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Research Article

Some Results on Neutrosophic Group

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We present the concept of a neutrosophic group without the use of an indeterminate element "*I*" in this paper. We also present a similar application to the fundamentals of group theory. We define and study the so-called level subgroups of a neutrosophic subgroup in order to characterize neutrosophic subgroups of finite cyclic groups in a similar way.

1. Introduction

In algebraic structures, groups play a vital role [1, 2] and have applications in several fields. The notion of neutrosophic sets (NS) was first put forth by Smarandache in [3]. Recently, it attracted significant attention from researchers working in several fields such as in analysis [4, 5], topology [6], algebraic structure [7–9], and graph theory [10] as well as applications ranging from sentiment analysis to medical diagnostics [11].

Smarandache and Kandasamy introduced the idea of neutrosophic algebraic structures using neutrosophic theory in [12] by putting an indeterminate element, "*I*," into the algebraic structure. The concept of the neutrosophic group was further studied by Agboola et al. [13].

So far, researchers have described the basic algebraic operations of neutrosophic sets from three distinct viewpoints [14]. In [15], Vildan and Halis introduced an approach to the neutrosophic subgroup according to the second viewpoint.

In this research, we introduced the neutrosophic group as the algebraic structure by not inserting an indeterminate element "*I*." Also, this approach corresponds with the first viewpoint and represents an extension of the fuzzy subgroup. Again, this approach represents an extension of the fuzzy subgroup.

The work has been conceived in the manner as follows.

In Section 2, some basic concepts are established. The novel definition of a neutrosophic group as an expansion of the fuzzy group is introduced in Section 3. Moreover, we obtain a characterization of neutrosophic subgroups of finite cyclic groups. In order to do this, we study the "level subgroups" of a neutrosophic subgroup. In Section 4, the conclusions and observations are made.

2. Basic Concepts

Here, we go through some of the concepts and results that we use in the following sections.

Definition 1 (see [16]). Assume that X is a set. Function $P\colon X \longrightarrow [0,1]$ is a fuzzy subset of X.

Definition 2 (see [17]). Assume that $\mathcal{G}$ is a group. Then, P is called a fuzzy subgroup of $\mathcal{G}$ if the following axioms are satisfied:

(i) $P(uv) \geqslant \min(P(u), P(v))$
(ii) $P(u^{-1}) \geqslant P(u)$

where $u, v \in \mathcal{G}$.

Definition 3 (see [16]). Let P be a fuzzy subset of X, then we can define the level subset of the fuzzy subset P as follows:

$$P_t = \{u \in X\colon P(u) \geq t\}, \quad (1)$$

where $t \in [0,1]$. Note that if P is a fuzzy subgroup of $\mathcal{G}$, then for all $u \in \mathcal{G}$ and e is an identity of $\mathcal{G}$, we have $P(u) \leq P(e)$.

Definition 4 (see [3]). A neutrosophic set $\mathcal{N}$ on the universe set $\mathbb{N}$ is defined as follows:

$$\mathcal{N} = \{<u, \mu(u), \gamma(u), \zeta(u)> : u \in \mathbb{N}\}, \quad (2)$$

with $\mu, \gamma, \zeta\colon \mathbb{N} \longrightarrow [0,1]$.

Now, we define the intersection and union of two neutrosophic sets according to the first viewpoint.

Definition 5 (see [3]). Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two neutrosophic sets on E. Then,

(1) $\mathcal{N}_1 \cap_1 \mathcal{N}_2 = \{<\kappa, \min(\mu_1(\kappa), \mu_2(\kappa)), \max(\gamma_1(\kappa), \gamma_2(\kappa)), \max(\zeta_1(\kappa), \zeta_2(\kappa))> : \kappa \in E\}$

(2) $\mathcal{N}_1 \cup_1 \mathcal{N}_2 = \{<\kappa, \max(\mu_1(\kappa), \mu_2(\kappa)), \min(\gamma_1(\kappa), \gamma_2(\kappa)), \min(\zeta_1(\kappa), \zeta_2(\kappa))> : \kappa \in E\}$

3. Main Result

First, we present a novel definition of a neutrosophic group in this section. In addition, we define the level set of the neutrosophic set.

Definition 6. Presume that $\mathcal{G}$ is a group. A neutrosophic subset $\mathcal{M} = \{<\kappa, \mu(\kappa), \gamma(\kappa), \zeta(\kappa)> : \kappa \in \mathcal{G}\}$ of $\mathcal{G}$ is called a neutrosophic subgroup of $\mathcal{G}$ if the following axioms are satisfied:

(i) $\mu(\kappa\iota) \geqslant \min(\mu(\kappa), \mu(\iota))$

(ii) $\mu(\kappa^{-1}) \geqslant \mu(\kappa)$

(iii) $\gamma(\kappa\iota) \leqslant \max(\gamma(\kappa), \gamma(\iota))$

(iv) $\gamma(\kappa^{-1}) \leqslant \gamma(\kappa)$

(v) $\zeta(\kappa\iota) \leqslant \max(\zeta(\kappa), \zeta(\iota))$

(vi) $\zeta(\kappa^{-1}) \leqslant \zeta(\kappa)$

where $\kappa, \iota \in \mathcal{G}$.

Proposition 7. *The intersection of the finite set of neutrosophic subgroups is a neutrosophic subgroup.*

Proof. We only verify (iii) and (iv) axioms in Definition 6, as the other axioms are well-known.

$$\begin{aligned}(iii)\quad [\cap_1 \gamma_i](ab) &= \inf[\gamma_i(ab)] \\ &\leq \sup[\max(\gamma_i(a), \gamma_i(b))] \\ &= \max(\sup\gamma_i(a), \sup\gamma_i(b)) \\ &= \max([\cap_1 \gamma_i](a), [\cap_1 \gamma_i](b)),\end{aligned} \quad (3)$$

$$\begin{aligned}(iv)\quad [\cap_1 \gamma_i]\left(a^{-1}\right) &= \sup\left[\gamma_i\left(a^{-1}\right)\right] \\ &\leq \sup[\gamma_i(a)] \\ &= [\cap_1 \gamma_i](a),\end{aligned}$$

where $i = 1, 2, \ldots, n$. Hence, the proposition is claimed. □

Proposition 8. *Let $\mathcal{M}$ be a neutrosophic subgroup of $\mathcal{G}$. Then,*

(i) $\mu(\kappa^{-1}) = \mu(\kappa)$ and $\mu(\kappa) \leq \mu(e)$

(ii) $\gamma(\kappa^{-1}) = \gamma(\kappa)$ and $\gamma(\kappa) \geq \gamma(e)$

(iii) $\zeta(\kappa^{-1}) = \zeta(\kappa)$ and $\zeta(\kappa) \geq \zeta(e)$

where e is an identity of $\mathcal{G}$.

Proof. We only explain (ii), as the other cases are widely known. Suppose that $\kappa \in \mathcal{G}$, then

$$\gamma(\kappa) = \gamma\left(\left(\kappa^{-1}\right)^{-1}\right) \leq \gamma\left(\kappa^{-1}\right) \leq \gamma(\kappa). \quad (4)$$

Therefore, $\gamma(\kappa) = \gamma(\kappa^{-1})$.

Also, we can prove $\gamma(\kappa) \leq \gamma(e)$ as follows:

$$\gamma(e) = \gamma\left(\kappa\kappa^{-1}\right) \leq \max\left(\gamma(\kappa), \gamma\left(\kappa^{-1}\right)\right) = \gamma(\kappa). \quad (5)$$

□

Proposition 9. *Let $\mathcal{M}$ be a neutrosophic subgroup of $\mathcal{G}$. Then,*

(i) $\mu(\kappa\iota^{-1}) = \mu(e) \Longrightarrow \mu(\kappa) = \mu(\iota)$

(ii) $\gamma(\kappa\iota^{-1}) = \gamma(e) \Longrightarrow \gamma(\kappa) = \gamma(\iota)$

(iii) $\zeta(\kappa\iota^{-1}) = \zeta(e) \Longrightarrow \zeta(\kappa) = \zeta(\iota)$

where $\kappa, \iota \in \mathcal{G}$, and e is an identity of $\mathcal{G}$.

Proof. We only explain (ii), as the other cases are widely known. Presume that $\kappa, \iota \in \mathcal{G}$, then

$$\begin{aligned}\gamma(\kappa) &= \gamma\left(\left(\kappa\iota^{-1}\right)\iota\right) \\ &\leq \max\left(\gamma\left(\kappa\iota^{-1}\right), \gamma(\iota)\right) \\ &= \gamma(\iota).\end{aligned} \quad (6)$$

On the other hand,

$$\begin{aligned}\gamma(\iota) &= \gamma\left(\left(\iota\kappa^{-1}\right)\kappa\right) \\ &\leq \max\left(\gamma\left(\iota\kappa^{-1}\right), \gamma(\kappa)\right) \\ &= \gamma(\kappa).\end{aligned} \quad (7)$$

Thus, $\gamma(\kappa) = \gamma(\iota)$. □

Proposition 10. *$\mathcal{M}$ is a neutrosophic subgroup of $\mathcal{G}$ if the following axioms are satisfied:*

(i) $\mu(\kappa\iota^{-1}) \geq \min(\mu(\kappa), \mu(\iota))$

(ii) $\gamma(\kappa\iota^{-1}) \leq \max(\gamma(\kappa), \gamma(\iota))$

(iii) $\zeta(\kappa\iota^{-1}) \leq \max(\zeta(\kappa), \zeta(\iota))$

where $\kappa, \iota \in \mathcal{G}$.

Proof. Let $\mathcal{M}$ be a neutrosophic subgroup of $\mathcal{G}$, then we have

(i) $\mu(\kappa\iota^{-1}) \geq \min(\mu(\kappa), \mu(\iota^{-1})) = \min(\mu(\kappa), \mu(\iota))$

(ii) $\gamma(\kappa\iota^{-1}) \leq \max(\gamma(\kappa), \gamma(\iota^{-1})) = \max(\gamma(\kappa), \gamma(\iota))$

(iii) $\zeta(\kappa\iota^{-1}) \le \max(\zeta(\kappa), \zeta(\iota^{-1})) = \max(\zeta(\kappa), \zeta(\iota))$

Conversely, assume that the above axioms are true, and let $\kappa = \iota$, then we obtain $\mu(e) \ge \mu(\kappa)$, $\gamma(e) \le \gamma(\kappa)$, and $\zeta(e) \le \zeta(\kappa)$. Therefore,

$$\begin{aligned} \mu(\kappa^{-1}) &= \mu(\kappa^{-1}e) \ge \min(\mu(\kappa), \mu(e)) = \mu(\kappa), \\ \gamma(\kappa^{-1}) &= \gamma(\kappa^{-1}e) \le \max(\gamma(\kappa), \gamma(e)) = \gamma(\kappa), \\ \zeta(\kappa^{-1}) &= \zeta(\kappa^{-1}e) \le \max(\zeta(\kappa), \zeta(e)) = \zeta(\kappa). \end{aligned} \tag{8}$$

Again, we have

$$\begin{aligned} \mu(\kappa\iota) &= \mu\left(\kappa(\iota^{-1})^{-1}\right) \ge \min\left(\mu(\kappa), \mu(\iota^{-1})\right) \\ &\ge \min(\mu(\kappa), \mu(\iota)), \\ \gamma(\kappa\iota) &= \gamma\left(\kappa(\iota^{-1})^{-1}\right) \le \max\left(\gamma(\kappa), \gamma(\iota^{-1})\right) \\ &\le \max(\gamma(\kappa), \gamma(\iota)), \\ \zeta(\kappa\iota) &= \zeta\left(\kappa(\iota^{-1})^{-1}\right) \le \max\left(\zeta(\kappa), \zeta(\iota^{-1})\right) \\ &\le \max(\zeta(\kappa), \zeta(\iota)). \end{aligned} \tag{9}$$

Hence, the proposition is claimed.

The following result is true as a fuzzy group. □

Proposition 11. *Any two neutrosophic subgroups that are union do not form a neutrosophic subgroup.*

We prove Proposition 11 by the following example.

Example 1. We consider the group $Z_8 = \{0, 1, \ldots, 7\}$ under addition modulo 8, and let $\mathscr{H}_1 = \{0, 1, 7\}$ be a subgroup with $\mu_1(0) = 1$, $\mu_1(1) = \mu_1(7) = 0.9$, $\gamma_1(0) = 0.8$, $\gamma_1(1) = \gamma_1(7) = 0.9$, $\zeta_1(0) = 0.7$, and $\zeta_1(1) = \zeta_1(7) = 0.8$. Also, let $\mathscr{H}_2 = \{0, 2, 6\}$ be a subgroup with $\mu_2(0) = 1$, $\mu_2(2) = \mu_2(6) = 0.8$, $\gamma_2(0) = 0.7$, $\gamma_2(2) = \gamma_2(6) = 0.8$, $\zeta_2(0) = 0.7$, and $\zeta_2(2) = \zeta_2(6) = 0.9$. Clearly, $(\mathscr{H}_1, \mu_1, \gamma_1, \zeta_1)$ and $(\mathscr{H}_2, \mu_2, \gamma_2, \zeta_2)$ are neutrosophic subgroups but the union of them is not a neutrosophic subgroup.

Proposition 12. *Let $\mathscr{G}_p$ be a cyclic group with order p (p is a prime), then $\mathscr{M}$ is a neutrosophic subgroup of $\mathscr{G}_p$ if the following is held: $\mu(\kappa) = \mu(1) \ge \mu(0)$, $\gamma(\kappa) = \gamma(1) \le \gamma(0)$, and $\zeta(\kappa) = \zeta(1) \le \zeta(0)$, for all $\kappa \in \mathscr{G}_p$ and $\kappa \ne 0$.*

Proof. Assume the above conditions are satisfied, then all axioms in Definition 6 are verified, so $\mathscr{M}$ is a neutrosophic subgroup of $\mathscr{G}_p$. Conversely, for any $\kappa \ne 0$ and $\iota \ne 0$ in $\mathscr{G}_p$, it holds that $\kappa = m\iota$ and $\iota = n\kappa$ where m, n are integers. Therefore, we have $\mu(\kappa) \ge \mu(\iota) \ge \mu(\kappa)$, $\gamma(\kappa) \le \gamma(\iota) \le \gamma(\kappa)$, and $\zeta(\kappa) \le \zeta(\iota) \le \zeta(\kappa)$. The proposition is claimed. □

3.1. Level Subgroups. Now, it is useful to define the terminology level set of the neutrosophic subset.

Definition 13. Assume that $\mathscr{M}$ is a neutrosophic subset of $\mathscr{N}$. For $\alpha \in [0, 1]$, the following set:

$$\begin{aligned} \mathscr{M}_\alpha = \{&<\kappa, \mu(\kappa), \gamma(\kappa), \zeta(\kappa)>, \kappa \in \mathscr{G}: \mu(\kappa) \\ &\ge \alpha, \gamma(\kappa) \le \alpha, \zeta(\kappa) \le \alpha\}, \end{aligned} \tag{10}$$

is labeled as a level subset of $\mathscr{M}$.

Theorem 14. *Presume that $\mathscr{G}$ is a group with identity e and $\mathscr{M}$ is a neutrosophic subgroup of $\mathscr{G}$, then the level subset $\mathscr{M}_\alpha$, for $\alpha \in [0, 1], \alpha \leqslant \mu(e), \alpha \geqslant \gamma(e)$, and $\alpha \geqslant \zeta(e)$, is a subgroup of $\mathscr{G}$.*

Proof. Clearly, $\mathscr{M}_\alpha$ is nonempty. Suppose that $\kappa, \iota \in \mathscr{M}_\alpha$, then $\mu(\kappa) \geqslant \alpha$, $\mu(\iota) \geqslant \alpha$, $\gamma(\kappa) \leqslant \alpha$, $\gamma(\iota) \leqslant \alpha$, $\zeta(\kappa) \leqslant \alpha$, and $\zeta(\iota) \leqslant \alpha$. Since $\mathscr{M}$ is a neutrosophic subgroup of $\mathscr{G}$, then the axioms (i), (iii), and (v) in Definition 6 are satisfied. This leads to $\mu(\kappa\iota) \geqslant \alpha$, $\gamma(\kappa\iota) \leqslant \alpha$, and $\zeta(\kappa\iota) \leqslant \alpha$. Hence, $<\kappa\iota, \mu(\kappa\iota), \gamma(\kappa\iota), \zeta(\kappa\iota)> \in \mathscr{M}_\alpha$. Also, since $\mathscr{M}$ is a neutrosophic subgroup of $\mathscr{G}$, then the axioms (ii), (iv), and (vi) in Definition 6 are satisfied, and this leads to $\mu(\kappa^{-1}) \geqslant \alpha$, $\gamma(\kappa^{-1}) \leqslant \alpha$, and $\zeta(\kappa^{-1}) \leqslant \alpha$. This means $<\kappa^{-1}, \mu(\kappa^{-1}), \gamma(\kappa^{-1}), \zeta(\kappa^{-1})> \in \mathscr{M}_\alpha$. Therefore, $\mathscr{M}_\alpha$ is a subgroup of $\mathscr{G}$. □

Example 2. Consider the classical group $Z_3 = \{0, 1, 2\}$ under addition modulo 3. We define a neutrosophic subgroup $\mathscr{M}$ of Z_3 as follows:

$$\mathscr{M} = \{<0, 1, 0.7, 0.3>, <1, 0.9, 0.8, 0.5>, <2, 0.9, 0.8, 0.7>\}. \tag{11}$$

Let $\alpha = 0.7$, then by Definition 13, we get a level subset of $\mathscr{M}$ as follows:

$$\mathscr{M}_\alpha = \{<0, 1, 0.7, 0.3>\}. \tag{12}$$

Since $0.7 \leqslant \mu(0) = 1$, $0.7 \geqslant \gamma(0) = 0.7$, and $0.7 \geqslant \zeta(0) = 0.3$, then $\mathscr{M}_\alpha$ is a subgroup of Z_3.

Theorem 15. *Presume that $\mathscr{G}$ is a group with identity e and $\mathscr{M}$ be a neutrosophic subset of $\mathscr{G}$ such that $\mathscr{M}_\alpha$ is a subgroup of $\mathscr{M}$ for all $\alpha \in [0, 1], \alpha \leqslant \mu(e), \alpha \geqslant \gamma(e)$, and $\alpha \geqslant \zeta(e)$, then $\mathscr{M}$ is a neutrosophic subgroup of $\mathscr{G}$.*

Proof. Suppose that $a, b \in \mathscr{G}$ with $\mathscr{M}(a) = \alpha_1$ and $\mathscr{M}(b) = \alpha_2$. Then, $a \in \mathscr{M}_{\alpha_1}$ and $b \in \mathscr{M}_{\alpha_2}$, i.e., $\mu(a) \ge \alpha_1$, $\zeta(a) \le \alpha_1$, $\gamma(a) \le \alpha_1$, $\mu(b) \ge \alpha_2$, $\zeta(b) \le \alpha_2$, and $\gamma(b) \le \alpha_2$. Let us assume $\alpha_1 < \alpha_2$. Then, it follows $\mathscr{M}_{\alpha_2} \subseteq \mathscr{M}_{\alpha_1}$. So, $b \in \mathscr{M}_{\alpha_1}$. Thus, a, $b \in \mathscr{M}_{\alpha_1}$, and since $\mathscr{M}_{\alpha_1}$ is a subgroup of $\mathscr{G}$, by hypothesis, $ab \in \mathscr{M}_{\alpha_1}$. Thus, $\mu(ab) \geqslant \alpha_1 = \min(\mu(a), \mu(b))$, $\gamma(ab) \leqslant \alpha_1 = \max(\gamma(a), \gamma(b))$, and $\zeta(ab) \leqslant \alpha_1 = \max(\zeta(a), \zeta(b))$. Then, suppose that $a \in \mathscr{G}$ and let $\mathscr{M}(a) = \alpha$. Then, $a \in \mathscr{M}_\alpha$. Since $\mathscr{M}_\alpha$ is a subgroup, $a^{-1} \in \mathscr{M}_\alpha$. Therefore, $\mu(a^{-1}) \geqslant \alpha$, $\gamma(a^{-1}) \leqslant \alpha$, and $\zeta(a^{-1}) \leqslant \alpha$ and hence $\mu(a^{-1}) \geqslant \mu(a)$, $\gamma(a^{-1}) \leqslant \gamma(a)$, and $\zeta(a^{-1}) \leqslant \zeta(a)$. Therefore, $\mathscr{M}$ is a fuzzy subgroup of $\mathscr{G}$. □

Example 3. Again, consider the classical group $Z_3 = \{0, 1, 2\}$ under addition modulo 3 and let $\mathscr{M}$ be a subset of Z_3 as follows:

$$\mathscr{M} = \{<0, 0.9, 0.1, 0.2>, <1, 0.8, 0.4, 0.3>, <2, 0.9, 0.6, 0.5>\}. \tag{13}$$

Let $\alpha = 0.4$, then by Definition 13, we get a level subset of $\mathscr{M}$ as follows:

$$\mathscr{M}_\alpha = \{<0, 1, 0.7, 0.3>\}. \tag{14}$$

It is easy to check that $\mathscr{M}_\alpha$ is a subgroup of $\mathscr{M}$. Also, we find $0.4 \leqslant \mu(0) = 0.9, 0.4 \geqslant \gamma(0) = 0.1$, and $0.4 \geqslant \zeta(0) = 0.2$. Thus, from Theorem 15, we find $\mathscr{M}$ is a neutrosophic subgroup of Z_3.

Definition 16. Presume that $\mathscr{G}$ is a group and $\mathscr{M}$ is a neutrosophic subgroup of $\mathscr{G}$. Then, the subgroups $\mathscr{M}_\alpha$ with $\alpha \in [0, 1]$ and $\alpha \leqslant \mu(e), \alpha \geqslant \gamma(e)$ and $\alpha \geqslant \zeta(e)$ are called level subgroups of $\mathscr{M}_\alpha$.

The number of subgroups of $\mathscr{G}$ must also be finite when $\mathscr{G}$ is a finite group. A neutrosophic subgroup $\mathscr{M}$ appears to have an infinite number of level subgroups. However, not all of these level subgroups are distinct because they are all in fact subgroups of $\mathscr{G}$. This property is characterized by the subsequent theorem.

Theorem 17. *Assume that $\mathscr{G}$ is a group and $\mathscr{M}$ be a neutrosophic subgroup of $\mathscr{G}$. Then, the two level subgroups $\mathscr{M}_{\alpha_1}$ and $\mathscr{M}_{\alpha_2}$ with $\alpha_1 < \alpha_2$ of $\mathscr{M}$ are equal if there is no $a \in \mathscr{G}$ such that $\alpha_1 < \mu(a) < \alpha_2$, $\alpha_2 > \alpha_1 > \gamma(a)$, and $\alpha_2 > \alpha_1 > \zeta(a)$.*

Proof. Suppose that $\mathscr{M}_{\alpha_1}$ and $\mathscr{M}_{\alpha_2}$ are equal, then $a \in \mathscr{G}$ with $\alpha_1 < \mu(a) < \alpha_2$, $\alpha_1 > \gamma(a), \gamma(a) < \alpha_2$, $\alpha_1 > \zeta(a)$, and $\zeta(a) < \alpha_2$. Thus, $\mathscr{M}_{\alpha_1} \subset \mathscr{M}_{\alpha_2}$, where $a \in \mathscr{M}_{\alpha_1}$ but $a \notin \mathscr{M}_{\alpha_2}$ which gives a contradiction. In the other direction, assume that there are no $a \in \mathscr{G}$ with $\alpha_1 < \mu(a) < \alpha_2$, $\alpha_1 > \gamma(a), \gamma(a) < \alpha_2$, $\alpha_1 > \zeta(a)$, and $\zeta(a) < \alpha_2$. Since $\alpha_1 < \alpha_2$, we have $\mathscr{M}_{\alpha_2} \subseteq \mathscr{M}_{\alpha_1}$. Let $a \in \mathscr{M}_{\alpha_1}$, then $\alpha_1 \leq \mu(a)$, $\alpha_1 \geq \gamma(a)$, and $\alpha_1 \geq \zeta(a)$ which gives $\alpha_2 \leq \mu(a)$, $\alpha_2 \geq \gamma(a)$, and $\alpha_2 \geq \zeta(a)$, where $\mu(a), \gamma(a)$, and $\zeta(a)$ do not lie between α_1 and α_2. Therefore, $a \in \mathscr{M}_{\alpha_2}$. So, we have $\mathscr{M}_{\alpha_1} \subseteq \mathscr{M}_{\alpha_2}$. Thus, $\mathscr{M}_{\alpha_2} = \mathscr{M}_{\alpha_1}$. □

Example 4. Let $\mathscr{M} = \{<0, 1, 0.3, 0.3>, <1, 0.9, 0.5, 0.5>, <2, 0.6, 0.2, 0.1>\}$ over Z_3. Consider $\alpha_1 = 0.5$ and $\alpha_2 = 0.6$. Since $\alpha_1 < \alpha_2$ and there are no $a \in Z_3$ with $\alpha_1 < \mu(a) < \alpha_2$, $\alpha_1 > \gamma(a), \gamma(a) < \alpha_2$, $\alpha_1 > \zeta(a)$, and $\zeta(a) < \alpha_2$, thus we get $\mathscr{M}_{\alpha_1} = \mathscr{M}_{\alpha_2} = \mathscr{M}$.

Theorem 18. *Let $\mathscr{H}$ be any subgroup of $\mathscr{G}$ which can be realized as a level subgroup of some neutrosophic subgroup of $\mathscr{G}$.*

Proof. Suppose that $\mathscr{H} = \{<\kappa, \mu(\kappa), \gamma(\kappa), \zeta(\kappa)>: \kappa \in \mathscr{G}\}$ is a neutrosophic subset of $\mathscr{G}$ defined as follows:

$$\mu(\kappa) = \begin{cases} \alpha, & \text{if } \kappa \in \mathscr{H}, \\ 0, & \text{if } \kappa \notin \mathscr{H}, \end{cases}$$
$$\gamma(\kappa) = \begin{cases} 0, & \text{if } \kappa \in \mathscr{H}, \\ \alpha, & \text{if } \kappa \notin \mathscr{H}, \end{cases} \tag{15}$$
$$\zeta(\kappa) = \begin{cases} 0, & \text{if } \kappa \in \mathscr{H}, \\ \alpha, & \text{if } \kappa \notin \mathscr{H}, \end{cases}$$

where $0 < \alpha < 1$. Now, we will prove that $\mathscr{H}$ is a neutrosophic subgroup of $\mathscr{G}$. Assume that $\kappa, \iota \in \mathscr{G}$. According to the above definition of $\mathscr{H}$, we have some cases which are discussed as follows:

Case 1: Let $\kappa, \iota \in \mathscr{H}$, then $\kappa\iota \in \mathscr{H}$. So, $\mu(\kappa) = \mu(\iota) = \alpha$ and $\mu(\kappa\iota) = \alpha$. Thus, (i) and (ii) in Definition 6 are verified. Again, $\gamma(\kappa) = \gamma(\iota) = 0$, $\gamma(\kappa\iota) = 0$, $\zeta(\kappa) = \zeta(\iota) = 0$, and $\zeta(\kappa\iota) = 0$. Therefore, (iii), (iv), (v), and (vi) in Definition 6 are satisfied.

Case 2: Let $\kappa \in \mathscr{H}$ and $\iota \notin \mathscr{H}$, then $\kappa\iota \notin \mathscr{H}$. So, $\mu(\kappa) = \alpha$, $\mu(\iota) = 0$, and $\mu(\kappa\iota) = 0$. Thus, (i) and (ii) in Definition 6 are satisfied. Again, $\gamma(\kappa) = 0$, $\gamma(\iota) = \alpha$, $\gamma(\kappa\iota) = \alpha$, $\zeta(\kappa) = 0$, $\zeta(\iota) = \alpha$, and $\zeta(\kappa\iota) = \alpha$. Therefore, (iii), (iv), (v), and (vi) in Definition 6 are satisfied.

Case 3: Let $\kappa, \iota \notin \mathscr{H}$, then $\kappa\iota$ may or may not be in $\mathscr{H}$. Now, in any case we find that all axioms of Definition 6 are held.

According to the above cases, we find that $\mathscr{H}$ is a neutrosophic subgroup of $\mathscr{G}$. □

Remark 19. The level subgroups of a neutrosophic group $\mathscr{M}$ form a string as a result of Theorem 17. But, $\mu(\kappa) \leq \mu(e), \gamma(\kappa) \geq \gamma(e)$, and $\zeta(\kappa) \geq \zeta(e), \kappa \in \mathscr{G}$. Thus, the smallest level subgroup is $\mathscr{M}_{\alpha_0} = (e)$, where $\mu(\kappa) = \gamma(\kappa) = \zeta(\kappa) = \alpha_0$. Also, we have the following string:

$$(e) = \mathscr{M}_{\alpha_0} \subset \mathscr{M}_{\alpha_1} \subset \mathscr{M}_{\alpha_2} \subset \cdots \subset \mathscr{M}_{\alpha_r} = \mathscr{G}, \tag{16}$$

where $\alpha_0 > \alpha_1 > \cdots > \alpha_r$. This string of level subgroups is denoted by $\mathscr{C}(\mathscr{M})$. Consequently, not all subgroups of $\mathscr{G}$ are level subgroups of a particular neutrosophic subgroup, as all subgroups of $\mathscr{G}$, generally speaking, do not form a string. Thus, an important problem is to find a neutrosophic subgroup of $\mathscr{G}$ that can accommodate as many subgroups of $\mathscr{G}$ in $\mathscr{C}(\mathscr{M})$ as possible.

Then, we construct a finite group that is a direct product of prime cyclic groups.

Theorem 20. *Assuming that $\mathscr{G}$ is a finite group with the property that $\mathscr{G} = \mathscr{C}_{p_1} \times \mathscr{C}_{p_2} \times \cdots \times \mathscr{C}_{p_k}$, where $\mathscr{C}_{p_j}$ and $j = 1, 2, \ldots, k$ are prime cyclic groups with p_j order, then there* *exists a neutrosophic subgroup $\mathscr{M}$ such that $\mathscr{C}(\mathscr{M})$ is a maximal string of length $k+1$.*

Proof. By using induction on k, we prove this theorem. Let $k=1$, then $\mathscr{G}=\mathscr{C}_{p_1}$, and then there exists a neutrosophic subgroup $\mathscr{M}$ of $\mathscr{G}$ with $\mathscr{M}(e)=\alpha_0$ and $\mathscr{M}(a)=\alpha_1$ for all $a\neq e\in\mathscr{G}$ and $\alpha_0>\alpha_1$. Then, $\mathscr{M}_{\alpha_0}=(e)$ and $\mathscr{M}_{\alpha_1}=\mathscr{G}$; thus, $\mathscr{M}_{\alpha_0}\subset\mathscr{M}_{\alpha_1}$ is a maximal string with length 2. So, for $k=1$, the theorem is valid. Assuming the theory is correct for the integers $\leq k-1$ with $k>1$, we now prove it for k. Since $\mathscr{G}=\mathscr{H}\times\mathscr{C}_{p_k}$, $\mathscr{H}=\mathscr{C}_{p_1}\times\mathscr{C}_{p_2}\times\cdots\times\mathscr{C}_{p_{k-1}}$. What follows is a definition of the neutrosophic set $\mathscr{M}$ by $\mathscr{M}(e)=\alpha_0$, $\mathscr{M}(\mathscr{C}_{p_1}-(e))=\alpha_1$, and $\mathscr{M}(\mathscr{C}_{p_1}\times\mathscr{C}_{p_2}-\mathscr{C}_{p_1})=\alpha_2,\cdots,\mathscr{M}(\mathscr{C}_{p_1}\times\mathscr{C}_{p_2}\times\cdots\times\mathscr{C}_{p_{k-1}}\times\mathscr{C}_{p_k}-\mathscr{C}_{p_{k-1}})=\alpha_k$, where $\alpha_0>\alpha_1>\cdots>\alpha_k$. Here, we will show that $\mathscr{M}$ is a neutrosophic subgroup of $\mathscr{G}$. Presume that $a,b\in\mathscr{G}$. Let $a,b\in\mathscr{H}$, then $ab\in\mathscr{H}$, and by induction, all axioms in Definition 6 are satisfied. Again, presume that $a\in\mathscr{H}$ and $b\notin\mathscr{H}$, then $\mathscr{M}(ab)=\alpha_k$, $\mu(a)\geq\alpha_{k-1},\gamma(a)\leq\alpha_{k-1},\zeta(a)\leq\alpha_{k-1}$, and $\mathscr{M}(b)=\alpha_k$. Therefore, all axioms in Definition 6 are satisfied. Finally, let $a,b\notin\mathscr{H}$. Then, it is easy for us to confirm that $\mathscr{M}$ is a neutrosophic subgroup. Now, $\mathscr{M}_{\alpha_0}=(e)$, $\mathscr{M}_{\alpha_1}=\mathscr{C}_{p_1}$, $\mathscr{M}_{\alpha_2}=\mathscr{C}_{p_1}\times\mathscr{C}_{p_1}$, $\cdots\mathscr{M}_{\alpha_k}=\mathscr{H}\times\mathscr{C}_{p_k}$. Thus, $\mathscr{M}_{\alpha_0}\subset\mathscr{M}_{\alpha_1}\subset\cdots\subset\mathscr{M}_{\alpha_k}$ is $\mathscr{C}(\mathscr{M})$ which is a maximal and has a length of $k+1$. □

3.2. Characterization of Neutrosophic Subgroup. In this subsection, we introduce the characterization of the neutrosophic subgroup of finite cyclic groups. In the following, we assume that $\mathscr{G}$ is a cyclic p-group with $|\mathscr{G}|=p^n$ and p is a prime number.

Theorem 21. *Let $\mathscr{M}$ be a neutrosophic subgroup of $\mathscr{G}$, then we get the following:*

(i) If $|\kappa|\geq|\iota|$, then $\mu(\kappa)\leq\mu(\iota),\gamma(\kappa)\geq\gamma(\iota)$, and $\zeta(\kappa)\geq\zeta(\iota)$

(ii) If $|\kappa|=|\iota|$, then $\mu(\kappa)=\mu(\iota),\gamma(\kappa)=\gamma(\iota)$, and $\zeta(\kappa)=\zeta(\iota)$

for all $\kappa,\iota\in\mathscr{G}$.

Proof. Let $|\mathscr{G}|=p^n$. By using induction on n, if $n=1$, then $|\mathscr{G}|=p$, and by Proposition 8, the theorem is claimed. The theorem is valid for all integers less than and equal to $n-1$ and $n>1$. Presume that $\mathscr{H}$ is a subgroup of $\mathscr{G}$ and $|\mathscr{H}|=p^{n-1}$. Assume that $\kappa,\iota\in\mathscr{H}$, then the result follows by induction. Assume $\kappa\in\mathscr{H}$ and $\iota\in\mathscr{H}$; this leads to $|\kappa|=p^n$ and $|\iota|=p^r$ with $r<n-1$. Therefore, $\mathscr{G}=<\kappa>$ and $y=\kappa^m$ for some integral number m.Thus,

$$\begin{aligned}\mu(\iota)&=\mu(\kappa)\geq\mu(\kappa),\\ \gamma(\iota)&=\gamma(\kappa)\leq\gamma(\kappa),\\ \zeta(\kappa)&=\zeta(\kappa)\leq\zeta(\kappa).\end{aligned}\quad(17)$$

Finally, let $\kappa,\iota\notin\mathscr{H}$, then $|\kappa|=|\iota|=p^n$. Again, we find $\mathscr{G}=<\kappa>=<\iota>$. So, $\kappa=\iota^m$ and $\iota=\kappa^n$ for some integral numbers m and n. Thus,

$$\begin{aligned}\mu(\kappa)&=\mu(\iota^m)\geq\mu(\iota),\\ \mu(\iota)&=\mu(\kappa^n)\geq\mu(\kappa),\\ \gamma(\kappa)&=\gamma(\iota^m)\leq\gamma(\iota),\\ \gamma(\iota)&=\gamma(\kappa^n)\leq\gamma(\kappa),\\ \zeta(\kappa)&=\zeta(\iota^m)\leq\zeta(\iota),\\ \zeta(\iota)&=\zeta(\kappa^n)\leq\zeta(\kappa).\end{aligned}\quad(18)$$

Hence, we have $\mu(\kappa)=\mu(\iota),\gamma(\kappa)=\gamma(\iota)$, and $\zeta(\kappa)=\zeta(\iota)$. □

Remark 22. In general, the above theorem is invalid as in the case fuzzy subgroup of cyclic p-group of order p^n (see [17]).

In the following, we characterize each neutrosophic subgroup of a finite cyclic group. Also, we suppose that $\mathscr{G}$ is a finite cyclic group.

Theorem 23. *Any neutrosophic subset of $\mathscr{G}$ is a neutrosophic subgroup if and only if there is a maximal string of subgroups $(e)\subset\mathscr{C}_0\subset\mathscr{C}_1\subset\mathscr{C}_2\subset\cdots\subset\mathscr{C}_k\subset\mathscr{G}$ with $\mathscr{M}(e)=\alpha_0$, $\mathscr{M}(\mathscr{C}_1-\mathscr{C}_0)=\alpha_1,\ldots,\mathscr{M}(\mathscr{C}_k-\mathscr{C}_{k-1})=\alpha_k$, where $\alpha_i=\mathscr{M}(a)$ for some $a\in\mathscr{G}$ and $i=0,1\cdots,k$.*

Proof. Suppose that $(e)\subset\mathscr{C}_0\subset\mathscr{C}_1\subset\mathscr{C}_2\subset\cdots\subset\mathscr{C}_k\subset\mathscr{G}$ is a maximal string of subgroups with $\mathscr{M}(e)=\alpha_0$, $\mathscr{M}(\mathscr{C}_1-\mathscr{C}_0)=\alpha_1,\ldots,\mathscr{M}(\mathscr{C}_k-\mathscr{C}_{k-1})=\alpha_k$. Now, we prove that $\mathscr{M}$ is a neutrosophic subgroup of $\mathscr{G}$. Let $a,b\in\mathscr{G}$ and $a,b\in\mathscr{C}_i$ and $a,b\notin\mathscr{C}_{i-1}$, then we have $\mathscr{M}(a)=\mathscr{M}(b)=\alpha_i$ (i.e., $\mu(a)=\gamma(a)=\zeta(a)=\alpha_i$) and either $ab\in\mathscr{C}_i$ or $\mathscr{C}_{i-1}$. Thus, the axioms of Definition 6 are held. Assume that $a\in\mathscr{C}_i$ but $a\notin\mathscr{C}_{i-1}$, $b\in\mathscr{C}_j$ but $b\notin\mathscr{C}_{j-1}$, and when $i>j$, we have $\mathscr{M}(a)=\alpha_i$ and $\mathscr{M}(b)=\alpha_j$. Thus, the axioms of Definition 6 are held. After discussing the above cases, we have $\mathscr{M}$ as a neutrosophic subgroup of $\mathscr{G}$. Conversely, let $\mathscr{M}$ be a neutrosophic subgroup of $\mathscr{G}$, then we have $\mathscr{M}_{\alpha_0},\mathscr{M}_{\alpha_1},\ldots,\mathscr{M}_{\alpha_k}$ are only level subgroups of $\mathscr{M}$ where $\alpha_i=\mathscr{M}(a)$ for some $a\in\mathscr{G}$, $i=0,1\cdots,k$, and $\alpha_0>\alpha_1>\ldots>\alpha_k$. Moreover, the level subgroups from a string $\mathscr{M}_{\alpha_0}\subset\mathscr{M}_{\alpha_1}\subset\ldots\subset\mathscr{M}_{\alpha_k}$ with $\mathscr{M}_{\alpha_0}=(e)$ and $\mathscr{M}_{\alpha_k}=\mathscr{G}$. Let $\mathscr{C}(\mathscr{M})$ be a maximal, then we take $\mathscr{C}_i=\mathscr{M}_{\alpha_j}$. If $\mathscr{C}(\mathscr{M})$ is not maximal, we expand $\mathscr{C}(\mathscr{M})$ by inserting subgroups of $\mathscr{G}$. This string is defined as $\mathscr{C}_0 \subset \mathscr{C}_1 \subset \mathscr{C}_2 \subset \cdots \subset \mathscr{C}_l \subset \mathscr{G}$, where $\mathscr{C}_0 = (e) = \mathscr{M}_{\alpha_0}$ and $\mathscr{C}_l = \mathscr{G} = \mathscr{M}_{\alpha_k}$. Now, $\mathscr{M}(\mathscr{C}_0) = \alpha_0$ and for all $\mathscr{C}_i$ between $\mathscr{C}_0 = \mathscr{M}_0$ and $\mathscr{C}_j = \mathscr{M}_{\alpha_1}$, we have $\mathscr{M}(\mathscr{C}_i - \mathscr{C}_{i-1}) = \alpha_1$, and also for all $\mathscr{C}_r$ between $\mathscr{M}_{\alpha_r}$ and $\mathscr{M}_{\alpha_{r+1}}$, we have $\mathscr{M}(\mathscr{C}_r - \mathscr{C}_{r-1}) = \alpha_r$. Similarly, $\mathscr{M}(\mathscr{C}_l - \mathscr{C}_{l-1}) = \alpha_k$. Thus, the theorem is claimed. □

4. Conclusions

The mathematical branches have recently found it useful and important to study neutrosophic sets. The definition of a neutrosophic group has been modified by the authors of this study as an extension of a fuzzy group. This notion has also been studied in a framework comparable to the basic theory of groups. Moreover, the notion of a level subgroup is used to characterize a neutrosophic subgroup of cyclic groups.

In future works, we will strive to define and explore neutrosophic subring and subfield according to the first viewpoint, following the footsteps of the current publication. Furthermore, we strive to continue researching neutrosophic groups and their applications in information technology and decision support systems, such as relational database systems, semantic web services, financial data set identification, new economic growth and decline analysis, and so on.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare no that there are conflicts of interest.

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