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Research Article

Number of Distinct Fragments in Coset Diagrams for $\mathrm{PSL}(2, \mathbb{Z})$

Muhammad Aamir,[1] **Awais Yousaf**,[1] **Ibtisam Masmali**,[2] **and Abdul Razaq**[3]

[1]*Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, Pakistan*
[2]*Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia*
[3]*Department of Mathematics, Division of Science and Technology, University of Education, Lahore, Pakistan*

Correspondence should be addressed to Muhammad Aamir; aamir.math.hed@gmail.com

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Coset diagrams [1, 2] are used to demonstrate the graphical representation of the action of the extended modular group $\mathrm{PGL}(2, \mathbb{Z})$ over $\mathrm{PL}(F_q) = F_q \cup \{\infty\}$. In these sorts of graphs, a closed path of edges and triangles is known as a circuit, and a fragment is emerged by the connection of two or more circuits. The coset diagram evolves through the joining of these fragments. If one vertex of the circuit is fixed by $(ax)^{\rho_1}(ax^{-1})^{\rho_2}(ax)^{\rho_3}\cdots(ax^{-1})^{\rho_k} \in \mathrm{PSL}(2, \mathbb{Z})$, then this circuit is termed to be a length $-\ k$ circuit, denoted by $(\rho_1, \rho_2, \rho_3, \cdots, \rho_k)$. In this study, we consider two circuits of length -6 as $\Omega_1 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ and $\Omega_2 = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ with the vertical axis of symmetry that is $\alpha_2 = \alpha_6, \alpha_3 = \alpha_5$ and $\beta_2 = \beta_6, \beta_3 = \beta_5$. It is supposed that Ω is a fragment formed by joining Ω_1 and Ω_2 at a certain point. The condition for existence of a fragment is given in [3] in the form of a polynomial in $\mathbb{Z}[z]$. If we change the pair of vertices and connect them, then the resulting fragment and the fragment Ω may coincide. In this article, we find the total number of distinct fragments by joining all the vertices of Ω_1 with the vertices of Ω_2 provided the condition $\beta_4 < \beta_3 < \beta_2 < \beta_1 < \alpha_4 < \alpha_3 < \alpha_2 < \alpha_1$.

1. Introduction

It is considered that $\mathbb{H} = \{p_1 + ip_2; p_1, p_2 \in \mathbb{R}, p_2 > 0\}$ is known as the Lobachevski plane model, the model of the upper-half plane of hyperbolic plane geometry. Then, the group of Mobius Transformations M [4], with $\lambda \longrightarrow (a_1\lambda + a_2)/(a_3\lambda + a_4)$ where $a_1, a_2, a_3, a_4 \in \mathbb{R}$ and $a_1a_4 - a_2a_3 = \pm 1$ is the group of isometries that preserve the orientation in $\mathbb{H}$. It is isomorphic to the quotient group $\mathrm{PSL}(2, \mathbb{R})$, which is called the projective special linear group. Geometrically, the group of isometries of $\mathbb{H}$ is the action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{H}$ [5] with left action (faithful) as

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\lambda = \frac{a_1\lambda + a_2}{a_3\lambda + a_4}. \quad (1)$$

The Mobius transformations of $\mathbb{H}$ with coefficients from the set of integers form a group known as a discrete group [6], a subgroup of $\mathrm{PSL}(2, \mathbb{R})$, symbolically written as $\mathrm{PSL}(2, \mathbb{Z})$, it is a quotient group of special linear group $SL(2, \mathbb{Z})$ by its center $\{I, -I\}$.

It is eminent that the transformations of linear fractions $a\colon \rho \longrightarrow -1/\rho$ and $x\colon \rho \longrightarrow (\rho - 1)/\rho$ are used to generate $\mathrm{PSL}(2, \mathbb{Z})$, so-called the modular group, with presentation

$$\langle a, x; a^2 = x^3 = 1\rangle. \quad (2)$$

By introducing a *new* generator $z\colon \rho \longrightarrow 1/\rho$ with a and x, we obtain a group $\mathrm{PGL}(2, \mathbb{Z})$, the extension of $\mathrm{PSL}(2, \mathbb{Z})$, using

$$a^2 = x^3 = z^2 = (az)^2 = (xz)^2 = 1, \quad (3)$$

as a relation.

The coset diagrams present the action of $\mathrm{PGL}(2, \mathbb{Z})$ on $F_q \cup \{\infty\}$, where F_q is a finite field and q shows a prime power. These graphs have a long and rich history [7, 8]. Small triangles are proposed for the cycle x^3, such that x permutes the vertices of triangles in the opposite direction of rotation of clock and an edge is attached to any two vertices that are interchanged by a. Heavy dots represent the fixed point of a and x. Note that, $(xz)^2 = 1$ equals $zxz = x^{-1}$, that means z reverses the triangle orientation proposed for the cycle x^3. For that reason, the diagram need not be made more intricate by inserting z − edges.

Definition 1. A coset diagram (subdiagram) Γ_1 is said to be a homomorphic image of the coset diagram (subdiagram) Γ_2 if and only if

(i) $|V(\Gamma_1)| < |V(\Gamma_2)|$

(ii) $\forall s \in V(\Gamma_2)$ with $(s)u = s$, where $u \in \mathrm{PSL}(2,\mathbb{Z})$, there exist a vertex t in $V(\Gamma_1)$ such that $(t)u = t$

(iii) a-edges map to a-edges

(iv) x-edges mapped to x-edges

Coset diagrams obtained from the action of $\mathrm{PSL}(2,\mathbb{Z})$ over Q_ϵ are infinite graphs, where $Q_\epsilon = \{a_1 + a_2\sqrt{\epsilon}\,;\, a_1, a_2 \in \mathbb{Q} \text{ and } \epsilon \in \mathbb{Z}^+ \text{ is a square} - \text{free}\}$, whereas coset diagrams for the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $PL(F_q)$ presents finite graphs. The number $(x_1 + \sqrt{\epsilon})/x_2$ is an expression of the number $a_1 + a_2\sqrt{\epsilon} \in Q_\epsilon$, where $(x_1, x_2, (x_1^2 - \epsilon)/x_2) = 1\cdot$ The finite coset diagrams are homomorphic images of the coset diagrams for $(x_1 + \sqrt{\epsilon})/x_2$, where $\epsilon \equiv z^2 \bmod p$ for some $z \in N$.

To explain more, the coset diagram in the following (Figure 1), illustrate the action on $\mathrm{PL}(F_{23})$ by $\mathrm{PGL}(2,\mathbb{Z})$ with permutation representations a, x, and z by $(\rho)a = -1/\rho$, $(\rho)x = (\rho - 1)/\rho$, and $(\rho)z = 1/\rho$, respectively, as

$$\begin{aligned} a&: (0\,\infty), (1\,22), (2\,11), (3\,15), (4\,17), (5\,9), (6\,19), (7\,13), (8\,20), (10\,16), (12\,21), (14\,18),\\ x&: (0\,\infty\,1), (2\,12\,22), (3\,16\,11), (4\,18\,15), (5\,10\,17), (6\,20\,9), (7\,14\,19), (8\,21\,13),\\ z&: (0\,\infty), (2\,12), (3\,8), (4\,6), (5\,14), (7\,10), (9\,18), (11\,21), (13\,16), (15\,20), (17\,19), (1), (22). \end{aligned} \tag{4}$$

For a comprehensive understanding of coset diagrams, we propose [9–13].

In the study of [3], it has been shown that for a fixed value of ϵ there are only a finite number of real quadratic irrational ambiguous numbers of the form $\alpha = (x_1 + \sqrt{\epsilon})/x_2$ and that part of the coset diagram containing ambiguous numbers forms a circuit and it is the only circuit in the orbit of α.

In a coset diagram, a closed path of triangles and edges is called a circuit. A circuit is said to be of length –, denoted by $(\rho_1, \rho_2, \rho_3, \cdots, \rho_k)$, if its one vertex is fixed by

$$(ax)^{\rho_1}\left(ax^{-1}\right)^{\rho_2}(ax)^{\rho_3}\cdots\left(ax^{-1}\right)^{\rho_k} \in \mathrm{PSL}(2,\mathbb{Z}). \tag{5}$$

Alternatively, it means that one vertex of the ρ_1 triangles lie outside of the circuit and one vertex of the ρ_2 triangles lies inside of the circuit and likewise. Since $(\rho_1, \rho_2, \rho_3, \cdots, \rho_k)$ is a cycle, so it does not matter if one vertex of the ρ_1 triangles lie inner of the circuit and one vertex of the ρ_2 triangles lies outer of the circuit and likewise. Note that, k is always even.

The circuit of the type $(\rho_1, \rho_2, \rho_3, \cdots, \rho_{l'}, \rho_1, \rho_2, \rho_3, \cdots, \rho_{l'}, \cdots, \rho_1, \rho_2, \rho_3, \cdots, \rho_{l'})$ is termed as a periodic circuit with the period of length l'.

For more on circuits in coset diagrams, we refer [14].

Consider two nonperiodic and simple circuits $C_1 = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_m)$ and $C_2 = (\beta_1, \beta_2, \beta_3, \cdots, \beta_n)$. Let s_1 and s_2 be any vertices from C_1 and C_2 fixed by the words u_1 and u_2 from $\mathrm{PSL}(2,\mathbb{Z})$, respectively. To connect C_1 and C_2 at s_1 and s_2, we arbitrarily choose the circuit C_2 and apply u_1 on s_2 in such a way that u_1 ends at s_2. Appropriately, a fragment Ω (say) emerge, consisting a vertex $s = s_1 = s_2$ fixed by the pair u_1, u_2.

Let Ω^* shows itself as the mirror image of Ω. Since the permutation z ensures that the coset diagram is symmetric along the vertical axis. This implies Ω^* will assuredly occur.

If $u = ax^{\pi_1}ax^{\pi_2}\cdots ax^{\pi_n}$ ($\pi_i = 1$ or -1) is a word then $u^* = ax^{-\pi_1}ax^{-\pi_2}\cdots ax^{-\pi_n}$. If the word u fixes the vertex s, then the vertex s^* is fixed by u^*.

There are two components involves in the action of $\mathrm{PGL}(2,\mathbb{Z})$ on $F_{q^2} \cup \{\infty\}$ and they are $F_q \cup \{\infty\}$ and $F_{q^2} \backslash F_q$. Let $\overline{F_q}$ denote itself as the complement $F_{q^2} \backslash F_q$. In what follows, by Ω, we shall mean a nonsimple fragment composed by connecting two nontrivial and nonperiodic circuits. Coset diagrams corresponding to the actions of $PGL(2,\mathbb{Z})$ on $PL(F_q)$ via a homomorphism α with parameter θ are denoted by $D(\theta, q)$ [3]. These diagrams are composed of fragments. There is a question that must revolve in minds when a fragment exists in $D(\theta, q)$. In [3], the response is found in the following way.

Theorem 1. *Given a fragment Ω, there is a polynomial f in $\mathbb{Z}[z]$ such that*

(i) if Ω occurs in $D(\theta, q)$, then $f(\theta) = 0$

(ii) if $f(\theta) = 0$ then Ω or Ω^* occurs in $D(\theta, q)$ or in $\overline{F_q}$

How to calculate a polynomial from Ω? The answer is given in [3].

Let $f(\theta)$ be a polynomial acquired from the fragment Ω, which is emerged in the connection of two nonperiodic circuits. Then, there exists a homomorphic image of Ω other than Ω corresponding to each zero of $f(\theta)$ in the appropriate coset diagrams. Thus, we are compromising with the fragments, which are set up by connecting a pair of nonperiodic circuits.

Remark 1. The direction of the triangles describing the three-cycles of x completely changed by the action of z (like as reflection). So if s is a vertex of Ω fixed by the pair u_1, u_2, then obviously the pair u_1^*, u_2^* fixed the vertex s^* of Ω^*. Since

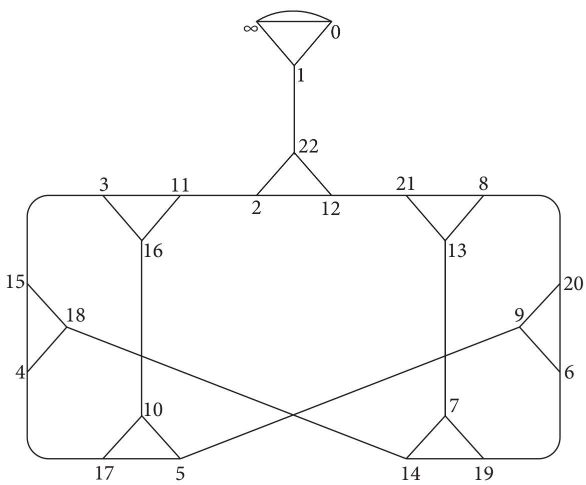

FIGURE 1: Coset diagram for the action of $PGL(2,\mathbb{Z})$ on $PL(F_{23})$.

a vertical axis of symmetry possesses by $D(\theta,q)$, therefore if Ω founds in $D(\theta,q)$, then $D(\theta,q)$ also contains Ω^*. So Ω and Ω^* have the same existence condition in $D(\theta,q)$ implying that, they give a unique polynomial. There are specific fragments that admit a vertical axis of symmetry. In this case, the orientation of the mirror image is the same as that of the fragment. These types of fragments may have fixed points of z. If u_1, u_2 is a couple of words fixing the vertex s of the fragment Ω, then the orientation of Ω is same as that of Ω^* if and only if there exists a vertex t in Ω which is fixed by a couple of words u_1^*, u_2^*.

2. Pairs of Connecting Vertices

If a fragment Ω emerged by connecting vertices s_i and s_j from C_1 and C_2, respectively; then s_i and s_j are not the single pair of joined vertices but there are many (depends upon s_i and s_j) pairs of vertices in C_1 and C_2, that are connected. That is, a finite number of pairs of connected vertices results the same fragment.

Definition 2. Let s_i, s_k and s_j, s_l be the vertices in C_1 and C_2 such that s_i, s_k, s_j, and s_l are fixed by u_i, u_k, u_j, and u_l, respectively. Let Ω be the fragment set up in the connection of s_i with s_j. Then, the pair of vertices $V(s_i,s_j)$ is identical to the pair of vertices $V(s_k,s_l)$ if and only if in the connection of s_i with s_j to produce Ω, s_k, and s_l also, get connected with each other. If two pairs of vertices $V(s_i,s_j)$ and $V(s_k,s_l)$ are identical, then we write $V(s_i,s_j) \sim V(s_k,s_l)$.

Let Ω be a fragment formed by connecting the vertex s_1 fixed by u_1 in C_1 with the vertex s_2 fixed by u_2 in C_2 and R denotes itself as the set of pairs of joining vertices that are identical to $V(s_1,s_2)$. Let P be the collection of words such that for all $u \in P$, the vertices $(s_1)u$ and $(s_2)u$ lies on C_1 and C_2, respectively.

The following theorems proved in [15] will help us to find all the pairs of joining vertices for the fragment acquired by the connection of C_1 and C_2.

Theorem 2. *For any $u \in P$, there is a pair of joining vertices in R.*

Theorem 3. *Corresponding to each pair of joining vertices $V(s_k,s_l) \in R$, there is a unique word $u \in P$ such that $(s_i)u = s_k$, $(s_j)u = s_l$.*

Theorem 4. *There is a one-to-one correspondence between R and P.*

3. Counting the Number of Pairs of Connecting Vertices for a Fragment

Each joining point gives a couple of words, which further ensures a polynomial. Since a polynomial acquired from a fragment is unique. Thus, for all pairs of joining vertices for a fragment, a unique polynomial is evolved. Therefore, all the pairs of joining vertices for a fragment must be identified.

Let us join the vertices s_i and s_j of C_1 and C_2 fixed by u_i and u_j, respectively, to acquire the fragment Ω. Let $S = |R|$, then there are at least S pairs of joining vertices in C_1 and C_2 to obtain Ω. Note that, S is not the total number of pairs of joining vertices in C_1 and C_2 to compose Ω.

Let s_1, s_2 be any two vertices from the circuit $C = (\Delta_1, \Delta_2, \Delta_3, \cdots, \Delta_k)$ fixed by the words u_1 and u_2 that is $(s_1)u_1 = s_1$ and $(s_2)u_2 = s_2$. Suppose u_3 is the word that maps s_1 to s_2 that is $(s_1)u_1^{-1}u_3 = s_2$. Note that, u_3 and $u_1^{-1}u_3$ are the only paths that assign s_2 to s_1. By contraction of vertices s_1 and s_2, we mean that s_1 and s_2 melt together to become one node $s = s_1 = s_2$ such that $(s)u_3 = (s)u_1^{-1}u_3 = s$. As a result of this contraction, a closed path Γ is created that containing the vertex s fixed by u_3 and $u_1^{-1}u_3$. This closed path Γ is the homomorphic image of the circuit C. Note that, s_1 and s_2 is not the only pair of contraction in C that creates homomorphic image Γ. There are also many pairs of contraction other than s_1 and s_2 that create the same homomorphic image Γ. It is therefore necessary to ask how many distinct homomorphic images are obtained if we contract all pairs of vertices of the circuit C? The answer to this question is given in [15]. The process to find the number of pairs of connecting vertices for a fragment is same as that the number of pairs of contracting vertices for a homomorphic image. To know, how many total pairs of joining vertices for a fragment, one has to be extra careful.

(i) If the orientation of Ω is different from Ω^*, then the pair of words u_i^*, u_j^* does not fix any vertex in Ω (Remark 1). That is, to produce Ω, with the joining of s_i and s_j, the pair of vertices s_i^*, s_j^* is not connected. So, there are total $2S$ pairs of joining vertices for Ω and Ω^*.

(ii) If the orientation of Ω is same as Ω^*, then there exists a vertex in Ω fixed by the pair of words u_i^*, u_j^* (Remark 1). That is, to produce Ω, with the joining of s_i and s_j, the pair of vertices s_i^*, s_j^* is also connected. So, there are total S pairs of joining vertices for Ω and Ω^*.

In the literature, the question that how many pairs of connecting vertices form the fragment Ω is answered for the pairs of circuits of length-2 [15] and contracting vertices produce the homomorphic image Γ is responded for the circuit of length-4 [11] under certain conditions. We have solved this problem for the pair of circuits of length 6.

4. Connection of Circuits

Consider two circuits of length-6 as $\Omega_1 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ (Figure 2) and $\Omega_2 = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ (Figure 3) with the vertical axis of symmetry that is $\alpha_2 = \alpha_6, \alpha_3 = \alpha_5$ and $\beta_2 = \beta_6, \beta_3 = \beta_5$ and impose a condition $\beta_4 < \beta_3 < \beta_2 < \beta_1 < \alpha_4 < \alpha_3 < \alpha_2 < \alpha_1$.

Let us connect Ω_1 and Ω_2 at a certain point and obtained a fragment Ω. Since there are finitely many pairs of connecting vertices that build same fragment; therefore, it is not necessary that if we altered the pair of connecting vertices of Ω_1 and Ω_2, we obtain a fragment other than Ω. But it is necessary to inquire that how many distinct fragments are obtained by joining the circuits Ω_1 and Ω_2 at all pairs of connecting vertices. In this article, we will not only answer this question but also identify the pairs of connecting vertices of Ω_1 and Ω_2 that are important. At those pairs of connecting vertices that are not stated as important, Ω_1 and Ω_2 need not to connect because if we join Ω_1 and Ω_2 at such pairs, we attain fragments that we already obtained by connecting important pairs. Since $\alpha_2 = \alpha_6, \alpha_3 = \alpha_5$ and $\beta_2 = \beta_6, \beta_3 = \beta_5$, therefore Ω_1 and Ω_2 have $3(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)$ and $3(\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4)$ number of vertices, respectively, implies that there are total $9(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)(\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4)$ pairs of connecting vertices of Ω_1 and Ω_2. We join a vertex of Ω_1 with the vertex of Ω_2 and compose a fragment.

From Figures 2 and 3, for each $l_1 \in \{1,2,3,4,5,6\}$; $i_1 = 1,2,3,\cdots,\alpha_{l_1}$; $j_1 = 1,2,3,\cdots,\beta_{l_1}$; $i_2 = 1,2,3,\cdots,\alpha_{l_1} - 1$; $j_2 = 1,2,3,\cdots,\beta_{l_1} - 1$ and $\underline{k} = \begin{cases} 6 & \text{if } k = 0,6 \\ k \,(\text{mod } 6) & \text{otherwise} \end{cases}$, we have

(i) $\alpha^{\underline{2-l_1}}_{3\alpha_{l_1}-(i_1-1)}$ is mirror image of the vertex $a^{l_1}_{i_1}$

(ii) $\beta^{\underline{2-l_1}}_{3\alpha_{l_1}-(j_1-1)}$ is the mirror image of the vertex $\beta^{l_1}_{j_1}$

(iii) The vertex $a^{l_1}_{3i_2+1}$ is fixed by the word

$$\left(ax^{-1}\right)^{i_2}(ax)^{\alpha_{\underline{l_1 \pm 5}}}\left(ax^{-1}\right)^{\alpha_{\underline{l_1 \pm 4}}}(ax)^{\alpha_{\underline{l_1 \pm 3}}}\left(ax^{-1}\right)^{\alpha_{\underline{l_1 \pm 2}}}(ax)^{\alpha_{\underline{l_1 \pm 1}}}\left(ax^{-1}\right)^{\alpha_{l_1} - i_2} \tag{6}$$

(iv) The vertex $x^{l_1}_{3j_2}$ is fixed by the word

$$(ax)^{\beta_{l_1} - j_2}\left(ax^{-1}\right)^{\beta_{\underline{l_1 \pm 1}}}(ax)^{\beta_{\underline{l_1 \pm 2}}}\left(ax^{-1}\right)^{\beta_{\underline{l_1 \pm 3}}}(ax)^{\beta_{\underline{l_1 \pm 4}}}\left(ax^{-1}\right)^{\beta_{\underline{l_1 \pm 5}}}(ax)^{j_2} \tag{7}$$

We take + at the place of $\pm$ if l_1 is odd and – if l_1 is even.

Before proving the results, we define some symbolic notations in the following:

$$l_1 \in \{1,2,3,4,5,6\}: l_2 \in \{2,3\}: l_3 \in \{1,4\},$$

$$\psi_1^{l_3} = \begin{cases} 0 \text{ if } \alpha_{l_3} + \beta_1 = 0\,(\text{mod } 2), \\ 1 \text{ if } \alpha_{l_3} + \beta_1 = 1\,(\text{mod } 2), \end{cases}$$

$$\psi_2^{l_3} = \begin{cases} 0 \text{ if } \alpha_{l_3} + \beta_4 = 0\,(\text{mod } 2), \\ 1 \text{ if } \alpha_{l_3} + \beta_4 = 1\,(\text{mod } 2), \end{cases}$$

$$\psi_3^{l_3} = \begin{cases} 0 \text{ if } \alpha_{l_3} = 0\,(\text{mod } 2), \\ 1 \text{ if } \alpha_{l_3} = 1\,(\text{mod } 2), \end{cases}$$

$$\psi_1 = \begin{cases} 0 \text{ if } \beta_1 = 0\,(\text{mod } 2), \\ 1 \text{ if } \beta_1 = 1\,(\text{mod } 2), \end{cases}$$

$$\psi_2 = \begin{cases} 0 \text{ if } \beta_4 = 0\,(\text{mod } 2), \\ 1 \text{ if } \beta_4 = 1\,(\text{mod } 2), \end{cases}$$

$$k_1 = 0,1,2,\cdots,\beta_2 - 1,$$

$$k_2 = 1,2,3,\cdots,\beta_1 - 1,$$

$$k_3 = 0,1,2,\cdots,\beta_4 - 1,$$

$$k_4 = 1,2,3,\cdots,\beta_3 - 1,$$

$$k_5 = 0,1,2,\cdots,\beta_6 - 1,$$

$$k_6 = 1,2,\cdots,\beta_5 - 1,$$

$$k_{(7,l_1)} = \beta_2 + 1, \beta_2 + 2, \cdots, \alpha_{l_1} - 1,$$

$$k_{(8,l_1)} = \beta_3 + 1, \beta_3 + 2, \cdots, \alpha_{l_1} - 1,$$

$$k_{(9,l_2)} = \beta_1 + 1, \beta_1 + 2, \cdots, \alpha_{l_2} - 1,$$

$$k_{(10,l_2)} = \beta_4 + 1, \beta_4 + 2, \cdots, \alpha_{l_2} - 1,$$

$$k_{(11,l_3)} = \beta_1 + 1, \beta_1 + 2, \cdots, \frac{\alpha_{l_3} + \beta_1 - \psi_1^{l_3}}{2},$$

$$k_{(12,l_3)} = \beta_4 + 1, \beta_4 + 2, \cdots, \frac{\alpha_{l_3} + \beta_4 - \psi_2^{l_3}}{2},$$

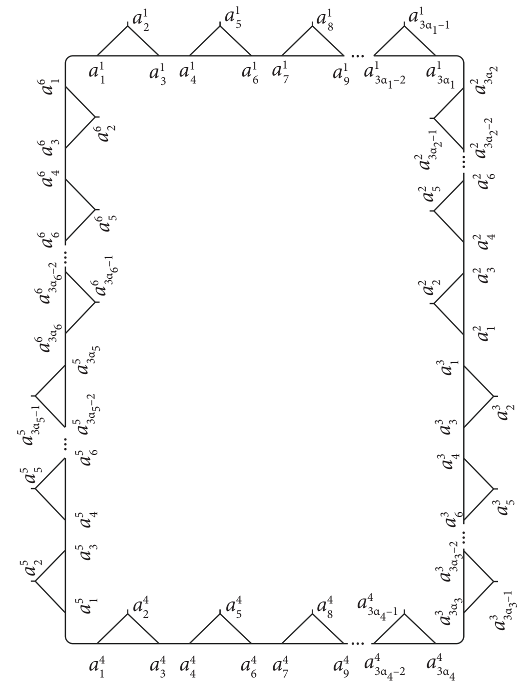

FIGURE 2: Circuit Ω_1.

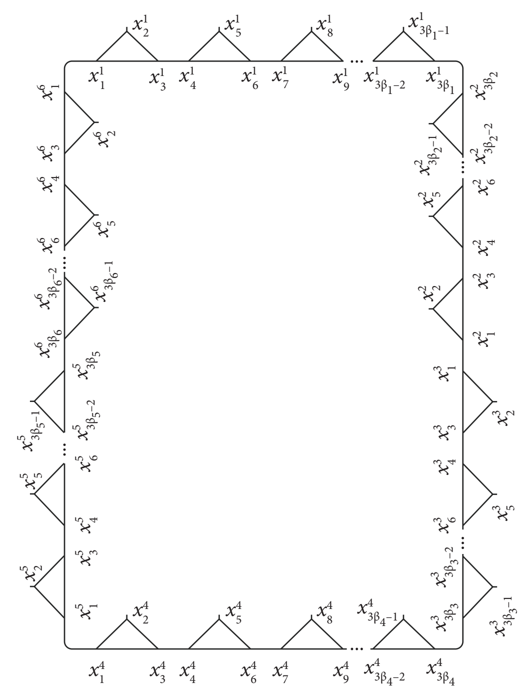

FIGURE 3: Circuit Ω_2.

$$k_{(13,l_3)} = \begin{cases} 1,2,3,\cdots,\dfrac{\alpha_{l_3}-\psi_3^{l_3}}{2} & \text{if } k_{14}=\dfrac{\beta_1}{2}\left(k_{15}=\dfrac{\beta_4}{2}\right), \\ 1,2,3,\cdots,\alpha_{l_3}-1 & \text{otherwise,} \end{cases}$$
$$k_{14} = 1,2,3,\cdots,\frac{\beta_1-\psi_1}{2},$$
$$k_{15} = 1,2,3,\cdots,\frac{\beta_4-\psi_2}{2},$$
$$k_{(16,l_2)} = 1,2,3,\cdots,\alpha_{l_2}-1,$$
$$k_{17} = 1,2,3,\cdots,\beta_4-1,$$
$$k_{(18,l_1)} = 1,2,3,\cdots,\alpha_{l_1}-1,$$
$$k_{19} = 1,2,3,\cdots,\beta_2-1. \tag{8}$$

Note. All lemma's presented in the following, we take $-ve$ sign at the place of $\pm$, if l_1, l_2 and l_3 are even, and $+ve$ otherwise.

Lemma 1. *If we join $x^1_{3\beta_1}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_1+1}$ from Ω_1, then for a fix l_1, there occurs β_2 number of distinct fragments and $3(k_1+2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total of $3(\beta_2^2+3\beta_2)$ pairs of connecting vertices produces all β_2 fragments and their mirror images.*

Proof. For a fix l_1, let us join $x^1_{3\beta_1}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_1+1}$ from Ω_1 and attain a set of fragments $F_1^{l_1} = \{\Gamma^1_{(k_1,l_1)};\ k_1 = 0,1,2,\cdots,\beta_2-1\}$ (Figure 4), where the word

$$\left(ax^{-1}\right)^{\beta_2}(ax)^{\beta_3}\left(ax^{-1}\right)^{\beta_4}(ax)^{\beta_5}\left(ax^{-1}\right)^{\beta_6}(ax)^{\beta_1}, \tag{9}$$

fixing the vertex $x^1_{3\beta_1}$ and the vertex $a^{l_1}_{3k_1+1}$ is fixed by the word

$$\left(ax^{-1}\right)^{k_1}(ax)^{\frac{\alpha_{l_1\pm5}}{}}\left(ax^{-1}\right)^{\frac{\alpha_{l_1\pm4}}{}}(ax)^{\frac{\alpha_{l_1\pm3}}{}}\left(ax^{-1}\right)^{\frac{\alpha_{l_1\pm2}}{}}(ax)^{\frac{\alpha_{l_1\pm1}}{}}\left(ax^{-1}\right)^{\alpha_{l_1}-k_1}. \tag{10}$$

Then, the set

$$D_1^{l_1} = \left\{\begin{array}{c} e,x,x^{-1},a,ax,ax^{-1},\left(ax^{-1}\right)^1a,\left(ax^{-1}\right)^1ax,\left(ax^{-1}\right)^2 \\ ,\cdots,\left(ax^{-1}\right)^{k_1}a,\left(ax^{-1}\right)^{k_1}ax,\left(ax^{-1}\right)^{k_1+1} \end{array}\right\}, \tag{11}$$

contains words such that if u is any word from $D_1^{l_1}$ implies $(a^{l_1}_{3k_1+1})u$ and $(x^1_{3\beta_1})u$ lies on Ω_1 and Ω_2, respectively. Thus, the fragment $\Gamma^1_{(k_1,l_1)}$ has $|D_1^{l_1}| = 3(k_1+2)$ pairs of connecting vertices in Ω_1 and Ω_2. In other words, for each $u \in D_1^{l_1}$, the connection of $(a^{l_1}_{3k_1+1})u$ and $(x^1_{3\beta_1})u$ give the same fragment $\Gamma^1_{(k_1,l_1)}$.

Now, we prove that all the elements in $F_1^{l_1}$ are distinct and no one is the mirror image of itself. For this, let $\Gamma^1_{(m,l_1)}$ and $\Gamma^1_{(n,l_1)}$ be any two fragments from $F_1^{l_1}$. Then, $\Gamma^1_{(m,l_1)}$ is set up by connecting $a^{l_1}_{3m+1}$ and $x^1_{3\beta_1}$ and $\Gamma^1_{(n,l_1)}$ is set up by $a^{l_1}_{3n+1}$ and $x^1_{3\beta_1}$. If $V(a^{l_1}_{3m+1}, x^1_{3\beta_1}) \sim V(a^{l_1}_{3n+1}, x^1_{3\beta_1})$, then there exists an element $u \in D_1^{l_1}$ such that $(a^{l_1}_{3m+1})u = a^{l_1}_{3n+1}$ and $(x^1_{3\beta_1})u = x^1_{3\beta_1}$. $e \in D$ is the only word such that $(x^1_{3\beta_1})e = x^1_{3\beta_1}$ but $(a^{l_1}_{3m+1})e \neq a^{l_1}_{3n+1}$. Thus, $V(a^{l_1}_{3m+1}, x^1_{3\beta_1})$ is disequivalent to $V(a^{l_1}_{3n+1}, x^1_{3\beta_1})$, that is by joining $a^{l_1}_{3m+1}$ with $x^1_{3\beta_1}$ to produce $\Gamma^1_{(m,l_1)}$, $x^1_{3\beta_1}$ is not connected with $a^{l_1}_{3n+1}$. Now, if $V(a^{l_1}_{3m+1}, x^1_{3\beta_1}) \sim V(a^{l_1*}_{3n+1}, x^{1*}_{3\beta_1})$, then there exists an element $u \in D_1^{l_1}$ such that $(a^{l_1}_{3m+1})u = a^{l_1*}_{3n+1}$ and $(x^1_{3\beta_1})u = x^{1*}_{3\beta_1}$. But there does not such an element exist in $D_1^{l_1}$. Thus, $V(a^{l_1}_{3m+1}, x^1_{3\beta_1})$ is disequivalent to $V(a^{l_1*}_{3n+1}, x^{1*}_{3\beta_1})$, that is by joining $a^{l_1}_{3m+1}$ with $x^1_{3\beta_1}$ to produce $\Gamma^1_{(m,l_1)}$, $x^{1*}_{3\beta_1}$ is not connected with $a^{l_1*}_{3n+1}$. Hence, all the elements in $F_1^{l_1}$ are distinct. Since, $F_1^{l_1} = \{\Gamma^1_{(k_1,l_1)};\ k_1 = 0,1,2,\cdots,\beta_2-1\}$ implying that $|F_1^{l_1}| = \beta_2$.

If $V(a^{l_1}_{3m+1}, x^1_{3\beta_1}) \sim V(a^{l_1*}_{3m+1}, x^{1*}_{3\beta_1})$, then there exists an element $u \in D_1^{l_1}$ such that $(a^{l_1}_{3m+1})u = a^{l_1*}_{3m+1}$ and $(x^1_{3\beta_1})u = x^{1*}_{3\beta_1}$. But there does not such an element exist in $D_1^{l_1}$. Thus, $V(a^{l_1}_{3m+1}, x^1_{3\beta_1})$ is disequivalent to $V(a^{l_1*}_{3m+1}, x^{1*}_{3\beta_1})$, that is by joining $a^{l_1}_{3m+1}$ with $x^1_{3\beta_1}$ to produce $\Gamma^1_{(m,l_1)}$, $x^{1*}_{3\beta_1}$ is not connected with $a^{l_1*}_{3m+1}$. Therefore, $\Gamma^1_{(m,l_1)}$ is orientally different from its mirror image $\Gamma^{1*}_{(m,l_1)}$. Alternatively, the vertical axis of symmetry does not possess by any of the elements in $F_1^{l_1}$. Hence, there are total

$$2\sum_{k_1=1}^{\beta_2-1}\left|D_1^{l_1}\right| = 6\sum_{k_1=1}^{\beta_2-1}(k_1+2) = 3\left(\beta_2^2+3\beta_2\right), \tag{12}$$

pairs of connecting vertices to produce fragments in $F_1^{l_1}$ and their mirror images. □

Lemma 2. *If we join $x^2_{3\beta_2}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_2+1}$ from Ω_1, then for a fix l_1, there occurs β_1-1 number of distinct fragments and $3(k_2+2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total of $3(\beta_1^2+3\beta_1-4)$ pairs of connecting vertices produces all β_1-1 fragments and their mirror images. (Figure 5).*

We can prove Lemma 2 in a similar way as Lemma 1 by replacing β_1, β_2, k_1, $F_1^{l_1}$, $\Gamma^1_{(k_1,l_1)}$, and $D_1^{l_1}$ by β_2, β_1, k_2, $F_2^{l_1}$, $\Gamma^2_{(k_2,l_1)}$, and $D_2^{l_1}$, respectively.

Lemma 3. *If we join $x^3_{3\beta_3}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_3+1}$ from Ω_1, then for a fix l_1, there occurs β_2 number of distinct fragments and $3(k_3+2)$ pairs of connecting vertices generate each (same) fragment. Furthermore,*

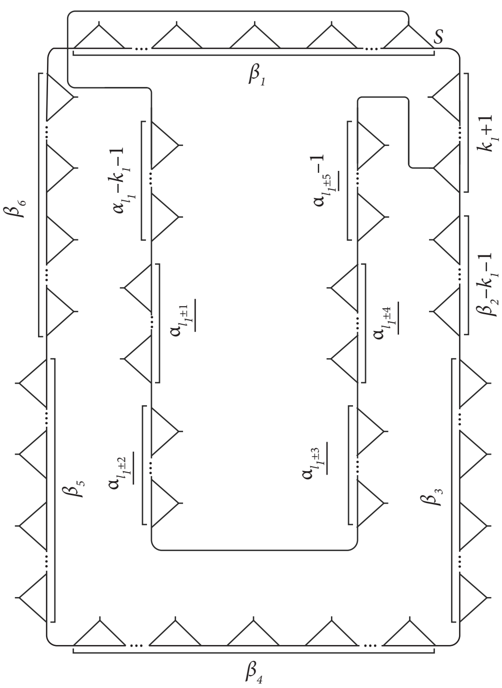

FIGURE 4: Fragments $\Gamma^1_{(k_1,l_1)}$.

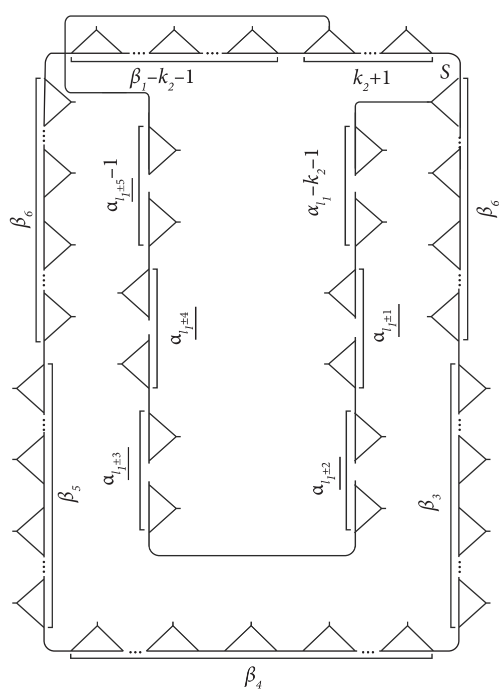

FIGURE 5: Fragments $\Gamma^2_{(k_2,l_1)}$.

a total of $3(\beta_4^2 + 3\beta_4)$ *pairs of connecting vertices produces all* β_4 *fragments and their mirror images (Figure 6).*

We can prove Lemma 3 in a similar way as Lemma 1 by replacing β_1, β_2, k_1, $F_1^{l_1}$, $\Gamma^1_{(k_1,l_1)}$, and $D_1^{l_1}$ by β_3, β_4, k_3, $F_3^{l_1}$, $\Gamma^3_{(k_3,l_1)}$, and $D_3^{l_1}$, respectively.

Lemma 4. *If we join* $x^4_{3\beta_4}$, *the vertex from* Ω_2, *with the vertices* $a^{l_1}_{3k_4+1}$ *from* Ω_1, *then for a fix* l_1, *there occurs* $\beta_3 - 1$ *number of distinct fragments and* $3(k_4 + 2)$ *pairs of connecting vertices generate each (same) fragment. Furthermore, a total of* $3(\beta_3^2 + 3\beta_3 - 4)$ *pairs of connecting vertices produces all* $\beta_3 - 1$ *fragment and their mirror images (Figure 7).*

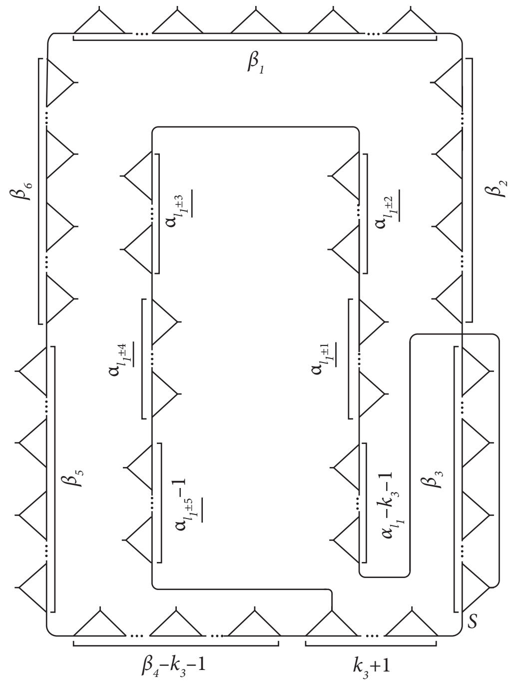

Figure 6: Fragments $\Gamma^{3}_{(k_3,l_1)}$.

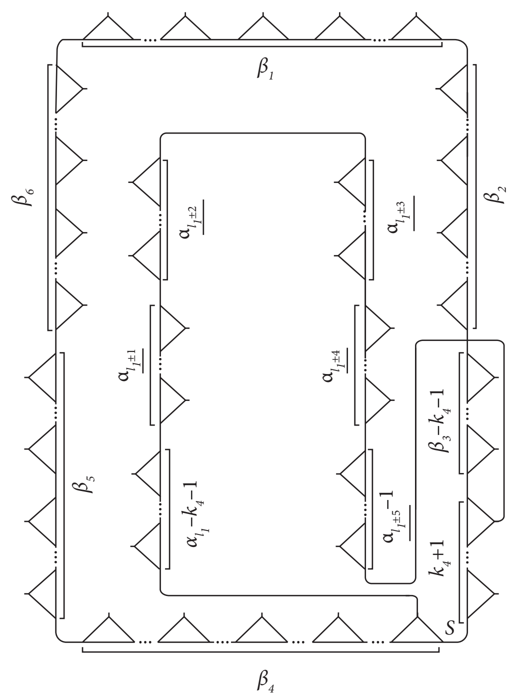

Figure 7: Fragments $\Gamma^{4}_{(k_4,l_1)}$.

We can prove Lemma 4 in a similar way as Lemma 1 by replacing β_1, β_2, k_1, $F_1^{l_1}$, $\Gamma^1_{(k_1,l_1)}$, and $D_1^{l_1}$ by β_4, β_3, k_4, $F_4^{l_1}$, $\Gamma^4_{(k_4,l_1)}$, and $D_4^{l_1}$, respectively.

Lemma 5. *If we join $x^5_{3\beta_5}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_5+1}$ from Ω_1, then for a fix l_1, there occurs β_6 number of distinct fragments and $3(k_5+2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total of $3(\beta_6^2+3\beta_6)$ pairs of connecting vertices produces all β_6 fragments and their mirror images (Figure 8).*

We can prove Lemma 5 in a similar way as Lemma 1 by replacing β_1, β_2, k_1, $F_1^{l_1}$, $\Gamma^1_{(k_1,l_1)}$, and $D_1^{l_1}$ by β_5, β_6, k_5, $F_5^{l_1}$, $\Gamma^5_{(k_5,l_1)}$, and $D_5^{l_1}$, respectively.

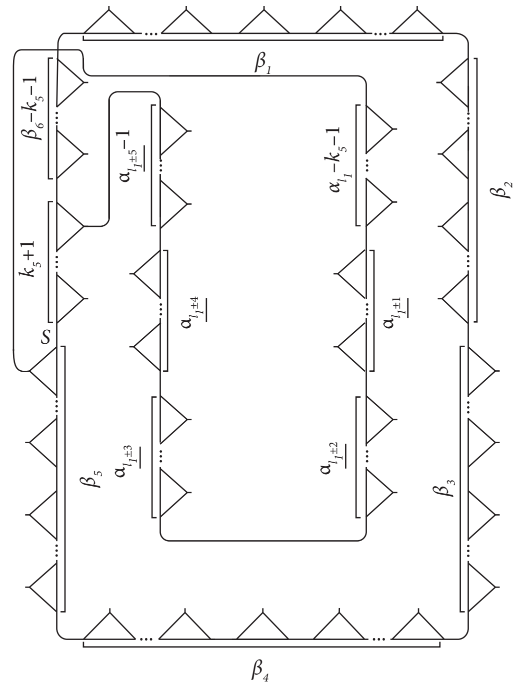

Figure 8: Fragments $\Gamma^5_{(k_5,l_1)}$.

Lemma 6. *If we join $x^6_{3\beta_6}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_6+1}$ from Ω_1, then for a fix l_1, there occurs β_5-1 number of distinct fragments and $3(k_6+2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total of $3(\beta_5^2+3\beta_5-4)$ pairs of connecting vertices produces all β_5-1 fragments and their mirror images (Figure 9).*

We can prove Lemma 6 in a similar way as Lemma 1 by replacing β_1, β_2, k_1, $F^{l_1}_1$, $\Gamma^1_{(k_1,l_1)}$, and $D^{l_1}_1$ by β_6, β_5, k_6, $F^{l_1}_6$, $\Gamma^6_{(k_6,l_1)}$, and $D^{l_1}_6$, respectively.

Lemma 7. *If we join $x^1_{3\beta_1}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_{(7,l_1)}+1}$ from Ω_1, then for a fix l_1, there occurs $\alpha_{l_1}-\beta_2-1$ number of distinct fragments and $3(\beta_2+2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total of $6(\beta_2+2)(\alpha_{l_1}-\beta_2-1)$ pairs of connecting vertices produces all $\alpha_{l_1}-\beta_2-1$ fragments and their mirror images.*

Proof. For a fix l_1, let us join $x^1_{3\beta_1}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_{(7,l_1)}+1}$ from Ω_1 and attain a set of fragments $F^{l_1}_7=\left\{\Gamma^7_{(k_{(7,l_1)},l_1)};k_{(7,l_1)}=\beta_2+1,\beta_2+2,\cdots,\alpha_{l_1}-1\right\}$ (Figure 10), where the word

$$\left(ax^{-1}\right)^{\beta_2}(ax)^{\beta_3}\left(ax^{-1}\right)^{\beta_4}(ax)^{\beta_5}\left(ax^{-1}\right)^{\beta_6}(ax)^{\beta_1}, \quad (13)$$

fixing the vertex $x^1_{3\beta_1}$ and the vertex $a^{l_1}_{3k_{(7,l_1)}+1}$ is fixed by the word

$$\left(ax^{-1}\right)^{k_{(7,l_1)}}(ax)^{\underline{\alpha_{l_1\pm5}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_1\pm4}}}(ax)^{\underline{\alpha_{l_1\pm3}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_1\pm2}}}(ax)^{\underline{\alpha_{l_1\pm1}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_1}}-k_{(7,l_1)}}. \quad (14)$$

Then, the set

$$D^{l_1}_7=\left\{\begin{array}{c} e,x,x^{-1},a,ax,ax^{-1},\left(ax^{-1}\right)^1a,\left(ax^{-1}\right)^1ax,\left(ax^{-1}\right)^2 \\ ,\cdots,\left(ax^{-1}\right)^{\beta_2}a,\left(ax^{-1}\right)^{\beta_2}ax,\left(ax^{-1}\right)^{\beta_2+1}\end{array}\right\}, \quad (15)$$

contains words such that if u is any word from $D^{l_1}_7$ implies $(a^{l_1}_{3k_{(7,l_1)}+1})u$ and $(x^1_{3\beta_1})u$ lies on Ω_1 and Ω_2, respectively. Thus, the fragment $\Gamma^7_{(k_{(7,l_1)},l_1)}$ has $|D^{l_1}_7|=3(\beta_2+2)$ pairs of connecting vertices in Ω_1 and Ω_2. In other words, for each $u\in D^{l_1}_7$, the connection of $(a^{l_1}_{3k_{(7,l_1)}+1})u$ and $(x^1_{3\beta_1})u$ give the same fragment $\Gamma^7_{((7,l_1),l_1)}$.

Now, we prove that all the elements in $F^{l_1}_7$ are distinct and no one is the mirror image of itself. For this, let $\Gamma^1_{(m,l_1)}$

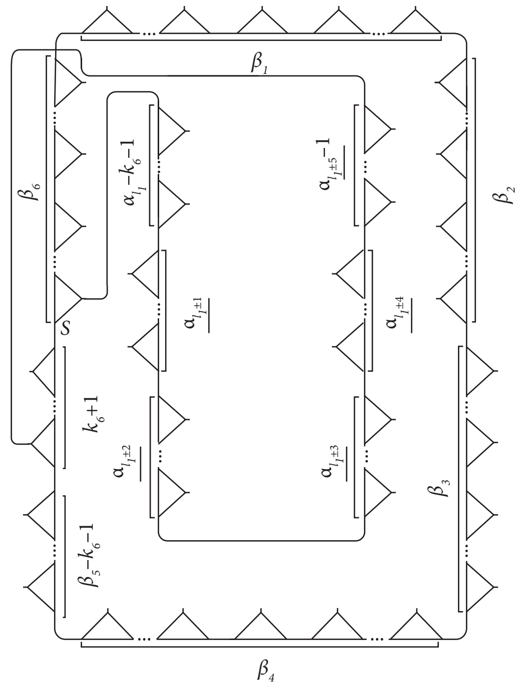

Figure 9: Fragments $\Gamma^{6}_{(k_6,l_1)}$.

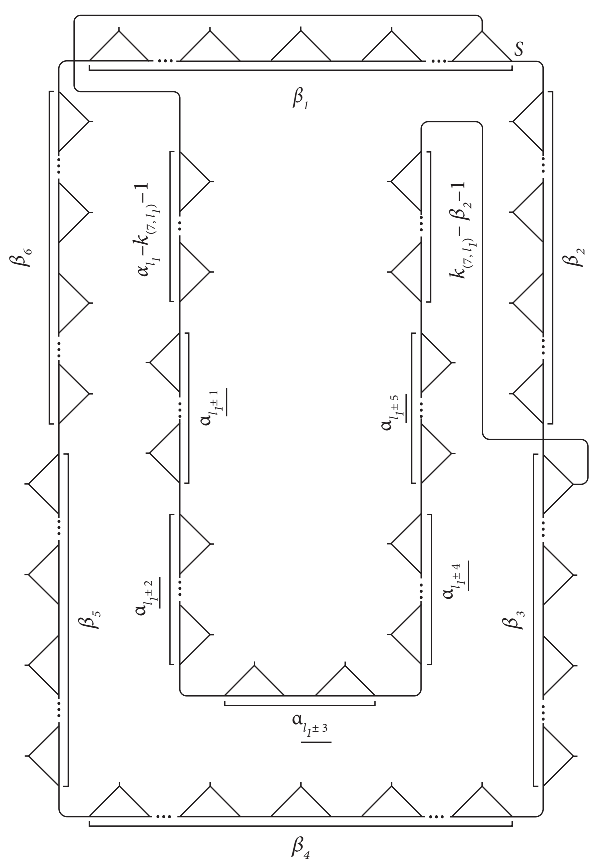

Figure 10: Fragments $\Gamma^{7}_{(k_{(7,l_1)},l_1)}$.

and $\Gamma^1_{(n,l_1)}$ be any two fragments from $F_7^{l_1}$. Then, $\Gamma^7_{(m,l_1)}$ is set up by connecting $a^{l_1}_{3m+1}$ and $x^1_{3\beta_1}$ and $\Gamma^7_{(n,l_1)}$ is set up by $a^{l_1}_{3n+1}$ and $x^1_{3\beta_1}$. If $V(a^{l_1}_{3m+1}, x^1_{3\beta_1}) \sim V(a^{l_1}_{3n+1}, x^1_{3\beta_1})$, then there exists an element $u \in D_7^{l_1}$ such that $(a^{l_1}_{3m+1})u = a^{l_1}_{3n+1}$ and $(x^1_{3\beta_1})u = x^1_{3\beta_1}$. $e \in D_7^{l_1}$ is the only word such that $(x^1_{3\beta_1})e = x^1_{3\beta_1}$ but $(a^{l_1}_{3m+1})e \neq a^{l_1}_{3n+1}$. Thus, $V(a^{l_1}_{3m+1}, x^1_{3\beta_1})$ is disequivalent to $V(a^{l_1}_{3n+1}, x^1_{3\beta_1})$, that is by joining $a^{l_1}_{3m+1}$ with $x^1_{3\beta_1}$ to produce $\Gamma^7_{(m,l_1)}$, $x^1_{3\beta_1}$ is not connected with $a^{l_1}_{3n+1}$. Now, if $V(a^{l_1}_{3m+1}, x^1_{3\beta_1}) \sim V(a^{l_1 *}_{3n+1}, x^{1*}_{3\beta_1})$, then there exists an element $u \in D_7^{l_1}$ such that $(a^{l_1}_{3m+1})u = a^{l_1 *}_{3n+1}$ and $(x^1_{3\beta_1})u = x^{1*}_{3\beta_1}$. But there does not such an element exist in $D_7^{l_1}$. Thus, $V(a^{l_1}_{3m+1}, x^1_{3\beta_1})$ is disequivalent to $V(a^{l_1 *}_{3n+1}, x^{1*}_{3\beta_1})$, that is by joining $a^{l_1}_{3m+1}$ with $x^1_{3\beta_1}$ to produce $\Gamma^7_{(m,l_1)}$, $x^{1*}_{3\beta_1}$ is not connected with $a^{l_1 *}_{3n+1}$. Hence, all elements in $F_7^{l_1}$ are distinct. Since, $F_7^{l_1} = \left\{\Gamma^1_{(k_{(7,l_1)},l_1)}; k_{(7,l_1)} = \beta_2 + 1, \beta_2 + 2, \cdots, \alpha_{l_1} - 1\right\}$ implying that $|F_7^{l_1}| = \alpha_{l_1} - \beta_2 - 1$.

If $V(a^{l_1}_{3m+1}, x^1_{3\beta_1}) \sim V(a^{l_1 *}_{3m+1}, x^{1*}_{3\beta_1})$, then there exists an element $u \in D_7^{l_1}$ such that $(a^{l_1}_{3m+1})u = a^{l_1 *}_{3m+1}$ and $(x^1_{3\beta_1})u = x^{1*}_{3\beta_1}$. But there does not such an element exist in $D_7^{l_1}$. Thus, $V(a^{l_1}_{3m+1}, x^1_{3\beta_1})$ is disequivalent to $V(a^{l_1 *}_{3m+1}, x^{1*}_{3\beta_1})$, that is by joining $a^{l_1}_{3m+1}$ with $x^1_{3\beta_1}$ to produce $\Gamma^7_{(m,l_1)}$, $x^{1*}_{3\beta_1}$ is not connected with $a^{l_1 *}_{3m+1}$. Therefore, $\Gamma^7_{(m,l_1)}$ is orientally different from its mirror image $\Gamma^{7*}_{(m,l_1)}$. Alternatively, the vertical axis of symmetry does not possess by any of the elements in $F_7^{l_1}$. Hence, there are total

$$2\left|F_7^{l_1}\right|\left|D_7^{l_1}\right| = 6(\beta_2 + 2)\left(\alpha_{l_1} - \beta_2 - 1\right), \tag{16}$$

pairs of connecting vertices to produce fragments in $F_7^{l_1}$ and their mirror images. □

Lemma 8. *If we join $x^4_{3\beta_4}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_{(8,l_1)}+1}$ from Ω_1, then for a fix l_1, there occurs $\alpha_{l_1} - \beta_3 - 1$ number of distinct fragments and $3(\beta_3 + 2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total of $6(\beta_3 + 2)(\alpha_{l_1} - \beta_3 - 1)$ pairs of connecting vertices produces all $\alpha_{l_1} - \beta_3 - 1$ fragments and their mirror images (Figure 11).*

We can prove Lemma 8 in a similar way as Lemma 7 by replacing $\beta_1, \beta_2, k_{(7,l_1)}, F_7^{l_1}, \Gamma^7_{(k_{(7,l_1)},l_1)}$, and $D_7^{l_1}$ by $\beta_4, \beta_3, k_{(8,l_1)}, F_8^{l_1}, \Gamma^8_{(k_{(8,l_1)},1)}$, and $D_8^{l_1}$, respectively.

Lemma 9. *If we join $x^2_{3\beta_2}$, the vertex from Ω_2, with the vertices $a^{l_2}_{3k_{(9,l_2)}+1}$ from Ω_1, then for a fix l_2, there occurs $\alpha_{l_2} - \beta_1 - 1$ number of distinct fragments and $3(\beta_1 + 2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total of $6(\beta_1 + 2)(\alpha_{l_2} - \beta_1 - 1)$ pairs of connecting vertices produces all $\alpha_{l_2} - \beta_1 - 1$ fragments and their mirror images (Figure 12).*

We can prove Lemma 9 in a similar way as Lemma 7 by replacing $l_1, \beta_1, \beta_2, k_{(7,l_1)}, F_7^{l_1}, \Gamma^7_{(k_{(7,l_1)},l_1)}$, and $D_7^{l_1}$ by $l_2, \beta_2, \beta_1, k_{(9,l_2)}, F_9^{l_2}, \Gamma^9_{(k_{(9,l_2)},l_2)}$, and $D_9^{l_2}$, respectively.

Lemma 10. *If we join $x^3_{3\beta_3}$, the vertex from Ω_2, with the vertices $a^{l_2}_{3k_{(10,l_2)}+1}$ from Ω_1, then for a fix l_2, there occurs $\alpha_{l_2} - \beta_4 - 1$ number of distinct fragments and $3(\beta_4 + 2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total of $6(\beta_4 + 2)(\alpha_{l_2} - \beta_4 - 1)$ pairs of connecting vertices produces all $\alpha_{l_2} - \beta_4 - 1$ fragments and their mirror images (Figure 13).*

We can prove Lemma 10 in a similar way as Lemma 7 by replacing $l_1, \beta_1, \beta_2, k_{(7,l_1)}, F_7^{l_1}, \Gamma^7_{(k_{(7,l_1)},l_1)}$, and $D_7^{l_1}$ by $l_2, \beta_3, \beta_4, k_{(10,l_2)}, F_{10}^{l_2}, \Gamma^{10}_{(k_{(10,l_2)},l_2)}$, and $D_{10}^{l_2}$, respectively.

Lemma 11. *If we join $x^2_{3\beta_2}$, the vertex from Ω_2, with the vertices $a^{l_3}_{3k_{(11,l_3)}+1}$ from Ω_1, then for a fix l_3, there occurs $(\alpha_{l_3} - \beta_1 - \psi_1^{l_3})/2$ number of distinct fragments and $3(\beta_1 + 2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total $3(\beta_1 + 2)(\alpha_{l_3} - \beta_1 - 1)$ pairs of connecting vertices produces all $(\alpha_{l_3} - \beta_1 - \psi_1^{l_3})/2$ fragments and their mirror images.*

Proof. For a fix l_3, let us join $x^2_{3\beta_2}$, the vertex from Ω_2, with the vertices $a^{l_3}_{3k_{(11,l_3)}+1}$ from Ω_1 and attain a set of fragments $F_{11}^{l_3} = \{\Gamma^{11}_{(k_{(11,l_3)},l_3)}; k_{(11,l_3)} = \beta_1 + 1, \beta_1 + 2, \cdots, (\alpha_{l_3} + \beta_1 - \psi_1^{l_3})/2\}$ (Figure 14), where the word

$$\left(ax^{-1}\right)^{\beta_1}(ax)^{\beta_6}\left(ax^{-1}\right)^{\beta_5}(ax)^{\beta_4}\left(ax^{-1}\right)^{\beta_3}(ax)^{\beta_2}, \tag{17}$$

fixing the vertex $x^2_{3\beta_2}$ and the vertex $a^{l_3}_{3k_{(11,l_3)}+1}$ is fixed by the word

$$\left(ax^{-1}\right)^{k_{(11,l_3)}}(ax)^{\underline{\alpha_{l_3 \pm 5}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_3 \pm 4}}}(ax)^{\underline{\alpha_{l_3 \pm 3}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_3 \pm 2}}}(ax)^{\underline{\alpha_{l_3 \pm 1}}}\left(ax^{-1}\right)^{\alpha_{l_3} - k_{(11,l_3)}}. \tag{18}$$

Then, the set

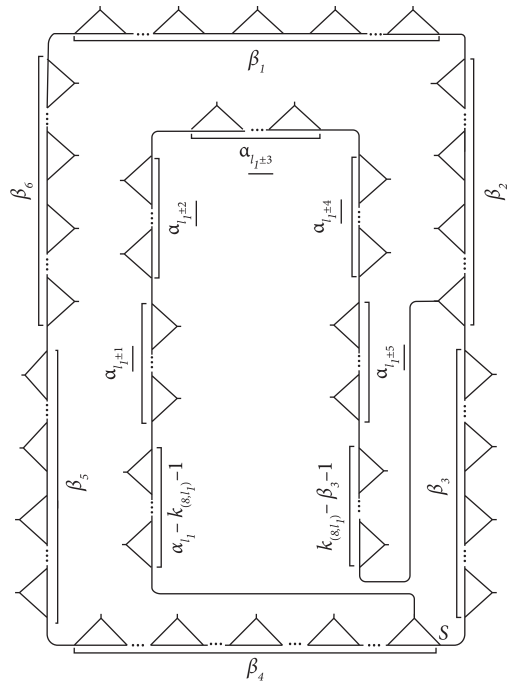

Figure 11: Fragments $\Gamma^{8}_{(k_{(8,l_1)},1)}$.

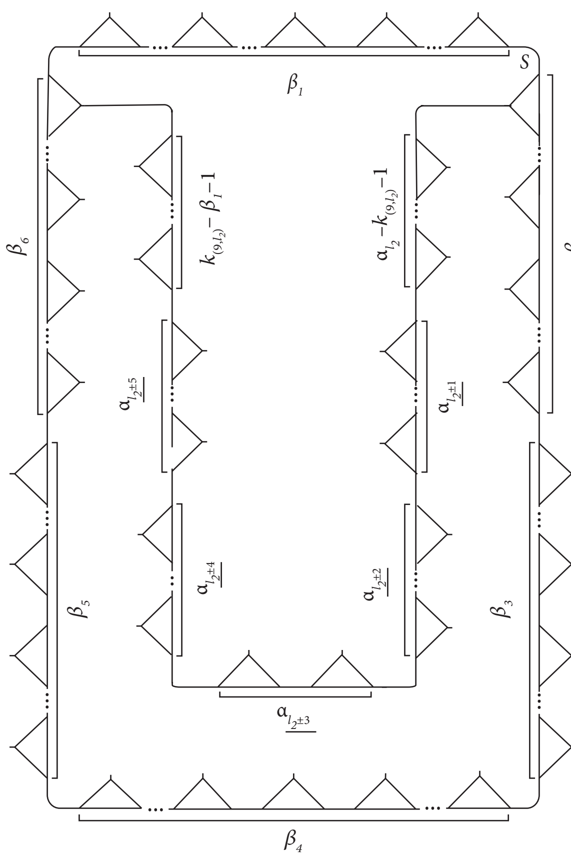

Figure 12: Fragments $\Gamma^{9}_{(k_{(9,l_2)},l_2)}$.

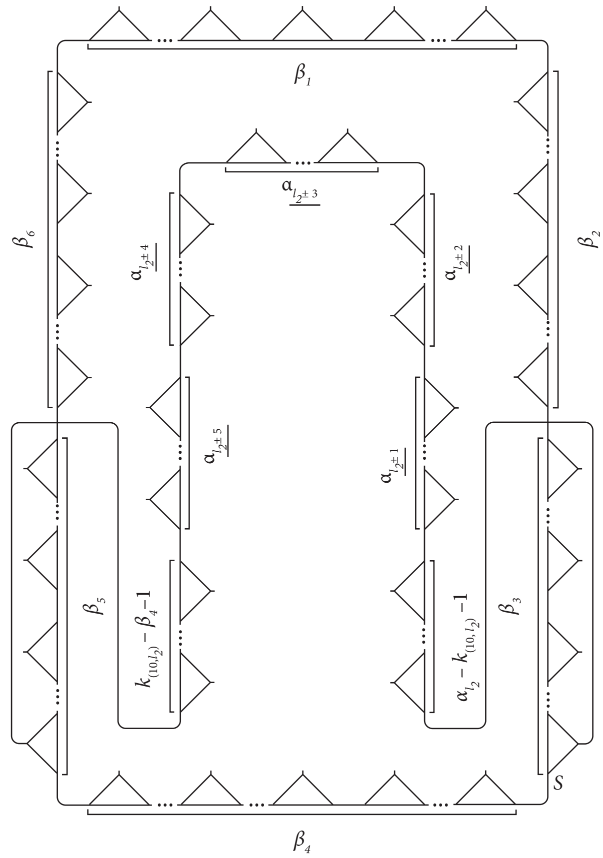

Figure 13: Fragments $\Gamma^{10}_{(k_{(10,l_2)},l_2)}$.

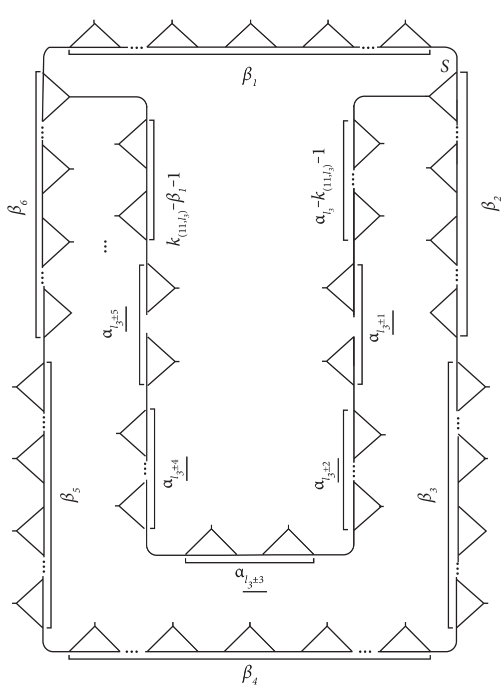

Figure 14: Fragments $\Gamma^{11}_{(k_{(11,l_3)},l_3)}$.

$$D_{11}^{l_3} = \left\{ \begin{array}{c} e, x, x^{-1}, a, ax, ax^{-1}, \left(ax^{-1}\right)^1 a, \left(ax^{-1}\right)^1 ax, \left(ax^{-1}\right)^2 \\ , \cdots, \left(ax^{-1}\right)^{\beta_1} a, \left(ax^{-1}\right)^{\beta_1} ax, \left(ax^{-1}\right)^{\beta_1+1} \end{array} \right\}. \tag{19}$$

contains words such that if u is any word from $D_{11}^{l_3}$ implies $(a_{3k_{(11,l_3)}+1}^{l_3})u$ and $(x_{3\beta_2}^2)u$ lies on Ω_1 and Ω_2 respectively. Thus, the fragment $\Gamma_{(k_{(11,l_3)},l_3)}^{11}$ has $|D_{11}^{l_3}| = 3(\beta_1 + 2)$ pairs of connecting vertices in Ω_1 and Ω_2. In other words, for each $u \in D_{11}^{l_3}$, the connection of $(a_{3k_{(11,l_3)}+1}^{l_3})u$ and $(x_{3\beta_2}^2)u$ give the same fragment $\Gamma_{(k_{(11,l_3)},l_3)}^{11}$.

Now, we prove that (i) for $\alpha_{l_3} + \beta_1 = 1 \pmod 2$, all the elements in $F_{11}^{l_3}$ are distinct and no one is the mirror image of itself and (ii) for $\alpha_{l_3} + \beta_1 = 0 \pmod 2$, all the elements in $F_{11}^{l_3}$ are distinct and $\Gamma_{((\alpha_{l_3}+\beta_1)/2,l_3)}^{11}$ is the only fragment which is orientally the same as its mirror image. For this, let $\Gamma_{(m,l_3)}^{11}$ and $\Gamma_{(n,l_3)}^{11}$ be any two fragments from $F_{11}^{l_3}$. Then, $\Gamma_{(m,l_3)}^{11}$ is set up by connecting $a_{3m+1}^{l_3}$ and $x_{3\beta_2}^2$ and $\Gamma_{(n,l_3)}^{11}$ is set up by $a_{3n+1}^{l_3}$ and $x_{3\beta_2}^2$. If $V(a_{3m+1}^{l_3}, x_{3\beta_2}^2) \sim V(a_{3n+1}^{l_3}, x_{3\beta_2}^2)$, then there exists an element $u \in D_{11}^{l_3}$ such that $(a_{3m+1}^{l_3})u = a_{3n+1}^{l_3}$ and $(x_{3\beta_2}^2)u = x_{3\beta_2}^2$. $e \in D_{11}^{l_3}$ is the only word such that $(x_{3\beta_2}^2)e = x_{3\beta_2}^2$ but $(a_{3m+1}^{l_3})e \neq a_{3n+1}^{l_3}$. Thus, $V(a_{3m+1}^{l_3}, x_{3\beta_2}^2)$ is disequivalent to $V(a_{3n+1}^{l_3}, x_{3\beta_2}^2)$, that is by joining $a_{3m+1}^{l_3}$ with $x_{3\beta_2}^2$ to produce $\Gamma_{(m,l_3)}^{11}$, $x_{3\beta_2}^2$ is not connected with $a_{3n+1}^{l_3}$. Now, if $V(a_{3m+1}^{l_3}, x_{3\beta_2}^2) \sim V(a_{3n+1}^{l_3 *}, x_{3\beta_2}^{2 *})$ then there exist an element $u \in D_{11}^{l_3}$ such that $(a_{3m+1}^{l_3})u = a_{3n+1}^{l_3 *}$ and $(x_{3\beta_2}^2)u = x_{3\beta_2}^{2 *}$. $(ax^{-1})^{\beta_1} a \in D_{11}^{l_3}$ is the only word such that $(x_{3\beta_2}^2)(ax^{-1})^{\beta_1} a = x_{3\beta_2}^{2 *}$ but $(a_{3m+1}^{l_3})(ax^{-1})^{\beta_1} a = a_{3(\alpha_{l_3}+\beta_1-m)+1}^{l_3 *}$. This implies that for $n = \alpha_{l_3} + \beta_1 - m$, $V(a_{3m+1}^{l_3}, x_{3\beta_2}^2)$ is equivalent to $V(a_{3n+1}^{l_3 *}, x_{3\beta_2}^{2 *})$, that is by joining $a_{3m+1}^{l_3}$ with $x_{3\beta_2}^2$ to produce $\Gamma_{(m,l_3)}^{11}$, $x_{3\beta_2}^{2 *}$ is connected with $a_{3n+1}^{l_3 *}$. So, the fragments $\Gamma_{(m,l_3)}^{11}$ and $\Gamma_{(n,l_3)}^{11}$ are the mirror images of each other if and only if $n = \alpha_{l_3} + \beta_1 - m$ and the fragment $\Gamma_{(m,l_3)}^{11}$ is the mirror image of itself if and only if $m = \alpha_{l_3} + \beta_1 - m$, that is $m = \alpha_{l_3} + \beta_1/2$. Now,

(1) if $\alpha_{l_3} + \beta_1$ is odd, then $\forall m \in \Gamma_{(k_{(11,l_3)},l_3)}^{11}$, $\alpha_{l_3} + \beta_1 - m > (\alpha_{l_3} + \beta_1 - 1)/2$ implying that $\Gamma_{(\alpha_{l_3}+\beta_1-m,l_3)}^{11} \notin F_{11}^{l_3}$. This indicates that no fragment in $F_{11}^{l_3}$ is the mirror image of others. Hence, all the elements in $F_{11}^{l_3}$ are distinct. Since

$$F_{11}^{l_3} = \left\{ \Gamma_{\left(k_{(11,l_3)},l_3\right)}^{11}; k_{(11,l_3)} = \beta_1 + 1, \beta_1 + 2, \cdots, \frac{\alpha_{l_3} + \beta_1 - \psi_1^{l_3}}{2} \right\}, \tag{20}$$

implying that

$$\left|F_{11}^{l_3}\right| = \frac{\alpha_{l_3} - \beta_1 - 1}{2}. \tag{21}$$

Also, $\alpha_{l_3} + \beta_1/2 > \alpha_{l_3} + \beta_1 - 1/2$ implies no fragment in $F_{11}^{l_3}$ is the mirror image of itself. Hence, there are

$$2\left|F_{11}^{l_3}\right|\left|D_{11}^{l_3}\right| = 3(\beta_1 + 2)\left(\alpha_{l_3} - \beta_1 - 1\right). \tag{22}$$

pairs of connecting vertices to produce fragments in $F_{11}^{l_3}$ and their mirror images.

(2) If $\alpha_{l_3} + \beta_1$ is even, then $\forall m \in k_{(11,l_3)}/\{(\alpha_{l_3} + \beta_1)/2\}$, $\alpha_{l_3} + \beta_1 - m \notin k_{(11,l_3)}$ implying that $\Gamma_{(\alpha_{l_3}+\beta_1-m,l_3)}^{11} \notin F_{11}^{l_3}$. This indicates that no fragment in $F_{11}^{l_3}$ is the mirror image of others. Hence, all the elements in $F_{11}^{l_3}$ are distinct. Since

$$F_{11}^{l_3} = \left\{ \Gamma_{\left(k_{(11,l_3)},l_3\right)}^{11}; k_{(11,l_3)} = \beta_1 + 1, \beta_1 + 2, \cdots, \frac{\alpha_{l_3} + \beta_1 - \psi_1^{l_3}}{2} \right\}, \tag{23}$$

implying that

$$\left|F_{11}^{l_3}\right| = \frac{\alpha_{l_3} - \beta_1}{2}. \tag{24}$$

For $m = (\alpha_{l_3} + \beta_1)/2$, we have $\alpha_{l_3} + \beta_1 - m = (\alpha_{l_3} + \beta_1)/2 \in k_{(11,l_3)}$ implies $\Gamma_{((\alpha_{l_3}+\beta_1)/2,l_3)}^{11}$ is the mirror image of itself. Hence, there are

$$2\left(\left|F_{11}^{l_3}\right| - 1\right)\left|D_{11}^{l_3}\right| + \left|D_{11}^{l_3}\right| = 3(\beta_1 + 2)\left(\alpha_{l_3} - \beta_1 - 1\right), \tag{25}$$

pairs of connecting vertices to produce fragments in $F_{11}^{l_3}$ and their mirror images. □

Lemma 12. *If we join $x_{3\beta_3}^3$, the vertex from Ω_2, with the vertices $a_{3k_{(12,l_3)}+1}^{l_3}$ from Ω_1, then for a fix l_3, there occurs $(\alpha_{l_3} -$*

$\beta_4 - \psi_2^{l_3})/2$ number of distinct fragments and $3(\beta_4+2)$ pairs of connecting vertices generate each (same) fragment. Furthermore, a total of $3(\beta_4+2)(\alpha_{l_3}-\beta_4-1)$ pairs of connecting vertices produces all $(\alpha_{l_3}-\beta_4-\psi_2^{l_3})/2$ fragments and their mirror images (Figure 15).

We can prove Lemma 12 in a similar way as Lemma 11 by replacing $\beta_1, \beta_2, \psi_1^{l_3}, k_{(11,l_3)}, F_{11}^{l_3}, \Gamma^{11}_{(k_{(11,l_3)},l_3)}$, and $D_{11}^{l_3}$ by β_4, β_3, $\psi_2^{l_3}, k_{(12,l_3)}, F_{12}^{l_3}, \Gamma^{12}_{(k_{(12,l_3)},l_3)}$, and $D_{12}^{l_3}$, respectively

Lemma 13. *If we join $x^1_{3k_{14}}$, the vertex from Ω_2, with the vertices $a^{l_3}_{3k_{(13,l_3)}+1}$ from Ω_1, then for a fix l_3, there occurs $\begin{cases} (1/2)(\alpha_{l_3}-1)(\beta_1-1) & \text{if } \beta_1 = 0 (\text{mod } 2) \\ (1/2)[(\alpha_{l_3}-1)(\beta_1-1)+1-\psi_3^{l_3}] & \text{if } \beta_1 = 1(\text{mod } 2) \end{cases}$, number of distinct fragments and 6 pairs of connecting vertices generates each (same) fragment. Furthermore, a total of $6(\alpha_{l_3}-1)(\beta_1-1)$ pairs of connecting vertices produces all these fragments and their mirror images.*

Proof. For a fix l_3, let us join $x^1_{3k_{14}}$, the vertex from Ω_2, with the vertices $a^{l_3}_{3k_{(13,l_3)}+1}$ from Ω_1 and attain a set of fragments $F_{13}^{l_3} = \{\Gamma^{13}_{(k_{(13,l_3)},k_{14},l_3)};$ $k_{(13,l_3)} = \begin{cases} 1,2,3,\cdots,\alpha_{l_3}-\psi_3^{l_3}/2 & if\ k_{14} = \beta_1/2\,(k_{15}=\beta_4/2) \\ 1,2,3,\cdots,\alpha_{l_3}-1 & \text{otherwise} \end{cases};$ $k_{14} = 1,2,3,\cdots,(\beta_1-\psi_1)/2\}$ (Figure 16), where the word

$$(ax)^{\beta_1-k_{14}}\left(ax^{-1}\right)^{\beta_2}(ax)^{\beta_3}\left(ax^{-1}\right)^{\beta_4}(ax)^{\beta_5}\left(ax^{-1}\right)^{\beta_6}(ax)^{k_{14}}. \tag{26}$$

fixing the vertex $x^1_{3k_{14}}$ and the vertex $a^{l_3}_{3k_{(13,l_3)}+1}$ is fixed by the word

$$\left(ax^{-1}\right)^{k_{(13,l_3)}}(ax)^{\underline{\alpha_{l_3\pm5}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_3\pm4}}}(ax)^{\underline{\alpha_{l_3\pm3}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_3\pm2}}}(ax)^{\underline{\alpha_{l_3\pm1}}}\left(ax^{-1}\right)^{\alpha_{l_3}-k_{(13,l_3)}}. \tag{27}$$

Then, the set

$$D_{13}^{l_3} = \left\{e, x, x^{-1}, a, ax, ax^{-1}\right\}, \tag{28}$$

contains words such that if u is any word from $D_{13}^{l_3}$ implies $(a^{l_3}_{3k_{(13,l_3)}+1})u$ and $(x^1_{3k_{14}})u$ lies on Ω_1 and Ω_2, respectively. Thus, the fragment $\Gamma^{13}_{(k_{(13,l_3)},k_{14},l_3)}$ has $|D_{13}^{l_3}| = 6$ pairs of connecting vertices in Ω_1 and Ω_2. In other words, for each $u \in D_{13}^{l_3}$, the connection of $(a^{l_3}_{3k_{(13,l_3)}+1})u$ and $(x^1_{3k_{14}})u$ gives the same fragment $\Gamma^{13}_{(k_{(13,l_3)},k_{14},l_3)}$.

Now, we prove that (i) for $\beta_1 = 0(\text{mod } 2)$, all the elements in $F_{13}^{l_3}$ are distinct and no one is the mirror image of itself and (ii) for $\beta_1 = 1(\text{mod } 2)$, $\Gamma^{13}_{(m_1,\beta_1/2,l_3)}$ is the mirror image of $\Gamma^{13}_{(\alpha_{l_3}-m_1,\beta_1/2,l_3)}$ for all $m_1 \in k_{(13,l_3)}$ and $\Gamma^{13}_{(\alpha_{l_3}/2,\beta_1/2,l_3)}$ is the mirror image of itself. Let, $\Gamma^{13}_{(m_1,m_2,l_3)}$ and $\Gamma^{13}_{(n_1,n_2,l_3)}$ be any two fragments from $F_{13}^{l_3}$. Then, $\Gamma^{13}_{(m_1,m_2,l_3)}$ is set up by connecting $a^{l_3}_{3m_1+1}$ and $x^1_{3m_2}$ and $\Gamma^{13}_{(n_1,n_2,l_3)}$ is set up by $a^{l_3}_{3n_1+1}$ and $x^1_{3n_2}$. If $V(a^{l_3}_{3m_1+1}, x^1_{3m_2}) \sim V(a^{l_3}_{3n_1+1}, x^1_{3n_2})$, then, there exists an element $u \in D_{13}^{l_3}$ such that $(a^{l_3}_{3m_1+1})u = a^{l_3}_{3n_1+1}$ and $(x^1_{3m_2})u = x^1_{3n_2}$. $ax \in D_{13}^{l_3}$ is the only word such that $(x^1_{3m_2})ax = x^1_{3(m_2+1)}$ but $(a^{l_3}_{3m_1+1})ax \neq a^{l_3}_{3n_1+1}$. Thus, $V(a^{l_3}_{3m_1+1}, x^1_{3m_2})$ is disequivalent to $V(a^{l_3}_{3n_1+1}, x^1_{3n_2})$, that is by joining $a^{l_3}_{3m_1+1}$ with $x^1_{3m_2}$ to produce $\Gamma^{13}_{(m_1,m_2,l_3)}$, $x^1_{3n_2}$ is not connected with $a^{l_3}_{3n_1+1}$.

If $V(a^{l_3}_{3m_1+1}, x^1_{3m_2}) \sim V(a^{l_3\,*}_{3n_1+1}, x^{1\,*}_{3n_2})$, then there exists an element $u \in D_{13}^{l_3}$ such that $(a^{l_3}_{3m_1+1})u = a^{l_3\,*}_{3n_1+1}$ and $(x^1_{3m_2})u = x^{1\,*}_{3n_2}$. $a \in D_{13}^{l_3}$ is the only word such that $(x^1_{3m_2})a = x^{1\,*}_{3(\beta_1-m_2)}$ and $(a^{l_3}_{3m_1+1})a = a^{l_3}_{3(\alpha_{l_3}-m_1)+1}$. This implies that for$n_1 = \alpha_{l_3}-m_1$ and $n_2 = \beta_1 - m_2$, $V(a^{l_3}_{3m_1+1}, x^1_{3m_2}) \sim V(a^{l_3\,*}_{3n_1+1}, x^{1\,*}_{3n_2})$, that is by joining $a^{l_3}_{3m_1+1}$ with $x^1_{3m_2}$ to produce $\Gamma^{13}_{(m_1,m_2,l_3)}$, $x^{1\,*}_{3n_2}$ get also connected with $a^{l_3\,*}_{3n_1+1}$. So, the fragments $\Gamma^{13}_{(m_1,m_2,l_3)}$ and $\Gamma^{13}_{(n_1,n_2,l_3)}$ are the mirror image of each other if and only if $n_1 = \alpha_{l_3}-m_1$ and $n_2 = \beta_1 - m_2$ and the fragment $\Gamma^{13}_{(m_1,m_2,l_3)}$ is the mirror image of itself if and only if $m_1 = \alpha_{l_3}-m_1$ and $m_2 = \beta_1 - m_2$, that is $m_1 = \alpha_{l_3}/2$ and $m_2 = \beta_1/2$. Now,

(1) If β_1 is odd, then for all $m_2 \in k_{14}$, $\beta_1 - m_2 > (\beta_1 - 1)/2$ gives $\Gamma^{13}_{(\alpha_{l_3}-m_1,\beta_1-m_2,l_3)} \notin F_{13}^{l_3}$. This indicates, no fragment in $F_{13}^{l_3}$ is the mirror image of others. Hence, all the elements in $F_{13}^{l_3}$ are distinct. Since,

$$F_{13}^{l_3} = \left\{\Gamma^{13}_{\left(k_{(13,l_3)},l_3\right)}; k_{(13,l_3)} = \begin{cases} 1,2,3,\ldots,\left(\alpha_1-\psi_3^{l_3}\right)/2, & \text{if } k_{14} = \dfrac{\beta_1}{2}\left(k_{15} = \dfrac{\beta_4}{2}\right), \\ 1,2,3,\ldots,\left(\alpha_{l_3}-1\right), & \text{otherwise}; \end{cases}; k_{14} = 1,2,3,\ldots,(\beta_1-\psi_1)/2, \right\}. \tag{29}$$

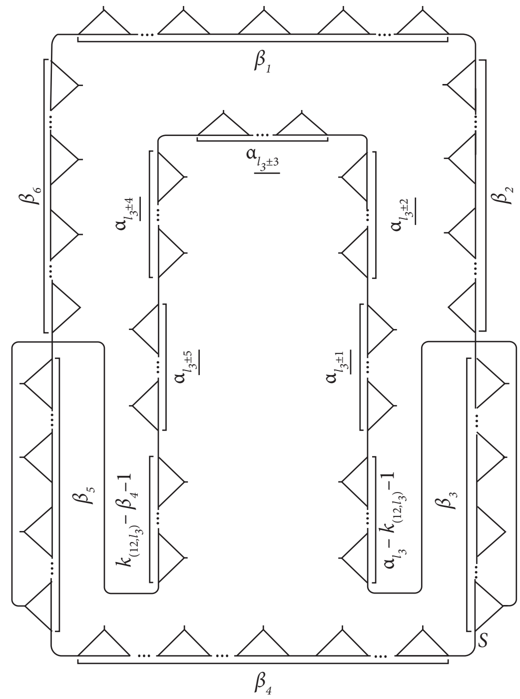

Figure 15: Fragments $\Gamma^{12}_{(k_{(12,l_3)},l_3)}$.

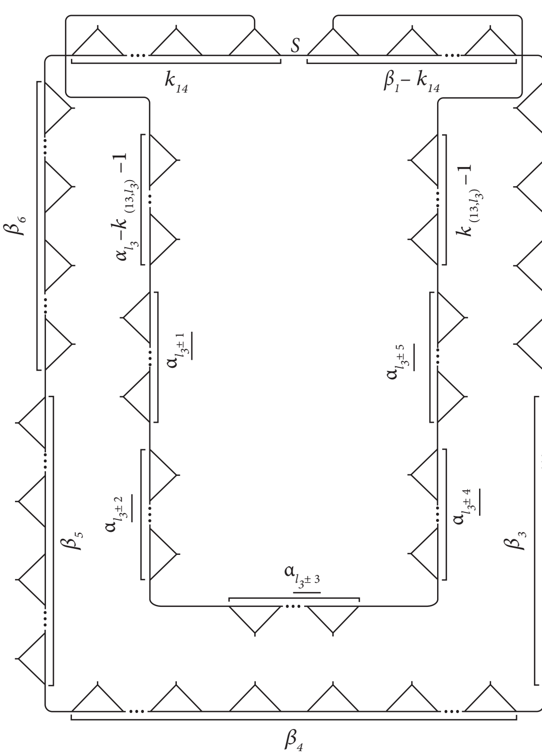

Figure 16: Fragments $\Gamma^{13}_{(k_{(13,l_3)},k_{14},l_3)}$.

implying that

$$\left|F_{13}^{l_3}\right| = \frac{1}{2}\left(\alpha_{l_3}-1\right)\left(\beta_1-\psi_1\right) = \frac{1}{2}\left(\alpha_{l_3}-1\right)\left(\beta_1-1\right). \quad (30)$$

Also, $\beta_1/2 > (\beta_1-1)/2$ implies no fragment in $F_{13}^{l_3}$ is the mirror image of itself. Hence, there are

$$2\left|D_{13}^{l_3}\right|\left|F_{13}^{l_3}\right| = 6\left(\alpha_{l_3}-1\right)\left(\beta_1-1\right), \quad (31)$$

pairs of connecting vertices to produce fragments in $F_{13}^{l_3}$ and their mirror images.

(2) If β_1 is even, then only for $m_2=\beta_1/2$, $\beta_1-m_2=\beta_1/2\in k_{14}$ gives, for all $m_1\in k_{(13,l_3)}\left\{\alpha_{l_3}/2\right\}$, $\alpha_{l_3}-m_1\notin k_{(13,l_3)}$ implying that $\Gamma^{13}_{(\alpha_{l_3}-m_1\beta_1/2,,l_3)}\notin F_{13}^{l_3}$. Hence, all the elements in $F_{13}^{l_3}$ are distinct. Since,

$$F_{13}^{l_3} = \left\{\Gamma^{13}_{\left(k_{(13,l_3)},k_{14},l_3\right)};\, k_{(13,l_3)} = \begin{cases} 1,2,3,\cdots,\dfrac{\alpha_{l_3}-\psi_3^{l_3}}{2} & if\, k_{14}=\dfrac{\beta_1}{2}\left(k_{15}=\dfrac{\beta_4}{2}\right) \\ 1,2,3,\cdots,\alpha_{l_3}-1 & \text{otherwise}\end{cases};\, k_{14}=1,2,3,\cdots,\frac{\beta_1-\psi_1}{2}\right\}, \quad (32)$$

implying that

$$\left|F_{13}^{l_3}\right| = \frac{1}{2}\left(\alpha_{l_3}-1\right)\left(\beta_1-2-\psi_1\right)+\frac{1}{2}\left(\alpha_{l_3}-\psi_3^{l_3}\right) = \frac{1}{2}\left[\left(\alpha_{l_3}-1\right)\left(\beta_1-1\right)+1-\psi_3^{l_3}\right]. \quad (33)$$

Now,

(a) If α_{l_3} is odd, then $\alpha_{l_3}/2\notin k_{(13,l_3)}$ implying that $\Gamma^{13}_{(\alpha_{l_3}/2,\beta_1/2,l_3)}\notin F_{13}^{l_3}$. So, no one is the mirror image of itself. Hence, there are

$$2\left|D_{13}^{l_3}\right|\left|F_{13}^{l_3}\right| = 6\left(\alpha_{l_3}-1\right)\left(\beta_1-1\right), \quad (34)$$

pairs of connecting vertices to produce fragments in $F_{13}^{l_3}$ and their mirror images.

(b) If α_{l_3} is even, then $\alpha_{l_3}/2\in k_{(13,l_3)}$ implying that $\Gamma^{13}_{(\alpha_{l_3}/2,\beta_1/2,l_3)}\in F_{13}^{l_3}$ is orientally the same as its mirror image. Hence, there are

$$2\left|D_{13}^{l_3}\right|\left(\left|F_{13}^{l_3}\right|-1\right)+\left|D_{13}^{l_3}\right|(1) = 6\left(\alpha_{l_3}-1\right)\left(\beta_1-1\right), \quad (35)$$

pairs of connecting vertices to produce fragments in $F_{13}^{l_3}$ and their mirror images. □

Lemma 14. *If we join $x^4_{3k_{15}}$, the vertex from Ω_2, with the vertices $a^{l_3}_{3k_{(13,l_3)}+1}$ from Ω_1, then for a fix l_3, there occurs* $\begin{cases}(1/2)(\alpha_{l_3}-1)(\beta_4-1) & \text{if } \beta_4=0\,(\text{mod }2)\\ (1/2)[(\alpha_{l_3}-1)(\beta_4-1)+1-\psi_3^{l_3}] & \text{if } \beta_4=1\,(\text{mod }2)\end{cases}$, *number of distinct fragments and 6 pairs of connecting vertices generates each (same) fragment. Furthermore, a total of $6(\alpha_{l_3}-1)(\beta_4-1)$ pairs of connecting vertices produces all these fragments and their mirror images (Figure 17).*

We can prove Lemma 14 in the same way as Lemma 13

Lemma 15. *If we join $x^1_{3k_2}$, the vertex from Ω_2, with the vertices $a^{l_2}_{3k_{(16,l_2)}+1}$ from Ω_1, then for a fix l_2, there occurs $(\alpha_{l_2}-1)(\beta_1-1)$ number of distinct fragments and 6 pairs of connecting vertices generates each (same) fragment. Furthermore, a total of $12(\alpha_{l_2}-1)(\beta_1-1)$ pairs of connecting vertices produces all these fragments and their mirror images.*

Proof. For a fix l_2, let us join $x^1_{3k_2}$, the vertex from Ω_2, with the vertices $a^{l_2}_{3k_{(16,l_2)}+1}$ from Ω_1, and attain a set of fragments $F_{15}^{l_2}=\left\{\Gamma^{15}_{(k_{(16,l_2)},k_2,l_2)};\, k_{(16,l_2)}=1,2,3,\cdots,\alpha_{l_2}-1;\, k_2=1,2,\cdots,\beta_1-1\right\}$ (Figure 18), where the word

$$(ax)^{\beta_1-k_2}\left(ax^{-1}\right)^{\beta_2}(ax)^{\beta_3}\left(ax^{-1}\right)^{\beta_4}(ax)^{\beta_5}\left(ax^{-1}\right)^{\beta_6}(ax)^{k_2}, \quad (36)$$

fixing the vertex $x^1_{3k_2}$ and the vertex $a^{l_2}_{3k_{(16,l_2)}+1}$ is fixed by the word

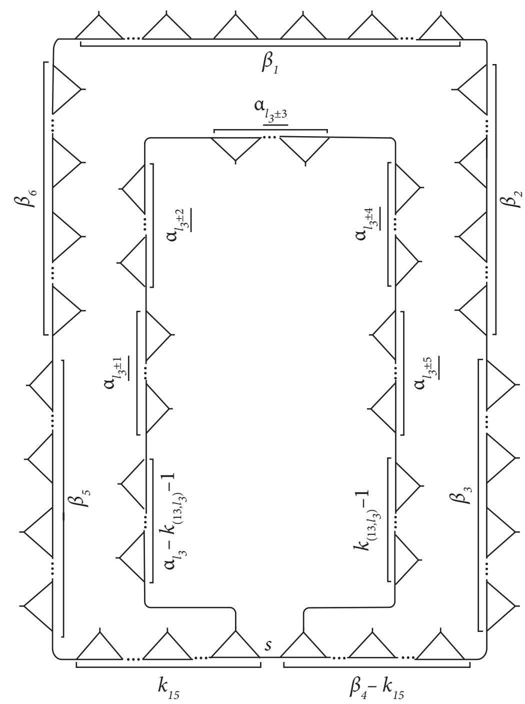

Figure 17: Fragments $\Gamma^{14}_{(k_{(13,l_3)},k_{15},l_3)}$.

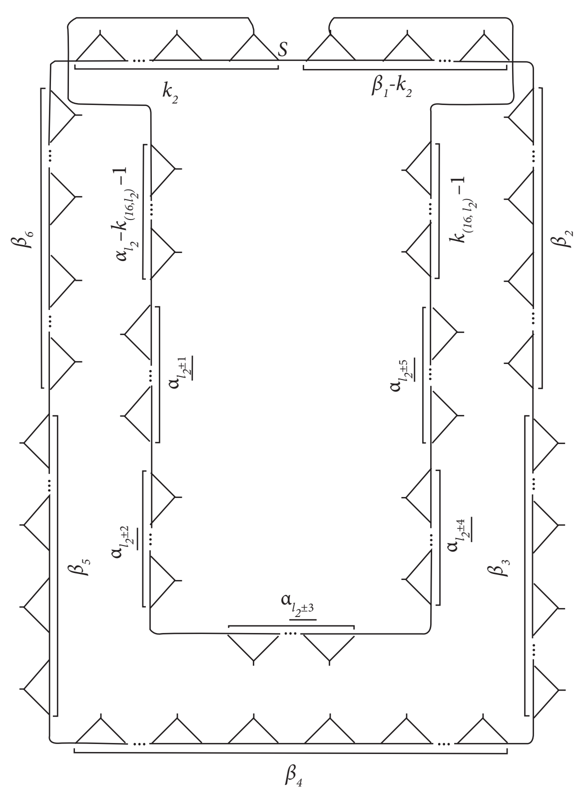

Figure 18: Fragments $\Gamma^{15}_{(k_{(16,l_2)},k_2,l_2)}$.

$$\left(ax^{-1}\right)^{k_{(16,l_2)}}(ax)^{\underline{\alpha_{l_2\pm5}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_2\pm4}}}(ax)^{\underline{\alpha_{l_2\pm3}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_2\pm2}}}(ax)^{\underline{\alpha_{l_2\pm1}}}\left(ax^{-1}\right)^{\alpha_{l_2}-k_{(16,l_2)}}. \tag{37}$$

Then, the set

$$D_{15}^{l_2}=\left\{e,x,x^{-1},a,ax,ax^{-1}\right\}, \tag{38}$$

contains words such that if u is any word from $D_{15}^{l_2}$ implies $(a^{l_2}_{3k_{(16,l_2)}+1})u$ and $(x^1_{3k_2})u$ lies on Ω_1 and Ω_2, respectively. Thus, the fragment $\Gamma^{15}_{(k_{(16,l_2)},k_2,l_2)}$ has $|D_{15}^{l_2}|=6$ pairs of connecting vertices in Ω_1 and Ω_2. In other words, for each $u\in D_{15}^{l_2}$, the connection of $(a^{l_2}_{3k_{(16,l_2)}+1})u$ and $(x^1_{3k_2})u$ gives the same fragment $\Gamma^{15}_{(k_{(16,l_2)},k_2,l_2)}$.

Now, we prove that all the elements in $F_{15}^{l_2}$ are distinct and no one is the mirror image of itself. Let $\Gamma^{15}_{(m_1,m_2,l_2)}$ and $\Gamma^{15}_{(n_1,n_2,l_2)}$ be any two fragments from $F_{15}^{l_2}$. Then, $\Gamma^{15}_{(m_1,m_2,l_2)}$ is set up by connecting $a^{l_2}_{3m_1+1}$ and $x^1_{3m_2}$ and $\Gamma^{12}_{(n_1,n_2,l_2)}$ is set up by $a^{l_2}_{3n_1+1}$ and $x^1_{3n_2}$. If $V(a^{l_2}_{3m_1+1}, x^1_{3m_2})\sim V(a^{l_2}_{3n_1+1}, x^1_{3n_2})$, then there exists an element $u\in D_{15}^{l_2}$ such that $(a^{l_2}_{3m_1+1})u=a^{l_2}_{3n_1+1}$ and $(x^1_{3m_2})u=x^1_{3n_2}$. $ax\in D_{15}^{l_2}$ is the only word such that $(x^1_{3m_2})ax=x^1_{3(m_2+1)}$ but $(a^{l_2}_{3m_1+1})ax\neq a^{l_2}_{3n_1+1}$. Thus, $V(a^{l_2}_{3m_1+1}, x^1_{3m_2})$ is disequivalent to $V(a^{l_2}_{3n_1+1}, x^1_{3n_2})$, that is by joining $a^{l_2}_{3m_1+1}$ with $x^1_{3m_2}$ to produce $\Gamma^{15}_{(m_1,m_2,l_2)}$, $x^1_{3n_2}$ is not connected with $a^{l_2}_{3n_1+1}$. Now, If $V(a^{l_2}_{3m_1+1}, x^1_{3m_2})\sim V(a^{l_2*}_{3n_1+1}, x^{1*}_{3n_2})$, then there exists an element $u\in D_{15}^{l_2}$ such that $(a^{l_2}_{3m_1+1})u=a^{l_2*}_{3n_1+1}$ and $(x^1_{3m_2})u=x^{1*}_{3n_2}$. $a\in D_{15}^{l_2}$ is the only word such that $(x^1_{3m_2})a=x^{1*}_{3(\beta_1-m_2)}$ but $(a^{l_2}_{3m_1+1})a\neq a^{l_2*}_{3n_1+1}$. Thus, $V(a^{l_2}_{3m_1+1}, x^1_{3m_2})$ is dis-equivalent to $V(a^{l_2*}_{3n_1+1}, x^{1*}_{3n_2})$, that is by joining $a^{l_2}_{3m_1+1}$ with $x^1_{3m_2}$ to produce $\Gamma^{15}_{(m_1,m_2,l_2)}$, $x^{1*}_{3n_2}$ is not connected with $a^{l_2*}_{3n_1+1}$. Hence, all the fragments in $F_{15}^{l_2}$ are distinct. Since

$$F_{15}^{l_2}=\left\{\Gamma^{15}_{\left(k_{(16,l_2)},k_2,l_2\right)};k_{(16,l_2)}=1,2,3,\cdots,\alpha_{l_2}-1;\ k_2=1,2,\cdots,\beta_1-1\right\}. \tag{39}$$

Implying that $|F_{15}^{l_2}|=(\alpha_{l_2}-1)(\beta_1-1)$.

Now, if $V(a^{l_2}_{3m_1+1}, x^1_{3m_2})\sim V(a^{l_2*}_{3m_1+1}, x^{1*}_{3m_2})$, then, there exists an element $u\in D_{15}^{l_2}$ such that $(a^{l_2}_{3m_1+1})u=a^{l_2*}_{3m_1+1}$ and $(x^1_{3m_2})u=x^{1*}_{3m_2}$. $a\in D_{15}^{l_2}$ is the only word such that $(x^1_{3m_2})a=x^{1*}_{3(\beta_1-m_2)}$ but $(a^{l_2}_{3m_1+1})a\neq a^{l_2*}_{3m_1+1}$. Thus, $V(a^{l_2}_{3m_1+1}, x^1_{3m_2})$ is dis-equivalent to $V(a^{l_2*}_{3m_1+1}, x^{1*}_{3m_2})$, that is by joining $a^{l_2}_{3m_1+1}$ with $x^1_{3m_2}$ to produce $\Gamma^{15}_{(m_1,m_2,l_2)}$, $x^{1*}_{3m_2}$ is not connected with $a^{l_2*}_{3m_1+1}$. Therefore, $\Gamma^{15}_{(m_1,m_2,l_2)}$ is orientally different from its mirror image $\Gamma^{15*}_{(m_1,m_2,l_2)}$. Alternatively, the vertical axis of symmetry does not possess by any of the elements in $F_{15}^{l_2}$. Hence there are total

$$2\left|D_{15}^{l_2}\right|\left|F_{15}^{l_2}\right|=12\left(\alpha_{l_2}-1\right)(\beta_1-1), \tag{40}$$

pairs of connecting vertices to produce fragments in $F_{15}^{l_2}$ and their mirror images. □

Lemma 16. *If we join $x^4_{3k_{17}}$, the vertex from Ω_2, with the vertices $a^{l_2}_{3k_{(16,l_2)}+1}$ from Ω_1, then for a fix l_2, there occurs $(\alpha_{l_2}-1)(\beta_4-1)$ number of distinct fragments and 6 pairs of connecting vertices generates each (same) fragment. Furthermore, a total of $12(\alpha_{l_2}-1)(\beta_4-1)$ pairs of connecting vertices produces all these fragments and their mirror images (Figure 19).*

We can prove Lemma 16 in a similar way as Lemma 15 by replacing k_2, β_1, $F_{15}^{l_2}$, $\Gamma^{15}_{(k_{(16,l_2)},k_2,l_2)}$, and $D_{15}^{l_2}$ by k_{17}, β_4, $F_{16}^{l_2}$, $\Gamma^{16}_{(k_{(16,l_2)},k_{17},l_2)}$, and $D_{16}^{l_2}$, respectively.

Lemma 17. *If we join $x^2_{3k_{19}}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_{(18,l_1)}+1}$ from Ω_1, then for a fix l_1, there occurs $(\alpha_{l_1}-1)(\beta_2-1)$ number of distinct fragments and 6 pairs of connecting vertices generates each (same) fragment. Furthermore, a total of $12(\alpha_{l_1}-1)(\beta_2-1)$ pairs of connecting vertices produces all these fragments and their mirror images (Figure 20).*

We can prove Lemma 17 in a similar way as Lemma 15 by replacing k_2, $k_{(16,l_2)}$, β_1, l_2, $F_{15}^{l_2}$, $\Gamma^{15}_{(k_{(16,l_2)},k_2,l_2)}$, and $D_{15}^{l_2}$ by k_{19}, $k_{(18,l_1)}$, β_2, l_1, $F_{17}^{l_1}$, $\Gamma^{17}_{(k_{(18,l_1)},k_{19},l_1)}$, and $D_{17}^{l_1}$, respectively.

Lemma 18. *If we join $x^3_{3k_4}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3k_{(18,l_1)}+1}$ from Ω_1, then for a fix l_1, there occurs $(\alpha_{l_1}-1)(\beta_3-1)$ number of distinct fragments and 6 pairs of connecting vertices generates each (same) fragment. Furthermore, a total of $12(\alpha_{l_1}-1)(\beta_3-1)$ pairs of connecting vertices produces all these fragments and their mirror images (Figure 21).*

We can prove Lemma 18 in a similar way as Lemma 15 by replacing k_2, $k_{(16,l_2)}$, β_1, l_2, $F_{15}^{l_2}$, $\Gamma^{15}_{(k_{(16,l_2)},k_2,l_2)}$, and $D_{15}^{l_2}$ by k_4, $k_{(18,l_1)}$, β_3, l_1, $F_{18}^{l_1}$, $\Gamma^{18}_{(k_{(18,l_1)},k_4,l_1)}$, and $D_{18}^{l_1}$, respectively.

Lemma 19. *For a fix l_1, there are $6(\beta_2+\beta_3+2)$ pairs of connecting vertices for the fragment $\Gamma^{19}_{l_1}$ and its mirror image by joining $x^1_{3\beta_1}$, the vertex from Ω_2, with the vertex $a^{l_1}_{3\beta_2+1}$ from Ω_1.*

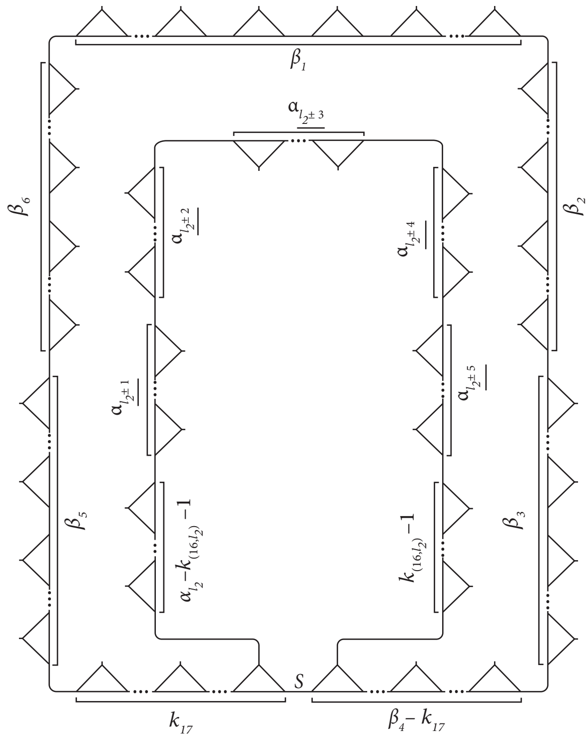

Figure 19: Fragments $\Gamma^{16}_{(k_{(16,l_2)},k_{17},l_2)}$.

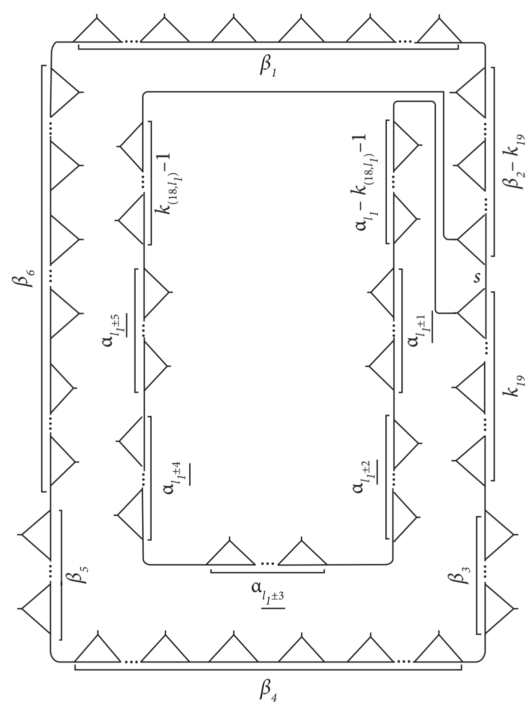

Figure 20: Fragments $\Gamma^{17}_{(k_{(18,l_1)},k_{19},l_1)}$.

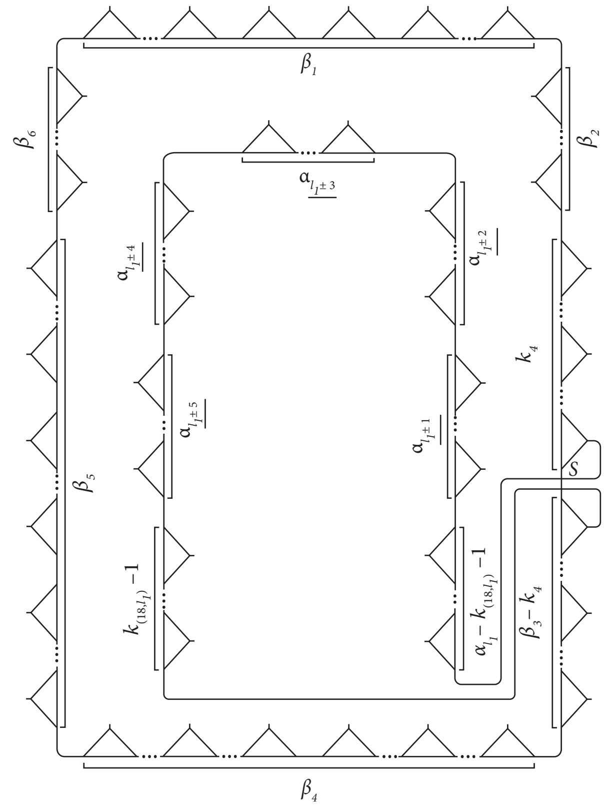

Figure 21: Fragments $\Gamma^{18}_{(k_{(18,l_1)},k_4,l_1)}$.

Proof. For a fix l_1, let us join $x^1_{3\beta_1}$, the vertex from Ω_2, with the vertices $a^{l_1}_{3\beta_2+1}$ from Ω_1 and attain a fragments $\Gamma^{19}_{l_1}$ (Figure 22), where the word

$$\left(ax^{-1}\right)^{\beta_2}(ax)^{\beta_3}\left(ax^{-1}\right)^{\beta_4}(ax)^{\beta_5}\left(ax^{-1}\right)^{\beta_6}(ax)^{\beta_1}, \tag{41}$$

fixing the vertex $x^1_{3\beta_1}$ and the vertex $a^{l_1}_{3\beta_2+1}$ is fixed by the word

$$\left(ax^{-1}\right)^{\beta_2}(ax)^{\underline{\alpha_{l_1\pm5}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_1\pm4}}}(ax)^{\underline{\alpha_{l_1\pm3}}}\left(ax^{-1}\right)^{\underline{\alpha_{l_1\pm2}}}(ax)^{\underline{\alpha_{l_1\pm1}}}\left(ax^{-1}\right)^{\alpha_{l_1}-\beta_2}. \tag{42}$$

Then, the set

$$D^{l_1}_{19}=\left\{\begin{array}{c} e,x,x^{-1},a,ax,ax^{-1},\left(ax^{-1}\right)^1a,\left(ax^{-1}\right)^1ax,\left(ax^{-1}\right)^2,\cdots,\left(ax^{-1}\right)^{\beta_2-1}a,\left(ax^{-1}\right)^{\beta_2-1}ax,\\ \left(ax^{-1}\right)^{\beta_2},\left(ax^{-1}\right)^{\beta_2}a,\left(ax^{-1}\right)^{\beta_2}ax,\left(ax^{-1}\right)^{\beta_2+1},\left(ax^{-1}\right)^{\beta_2}(ax)^1a,\left(ax^{-1}\right)^{\beta_2}(ax)^2,\\ \left(ax^{-1}\right)^{\beta_2}(ax)^1\left(ax^{-1}\right),\cdots\left(ax^{-1}\right)^{\beta_2}(ax)^{\beta_3}a,\left(ax^{-1}\right)^{\beta_2}(ax)^{\beta_3+1},\left(ax^{-1}\right)^{\beta_2}(ax)^{\beta_3}\left(ax^{-1}\right)\end{array}\right\}, \tag{43}$$

contains words such that if u is any word from $D^{l_1}_{19}$ implies $(a^{l_1}_{3\beta_2+1})u$ and $(x^1_{3\beta_1})u$ lies on Ω_1 and Ω_2, respectively. Thus, the fragment $\Gamma^{19}_{l_1}$ has $|D^{l_1}_{19}|=3(\beta_2+\beta_3+2)$ pairs of connecting vertices in Ω_1 and Ω_2. In other words, for each $u\in D^{l_1}_{19}$, the connection of $(a^{l_1}_{3\beta_2+1})u$ and $(x^1_{3\beta_1})u$ give the same fragment $\Gamma^{13}_{l_1}$.

Now, we show that the fragment $\Gamma^{19}_{l_1}$ is orientally different from its mirror image $\Gamma^{19*}_{l_1}$. If

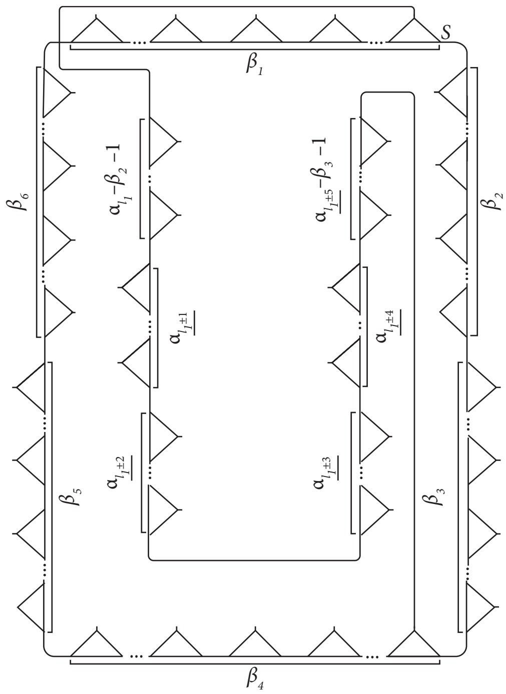

FIGURE 22: Fragment $\Gamma_{l_1}^{19}$.

$V(a_{3\beta_2+1}^{l_1}), x_{3\beta_1}^{1}) \sim V(a_{3\beta_2+1}^{l_1 *}), x_{3\beta_1}^{1 *})$, then there exists an element $u \in D_{19}^{l_1}$ such that $(a_{3\beta_2+1}^{l_1})u = a_{3\beta_2+1}^{l_1 *}$ and $(x_{3\beta_1}^{1})u = x_{3\beta_1}^{1 *}$. But there does not such an element exist in $D_{19}^{l_1}$. Thus, $V(a_{3\beta_2+1}^{l_1}, x_{3\beta_1}^{1})$ is disequivalent to $V(a_{3\beta_2+1}^{l_1 *}, x_{3\beta_1}^{1 *})$, that is by joining $a_{3\beta_2+1}^{l_1}$ with $x_{3\beta_1}^{1}$ to produce $\Gamma_{l_1}^{19}$, $x_{3\beta_1}^{1 *}$ is not connected with $a_{3\beta_2+1}^{l_1 *}$. Therefore, $\Gamma_{l_1}^{19}$ is orientally different from its mirror image $\Gamma_{l_1}^{19 *}$. Hence, there are total

$$2\left|D_{19}^{l_1}\right| = 3(\beta_2 + \beta_3 + 2), \quad (44)$$

pairs of connecting vertices to produce fragment $\Gamma_{l_1}^{19}$ and its mirror images. □

Lemma 20. *For a fix l_1, there are $6(\beta_1 + \beta_6 + 2)$ pairs of connecting vertices for the fragment $\Gamma_{l_1}^{20}$ and its mirror image by joining $x_{3\beta_2}^{2}$, the vertex from Ω_2, with the vertex $a_{3\beta_1+1}^{l_1}$ from Ω_1 (Figure 23).*

We can prove Lemma 20 in a similar way as Lemma 19 by replacing $\beta_1, \beta_2, \beta_3, \Gamma_{l_1}^{19}$, and $D_{19}^{l_1}$ by $\beta_2, \beta_1, \beta_6, \Gamma_{l_1}^{20}$, and $D_{20}^{l_1}$, respectively.

Lemma 21. *For a fix l_1, there are $6(\beta_4 + \beta_5 + 2)$ pairs of connecting vertices for the fragment $\Gamma_{l_1}^{21}$ and its mirror image by joining $x_{3\beta_3}^{3}$, the vertex from Ω_2, with the vertex $a_{3\beta_4+1}^{l_1}$ from Ω_1 (Figure 24).*

We can prove Lemma 21 in a similar way as Lemma 19 by replacing $\beta_1, \beta_2, \beta_3, \Gamma_{l_1}^{19}$, and $D_{19}^{l_1}$ by $\beta_3, \beta_4, \beta_5, \Gamma_{l_1}^{21}$, and $D_{21}^{l_1}$, respectively.

Now, to prove our main result, we define some symbolic notations as follows:

$$\begin{aligned} S_1 &= \{(E,E,E,O), (E,E,O,E), (O,O,O,O), (E,O,E,E), (O,E,E,E)\}, \\ S_2 &= \{(E,E,O,O), (O,O,E,E), (E,O,O,E), (O,E,E,O)\}, \\ S_3 &= \{(O,O,E,O), (O,O,O,E), (E,O,O,O), (O,E,O,O)\}, \end{aligned} \quad (45)$$

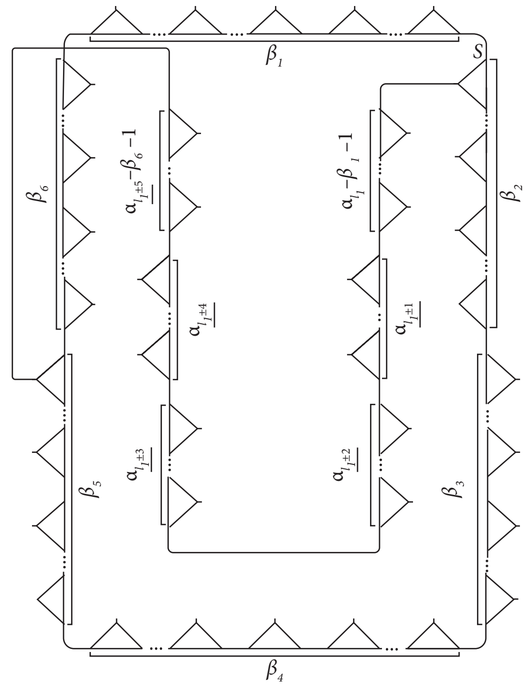

Figure 23: Fragment $\Gamma_{l_1}^{20}$.

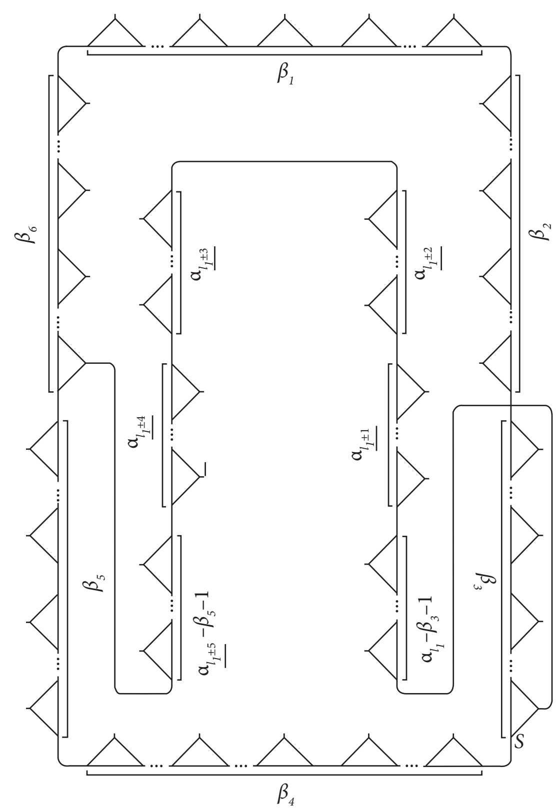

Figure 24: Fragment $\Gamma_{l_1}^{21}$.

where E and O stand for even and odd positive integers, respectively.

Let

$$\Theta = \begin{cases} 8, & \text{if } (\alpha_1, \beta_1, \alpha_4, \beta_4) = (E, E, E, E), \\ 4, & \text{if } (\alpha_1, \beta_1, \alpha_4, \beta_4) \in S_1, \\ 3, & \text{if } (\alpha_1, \beta_1, \alpha_4, \beta_4) \in S_2, \\ 2, & \text{if } (\alpha_1, \beta_1, \alpha_4, \beta_4) \in S_3, \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

Theorem 5. *If we connect the circuit Ω_1 with the circuit Ω_2 at all pairs of vertices then the total number of distinct fragments are* $(1/2)[(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)(\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4) + \Theta]$.

Proof. Let us collect all the pairs of connecting vertices of Ω_1 and Ω_2 mentioned in Lemma 1 to Lemma 21 in the form of set S as

$$S = \left\{ \begin{array}{c} \left(a^{l_1}_{3k_1+1}, x^1_{3\beta_1}\right), \left(a^{l_1}_{3k_2+1}, x^2_{3\beta_2}\right), \left(a^{l_1}_{3k_3+1}, x^3_{3\beta_3}\right), \left(a^{l_1}_{3k_4+1}, x^4_{3\beta_4}\right), \left(a^{l_1}_{3k_5+1}, x^5_{3\beta_5}\right) \\ \left(a^{l_1}_{3k_6+1}, x^6_{3\beta_6}\right), \left(a^{l_1}_{3k_{(7,l_1)}+1}, x^1_{3\beta_1}\right), \left(a^{l_1}_{3k_{(8,l_1)}+1}, x^4_{3\beta_4}\right), \left(a^{l_2}_{3k_{(9,l_2)}+1}, x^2_{3\beta_2}\right) \\ \left(a^{l_2}_{3k_{(10,l_2)}+1}, x^3_{3\beta_3}\right), \left(a^{l_3}_{3k_{(11,l_3)}+1}, x^2_{3\beta_2}\right), \left(a^{l_3}_{3k_{(12,l_3)}+1}, x^3_{3\beta_3}\right), \left(a^{l_3}_{3k_{(13,l_3)}+1}, x^1_{3k_{14}}\right) \\ \left(a^{l_3}_{3k_{(13,l_3)}+1}, x^4_{3k_{15}}\right), \left(a^{l_2}_{3k_{(16,l_2)}+1}, x^1_{3k_2}\right), \left(a^{l_2}_{3k_{(16,l_2)}+1}, x^4_{3k_{17}}\right), \left(a^{l_1}_{3k_{(18,l_1)}+1}, x^2_{3k_{19}}\right) \\ \left(a^{l_1}_{3k_{(18,l_1)}+1}, x^3_{3k_4}\right), \left(a^{l_1}_{3\beta_2+1}, x^1_{3\beta_1}\right), \left(a^{l_1}_{3\beta_1+1}, x^2_{3\beta_2}\right), \left(a^{l_1}_{3\beta_4+1}, x^2_{3\beta_3}\right) \end{array} \right\}. \tag{47}$$

Let F be the set of fragments obtained by joining each element of set S, then

$$F = \left\{ \begin{array}{c} \Gamma^1_{(k_1,l_1)}, \Gamma^2_{(k_2,l_1)}, \Gamma^3_{(k_3,l_1)}, \Gamma^4_{(k_4,l_1)}, \Gamma^5_{(k_5,l_1)}, \Gamma^6_{(k_6,l_1)}, \Gamma^7_{\left(k_{(7,l_1)},l_1\right)}, \Gamma^8_{\left(k_{(8,l_1)},l_1\right)}, \Gamma^9_{\left(k_{(9,l_2)},l_2\right)}, \\ \Gamma^{10}_{\left(k_{(10,l_2)},l_2\right)}, \Gamma^{11}_{\left(k_{(11,l_3)},l_3\right)}, \Gamma^{12}_{\left(k_{(12,l_3)},l_3\right)}, \Gamma^{13}_{\left(k_{(13,l_3)},k_{14},l_3\right)}, \Gamma^{14}_{\left(k_{(13,l_3)},k_{15},l_3\right)}, \Gamma^{15}_{\left(k_{(16,l_2)},k_2,l_2\right)}, \\ \Gamma^{16}_{\left(k_{(16,l_2)},k_{17},l_2\right)}, \Gamma^{17}_{\left(k_{(18,l_1)},k_{19},l_1\right)}, \Gamma^{18}_{\left(k_{(18,l_1)},k_4,l_1\right)}, \Gamma^{19}_{l_1}, \Gamma^{20}_{l_1}, \Gamma^{21}_{l_1} \end{array} \right\}. \tag{48}$$

This implies

$$\begin{aligned} |S| = {} & \sum_{l_1=1}^{6} \Big[3\left(\beta_2^2 + 3\beta_2\right) + 3\left(\beta_1^2 + 3\beta_1 - 4\right) + 3\left(\beta_4^2 + 3\beta_4\right) + 3\left(\beta_3^2 + 3\beta_3 - 4\right) + 3\left(\beta_6^2 + 3\beta_6\right) \\ & +3\left(\beta_5^2 + 3\beta_5 - 4\right) + 6(\beta_2 + 2)\left(\alpha_{l_1} - \beta_2 - 1\right) + 6(\beta_3 + 2)\left(\alpha_{l_1} - \beta_3 - 1\right) + 12\left(\alpha_{l_1} - 1\right)(\beta_2 - 1) \\ & +12\left(\alpha_{l_1} - 1\right)(\beta_3 - 1) + 6(\beta_2 + \beta_3 + 2) + 6(\beta_1 + \beta_6 + 2) + 6(\beta_4 + \beta_5 + 2)\Big] \\ & + \sum_{l_2=2}^{3} \Big[6(\beta_1 + 2)\left(\alpha_{l_2} - \beta_1 - 1\right) + 6(\beta_4 + 2)\left(\alpha_{l_2} - \beta_4 - 1\right) + 12\left(\alpha_{l_2} - 1\right)(\beta_1 - 1) \\ & +12\left(\alpha_{l_2} - 1\right)(\beta_4 - 1)\Big] + \sum_{l_3=1,4} \Big[3(\beta_1 + 2)\left(\alpha_{l_3} - \beta_1 - 1\right) + 3(\beta_4 + 2)\left(\alpha_{l_3} - \beta_4 - 1\right) \\ & + 6\left(\alpha_{l_3} - 1\right)(\beta_1 - 1) + 6\left(\alpha_{l_3} - 1\right)(\beta_4 - 1)\Big] = 9(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4)(\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4), \end{aligned} \tag{49}$$

shows that each pair of vertices in $\Omega_1 \times \Omega_2$ is connected, where $\Omega_1 \times \Omega_2$ is the Cartesian product of the vertices of Ω_1 and Ω_2.

Now,

$$\begin{aligned}
&\sum_{l_1=1}^{6}\Big[\beta_2+(\beta_1-1)+\beta_4+(\beta_3-1)+\beta_6+(\beta_5-1)+\left(\alpha_{l_1}-\beta_2-1\right)+\left(\alpha_{l_1}-\beta_3-1\right)\\
&\quad+12\left(\alpha_{l_1}-1\right)(\beta_2-1)+12\left(\alpha_{l_1}-1\right)(\beta_3-1)\Big]+\sum_{l_2=2}^{3}\Big[\left(\alpha_{l_2}-\beta_1-1\right)+\left(\alpha_{l_2}-\beta_4-1\right)\\
|F| &= +12\left(\alpha_{l_2}-1\right)(\beta_1-1)+12\left(\alpha_{l_2}-1\right)(\beta_4-1)\Big]+\sum_{l_3=1,4}\left[\frac{\alpha_{l_3}-\beta_1-\psi_1^{l_3}}{2}+\frac{\alpha_{l_3}-\beta_4-\psi_2^{l_3}}{2}\right]\\
&\quad+\sum_{l_3=1,4}\left[\begin{cases}\frac{1}{2}\left(\alpha_{l_3}-1\right)(\beta_1-1) & \text{if } \beta_1=1\,(\text{mod } 2)\\ \frac{1}{2}\left[\left(\alpha_{l_3}-1\right)(\beta_1-1)+1-\psi_3^{l_3}\right] & \text{if } \beta_1=0\,(\text{mod } 2)\end{cases}+\begin{cases}\frac{1}{2}\left(\alpha_{l_3}-1\right)(\beta_4-1) & \text{if } \beta_4=1\,(\text{mod } 2)\\ \frac{1}{2}\left[\left(\alpha_{l_3}-1\right)(\beta_4-1)+1-\psi_3^{l_3}\right] & \text{if } \beta_4=0\,(\text{mod } 2)\end{cases}\right]\\
&=\frac{1}{2}\left[(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)(\beta_1+2\beta_2+2\beta_3+\beta_4)+\Theta\right].
\end{aligned} \tag{50}$$

The value of $|F|$ gives the total number of distinct fragments produced in the connection of Ω_1 and Ω_2 at all pairs of vertices. The value of $|S|$ promised that the set F contains all fragments obtained by the connection of Ω_1 and Ω_2. A unique polynomial is obtained by a fragment, so there are $(1/2)[(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)(\beta_1+2\beta_2+2\beta_3+\beta_4)+\Theta]$ polynomials obtained in the connection of Ω_1 and Ω_2. □

5. Conclusion

Since there are total $9(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)(\beta_1+2\beta_2+2\beta_3+\beta_4)$ pairs of vertices in $\Omega_1 \times \Omega_2$. To find all the fragments, we do not need to connect each pair of $\Omega_1 \times \Omega_2$. We have to join only those pairs of vertices, which are in the set S and they are

$$\frac{1}{2}\left[(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)(\beta_1+2\beta_2+2\beta_3+\beta_4)+\Theta\right], \tag{51}$$

in numbers because if we connect the pair which do not belong to set S, we will attain a fragment, which has already been acquired earlier by connecting the elements of set ?.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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