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Research Article

A Generalized Approach of Triple Integral Transforms and Applications

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In this study, we introduce a novel generalization of triple integral transforms, which is called a general triple transform. We present the definition of the new approach and prove the main properties related to the existence, uniqueness, shifting, scaling, and inverse. Moreover, relations between the new general triple transform and other transforms are presented, and new results related to partial derivatives and the triple convolution theorem are established. We apply the general triple transform to solve some applications of various types of partial differential equations. The strength of the new approach is that it covers almost all integral transforms of order one, two, and three, and hence no need to find new formulas of triple integral transforms or to study the basic properties.

1. Introduction

Many researchers around the world have studied partial differential equations, partial integrodifferential equations, and the methods of solution due to their ability in describing such formulas and phenomena. Mathematical formulas for real-world applications involving numerous variables are difficult to formulate, several mathematicians now have focused most of their attention on finding approximate and exact solutions to partial differential and integral equations, and many approaches to doing so have been proposed [1–7].

Differential equations had several applications in different industries. Dehghan and Shakeri [6] employed the variational iteration technique to resolve PIDEs that came up during the heat conduction of materials with memory. To address PIDEs that arose in viscoelasticity, Dehghan [7] offers a variety of numerical approaches. Pao [8] and Pachpatte [9] solved the nonlinear PIDEs that arose in nuclear reactor dynamics. For the purpose of pricing derivatives in the financial industry, PIDEs were used [10]. In financial modeling, Abergel and Tachet [11] employed a nonlinear PIDE. In the paradigm of electricity swaptions, Hepperger [12] suggested a PDE. In [13], Zadeh suggested a PIDE that controls biofluid flow in cracked biomaterials.

Several analytical approaches have been developed by numerous researchers to solve linear and nonlinear partial differential equations, such as the Homotopy analysis method [14], Adomian decomposition method [15], the variational iteration method [16], power series method [17], residual power series method [18], Laplace residual power series [19], ARA residual power series method [20], and others [21, 22].

One of the most promising tools for solving linear equations are integral transforms method. Mathematicians have proposed many types of integral transforms in literature, Laplace transform [23], Sumudu transform [24], Elzaki transform [25], Natural transform [26], ARA transform [27], formable transform [28], and others. Most of these transforms were studied and combined with other numerical methods to solve linear and nonlinear problems. Moreover, new approaches have appeared that generalizes the idea of single transform to double and triple transforms, such as double and triple Laplace transform, double and triple Sumudu transform, double and triple Elzaki transform, double and triple

ARA transform, triple Laplace ARA Sumudu transform, and others [29–36].

In 2020, Jafari [37] introduced the new general transform (single transform) which is defined for a continuous function as

$$G_S(s) = \mathcal{T}_1[g(x)] = p(s)\int_0^\infty g(x)e^{-q(s)x}\mathrm{d}x, \quad x > 0, \tag{1}$$

provided the integral exists. This transform is important due to its ability in generalizing large numbers of single transforms. Recently, Meddahi and others [38] introduced a new general double integral transform that generalizes the idea of double transforms, that is given for a continuous function of two variables as

$$\begin{aligned} G_D(s,u) &= \mathcal{T}_2[g(x,y)] \\ &= p_1(s)p_2(u)\int_0^\infty\int_0^\infty g(x,y)e^{-q_1(s)x-q_2(u)y} \\ &\quad \mathrm{d}x\mathrm{d}y, \quad x,y > 0, \end{aligned} \tag{2}$$

provided the integral exists. Also, this new formula mostly covers double integral transforms.

The aim of this work is to generalize a new formula for triple integral transforms, that covers the previous two approaches considering dimensions one and two of the new proposed transform of dimension three and to establish a new formula that covers almost all integral transforms of different dimensions 1, 2, and 3. We define a new general triple transform and introduce the basic properties and relations to other transforms, following that we present new general formulas related to partial derivatives and the triple convolution theorem. Moreover, we present some applications of partial differential equations and discuss the solution by the new triple transform. It is worth mentioning here that the new general triple transform generalizes mostly all triple integral transforms that have a duality to Laplace transform.

The layout of this article is as follows. Section 2 presents the new definition of the new general triple transform and some basic properties with proofs. In Section 3, more results related to partial derivatives and the convolution theorem are illustrated. Section 4 introduces some applications using the new approach in solving partial differential equations and integral equations. Section 5 concludes the paper.

2. A New General Formula of Triple Transform

This section includes a new general formula that generalizes mostly, all three-dimensional integral transforms. Fundamental definitions, properties, and theorems related to the new approach are all covered. For readers who want to see more properties about the single and double general transforms, we add Table 1.

2.1. Basic Definition of General Triple Transform

Definition 1. The general triple transform of the continuous function $g(x,y,z)$ denoted by $G_T(s,u,v)$ is defined by the following equation:

$$\begin{aligned} G_T(s,u,v) &= \mathcal{T}_3[g(x,y,z),(s,u,v)] \\ &= \mathcal{T}_x\left[\mathcal{T}_y[\mathcal{T}_z[g(x,y,z);z\longrightarrow v];y\longrightarrow u];x\longrightarrow s\right] \\ &= p_1(s)\int_0^\infty e^{-q_1(s)x}\left[p_2(u)\int_0^\infty e^{-q_2(u)y}\left[p_3(v)\int_0^\infty g(x,y,z)e^{-q_3(v)z}\right]dz\right]dydx, \end{aligned} \tag{3}$$

where $g(x,y,z)$ is a function defined for all $x>0, y>0, z>0$ the given functions $p_1(s), p_2(u), p_3(v), q_1(s)q_2(u)q_3(v)$ are complex functions under the condition $p_1(s), p_2(u)$ and $p_3(v) \neq 0$.

Using Fubini's theorem, one can obtain the following equation:

$$\begin{aligned} G_T(s,u,v) &= \mathcal{T}_3[g(x,y,z)] \\ &= p_1(s)p_2(u)p_3(v)\int_0^\infty\int_0^\infty \\ &\quad \int_0^\infty g(x,y,z)e^{-q_1(s)x-q_2(u)y-q_3(v)z}dx\,dy\,dz. \end{aligned} \tag{4}$$

The authors in [26] introduced the following result that allows us to define the inverse of the general triple transform as stated follows.

Theorem 1. *Assume that $g(x)$ is an integrable function on $x>0$, and if*

$$\begin{aligned} G_S(s) &= \mathcal{T}_1[g(x)] \\ &= p(s)\int_0^\infty g(x)e^{-q(s)x}dx, \end{aligned} \tag{5}$$

where $p(s)$ and $q(s)$ are both complex functions, and the Fourier transform of $e^{-ax}g(x)$ is also integrable, where $a\in\mathbb{R}$. Then, the inverse of the transform $G_S(s)$ is given by the following equation:

$$g(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{p(s)}e^{q(s)x}q'(s)G_S(s)ds, \tag{6}$$

provided that $p(s)\neq 0$ for $s\in\mathbb{C}$.

Thus, we can define the inverse general triple transform as follows:

Table 1: Some properties of the general single and double transform [40, 41].

$g(x)$	$\mathcal{T}_1[g(x)] = G_S(s)$	$g(x,y)$	$\mathcal{T}_2[g(x,y)] = G_D(s,u)$
$a \in R$	$ap_1(s)/q_1(s)$	$a \in R$	$ap_1(s)/q_1(s)p_2(u)/q_2(u)$
x	$p_1(s)/q_1^2(s)$	xy	$p_1(s)/q_1^2(s)p_2(u)/q_2^2(u)$
e^{ax}	$p_1(s)/q_1(s) - a$	e^{ax+by}	$p_1(s)/q_1(s) - ap_2(u)/q_2(u) - b, q_1(u) > a, q_2(u) > b,$
$\cos ax$	$p_1(s)q_1(s)/q_1^2(s) + a^2$	$\cos ax \cos by$	$p_1(s)q_1(s)/q_1^2(s) + a^2p_2(u)q_2(u)/q_2^2(u) + b^2$
$\sin ax$	$ap_1(s)/q_1^2(s) + a^2$	$\sin ax \sin by$	$ap_1(s)/q_1^2(s) + a^2bp_2(u)/q_2^2(u) + b^2$
x^α	$p_1(s)\Gamma(\alpha+1)/q_1^{\alpha+1}(s)$	$x^\alpha y^\beta$	$p_1(s)p_2(u)\Gamma(\alpha+1)/q_1^{\alpha+1}(s)\Gamma(\beta+1)/q_2^{\beta+1}(u)\forall\alpha,\beta > -1$
$g'(x)$	$q_1(s)G_S(s) - p_1(s)g(0)$	$\partial g(x,y)/\partial x$	$q_1^2(s)G_D(s,u) - q_1(s)p_1(s)G_S(0,u) - p_1(s)\partial G_S(0,u)/\partial x$

$$\begin{aligned}\mathcal{T}_3^{-1}[G(s,u,v)] = \mathcal{T}_x^{-1}\mathcal{T}_y^{-1}\mathcal{T}_z^{-1}[G_T(s,u,v)] = g(x,y,z)\\ = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{p_1(s)}e^{q_1(s)x}q_1'(s)ds\\ \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{1}{p_2(u)}e^{q_2(u)y}q_2'(u)du\\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p_3(v)}e^{q_3(v)z}q_3'(v)G_T(s,u,v)dv,\end{aligned} \tag{7}$$

where a, b and c are real constants and $p_1(s) \neq 0, p_2(u) \neq 0,$ and $p_3(v) \neq 0,$ for $s \in \mathbb{C}$.

In the following theorem, we introduce the existence and uniqueness conditions of the new approach.

Theorem 2. *Let $g(x,y,z)$ be a continuous function defined on J^3, where $J = (0,\infty)$, and suppose that $g(x,y,z)$ is a function of exponential orders $a,b,c > 0$, which implies $|g(x,y,z)| \leq Qe^{ax+by+cz}$ for some $Q > 0$. Then, the general triple transform $G_T(s,u,v)$ exists for all $Re(q_1(s)) > a, Re(q_2(u)) > b, Re(q_3(v)) > c$. Moreover, if $g(x,y,z)$ and $h(x,y,z)$ are both continuous functions possessing the general triple transforms $G_T(s,u,v)$ and $H_T(s,u,v)$, respectively, where $G_T(s,u,v) = H_T(s,u,v)$, then $g(x,y,z) = h(x,y,z)$.*

Proof. First, we prove the existing conditions.

The definition of the general triple transform implies that

$$\begin{aligned}\left\|\mathcal{T}_3[g(x,y,z),(s,u,v)]\right\| \leq \left|p_1(s)p_2(u)p_3(v)\right|\int_0^\infty\int_0^\infty\\ \int_0^\infty |g(x,y,z)|\\ e^{-\operatorname{Re}[q_1(s)]x-\operatorname{Re}[q_2(u)]y-\operatorname{Re}[q_3(v)]z}\\ dxdydz.\end{aligned} \tag{8}$$

Now, from the assumptions, that $\operatorname{Re}(q_1(s)) > a, \operatorname{Re}(q_2(u)) > b$ and $\operatorname{Re}(q_3(v)) > c$, we have the following equation:

$$\left\|\mathcal{T}_3[g(x,y,z),(s,u,v)]\right\| \leq Q\frac{|p_1(s)|}{a-\operatorname{Re}[q_1(s)]}\frac{|p_2(u)|}{b-\operatorname{Re}[q_2(u)]}\frac{|p_3(v)|}{c-\operatorname{Re}[q_3(v)]}. \tag{9}$$

In addition, if $G_T(s,u,v) = H_T(s,u,v)$ the definition of inverse triple transform implies that

TABLE 2: Relations of general triple transform and other transforms.

$p_1(s)$	$p_2(u)$	$p_3(v)$	$q_1(s)$	$q_2(u)$	$q_3(v)$	$\mathcal{T}_3[g(x,y,z)] = G_T(s,u,v)$
1	1	1	s	u	v	Triple laplace transform
$1/s$	$1/u$	$1/v$	$1/s$	$1/u$	$1/v$	Triple sumudu transform
s	u	v	$1/s$	$1/u$	$1/v$	Triple elzaki transform
1	1	1	s/s'	u/u'	v/v'	Triple sheuh transform
$1/s$	$1/u$	$1/v$	s	u	v	Triple aboodh transform
s	u	v	s	u	v	Triple ARA transform
1	u	$1/v$	s	u	$1/v$	Triple laplace-ARA-sumudu transform

$$
\begin{aligned}
g(x,y,z) &= \\
&= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{p_1(s)}e^{q_1(s)x}q_1'(s)ds \\
&\quad \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{1}{p_2(u)}e^{q_2(u)y}q_2'(u)du \\
&\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p_3(v)}e^{q_3(v)z}q_3'(v)G_T(s,u,v)dv \\
&= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{p_1(s)}e^{q_1(s)x}q_1'(s)ds \\
&\quad \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{1}{p_2(u)}e^{q_2(u)y}q_2'(u)du \\
&\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p_3(v)}e^{q_3(v)z}q_3'(v)H_T(s,u,v)dv, \\
&= h(x,y,z),
\end{aligned}
\tag{10}
$$

which ends the proof of the uniqueness.

The following remark illustrates the linearity of the new triple transform. □

Remark 1. The general triple transform is linear, in fact for any two constants $\alpha, \beta \in \mathbb{R}$, we have the following equation:

$$
\begin{aligned}
\mathcal{T}_3[\alpha g(x,y,z) + \beta h(x,y,z)] &= \alpha \mathcal{T}_3[g(x,y,z)] + \beta \mathcal{T}_3[h(x,y,z)] \\
&= \alpha G_T(s,u,v) + \beta H_T(s,u,v).
\end{aligned}
\tag{11}
$$

In addition, the inverse general triple transform is linear;

$$
\mathcal{T}_3^{-1}[\alpha G_T(s,u,v) + \beta H_T(s,u,v)] = \alpha g(x,y,z) + \beta h(x,y,z).
\tag{12}
$$

2.2. Dualities to Other Triple Transforms. The aim of this section is to illustrate the relation between the general triple transform and some other popular transforms. We study these dualities and state them in the following table (Table 2).

Moreover, we mention the relationship between the new general triple transform and the single and double transforms.

(i) $G_T(s,0,0) = G_S(s)$

(ii) $G_T(s,u,0) = G_D(s,u)$.

3. Derivatives and Convolution Results

In this section, we establish and prove some important results of the general triple transform, including the value of the proposed transform to partial derivatives, the triple convolution property, and others.

Theorem 3. *Let $G_T(s,u,v)$ be the general triple transform of the function $g(x,y,z)$. Then,*

(i) $\mathcal{T}_3[\partial g(x,y,z)/\partial x] = q_1(s)G_T(s,u,v) - p_1(s)G_D(0,u,v)$

(ii) $\mathcal{T}_3[\partial^2 g(x,y,z)/\partial x^2] = q_1^2(s)G_T(s,u,v) - q_1(s)p_1(s)G_D(0,u,v) - p_1(s)\partial G_D(0,u,v)/\partial x$

(iii) $\mathcal{T}_3[\partial^2 g(x,y,z)/\partial x \partial y] = q_1(s)q_2(u)G_T(s,u,v) - q_2(u)p_1(s)G_D(0,u,v) - q_1(u)p_2(s)G_D(s,0,v) + p_1(s)p_2(u)G_D(0,0,v)$

(iv) $\mathcal{T}_3[\partial^3 g(x,y,z)/\partial x\partial y\partial z] = q_1(s)q_2(u)q_3(v)G_T(s,u,v) - q_1(s)q_2(u)p_3(v)G_D(s,u,0) - q_1(s)q_3(v)p_2(u)G_D(s,0,v) - q_2(u)q_3(v)p_1(s)G_D(0,u,v) + q_1(s)p_2(u)p_3(v)G_S(s,0,0) + q_2(u)p_1(s)p_3(v)G_S(0,u,0) + q_3(v)p_1(s)p_2(u)G_S(0,0,v) - p_1(s)p_2(u)p_3(v)g(0,0,0)$.

Here, $G_D(0,u,v) = \mathcal{T}_2[g(0,y,z)], \partial G_D(0,u,v)/\partial x = \mathcal{T}_2[\partial g(0,y,z)/\partial x], G_S(s,0,0) = \mathcal{T}_1[g(x,0,0)]$.

Proof. For part (i), we get from the definition of the general triple transform,

$$\begin{aligned}\mathcal{T}_3\left[\frac{\partial g(x,y,z)}{\partial x}\right] &= p_1(s)p_2(u)p_3(v)\int_0^\infty\int_0^\infty\int_0^\infty \frac{\partial g(x,y,z)}{\partial x}e^{-q_1(s)x-q_2(u)y-q_3(v)z}dx\,dy\,dz\\ &= p_2(u)p_3(v)\int_0^\infty\int_0^\infty\left(p_1(s)\int_0^\infty\frac{\partial g(x,y,z)}{\partial x}e^{-q_1(s)x}dx\right)e^{-q_2(u)y-q_3(v)z}dy\,dz.\end{aligned} \tag{13}$$

Using the integration by parts for the integral in the bracket, with simple computations, equation (13) becomes the following equation:

$$\mathcal{T}_3[\partial g(x,y,z)/\partial x] = p_2(u)p_3(v)\int_0^\infty\int_0^\infty (q_1(s)G_S(s,y,z) - p_1(s)g(0,y,z))\, e^{-q_2(u)y-q_3(v)z}dy\,dz. \tag{14}$$

Hence, we obtain the following equation:

$$\mathcal{T}_3\left[\frac{\partial g(x,y,z)}{\partial x}\right] = q_1(s)G_T(s,u,v) - p_1(s)G_D(0,u,v). \tag{15}$$

To prove part (ii), the definition of the general triple transform yields

$$\mathcal{T}_3\left[\frac{\partial^2 g(x,y,z)}{\partial x^2}\right] = \mathcal{T}_3\left[\frac{\partial}{\partial x}\left(\frac{\partial g(x,y,z)}{\partial x}\right)\right]. \tag{16}$$

Using part (i) and with simple computations, equation (16) becomes the following equation:

$$\begin{aligned}\mathcal{T}_3\left[\frac{\partial^2 g(x,y,z)}{\partial x^2}\right] &= q_1(s)\mathcal{T}_3\left[\frac{\partial g(x,y,z)}{\partial x}\right] - p_1(s)\mathcal{T}_D\left(\frac{\partial g(0,y,z)}{\partial x}\right)\\ &= q_1^2(s)G_T(s,u,v) - q_1(s)p_1(s)G_D(0,u,v) - p_1(s)\frac{\partial G_D(0,u,v)}{\partial x}.\end{aligned} \tag{17}$$

which is the desired result. The proof of parts (iii) and (iv) can be obtained by similar arguments.

Now, we can present the following corollary which can be gained by replacing the variables in the previous theorem. □

Corollary 1. *Let* $G_T(s,u,v)$ *be the general triple transform of the function* $g(x,y,z)$. *Then,*

(i) $\mathcal{T}_3[\partial g(x,y,z)/\partial y] = q_2(u)G_T(s,u,v) - p_2(u)G_D(s,0,v)$

(ii) $\mathcal{T}_3[\partial g(x,y,z)/\partial z] = q_3(v)G_T(s,u,v) - p_3(u)G_D(s,u,0)$

(iii) $\mathcal{T}_3[\partial^2 g(x,y,z)/\partial y^2] = q_2^2(u)G_T(s,u,v) - q_2(u)p_2(u)G_D(s,0,v) - p_2(u)\partial G_D(s,0,v)/\partial y$

(iv) $\mathcal{T}_3[\partial^2 g(x,y,z)/\partial z^2] = q_3^2(v)G_T(s,u,v) - q_3(v)p_1(v)G_D(s,u,0) - p_3(v)\partial G_D(s,u,0)/\partial z$

(v) $\mathcal{T}_3[\partial^2 g(x,y,z)/\partial x\partial z] = q_1(s)q_3(v)G_T(s,u,v) - q_3(v)p_1(s)G_D(0,u,v) - q_1(s)p_3(v)G_D(s,u,0) + p_1(s)p_3(v)G_S(0,u,0)$.

(vi) $\mathcal{T}_3[\partial^2 g(x,y,z)/\partial y\partial z] = q_2(u)q_3(v)G_T(s,u,v) - q_3(v)p_2(u)G_D(s,0,v) - q_2(u)p_3(v)G_D(s,u,0) + p_2(u)p_3(v)G_S(s,0,0)$

Corollary 2. *Let* $G_T(s,u,v)$ *be the general triple transform of the function* $g(x,y,z)$. *Then, we have the following results:*

$$\begin{aligned}&\mathcal{T}_3\left[\int_0^x\int_0^y\int_0^z g(x_1,y_1,z_1)dz_1dy_1dx_1\right]\\ &= \frac{1}{q_1(s)q_2(u)q_3(v)}G_T(s,u,v).\end{aligned} \tag{18}$$

Proof. Assume that

$$k(x,y,z) = \int_0^x\int_0^y\int_0^z g(x_1,y_1,z_1)dz_1dy_1dx_1. \tag{19}$$

It is obvious that

$$\frac{\partial^3 k(x,y,z)}{\partial x\partial y\partial z} = g(x,y,z),\quad k(0,0,0)=0. \tag{20}$$

Hence,

$$\mathcal{T}_3\left[\frac{\partial^3 k(x,y,z)}{\partial x\partial y\partial z}\right] = \mathcal{T}_3[g(x,y,z)] = G_T(s,u,v). \tag{21}$$

□

Theorem 4. *part (iv) implies that*

$$\begin{aligned} G_T(s,u,v) = {} & q_1(s)q_2(u)q_3(v)K_T(s,u,v) - q_1(s)q_2(u)p_3(v)K_D(s,u,0)\\ & - q_1(s)q_3(v)p_2(u)K_D(s,0,v) - q_2(u)q_3(v)p_1(s)K_D(0,u,v)\\ & + q_1(s)p_2(u)p_3(v)K_S(s,0,0) + q_2(u)p_1(s)p_3(v)K_S(0,u,0)\\ & + q_3(v)p_1(s)p_2(u)K_S(0,0,v) - p(s)p_2(u)p_3(v)k(0,0,0). \end{aligned} \tag{22}$$

But, we have $K_D(s,u,0) = K_D(s,0,v) = K_D\ (0,u,v) = K_S(s,0,0) = K_S(0,u,0) = K_S\ (0,0,v) = k(0,0,0) = 0$.

Thus, we obtain

$$K_T(s,u,v) = \frac{1}{q_1}$$
$(s)q_2(u)q_3(v)G_T(s,u,v)$.

Hence, the proof is complete.

Theorem 5. *Let $G_T(s,u,v)$ be the general triple transform of the function $g(x,y,z)$. Then,*

(i) $\mathcal{T}_3[xg(x,y,z)] = -p_1(s)/q_1'(s)\partial/\partial s(G_T(s,u,v)/p_1(s))$

(ii) $\mathcal{T}_3[yg(x,y,z)] = -p_2(u)/q_2'(u)\partial/\partial u(G_T(s,u,v)/p_2(u))$

(iii) $\mathcal{T}_3[zg(x,y,z)] = -p_3(v)/q_3'(v)\partial/\partial v(G_T(s,u,v)/p_3(v))$

Proof. We state the proof of part (i); for parts (ii) and (iii), we can get the proof by similar arguments. We get from the definition of the general triple transform, that

$$\mathcal{T}_3[g(x,y,z)] = p_1(s)p_2(u)p_3(v)\int_0^\infty\int_0^\infty\int_0^\infty g(x,y,z)\, e^{-q_1(s)x-q_2(u)y-q_3(v)z}dx\,dy\,dz. \tag{24}$$

Thus,

$$\frac{\mathcal{T}_3[g(x,y,z)]}{p_1(s)p_2(u)p_3(v)} = \int_0^\infty\int_0^\infty\int_0^\infty g(x,y,z)\, e^{-q_1(s)x-q_2(u)y-q_3(v)z}dxdydz. \tag{25}$$

Differentiating both sides of equation (25) with respect to s, we have the following equation:

$$\begin{aligned} \frac{\partial}{\partial s}\left(\frac{\mathcal{T}_3[g(x,y,z)]}{p_1(s)p_2(u)p_3(v)}\right) &= -q_1'(s)\int_0^\infty\int_0^\infty\int_0^\infty xg(x,y,z)\, e^{-q_1(s)x-q_2(u)y-q_3(v)z}dx\,dy\,dz\\ &= \frac{-q_1'(s)}{p_1(s)\,p_2(u)p_3(v)}\mathcal{T}_3[xg(x,y,z)]. \end{aligned} \tag{26}$$

Thus,

$$\mathcal{T}_3[xg(x,y,z)] = -\frac{p_1(s)}{q_1'(s)}\frac{\partial}{\partial s}\left(\frac{\mathcal{T}_3[g(x,y,z)]}{p_1(s)}\right). \tag{27}$$

□

Theorem 6. *Let $\mathcal{T}_3[g(x,y,z)]$ be the general triple transform of the function $g(x,y,z)$. Then,*

$$\mathcal{T}_3[g(x-a,y-b,z-c)H(x-a,y-b,z-c)] = e^{-q_1(s)a-q_2(u)b-q_3(v)c}G_T(s,u,v), \tag{28}$$

where $H(x,y,z)$ is the Heaviside unit step function that is defined by: $H(x-a,y-b,z-c)=1$ when $x>a, y>b, z>c$ and $H(x-a,y-b,z-c)=0$ when $x<a, y<b, z<c$.

Proof. Using the definition of the general triple transform, we obtain the following equation:

$$\begin{aligned} &\mathcal{T}_3[g(x-a,y-b,z-c)H(x-a,y-b,z-c)]\\ &= p_1(s)p_2(u)p_3(v)\int_0^\infty\int_0^\infty\int_0^\infty g(x-a,y-b,z-c)H(x-a,y-b,z-c)e^{-q_1(s)x-q_2(u)y-q_3(v)z}dx\,dy\,dz\\ &= p_1(s)p_2(u)p_3(v)\int_a^\infty\int_b^\infty\int_c^\infty g(x-a,y-b,z-c)\, e^{-q_1(s)x-q_2(u)y-q_3(v)z}dx\,dy\,dz. \end{aligned} \tag{29}$$

Putting $x-a=w_1, y-b=w_2, z-c=w_3$, we obtain the following equation:

$$\begin{aligned}
&\mathcal{T}_3[g(x-a,y-b,z-c)H(x-a,y-b,z-c)] = \\
&= p_1(s)p_2(u)p_3(v)e^{-q_1(s)a-q_2(u)b-q_3(v)c}\int_0^\infty\int_0^\infty\int_0^\infty e^{-q_1(s)w_1-q_2(u)w_2-q_3(v)w_3}g(w_1,w_2,w_2)dw_1dw_2dw_3 \\
&= e^{-q_1(s)a-q_2(u)b-q_3(v)c}G_T(s,u,v).
\end{aligned} \tag{30}$$

□

Definition 2. Let $g(x,y,z)$ and $h(x,y,z)$ be two integrable functions of three variables, then the triple convolution of $g(x,y,z)$ and $h(x,y,z)$ is given by the following equation:

$$\begin{aligned}
&g(x,y,z)***h(x,y,z) = \int_0^x\int_0^y\int_0^z g(x-a,y-b,z-c) \\
&g(a,b,c)da\,db\,dc,
\end{aligned} \tag{31}$$

where the symbol $***$ denotes the triple convolution with respect to x,y,z.

Theorem 7. *Let $G_T(s,u,v)$ and $H_T(s,u,v)$ be the general triple transforms of the functions $g(x,y,z)$ and $h(x,y,z)$, respectively; we suppose that $p_1(s)p_2(u)p_3(v)\neq 0$ for all $s,u,v>0$. Then,*

$$\begin{aligned}
&\mathcal{T}_3[g(x,y,z)***h(x,y,z)] = \frac{1}{p_1(s)p_2(u)p_3(v)} \\
&G_T(s,u,v)H_T(s,u,v).
\end{aligned} \tag{32}$$

Proof. Using the definition of the general triple transform and the triple convolution property, we have the following equation:

$$\begin{aligned}
\mathcal{T}_3[g(x,y,z)***h(x,y,z)] &= p_1(s)p_2(u)p_3(v)\int_0^\infty\int_0^\infty\int_0^\infty e^{-q_1(s)x-q_2(u)y-q_3(v)z}(g(x,y,z)***h(x,y,z))dx\,dy\,dz \\
&= p_1(s)p_2(u)p_3(v)\int_0^\infty\int_0^\infty\int_0^\infty e^{-q_1(s)x-q_2(u)y-q_3(v)z} \\
&\quad\cdot\left[\int_0^x\int_0^y\int_0^z g(x-a,y-b,z-c)h(a,b,c)da\,db\,dc\right]dx\,dy\,dz.
\end{aligned} \tag{33}$$

Using the Heaviside unit step function, we obtain the following equation:

$$\begin{aligned}
&\mathcal{T}_3[g(x,y,z)***h(x,y,z)] = p_1(s)p_2(u)p_3(v)\int_0^\infty\int_0^\infty\int_0^\infty e^{-q_1(s)x-q_2(u)y-q_3(v)z}\left[\int_0^\infty\int_0^\infty\int_0^\infty\right. \\
&\left. g(x-a,y-b,z-c)H(x-a,y-b,z-c)h(a,b,c)da\,db\,dc\right]dx\,dy\,dz.
\end{aligned} \tag{34}$$

Thus, with simple computations, we conclude that

$$\begin{aligned}
\mathcal{T}_3[g(x,y,z)***h(x,y,z)] &= \int_0^\infty\int_0^\infty h(a,b,c)e^{-q_1(s)a-q_2(u)b-q_3(v)c}G_T(s,u,v)da\,db\,dc \\
&= G_T(s,u,v)\int_0^\infty\int_0^\infty\int_0^\infty h(a,b,c)e^{-q_1(s)a-q_2(u)b-q_3(v)c}da\,db\,dc \\
&= \frac{1}{p_1(s)\,p_2(u)\,p_3(v)}G_T(s,u,v)H_T(s,u,v).
\end{aligned} \tag{35}$$

□

4. Case Studies

In this section, some applications of partial differential equations are discussed and solved by the new general triple transform. In addition, we mention herein that the values of the general triple transform of the discussed functions can be found in Tables 1 and 3.

Example 1. Consider the following homogeneous partial differential equation:

$$\frac{\partial^3 g(x,y,z)}{\partial x \partial y \partial z} + g(x,y,z) = 0, \tag{36}$$

subject to the conditions:

$$\begin{aligned} g(x,y,0) &= e^{x+y}, \\ g(x,0,z) &= e^{x-z}, \\ g(0,y,z) &= e^{y-z}, \\ g(0,0,z) &= e^{-z}, \\ g(0,y,0) &= e^{y}, \\ g(x,0,0) &= e^{x}, \\ g(0,0,0) &= 1. \end{aligned} \tag{37}$$

Applying the general triple transform $\mathcal{T}_3$ to both sides of equation (36), we obtain the following equation:

$$\mathcal{T}_3\left[\frac{\partial^3 g(x,y,z)}{\partial x \partial y \partial z}\right] + \mathcal{T}_3[g(x,y,z)] = 0. \tag{38}$$

Running the triple integral transform on equation (38), we obtain the following equation:

$$q_1(s)q_2(u)q_3(v)G_T(s,u,v) + G_T(s,u,v) = H_T(s,u,v), \tag{39}$$

where

$$\begin{aligned} H_T(s,u,v) = {} & q_1(s)q_2(u)p_3(v)G_D(s,u,0) + q_1(s)q_3(v)p_2(u)G_D(s,0,v) \\ & + q_2(u)q_3(v)p_1(s)G_D(0,u,v) - q_1(s)p_2(u)p_3(v)G_S(s,0,0) \\ & - q_2(u)p_1(s)p_3(v)G_S(0,u,0) - q_3(v)p_1(s)p_2(u)G_S(0,0,v) \\ & + p_1(s_1)p_2(s_2)p_3(s_3)g(0,0,0). \end{aligned} \tag{40}$$

Simplifying equation (39), we have the following equation:

$$G_T(s,u,v) = \frac{H_T(s,u,v)}{1 + q_1(s)q_2(u)q_3(v)}. \tag{41}$$

Applying the inverse triple transform $\mathcal{T}_3^{-1}$, we get the solution of equation (36) in the original space

$$\begin{aligned} g(x,y,z) &= \mathcal{T}_3^{-1}\left[\frac{H_T(s,u,v)}{1 + q_1(s)q_2(u)q_3(v)}\right] \\ &= \mathcal{T}_3^{-1}\left[\frac{1}{q_1(s)-1}\,\frac{1}{q_2(u)-1}\,\frac{1}{q_3(v)+1}\right] = e^{x+y-z}, \end{aligned} \tag{42}$$

which is the exact solution of equations (36) and (37).

Example 2. Consider the following partial differential equation:

$$\frac{\partial^3 g(x,y,z)}{\partial x \partial y \partial z} + g(x,y,z) = -e^{x-2y+z}, \tag{43}$$

subject to the conditions:

$$\begin{aligned} g(x,y,0) &= e^{x-2y}, \\ g(x,0,z) &= e^{x+z}, \\ g(0,y,z) &= e^{-2y+z}, \\ g(0,0,z) &= e^{z}, \\ g(0,y,0) &= e^{-2y}, \\ g(x,0,0) &= e^{x}, \\ g(0,0,0) &= 1. \end{aligned} \tag{44}$$

Applying the general triple transform $\mathcal{T}_3$ to both sides of equation (43) to obtain the following equation:

$$\mathcal{T}_3\left[\frac{\partial^3 g(x,y,z)}{\partial x \partial y \partial z}\right] + \mathcal{T}_3[g(x,y,z)] = \mathcal{T}_3\left[-e^{x-2y+z}\right]. \tag{45}$$

Running the general triple transform in equation (45), we obtain the following equation:

$$q_1(s)q_2(u)q_3(v)G_T(s,u,v) + G_T(s,u,v) = H_T(s,u,v) - \frac{p_1(s)}{q_1(s)-1}\frac{p_2(u)}{q_2(u)+2}\frac{p_3(v)}{q_3(v)-1}. \tag{46}$$

Simplifying equation (46), we have the following equation:

$$G_T(s,u,v) = \frac{p_1(s_1)}{q_1(s_1)-1}\frac{p_2(s_2)}{q_2(s_2)+2}\frac{p_3(s_3)}{q_3(s_3)-1}. \tag{47}$$

Applying the inverse triple transform $\mathcal{T}_3^{-1}$, and after simple calculations, we get the solution of equation (43) in the original space

$$g(x,y,z) = \mathcal{T}_3^{-1}\left[\frac{p_1(s_1)}{q_1(s_1)-1}\frac{p_2(s_2)}{q_2(s_2)+2}\frac{p_3(s_3)}{q_3(s_3)-1}\right] = e^{x-2u+v}, \tag{48}$$

which is the exact solution of equation (43).

Example 3. Consider the following partial differential equation:

$$\frac{\partial^3 g(x,y,z)}{\partial x\partial y\partial z} + g(x,y,z) = \cos x\cos y\cos z - \sin x\sin y\sin z, \tag{49}$$

subject to the conditions:

$$\begin{aligned} g(x,y,0) &= \cos x\cos y,\\ g(x,0,z) &= \cos x\cos z,\\ g(0,y,z) &= \cos y\cos z,\\ g(0,0,z) &= \cos z,\\ g(0,y,0) &= \cos y,\\ g(x,0,0) &= \cos x,\\ g(0,0,0) &= 1. \end{aligned} \tag{50}$$

Applying the general triple transform $\mathcal{T}_3$ to both sides of equation (49), we obtain the following equation:

$$\mathcal{T}_3\left[\frac{\partial^3 g(x,y,z)}{\partial x\partial y\partial z}\right] + \mathcal{T}_3[g(x,y,z)] = \mathcal{T}_3[\cos x\cos y\cos z - \sin x\sin y\sin z]. \tag{51}$$

Running $\mathcal{T}_3$ in equation (51), to obtain the following equation:

$$q_1(s)q_2(u)q_3(v)G_T(s,u,v) + G_T(s,u,v) = H_T(s,u,v) + \frac{p_1(s)q_1(s)}{q_1^2(s)+1}\frac{p_2(u)q_2(u)}{q_2^2(u)+2}\frac{p_3(v)q_3(v)}{q_3^2(v)+1} - \frac{p_1(s)}{q_1^2(s)+1}\frac{p_2(u)}{q_2^2(u)+2}\frac{p_3(v)}{q_3^2(v)+1}. \tag{52}$$

Simplifying equation (52), we have the following equation:

$$G_T(s,u,v) = \frac{p_1(s)q_1(s)}{q_1^2(s)+1}\frac{p_2(u)q_2(u)}{q_2^2(u)+2}\frac{p_3(v)q_3(v)}{q_3^2(v)+1}. \tag{53}$$

Applying the inverse triple transform $\mathcal{T}_3^{-1}$, and after simple calculations, we get the solution of equation (49) in the original space

$$g(x,y,z) = \mathcal{T}_3^{-1}\left[\frac{p_1(s)q_1(s)}{q_1^2(s)+1}\frac{p_2(u)q_2(u)}{q_2^2(u)+2}\frac{p_3(v)q_3(v)}{q_3^2(v)+1}\right] = \cos x\cos y\cos z, \tag{54}$$

which is the exact solution of equation (49).

Example 4. Consider the following Volterra partial integro-differential equations:

$$\frac{\partial g(x,y,z)}{\partial x} + \frac{\partial g(x,y,z)}{\partial y} + \frac{\partial g(x,y,z)}{\partial z} = -(xyz)^2 + 4xy + 4xz + 4yz + 2\int_0^z\int_0^y\int_0^x g(a,b,c)\,da\,db\,dc, \tag{55}$$

subject to the conditions:

$$g(x,y,0) = g(x,0,z) = g(0,y,z) = 0. \tag{56}$$

Applying the general triple transform $\mathcal{T}_3$, to both sides of equation (55), we obtain the following equation:

$$\begin{aligned} &q_1(s)G_T(s,u,v) - p_1(s)G_D(0,u,v) + q_2(u)G_T(s,u,v) - p_2(u)G_D(s,0,v) \\ &+ q_3(v)G_T(s,u,v) - p_3(v)G_D(s,u,0) = -8\frac{p_1(s)p_2(u)p_3(v)}{q_1^3(s)q_2^3(u)q_3^3(v)} \\ &+ 4\frac{p_1(s)p_2(u)p_3(v)}{q_1^2(s)q_2^2(u)q_3(v)} + 4\frac{p_1(s)p_2(u)p_3(v)}{q_1^2(s)q_2(u)q_3^2(v)} \\ &+ 4\frac{p_1(s)p_2(u)p_3(v)}{q_1(s)q_2^2(u)q_3^2(v)} + 2\frac{G_T(s,u,v)}{q_1(s)q_2(u)q_3(v)}. \end{aligned} \tag{57}$$

Simplifying equation (57), and substituting the transformed conditions, we have the following equation:

$$G_T(s,u,v) = 4\frac{p_1(s)p_2(u)p_3(v)}{q_1^2(s)q_2^2(u)q_3^2(v)}. \tag{58}$$

Applying the inverse triple transform $\mathcal{T}_3^{-1}$ on equation (58), we get the solution of equation (55) in the original space

$$g(x,y,z) = \mathcal{T}_3^{-1}\left[4\frac{p_1(s)p_2(u)p_3(v)}{q_1^2(s)q_2^2(u)q_3^2(v)}\right] = 4xyz, \tag{59}$$

which is the exact solution of equation (55).

Example 5. Consider the following three-dimensional Poisson equation:

$$\frac{\partial^2 g(x,y,z)}{\partial x^2} + \frac{\partial^2 g(x,y,z)}{\partial y^2} + \frac{\partial^2 g(x,y,z)}{\partial z^2} = 2\sin x\cos y\sinh 2z, \tag{60}$$

where x, y and z are positive real numbers. Subject to the initial conditions

$$\begin{aligned} &g(0,y,z) = 0, \\ &g(x,0,z) = \sin x\sinh 2z, \\ &g(x,y,0) = 0, \\ &\frac{\partial g(0,y,z)}{\partial x} = \cos y\sinh 2z, \\ &\frac{\partial g(x,0,z)}{\partial y} = 0, \\ &\frac{\partial g(x,y,0)}{\partial z} = 2\sin x\cos y. \end{aligned} \tag{61}$$

We apply the general triple transform to equation (60) and compute the transformed values of the initial conditions, with simple calculations, one can get the following equation:

$$G_T(s,u,v) = \frac{p_1(s)}{q_1^2(s)+1}\,\frac{p_2(u)q_2(u)}{q_2^2(u)+1}\,\frac{2p_3(v)}{q_3^2(v)+4}. \tag{62}$$

Following that, we get the solution of Example 5 in the original space, after applying the inverse triple transform to equation (62), as follows:

$$\begin{aligned} g(x,y,z) &= \mathcal{T}_3^{-1}\left[\frac{p_1(s)}{q_1^2(s)+1}\,\frac{p_2(u)q_2(u)}{q_2^2(u)+1}\,\frac{2p_3(v)}{q_3^2(v)+4}\right] \\ &= \sin x\cos y\sinh 2z. \end{aligned} \tag{63}$$

Example 6. Consider the heat equation of the form

$$\frac{\partial g(x,y,z)}{\partial z} = \frac{\partial^2 g(x,y,z)}{\partial x^2} + \frac{\partial^2 g(x,y,z)}{\partial y^2} + 2\cos(x+y), \tag{64}$$

where x, y are real numbers and $z > 0$. Subject to the conditions:

$$\begin{aligned} &g(0,y,z) = e^{-2z}\sin y + \cos y, \\ &g(x,0,z) = e^{-2z}\sin x + \cos x, \\ &g(x,y,0) = \sin(x+y) + \cos(x+y), \\ &\frac{\partial g(0,y,z)}{\partial x} = e^{-2z}\cos y - \sin y \end{aligned} \tag{65}$$

We apply the general triple transform to equation (64) and compute the transformed values of the initial conditions (65), with simple calculations, one can obtain the following equation:

$$\begin{aligned} G_T(s,u,v) = &\frac{p_3(v)}{q_3(v)+2}\,\frac{p_1(s)p_2(u)\left(q_1(s)+q_2(u)\right)}{\left(q_1^2(s)+1\right)\left(q_2^2(u)+1\right)} \\ &+ \frac{p_1(s)p_2(u)\left(q_1(s)q_2(u)-1\right)}{\left(q_1^2(s)+1\right)\left(q_2^2(u)+1\right)}. \end{aligned} \tag{66}$$

Following that, we get the solution of Example 6 in the original space, after applying the inverse triple transform to equation (66), as follows:

Table 3: The general triple transforms for some functions.

$g(x,y,z)$	$\mathcal{T}_3[g(x,y,z)] = G_T(s,u,v)$
$a \in R$	$a p_1(s)/q_1(s) p_2(u)/q_2(u) p_3(v)/q_3(v)$
xyz	$p_1(s)/q_1^2(s) p_2(u)/q_2^2(u) p_3(v)/q_3^2(v)$
$e^{ax+by+cz}$	$p_1(s)/q_1(s) - a p_2(u)/q_2(u) - b\, p_3(v)/q_3(v) - c, q_1(u) > a, q_2(u) > b, q_3(v) > c$
$\cos ax \cos by \cos cz$	$p_1(s) q_1(s)/q_1^2(s) + a^2\ p_2(u) q_2(u)/q_2^2(u) + b^2\ p_3(v) q_3(v)/q_3^2(v) + c^2$
$\sin ax \sin by \sin cz$	$a\, p_1(s)/q_1^2(s) + a^2 b\, p_2(u)/q_2^2(u) + b^2 c\, p_3(v)/q_3^2(v) + c^2$
$x^\alpha y^\beta z^\gamma$	$p_1(s) p_2(u) p_3(v) \Gamma(\alpha+1)/q_1^{\alpha+1}(s) \Gamma(\beta+1)/q_2^{\beta+1}(u) \Gamma(\gamma+1)/q_3^{\gamma+1}(v), \forall \alpha, \beta, \gamma > -1$
$f(x) g(y) h(z)$	$F_S(s) G_S(u) H_S(v)$
$f(x)$	$F_S(s) p_2(u)/q_2(u)\, p_3(v)/q_3(v)$
$f(x) g(y)$	$F_S(s) G_S(u)\, p_3(v)/q_3(v)$

$$g(x,y,z) = \mathcal{T}_3^{-1}\left[\frac{p_3(v)}{q_3(v)+2} \frac{p_1(s) p_2(u)\left(q_1(s) + q_2(u)\right)}{\left(q_1^2(s)+1\right)\left(q_2^2(u)+1\right)} + \frac{p_1(s) p_2(u)\left(q_1(s) q_2(u) - 1\right)}{\left(q_1^2(s)+1\right)\left(q_2^2(u)+1\right)}\right] = e^{-2z} \sin(x+y) + \cos(x+y). \tag{67}$$

5. Conclusion

In this research, we introduced a new general triple integral transform, that generalizes almost all three-dimensional integral transforms. Several properties of the proposed triple transform were presented, discussed, and proved; also, the dualities to other transforms were illustrated. New relations related to partial derivatives and triple convolution were established. To show the applicability of the new approach some examples on partial differential equations were solved via the new transform. Our goal in this research is achieved, and in the future, we intend to solve fractional partial differential equations and fractional integral equations and generalize new formulas related to fractional operators and vector-valued solutions with applications on matrices such as [39–42].

Data Availability

No underlying data are applicable in this study.

Conflicts of Interest

The author declares there are no conflicts of interest.

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