, the equation* $\mathfrak{x}^n = \mathfrak{s}_R$ *has the following solutions as follows:*

*(i) If n is an even number and* $sgn(w) > 0$ *, there exist 2– solutions in the following forms:*

$$\mathfrak{x}_0 = \sqrt[n]{w}\begin{bmatrix} 1 & \frac{\lambda_1}{n} & \frac{\lambda_2}{n} & \frac{\lambda_3}{n} & -\frac{x_1}{n} & -\frac{x_2}{n} & -\frac{x_3}{n} & \frac{ct}{n} \\ \frac{\lambda_1}{n} & 1 & -\frac{x_3}{n} & \frac{x_2}{n} & \frac{ct}{n} & -\frac{\lambda_3}{n} & \frac{\lambda_2}{n} & -\frac{x_1}{n} \\ \frac{\lambda_2}{n} & \frac{x_3}{n} & 1 & -\frac{x_1}{n} & \frac{\lambda_3}{n} & \frac{ct}{n} & -\frac{\lambda_1}{n} & -\frac{x_2}{n} \\ \frac{\lambda_3}{n} & -\frac{x_2}{n} & \frac{x_1}{n} & 1 & -\frac{\lambda_2}{n} & \frac{\lambda_1}{n} & \frac{ct}{n} & -\frac{x_3}{n} \\ \frac{x_1}{n} & \frac{ct}{n} & \frac{\lambda_3}{n} & -\frac{\lambda_2}{n} & 1 & \frac{x_3}{n} & -\frac{x_2}{n} & -\frac{\lambda_1}{n} \\ \frac{x_2}{n} & -\frac{\lambda_3}{n} & \frac{ct}{n} & \frac{\lambda_1}{n} & -x_3 & 1 & \frac{x_1}{n} & -\frac{\lambda_2}{n} \\ \frac{x_3}{n} & \frac{\lambda_2}{n} & -\frac{\lambda_1}{n} & \frac{ct}{n} & \frac{x_2}{n} & -\frac{x_1}{n} & 1 & -\frac{\lambda_3}{n} \\ \frac{ct}{n} & \frac{x_1}{n} & \frac{x_2}{n} & \frac{x_3}{n} & -\frac{\lambda_1}{n} & -\frac{\lambda_2}{n} & -\frac{\lambda_3}{n} & 1 \end{bmatrix},$$

$$\mathfrak{x}_1 = -\sqrt[n]{w}\begin{bmatrix} 1 & \frac{\lambda_1}{n} & \frac{\lambda_2}{n} & \frac{\lambda_3}{n} & -\frac{x_1}{n} & -\frac{x_2}{n} & -\frac{x_3}{n} & \frac{ct}{n} \\ \frac{\lambda_1}{n} & 1 & -\frac{x_3}{n} & \frac{x_2}{n} & \frac{ct}{n} & -\frac{\lambda_3}{n} & \frac{\lambda_2}{n} & -\frac{x_1}{n} \\ \frac{\lambda_2}{n} & \frac{x_3}{n} & 1 & -\frac{x_1}{n} & \frac{\lambda_3}{n} & \frac{ct}{n} & -\frac{\lambda_1}{n} & -\frac{x_2}{n} \\ \frac{\lambda_3}{n} & -\frac{x_2}{n} & \frac{x_1}{n} & 1 & -\frac{\lambda_2}{n} & \frac{\lambda_1}{n} & \frac{ct}{n} & -\frac{x_3}{n} \\ \frac{x_1}{n} & \frac{ct}{n} & \frac{\lambda_3}{n} & -\frac{\lambda_2}{n} & 1 & \frac{x_3}{n} & -\frac{x_2}{n} & -\frac{\lambda_1}{n} \\ \frac{x_2}{n} & -\frac{\lambda_3}{n} & \frac{ct}{n} & \frac{\lambda_1}{n} & -x_3 & 1 & \frac{x_1}{n} & -\frac{\lambda_2}{n} \\ \frac{x_3}{n} & \frac{\lambda_2}{n} & -\frac{\lambda_1}{n} & \frac{ct}{n} & \frac{x_2}{n} & -\frac{x_1}{n} & 1 & -\frac{\lambda_3}{n} \\ \frac{ct}{n} & \frac{x_1}{n} & \frac{x_2}{n} & \frac{x_3}{n} & -\frac{\lambda_1}{n} & -\frac{\lambda_2}{n} & -\frac{\lambda_3}{n} & 1 \end{bmatrix}. \tag{78}$$

But there is no solution for $sgn(w) < 0$.

*(ii) If n is an odd number, there exists only 1− solution in the following form:*

$$\mathfrak{x}_0 = \sqrt[n]{w}\begin{bmatrix} 1 & \frac{\lambda_1}{n} & \frac{\lambda_2}{n} & \frac{\lambda_3}{n} & -\frac{x_1}{n} & -\frac{x_2}{n} & -\frac{x_3}{n} & \frac{ct}{n} \\ \frac{\lambda_1}{n} & 1 & -\frac{x_3}{n} & \frac{x_2}{n} & \frac{ct}{n} & -\frac{\lambda_3}{n} & \frac{\lambda_2}{n} & -\frac{x_1}{n} \\ \frac{\lambda_2}{n} & \frac{x_3}{n} & 1 & -\frac{x_1}{n} & \frac{\lambda_3}{n} & \frac{ct}{n} & -\frac{\lambda_1}{n} & -\frac{x_2}{n} \\ \frac{\lambda_3}{n} & -\frac{x_2}{n} & \frac{x_1}{n} & 1 & -\frac{\lambda_2}{n} & \frac{\lambda_1}{n} & \frac{ct}{n} & -\frac{x_3}{n} \\ \frac{x_1}{n} & \frac{ct}{n} & \frac{\lambda_3}{n} & -\frac{\lambda_2}{n} & 1 & \frac{x_3}{n} & -\frac{x_2}{n} & -\frac{\lambda_1}{n} \\ \frac{x_2}{n} & -\frac{\lambda_3}{n} & \frac{ct}{n} & \frac{\lambda_1}{n} & -x_3 & 1 & \frac{x_1}{n} & -\frac{\lambda_2}{n} \\ \frac{x_3}{n} & \frac{\lambda_2}{n} & -\frac{\lambda_1}{n} & \frac{ct}{n} & \frac{x_2}{n} & -\frac{x_1}{n} & 1 & -\frac{\lambda_3}{n} \\ \frac{ct}{n} & \frac{x_1}{n} & \frac{x_2}{n} & \frac{x_3}{n} & -\frac{\lambda_1}{n} & -\frac{\lambda_2}{n} & -\frac{\lambda_3}{n} & 1 \end{bmatrix} \tag{79}$$

*Proof.* The proof can be given similarly to Theorem 16. □

## 7. An Application

*Example 1.* Let us consider a split octonion as

$$s = \frac{1}{\sqrt{2}} + \sqrt{2}J_1 + \frac{1}{\sqrt{2}}J_2 + \frac{1}{\sqrt{2}}J_3 + \frac{1}{\sqrt{2}}j_1 + \frac{1}{\sqrt{2}}j_2 + \frac{1}{\sqrt{2}}j_3. \tag{80}$$

Since $\mathbb{I}(s) = -1$ and $V^2 = 3/2$, $s$ is a split octonion with the inner product having a negative value and timelike vector part. So the split octonion $s$ is written in the polar form

$$s = \sinh \Upsilon + \mathfrak{U} \cosh \Upsilon, \tag{81}$$

where

$$\mathfrak{U} = \frac{\sqrt{2}J_1 + 1/\sqrt{2}J_2 + 1/\sqrt{2}J_3 + 1/\sqrt{2}j_1 + 1/\sqrt{2}j_2 + 1/\sqrt{2}j_3}{\sqrt{3/2}} \tag{82}$$

and $\sinh\Upsilon = 1/\sqrt{2}$, $\cosh\Upsilon = \sqrt{3/2}$. Then, from Theorem 1, the $7^{th}-$ power of the split octonion $s$ is obtained as

$$\begin{aligned} s^7 &= \sinh 7\,\Upsilon + \mathfrak{U} \cosh 7\,\Upsilon, \\ &= 50.2046 + 57.9828J_1 + 28.9914J_2 + 28.9914J_3 + 28.9914j_1 + 28.9914j_2 + 28.9914j_3. \end{aligned} \tag{83}$$

And by Theorem 6, the $7^{th}-$ root of the split octonion $s$ is calculated as

$$\begin{aligned} \sqrt[7]{s} &= \sinh\frac{\Upsilon}{7} + \mathfrak{U} \cosh\frac{\Upsilon}{7}, \\ &= 0.0942 + 1.1598J_1 + 0.5799J_2 + 0.5799J_3 + 0.5799j_1 + 0.5799j_2 + 0.5799j_3. \end{aligned} \tag{84}$$

The right representation of the split octonion $s$ denoted by $\mathfrak{s}_R$ can be written as follows:

$$\mathfrak{s}_R = \begin{bmatrix} 1/\sqrt{2} & \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ \sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} & 0 & -\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & \sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & -\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & \sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & -\sqrt{2} & 0 & 1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \tag{85}$$

or in the polar form

$$\begin{bmatrix} \sinh\Upsilon & \frac{\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & 0 \\ \frac{\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \sinh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & 0 & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon \\ \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \sinh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & 0 & -\frac{\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon \\ \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \sinh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & 0 & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon \\ \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & 0 & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \sinh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon \\ \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & 0 & \frac{\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \sinh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon \\ \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{\sqrt{2}}{3/2}\cosh\Upsilon & 0 & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \sinh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon \\ 0 & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & -\frac{1/\sqrt{2}}{\sqrt{3/2}}\cosh\Upsilon & \sinh\Upsilon \end{bmatrix}. \quad (86)$$

In addition, the $7^{th}-$ power of the matrix $\mathfrak{s}_R$ by Theorem 11 is

$$\begin{bmatrix} 50.2046 & 57.9828 & 28.9914 & 28.9914 & -28.9914 & -28.9914 & -28.9914 & 0.0000 \\ 57.9828 & 50.2046 & -28.9914 & 28.9914 & 0.0000 & -28.9914 & 28.9914 & -28.9914 \\ 28.9914 & 28.9914 & 50.2046 & -28.9914 & 28.9914 & 0.0000 & -57.9828 & -28.9914 \\ 28.9914 & -28.9914 & 28.9914 & 50.2046 & -28.9914 & 57.9828 & 0.0000 & -28.9914 \\ 28.9914 & 0.0000 & 28.9914 & -28.9914 & 50.2046 & 28.9914 & -28.9914 & -57.9828 \\ 28.9914 & -28.9914 & 0.0000 & 57.9828 & -28.9914 & 50.2046 & 28.9914 & -28.9914 \\ 28.9914 & 28.9914 & -57.9828 & 0.0000 & 28.9914 & -28.9914 & 50.2046 & -28.9914 \\ 0.0000 & 28.9914 & 28.9914 & 28.9914 & -57.9828 & -28.9914 & -28.9914 & 50.2046 \end{bmatrix}. \quad (87)$$

However, if we want to find the $7^{th}-$ root of $\mathfrak{s}_R$ from Theorem 16, we find

$$\begin{bmatrix} 0.0942 & 1.1598 & 0.5799 & 0.5799 & -0.5799 & -0.5799 & -0.5799 & 0 \\ 1.1598 & 0.0942 & -0.5799 & 0.5799 & 0 & -0.5799 & 0.5799 & -0.5799 \\ 0.5799 & 0.5799 & 0.0942 & -0.5799 & 0.5799 & 0 & -1.1598 & -0.5799 \\ 0.5799 & -0.5799 & 0.5799 & 0.0942 & -0.5799 & 1.1598 & 0 & -0.5799 \\ 0.5799 & 0 & 0.5799 & -0.5799 & 0.0942 & 0.5799 & -0.5799 & 1.1598 \\ 0.5799 & -0.5799 & 0 & 1.1598 & -0.5799 & 0.0942 & 0.5799 & -0.5799 \\ 0.5799 & 0.5799 & -1.1598 & 0 & 0.5799 & -0.5799 & 0.0942 & -0.5799 \\ 0 & 0.5799 & 0.5799 & 0.5799 & -1.1598 & -0.5799 & -0.5799 & 0.0942 \end{bmatrix}. \quad (88)$$

The accuracy of each of the above calculations for $\Upsilon = \sinh^{-1}(1/\sqrt{2})$ can be checked with the following Matlab codes.

```
clc;
clear all;
syms g
w= 1/sqrt(2) %also we can use code input('input w ')
l1=sqrt(2) %input('input l1')
l2=1/sqrt(2) %input('input l2')
l3=1/sqrt(2) %input('input l3')
x1=1/sqrt(2) %input('input x1')
x2=1/sqrt(2) %input('input x2')
x3=1/sqrt(2) %input('input x3')
ct=0 %input('input ct')
n=7 %input('input power n')
m=7 %input('input root n')
A=[w l1 l2 l3 -x1 -x2 -x3 ct;
    l1 w -x3 x2 ct -l3 l2 -x1;
    l2 x3 w -x1 l3 ct -l1 -x2;
    l3 -x2 x1 w -l2 l1 ct -x3;
    x1 ct l3 -l2 w x3 -x2 -l1;
    x2 -l3 ct l1 -x3 w x1 -l2;
    x3 l2 -l1 ct x2 -x1 w -l3;
    ct x1 x2 x3 -l1 -l2 -l3 w]
nthpA=A^(n);
N=sqrt(abs(w^2-((l1)^2+(l2)^2+(l3)^2-(x1)^2-(x2)^2-(x3)^2+(ct)^2)));
g=asinh(w/N);
v2=(l1)^2+(l2)^2+(l3)^2-(x1)^2-(x2)^2-(x3)^2+(ct)^2;
v=sqrt(v2);
U=[0 l1 l2 l3 x1 x2 x3 ct]/v;
sinehyperbolicng=sinh(n*g)*[1 0 0 0 0 0 0 0];
cosinehyperbolicng=cosh(n*g);

nthPowers=sinehyperbolicng+U*cosinehyperbolicng

% Also the following code give us nth power of any split octonions
nthPowerrs=[nthpA(1,1) nthpA(2,1) nthpA(3,1) nthpA(4,1) nthpA(5,1)...
    nthpA(6,1) nthpA(7,1) nthpA(8,1)]

sinehyperbolicng=sinh(1/m*g)*[1 0 0 0 0 0 0 0];
cosinehyperbolicng=cosh(1/m*g);

nthroots=sinehyperbolicng+U*cosinehyperbolicng
```

```
nthPowerA=[sinh(n*g) l1/v*cosh(n*g) l2/v*cosh(n*g) l3/v*cosh(n*g)...
    -x1/v*cosh(n*g) -x2/v*cosh(n*g) -x3/v*cosh(n*g) ct/v*cosh(n*g);
    l1/v*cosh(n*g) sinh(n*g) -x3/v*cosh(n*g)  x2/v*cosh(n*g)...
    ct/v*cosh(n*g) -l3/v*cosh(n*g) l2/v*cosh(n*g) -x1/v*cosh(n*g);
    l2/v*cosh(n*g) x3/v*cosh(n*g) sinh(n*g) -x1/v*cosh(n*g)...
    l3/v*cosh(n*g) ct/v*cosh(n*g) -l1/v*cosh(n*g) -x2/v*cosh(n*g);
    l3/v*cosh(n*g) -x2/v*cosh(n*g) x1/v*cosh(n*g) sinh(n*g)...
    -l2/v*cosh(n*g)  l1/v*cosh(n*g) ct/v*cosh(n*g)  -x3/v*cosh(n*g);
    x1/v*cosh(n*g) ct/v*cosh(n*g) l3/v*cosh(n*g) -l2/v*cosh(n*g)...
    sinh(n*g) x3/v*cosh(n*g) -x2/v*cosh(n*g) l1/v*cosh(n*g);
    x2/v*cosh(n*g) -l3/v*cosh(n*g) ct/v*cosh(n*g) l1/v*cosh(n*g)...
    -x3/v*cosh(n*g) sinh(n*g) x1/v*cosh(n*g) -l2/v*cosh(n*g);
    x3/v*cosh(n*g) l2/v*cosh(n*g) -l1/v*cosh(n*g) ct/v*cosh(n*g)...
    x2/v*cosh(n*g) -x1/v*cosh(n*g) sinh(n*g) -l3/v*cosh(n*g);
    ct/v*cosh(n*g) x1/v*cosh(n*g) x2/v*cosh(n*g) x3/v*cosh(n*g)...
    -l1/v*cosh(n*g) -l2/v*cosh(n*g) -l3/v*cosh(n*g)  sinh(n*g)]

n=1/m;
nthRootA=[sinh(n*g) l1/v*cosh(n*g) l2/v*cosh(n*g) l3/v*cosh(n*g) ...
    -x1/v*cosh(n*g) -x2/v*cosh(n*g) -x3/v*cosh(n*g) ct/v*cosh(n*g);
    l1/v*cosh(n*g) sinh(n*g) -x3/v*cosh(n*g)  x2/v*cosh(n*g)...
    ct/v*cosh(n*g) -l3/v*cosh(n*g) l2/v*cosh(n*g) -x1/v*cosh(n*g);
    l2/v*cosh(n*g) x3/v*cosh(n*g) sinh(n*g) -x1/v*cosh(n*g)...
    l3/v*cosh(n*g) ct/v*cosh(n*g) -l1/v*cosh(n*g) -x2/v*cosh(n*g);
    l3/v*cosh(n*g) -x2/v*cosh(n*g) x1/v*cosh(n*g) sinh(n*g)...
    -l2/v*cosh(n*g)  l1/v*cosh(n*g) ct/v*cosh(n*g) -x3/v*cosh(n*g);
    x1/v*cosh(n*g) ct/v*cosh(n*g) l3/v*cosh(n*g) -l2/v*cosh(n*g) ...
    sinh(n*g) x3/v*cosh(n*g) -x2/v*cosh(n*g) l1/v*cosh(n*g);
    x2/v*cosh(n*g) -l3/v*cosh(n*g) ct/v*cosh(n*g) l1/v*cosh(n*g)...
    -x3/v*cosh(n*g) sinh(n*g) x1/v*cosh(n*g) -l2/v*cosh(n*g);
    x3/v*cosh(n*g) l2/v*cosh(n*g) -l1/v*cosh(n*g) ct/v*cosh(n*g)...
    x2/v*cosh(n*g) -x1/v*cosh(n*g) sinh(n*g) -l3/v*cosh(n*g);
    ct/v*cosh(n*g) x1/v*cosh(n*g) x2/v*cosh(n*g) x3/v*cosh(n*g)...
    -l1/v*cosh(n*g) -l2/v*cosh(n*g) -l3/v*cosh(n*g)  sinh(n*g)]
```

## 8. Conclusions

In this article, we present polar forms for the split octonions with the inner product being zero and the split octonions with the lightlike vector part. By using the polar forms of the split octonions, we give De Moivre's formulas for split octonions and matrices related with split octonions. Thus, we obtain the $n^{th}-$ powers of split octonions and matrices related with split octonions. In addition to this, we calculate the $n^{th}-$ roots of the split octonions and the matrix representations of split octonions. Finally, we give an example to illustrate the obtained results.

## Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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