Hindawi
Journal of Mathematics
Volume 2023, Article ID 4573332, 22 pages
https://doi.org/10.1155/2023/4573332

Research Article

On a New Stochastic Space of Solution for the Volterra-Type Summable Equations of Fuzzy Functions

Mustafa M. Mohammed and Awad A. Bakery

University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia

Correspondence should be addressed to Mustafa M. Mohammed; mustasta@gmail.com

Received 7 February 2023; Revised 16 August 2023; Accepted 11 October 2023; Published 6 December 2023

Academic Editor: Ammar Alsinai

This article explains the sufficient requirements for a newly constructed stochastic space by Poisson-like matrices and weighted variable exponent sequence spaces of fuzzy functions, as well as the ideals of their operators, for the Kannan contraction operator to have a unique fixed point. Moreover, we investigate some examples and the numerous applications of solutions to Volterra-type summable equations of fuzzy functions.

1. Introduction

Summable equations come up in many situations in the critical point theory for nonsmooth energy functionals, mathematical physics, control theory, biomathematics, difference variational inequalities, fuzzy set theory [1], probability theory [2], and traffic problems, to mention but a few. In particular, Volterra-type summable equations are known to be of great importance in investigating dynamical systems [3] and stochastic processes [4, 5]. Some instances are in the fields of granular systems, sweeping processes, oscillation problems, control problems, decision-making problems [6], and so on. The solution of summable equations is contained in a certain sequence space. So, there is a great interest in mathematics to construct new sequence spaces, see [7]. Mursaleen and Noman [8] examined some new sequence spaces of nonabsolute type related to the spaces ℓ_p and ℓ_∞, and Mursaleen and Başar [9] constructed and investigated the domain of the C_1 matrix in some spaces of double sequences. Mustafa and Bakery [10] introduced the concept of the private sequence space of fuzzy functions ($\mathfrak{pssff}$). Suppose $\mathscr{R}$ is the set of real numbers and $\mathbb{N}_0$ is the set of nonnegative integers. We have constructed the space $(\mathbb{P}_F^{qw})_\varsigma$ equipped with a definite function ς by the domain of Poisson-like operator defined in $\ell^F_{((q_b),(w_b))}$, where the Poisson-like matrix, $\mathbb{P} = (\gamma_{\mathrm{ba}})$, is defined as follows:

$$\gamma_{\mathrm{ba}} = \begin{cases} \dfrac{\mu^a/a!}{\sum_{a=0}^{b} \mu^a/a!}, & 0 \le a \le b, \\ 0, & a > b, \end{cases} \tag{1}$$

where $\mu \in (0,1]$ and $q_a, w_a \in (0,\infty)$, for all $a \in \mathbb{N}_0$.

For $0 < r < 1$, Matloka [11] introduced the r-level set of a fuzzy real number δ as follows:

$$\delta^r = \{m \in \mathscr{R}\colon \delta(m) \ge r\}. \tag{2}$$

Let us denote $\mathscr{R}([0,1])$ to the space of every δ^r is normal, compact, fuzzy convex, and upper semicontinuous. For $\delta \in \mathscr{R}([0,1])$, one has

$$\overline{\delta}(m) = \begin{cases} 1, & m = \delta, \\ 0, & m \ne \delta. \end{cases} \tag{3}$$

If $\Pi\colon \mathbb{N}_0^2 \longrightarrow \mathscr{R}$, $g\colon \mathbb{N}_0 \times \mathscr{R}([0,1]) \longrightarrow \mathscr{R}([0,1])$, $\overline{J}\colon \mathbb{N}_0 \longrightarrow \mathscr{R}([0,1])$, and $\overline{r}\colon \mathbb{N}_0 \longrightarrow \mathscr{R}([0,1])$, for every $\overline{J} \in \mathbb{P}_F^{qw}$. Consider the following Volterra-type summable equations of fuzzy functions [12]:

$$\overline{J_a} = \overline{r_a} + \sum_{p=0}^{\infty} \prod(a,p)\, g\left(p, \overline{J_p}\right), \tag{4}$$

and presume $L: (\mathbb{P}_F^{qw})_\varsigma \longrightarrow (\mathbb{P}_F^{qw})_\varsigma$, for certain functional ς, is defined as follows:

$$L(\overline{J_a})_{a\in\mathbb{N}_0} = \left(\overline{r_a} + \sum_{p=0}^{\infty}\prod(a,p)g\left(p,\overline{J_p}\right)\right)_{a\in\mathbb{N}_0}. \quad (5)$$

Mustafa and Bakery [10] investigated the unique solution of (4) of Kannan-type (5) in the operators' ideal (or in short **O.I**) formed by a weighted binomial matrix in the variable exponent sequence space of extended s-fuzzy functions. Bakery and Mohammed [13] explained Kannan nonexpansive mappings on the variable exponent Cesàro sequence space of fuzzy functions. We can use the newly constructed stochastic space to explore more spaces of solutions for the fuzzy fractional evolution equations; see the following interesting articles: Abuasbeh et al. [14], Niazi et al. [15], and Iqbal et al.'s [16] studies. In this paper, geometric and topological properties are used to define the space of fuzzy functions ($(\mathbb{P}_F^{qw})_\varsigma$) and the ideal space of its operators. The fixed point for the Kannan contraction operator is demonstrated, and its prequasi operator ideal is proven to be confirmed in this space. In the final section of this article, we discuss the various applications of solutions to Volterra-type summable equations of fuzzy functions and demonstrate the practical relevance of our findings.

2. Structures of $(\mathbb{P}_F^{\mathbf{qw}})_\varsigma$ and Its O.I

A few topological and geometric characteristics of $(\mathbb{P}_F^{qw})_\varsigma$ and the corresponding **O.I** have been examined here.

Notation 1

(1) A and B: infinite-dimensional Banach spaces.

(2) ${}_A\mathbb{B}_B$: the space of all bounded linear operators from A into B.

(3) $\mathbb{B}_A$: the space of all bounded linear operators from A into itself.

(4) ${}_A\mathbb{F}_B$: the space of finite rank linear operators from A into B.

(5) ${}_A\mathbb{R}_B$: the space of approximable operators from A into B.

(6) ${}_A\mathbb{K}_B$: the space of compact operators from A into B.

(7) $\mathbb{B}$: the ideal of bounded operators between any two Banach spaces.

(8) $\mathbb{F}$: the ideal of finite rank operators between any two Banach spaces.

(9) $\mathbb{R}$: the ideal of approximable operators between any two Banach spaces.

(10) $\mathbb{K}$: the ideal of compact operators between any two Banach spaces.

(11) ϖ_F: the space of all sequences of fuzzy reals.

(12) $\mathscr{R}^{+\mathbb{N}_0}$: the space of all sequences of positive reals.

(13) $\mathscr{H}_F$: the linear space of sequences of fuzzy functions.

(14) $[z]$: the integral part of the real number z.

(15) $\aleph := \max\{1, \sup_p w_p\}$.

(16) I_l: the unit operator on the q-dimensional Hilbert space ℓ_2^l.

(17) $\overline{e_l} := (\overline{0}, \overline{0}, \ldots, \overline{1}, \overline{0}, \overline{0}, \cdots)$, where $\overline{1}$ and $\overline{0}$ are the multiplicative and additive identity in $\mathscr{R}[0,1]$, respectively, while $\overline{1}$ marks at the l^{th} place.

(18) $\overline{\theta} := (\overline{0}, \overline{0}, \overline{0}, \ldots)$.

(19) $\mathscr{F}$: the space of finite sequences of fuzzy numbers.

(20) $\mathscr{I}$: the space of all monotonic increasing sequences of positive reals.

(21) $\mathscr{D}$: the space of all monotonic decreasing sequences of positive reals.

(22) $\mathscr{D}^F$: the space of each monotonic decreasing sequence of fuzzy functions.

(23) Γ: the Banach space of one dimension.

Definition 2. Presume $(q_l), (w_l) \in \mathscr{R}^{+\mathbb{N}_0}$.
$(\mathbb{P}_F^{qw})_\varsigma := \{\overline{m} = (\overline{m_l}) \in \varpi_F: \ \varsigma(\gamma\overline{m}) < \infty, \text{for some } \gamma > 0\}$, *where* $\varsigma(\overline{m}) = \sum_{l=0}^{\infty}(q_l\overline{\tau}(\sum_{n=0\mu^n/n!}^{l}\overline{m_n}, \overline{0})/\sum_{n=0}^{l}\mu^n/n!)^{w_l}$ and $\overline{\tau}(a,b) = \sup_{0\le\varepsilon\le 1}\tau(a^\varepsilon, b^\varepsilon)$.

Lemma 3 (see [17]).

$$\left|q_u + p_u\right|^{w_u} \le 2^{\aleph-1}\left(\left|q_u\right|^{w_u} + \left|p_u\right|^{w_u}\right), \quad (6)$$

where $w_u > 0$ *and* $q_u, p_u \in \mathscr{R}$, *for every* $a \in \mathbb{N}_0$.

Theorem 4. *Presume* $(w_p) \in \ell_\infty \cap \mathscr{R}^{+\mathbb{N}_0}$, *then*

$$(\mathbb{P}_F^{qw})_\varsigma = \left\{\overline{m} = \left(\overline{m_p}\right) \in \varpi_F: \varsigma(\eta\overline{m}) < \infty, \text{for every } \varepsilon > 0\right\}. \quad (7)$$

Proof. Since $(w_p) \in \ell_\infty \cap \mathscr{R}^{+\mathbb{N}_0}$, the proof follows. □

Theorem 5. *Presume* $(w_p) \in [1,\infty)^{\mathbb{N}_0} \cap \ell_\infty$, *then* $(\mathbb{P}_F^{qw})_\varsigma$ *is a nonabsolute type.*

Proof. Evidently, since

$$\varsigma(\overline{1}, -\overline{1}, \overline{0}, \overline{0}, \overline{0}, \ldots) = (q_0)^{w_0} + \left(\frac{q_1|1-\mu|}{1+\mu}\right)^{w_1} + \left(\frac{q_2|1-\mu|}{1+\mu+1/2\mu^2}\right)^{w_2} +, \ldots, \neq (q_0)^{w_0} + (q_1)^{w_1} + \left(\frac{q_2(1+\mu)}{1+\mu+1/2\mu^2}\right)^{w_2} + \ldots = \varsigma(\overline{1}, \overline{1}, \overline{0}, \overline{0}, \overline{0}, \ldots). \quad (8)$$

Definition 6. Presume $(q_n), (w_n) \in \mathscr{R}^{+\mathbb{N}_0}$ and $w_n \geq 1$, for all $n \in \mathbb{N}_0$.

$$\left(|\mathbb{P}_F^{qw}|\right)_{\wp} := \{\overline{m} = (\overline{m_n}) \in \varpi_F : \wp(\eta\overline{m}) < \infty, \text{for some } \varepsilon > 0\}, \tag{9}$$

where $\wp(\overline{m}) = \sum_{l=0}^{\infty} (q_l \overline{\tau}(\sum_{n=0}^{l} \mu^n/n! |\overline{m_n}|, \overline{0}) / \sum_{n=0}^{l} \mu^n/n!)^{w_l}$.

Theorem 7. *Suppose* $(w_a) \in (1,\infty)^{\mathbb{N}_0} \cap \ell_\infty$ *so that* $(q_b/\sum_{a=0}^{b} \mu^a/a!) \in \ell_{(w_b)}$ *and* $(q_b(b+1)/\sum_{a=0}^{b} \mu^a/a!) \notin \ell_{(w_b)}$, *then* $(|\mathbb{P}_F^{qw}|)_{\wp} \subsetneq (\mathbb{P}_F^{qw})_{\varsigma}$.

Proof. If $\overline{j} \in (|\mathbb{P}_F^{qw}|)_{\wp}$, then

$$\sum_{b=0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a! \overline{j_a}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} \leq \sum_{b=0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a! |\overline{j_a}|, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} < \infty. \tag{10}$$

Then, $\overline{j} \in (\mathbb{P}_F^{qw})_{\varsigma}$. By taking $\overline{i} = ((-\overline{1})^a a!/\mu^a)_{a\in\mathbb{N}_0}$, then $\overline{i} \in (\mathbb{P}_F^{qw})_{\varsigma}$ and $\overline{i} \notin (|\mathbb{P}_F^{qw}|)_{\wp}$. □

Definition 8 (see [10]). The space $\mathscr{H}_F$ is called a $\mathfrak{pssff}$ if it verifies the next conditions.

(c1) Presume $t \in \mathbb{N}_0$, then $\overline{e_t} \in \mathscr{H}_F$,

(c2) if $\overline{m} = (\overline{m_l}) \in \varpi_F$, $|\overline{r}| = (|\overline{r_l}|) \in \mathscr{H}_F$ and $|\overline{m_l}| \leq |\overline{r_l}|$, for all $l \in \mathbb{N}_0$, then $|\overline{m}| \in \mathscr{H}_F$,

(c3) if $(|\overline{r}_p|)_{p=0}^{\infty} \in \mathscr{H}_F$, then $(|\overline{r_{[p/2]}}|)_{p=0}^{\infty} \in \mathscr{H}_F$.

Definition 9 (see [18]). The sequence $(s_q(A))_{q=0}^{\infty}$, for all $J \in {}_A\mathbb{B}_B$, verifies the following conditions:

(a) $\|J\| = s_0(J) \geq s_1(J) \geq s_2(J) \geq \ldots \geq 0$, for every $J \in {}_A\mathbb{B}_B$,

(b) Suppose A_0 and B_0 are arbitrary Banach spaces. If $M \in {}_{A_0}\mathbb{B}_A$, $L \in {}_A\mathbb{B}_B$, and $J \in {}_B\mathbb{B}_{B_0}$, then $s_q(\text{JLM}) \leq \|J\| s_q(L) \|M\|$,

(c) $s_{r+n-1}(J_1 + J_2) \leq s_r(J_1) + s_n(J_2)$,

(d) Presume $J \in {}_A\mathbb{B}_B$ and $\delta \in \mathscr{R}$, then $s_q(\delta J) = |\delta| s_q(J)$,

(e) $s_r(J) = 0$, whenever $\text{rank}(J) \leq r$,

(f) $s_{r\geq m}(I_m) = 0$ or $s_{r<m}(I_m) = 1$.

Notations 10 (see [10]).

$$\begin{aligned}
\alpha_r(J) &= \inf\{\|J - M\| : M \in {}_A\mathbb{B}_B \text{ and } \text{rank}(M) \leq r\}, \\
d_r(J) &= \inf_{\dim(D)\leq r} \sup_{\|m\|\leq 1} \inf_{l\in D} \|Jm - l\|, \\
\overline{\mathbb{B}^s}_{\mathscr{H}_F} &:= \left\{{}_A\left[\overline{\mathbb{B}^s}_{\mathscr{H}_F}\right]_B\right\}, \text{where } {}_A\left[\overline{\mathbb{B}^s}_{\mathscr{H}_F}\right]_B := \left\{J \in {}_A\mathbb{B}_B : \left(\left(\overline{s_r(J)}\right)_{r=0}^{\infty} \in \mathscr{H}_F\right\}, \\
\overline{\mathbb{B}^\alpha}_{\mathscr{H}_F} &:= \left\{{}_A\left[\overline{\mathbb{B}^\alpha}_{\mathscr{H}_F}\right]_B\right\}, \text{where } {}_A\left[\overline{\mathbb{B}^\alpha}_{\mathscr{H}_F}\right]_B := \left\{J \in {}_A\mathbb{B}_B : \left(\left(\overline{\alpha_r(J)}\right)_{r=0}^{\infty} \in \mathscr{H}_F\right\}, \\
\overline{\mathbb{B}^d}_{\mathscr{H}_F} &:= \left\{{}_A\left[\overline{\mathbb{B}^d}_{\mathscr{H}_F}\right]_B\right\}, \text{where } {}_A\left[\overline{\mathbb{B}^d}_{\mathscr{H}_F}\right]_B := \left\{J \in {}_A\mathbb{B}_B : \left(\left(\overline{d_r(J)}\right)_{r=0}^{\infty} \in \mathscr{H}_F\right\}, \\
\left(\overline{\mathbb{B}^s}_{\mathscr{H}_F}\right)^\gamma &:= \left\{{}_A\left[\left(\overline{\mathbb{B}^s}_{\mathscr{H}_F}\right)^\gamma\right]_B\right\}, \text{where} \\
{}_A\left[\left(\overline{\mathbb{B}^s}_{\mathscr{H}_F}\right)^\gamma\right]_B &:= \left\{J \in {}_A\mathbb{B}_B : \left(\left(\overline{\gamma_r(J)}\right)_{r=0}^{\infty} \in \mathscr{H}_F \text{ and } \left\|J - \overline{\tau}\left(\overline{\gamma_r(J)}, \overline{0}\right) I\right\| = 0\right\}.
\end{aligned} \tag{11}$$

Theorem 11 (see [10]). *The space* $\overline{\mathbb{B}^s}_{\mathscr{H}_F}$ *is an* ***O.I***, *if* $\mathscr{H}_F$ *is a* $\mathfrak{pssff}$.

Definition 12 (see [19]). A subspace of the $\mathfrak{pssff}$ is said to be a premodular $\mathfrak{pssff}$ (**p-m-**$\mathfrak{pssff}$) if one has a function $\varsigma: \mathscr{H}_F \longrightarrow [0,\infty)$ which satisfies the following parts:

(d1) If $\overline{m} \in \mathscr{H}_F$, $\overline{m} = \overline{\theta} \Leftrightarrow \varsigma(|\overline{m}|) = 0$ and $\varsigma(\overline{m}) \geq 0$,

(d2) presume $\overline{m} \in \mathscr{H}_F$ and $\delta \in \mathscr{R}$, then $E_0 \geq 1$ with $\varsigma(\delta\overline{m}) \leq |\delta| E_0 \varsigma(\overline{m})$,

(d3) there are $G_0 \geq 1$ such that $\varsigma(\overline{m} + \overline{r}) \leq G_0 (\varsigma(\overline{m}) + \varsigma(\overline{r}))$, for every $\overline{m}, \overline{r} \in \mathscr{H}_F$,

(d4) if $|\overline{i_q}| \leq |\overline{j_q}|$, for every $q \in \mathbb{N}_0$, then $\varsigma(|\overline{i_q}|) \leq \varsigma(|\overline{j_q}|)$,

(d5) one has $D_0 \geq 1$ under $\varsigma(|\overline{r}|) \leq \varsigma(|\overline{r_{[.]}}|) \leq D_0 \varsigma(|\overline{r}|)$,

(d6) the closure of $\mathscr{F} = [\mathscr{H}_F]_{\varsigma}$,

(d7) one obtains $\iota > 0$ with $\varsigma(\overline{m}, \overline{0}, \overline{0}, \ldots) \geq \iota |m| \varsigma(\overline{1}, \overline{0}, \overline{0}, \ldots)$.

Definition 13 (see [19]). The $\mathfrak{pssff}$ $[\mathscr{H}_F]_{\varsigma}$ is called a prequasi normed $\mathfrak{pssff}$ (**p-qN-**$\mathfrak{pssff}$) if ς holds the parts (d1)-(d3) of Definition 12. The space $[\mathscr{H}_F]_{\varsigma}$ is called a prequasi Banach $\mathfrak{pssff}$ (**p-qB-**$\mathfrak{pssff}$); assume $\mathscr{H}_F$ is complete under ς.

Theorem 14 (see [10]). *The space* $[\mathscr{H}_F]_{\varsigma}$ *is a* ***p-qN-***$\mathfrak{pssff}$; *assume it is* ***p-m-***$\mathfrak{pssff}$.

Theorem 15. $(\mathbb{P}_F^{qw})_\varsigma$ *is a* **p-qB**-$\mathfrak{pssff}$, *if the next parts are established.*

(f1) $(w_a) \in \mathscr{I} \cap \ell_\infty$.

(f2) $(q_b/\sum_{a=0}^{b} \mu^a/a!)_{b=0}^{\infty} \in \ell_{(w_b)} \cap \mathscr{D}$.

Proof. First is to show that $(\mathbb{P}_F^{qw})_\varsigma$ is a **p-m**-$\mathfrak{pssff}$.

The condition (d1): obviously, $\varsigma(|\overline{m}|) = 0 \Leftrightarrow \overline{m} = \overline{\theta}$ and $\varsigma(\overline{m}) \geq 0$.

The conditions (c1) and (d3): presuming $\overline{i}, \overline{j} \in (\mathbb{P}_F^{qw})_\varsigma$, one has

$$\begin{aligned}\varsigma(\overline{m} + \overline{r}) &= \sum_{b=0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a!\,(\overline{m_a} + \overline{r_a}), \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} \\ &\leq 2^{\aleph-1} \left(\sum_{b=0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a!\overline{m_a}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} + \sum_{b=0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a!\overline{r_a}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} \right) = 2^{\aleph-1}(\varsigma(\overline{m}) + \varsigma(\overline{r})) < \infty,\end{aligned} \tag{12}$$

hence $\overline{m} + \overline{r} \in (\mathbb{P}_F^{qw})_\varsigma$.

The condition (d2): suppose $\zeta \in \mathscr{R}$, $\overline{j} \in (\mathbb{P}_F^{qw})_\varsigma$, and as $(w_b) \in \mathscr{I} \cap \ell_\infty$, then

$$\varsigma(\zeta\overline{j}) = \sum_{b=0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a!\zeta\overline{j_a}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} \leq |\zeta| \text{wb} \sup_b \sum_{b=0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a!\overline{j_a}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} \leq E_0 |\zeta| \varsigma(\overline{j}) < \infty, \tag{13}$$

where $E_0 = \max\left\{1, \sup_b |\zeta|^{w_b - 1}\right\} \geq 1$. Hence, $\zeta\overline{j} \in (\mathbb{P}_F^{qw})_\varsigma$.

As $(w_b) \in \mathscr{I} \cap \ell_\infty$, hence

$$\sum_{m=0}^{\infty} \left(\frac{q_m \overline{\tau}\left(\sum_{a=0}^{m} \mu^a/a!\overline{(e_b)_a}, \overline{0}\right)}{\sum_{a=0}^{m} \mu^a/a!} \right)^{w_m} = \sum_{m=b}^{\infty} \left(\frac{\mu^a/a! q_m}{\sum_{a=0}^{m} \mu^a/a!} \right)^{w_m} \leq \sup_{m=b}^{\infty} \left(\frac{\mu^b}{b!} \right)^{w_m} \sum_{m=b}^{\infty} \left(\frac{q_m}{\sum_{a=0}^{m} \mu^a/a!} \right)^{w_m} < \infty. \tag{14}$$

So, $\overline{e_b} \in (\mathbb{P}_F^{qw})_\varsigma$.

The conditions (c2) and (d4): presuming $|\overline{i_a}| \leq |\overline{j_a}|$, for all $a \in \mathbb{N}_0$ and $|\overline{j}| \in (\mathbb{P}_F^{qw})_\varsigma$, one can see that

$$\varsigma(|\overline{i}|) = \sum_{m=0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a!|\overline{i_a}|, \overline{0}\right)}{\sum_{a=0}^{m} \mu^a/a!} \right)^{w_m} \leq \sum_{m=0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a!|\overline{j_a}|, \overline{0}\right)}{\sum_{a=0}^{m} \mu^a/a!} \right)^{w_m} = \varsigma(|\overline{j}|) < \infty, \tag{15}$$

hence $|\overline{i}| \in (\mathbb{P}_F^{qw})_\varsigma$.

The conditions (c3) and (d5): if $(|\overline{j_a}|) \in (\mathbb{P}_F^{qw})_\varsigma$, with $(w_b) \in \mathscr{I} \cap \ell_\infty$ and $\left(\begin{bmatrix} a + p - 1 \\ a \end{bmatrix} q_a \right)_{a=0}^{\infty} \in \mathscr{D}$, then

$$\begin{aligned}
\varsigma\left(\left|\overline{j_{[a/2]}}\right|\right) &= \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left|\overline{j_{[a/2]}}\right|,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&= \sum_{b=0}^{\infty}\left(\frac{q_{2b}\overline{\tau}\left(\sum_{a=0}^{2b}\mu^a/a!\left|\overline{j_{[a/2]}}\right|,\overline{0}\right)}{\sum_{a=0}^{2b}\mu^a/a!}\right)^{w_{2b}} + \sum_{b=0}^{\infty}\left(\frac{q_{2b+1}\overline{\tau}\left(\sum_{a=0}^{2b+1}\mu^a/a!\left|\overline{j_{[a/2]}}\right|,\overline{0}\right)}{\sum_{a=0}^{2b+1}\mu^a/a!}\right)^{w_{2b+1}} \\
&\le \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{2b}\mu^a/a!\left|\overline{j_{[a/2]}}\right|,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{2b+1}\mu^a/a!\left|\overline{j_{[a/2]}}\right|,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&\le \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\mu^{2b}/(2b)!\left|\overline{j_b}\right| + \sum_{a=0}^{b}\left(\mu^{2a}/(2a)! + \mu^{2a+1}/(2a+1)!\right)\left|\overline{j_a}\right|,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&\quad + \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\left(\mu^{2a}/(2a)! + \mu^{2a+1}/(2a+1)!\right)\left|\overline{j_a}\right|,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&\le 2^{\aleph-1}\left(\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left|\overline{j_a}\right|,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \sum_{b=0}^{\infty}\left(\frac{2q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left|\overline{j_a}\right|,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right) \\
&\quad + \sum_{b=0}^{\infty}\left(\frac{2q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left|\overline{j_a}\right|,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \le D_0\varsigma(|\overline{j}|) < \infty,
\end{aligned} \tag{16}$$

where $D_0 \ge (2^{2\aleph-1} + 2^{\aleph-1} + 2^{\aleph}) \ge 1$. Hence, $(|\overline{j_{[a/2]}}|) \in (\mathbb{P}_F^{qw})_\varsigma$.

The condition (d6): the closure of $\mathscr{F} = \mathbb{P}_F^{qw}$.

The condition (d7): there are $0 < \iota \le \sup_l |v|^{w_l - 1}$ with $\varsigma(\overline{v},\overline{0},\overline{0},\overline{0},\ldots) \ge \iota|v|\varsigma(\overline{1},\overline{0},\overline{0},\overline{0},\ldots)$, for all $v \ne 0$ and $\iota > 0$, if $v = 0$.

From Theorem 14, the space $(\mathbb{P}_F^{qw})_\varsigma$ is a **p-qN-𝔭𝔰𝔰𝔣𝔣**. If $\overline{J^m} = (\overline{J_a^m})_{a=0}^{\infty}$ is a Cauchy sequence in $(\mathbb{P}_F^{qw})_\varsigma$, hence for all $\gamma \in (0,1)$, we have $m_0 \in \mathbb{N}_0$ with $m, n \ge m_0$, one has

$$\varsigma\left(\overline{J^m} - \overline{J^n}\right) = \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(\overline{J_a^m} - \overline{J_a^n}\right),\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} < \gamma^{\aleph}. \tag{17}$$

Hence, $q_b\overline{\tau}(\sum_{a=0}^{b}\mu^a/a!(\overline{J_a^m} - \overline{J_a^n}),\overline{0}) < \gamma$, as $(\mathscr{R}([0,1]),\overline{\tau})$ is a complete metric space. Therefore, $(\overline{J_a^n})$ is a Cauchy sequence in $\mathscr{R}([0,1])$, for fixed $a \in \mathbb{N}_0$. So, it is convergent to $\overline{J_a^0} \in \mathscr{R}([0,1])$. So, $\varsigma(\overline{J^m} - \overline{J^0}) < \gamma^{\aleph}$, for each $m \ge m_0$. Clearly, part (d3) that $\overline{J^0} \in (\mathbb{P}_F^{qw})_\varsigma$ implies that $(\mathbb{P}_F^{qw})_\varsigma$ is a Banach space.

By Theorems 11 and 15, we obtain the following theorem. □

Theorem 16. *The space $\overline{\mathbb{B}}^s_{\mathbb{P}_F^{qw}}$ is an **O.I**, when the parts of Theorem 15 are established.*

Theorem 17 (see [10]). *If s− type, $[\mathscr{H}_F]_\varsigma := \{\overline{j} = (\overline{s_q(V)}) \in \mathscr{R}^{\mathbb{N}_0}: V \in {}_A\mathbb{B}_B \text{ and } \varsigma(\overline{j}) < \infty\}$. Presume $\overline{\mathbb{B}}^s_{\mathscr{E}_\varsigma}$ is an **O.I**, then the next parts are established.*

(a) s− type $[\mathscr{H}_F]_\varsigma \supset \mathscr{F}$.

(b) When $(\overline{s_q(V_1)})_{q=0}^{\infty} \in$ s− type $[\mathscr{H}_F]_\varsigma$ and $(\overline{s_q(V_2)})_{q=0}^{\infty} \in$ s− type $[\mathscr{H}_F]_\varsigma$, then $(\overline{s_q(V_1 + V_2)})_{q=0}^{\infty} \in$ s− type $[\mathscr{H}_F]_\varsigma$.

(c) Presume $\xi \in \mathscr{R}$ and $(\overline{s_q(V)})_{q=0}^{\infty} \in$ s− type $[\mathscr{H}_F]_\varsigma$, then $|\xi|(\overline{s_q(V)})_{q=0}^{\infty} \in$ s− type $[\mathscr{H}_F]_\varsigma$.

(d) Let $(\overline{s_q(A)})_{q=0}^{\infty} \in$ s− type $[\mathscr{H}_F]_\varsigma$ and $\overline{s_q(B)} \le \overline{s_q(A)}$, for every $j \in \mathbb{N}_0$ and $B, A \in {}_A\mathbb{B}_B$, then $(\overline{s_q(B)})_{q=0}^{\infty} \in$ s− type $[\mathscr{H}_F]_\varsigma$. i.e., $[\mathscr{H}_F]_\varsigma$ is a solid space.

Some properties of s− type $(\mathbb{P}_F^{qw})_\varsigma$ are explained in the next theorem in view of Theorems 16 and 17.

Theorem 18

(a) s− type $(\mathbb{P}_F^{qw})_\varsigma \supset \mathscr{F}$.

(b) If $(\overline{s_q(W_1)})_{q=0}^{\infty} \in$ s− type $(\mathbb{P}_F^{qw})_\varsigma$ and $(\overline{s_q(W_2)})_{q=0}^{\infty} \in$ s− type $(\mathbb{P}_F^{qw})_\varsigma$, then $(\overline{s_q(W_1 + W_2)})_{q=0}^{\infty} \in$ s− type $(\mathbb{P}_F^{qw})_\varsigma$.

(c) Presume $\xi \in \mathscr{R}$ and $(\overline{s_q(W)})_{q=0}^{\infty} \in$ s− type $(\mathbb{P}_F^{qw})_\varsigma$, then $|\xi|(\overline{s_q(W)})_{q=0}^{\infty} \in$ s− type $(\mathbb{P}_F^{qw})_\varsigma$.

(d) s− type $(\mathbb{P}_F^{qw})_\varsigma$ is a solid space.

Definition 19 (see [20]). A subclass $\mathscr{V}$ of $\mathbb{B}$ is called an **O.I**, if each ${}_A\mathscr{V}_B = \mathscr{V} \cap {}_A\mathbb{B}_B$ holds the following parts:

(1) $I_\Gamma \in \mathscr{V}$.

(2) The space $A\mathscr{V}_B$ is linear over $\mathscr{R}$.

(3) Assume $W \in {}_{A_0}\mathbb{B}_A$, $X \in A\mathscr{V}_B$, and $Y \in {}_B\mathbb{B}_{B_0}$, then $YXW \in {}_{A_0}\mathscr{V}_{B_0}$.

Definition 20 (see [21]). A function $\Upsilon \in [0,\infty)^{\mathscr{V}}$ is said to be a **p-qN** on the ideal $\mathscr{V}$ when the next parts are established.

(1) Presume $J \in_A \mathbb{B}_B$, $\Upsilon(J) \geq 0$ and $\Upsilon(J) = 0 \Leftrightarrow J = 0$,

(2) there are $Q \geq 1$ with $\Upsilon(\gamma J) \leq Q|\gamma|\Upsilon(J)$, for all $J \in_A \mathscr{V}_B$ and $\gamma \in \mathscr{R}$,

(3) one has $P \geq 1$ such that $\Upsilon(J_1 + J_2) \leq P[\Upsilon(J_1) + \Upsilon(J_2)]$, for every $J_1, J_2 \in_A \mathscr{V}_B$,

(4) there are $\varepsilon \geq 1$ such that if $Y \in_{A_0} \mathbb{B}_A$, $X \in_A \mathscr{V}_B$, and $Z \in_B \mathbb{B}_{B_0}$, then $\Upsilon(ZXY) \leq \varepsilon \|Z\| \Upsilon(Y) \|X\|$.

Theorem 21. *(see [21]). Every* ***qN*** *on the ideal $\mathscr{V}$ is a* ***p-qN****.*

We have offered some properties of the ideal generated by our fuzzy space and extended $s-$ numbers in this part, presuming that the parts of Theorem 15 are satisfied.

Theorem 22. *The conditions of Theorem 15 are sufficient only for ${}_A[\overline{\mathbb{B}^{\alpha}}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ = the closure of ${}_A\mathbb{I}_B$.*

Proof. Obviously, the closure of ${}_A\mathbb{I}_B \subseteq {}_A[\overline{\mathbb{B}^{\alpha}}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ by the linearity of $(\mathbb{P}_F^{qw})_\varsigma$ and $\overline{e_b} \in (\mathbb{P}_F^{qw})_\varsigma$, for all $b \in \mathbb{N}_0$. After that, to show ${}_A[\overline{\mathbb{B}^{\alpha}}_{(\mathbb{P}_F^{qw})_\varsigma}]_B \subseteq$ the closure of ${}_A\mathbb{I}_B$, presume $J \in_A [\overline{\mathbb{B}^{\alpha}}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$, hence $(\overline{\alpha_b(J)})_{b=0}^{\infty} \in (\mathbb{P}_F^{qw})_\varsigma$. As $\varsigma(\overline{\alpha_b(J)})_{b=0}^{\infty} < \infty$, let $\gamma \in (0,1)$, there exists $b_0 \in \mathbb{N}_0 - \{0\}$ such that $\varsigma((\overline{\alpha_m(J)})_{m=b_0}^{\infty}) < \gamma / 2^{\aleph+3}\delta j$, for some $j \geq 1$ and $\delta = \max\left\{1, \sum_{b=b_0}^{\infty} (q_b / \sum_{a=0}^{b} \mu^a/a!)^{w_b}\right\}$. Since $\overline{\alpha_b(J)} \in \mathscr{D}^F$, hence

$$\sum_{b=b_0+1}^{2b_0} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a! \overline{\alpha_{2b_0}(J)}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} \leq \sum_{b=b_0+1}^{2b_0} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a! \overline{\alpha_a(J)}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} \leq \sum_{b=b_0}^{\infty} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a! \overline{\alpha_a(J)}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} < \frac{\gamma}{2^{\aleph+3}\delta j}. \tag{18}$$

Then, $G \in_A [\mathbb{I}_{2b_0}]_B$ such that rank $(G) \leq 2b_0$ and

$$\sum_{b=2b_0+1}^{3b_0} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a! \overline{\|J-G\|}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} \leq \sum_{b=b_0+1}^{2b_0} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a! \overline{\|J-G\|}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} < \frac{\gamma}{2^{\aleph+3}\delta j}, \tag{19}$$

since $(w_b) \in \mathscr{I} \cap \ell_\infty$, we get

$$\sup_{b=b_0}^{\infty} \overline{\tau}^{w_b} \left(\sum_{a=0}^{b_0} \mu^a/a! \overline{\|J-G\|}, \overline{0} \right) < \frac{\gamma}{2^{2\aleph+2}\delta}. \tag{20}$$

as one has

$$\sum_{b=0}^{b_0} \left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b} \mu^a/a! \overline{\|J-G\|}, \overline{0}\right)}{\sum_{a=0}^{b} \mu^a/a!} \right)^{w_b} < \frac{\gamma}{2^{\aleph+3}\delta j}. \tag{21}$$

Given inequalities (18)–(21), one obtains

$$
\begin{aligned}
d(J,G) &= \varsigma\left(\overline{\alpha_b(J-G)}\right)_{b=0}^{\infty} \\
&= \sum_{b=0}^{3b_0-1}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\alpha_a(J-G)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \sum_{b=3b_0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\alpha_a(J-G)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&\leq \sum_{b=0}^{3b_0}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\|J-G\|},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \sum_{b=b_0}^{\infty}\left(\frac{q_{b+2b_0}\overline{\tau}\left(\sum_{a=0}^{b+2b_0}\mu^a/a!\overline{\alpha_a(J-G)},\overline{0}\right)}{\sum_{a=0}^{b+2b_0}\mu^a/a!}\right)^{w_{b+2b_0}} \\
&\leq \sum_{b=0}^{3b_0}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\|J-G\|},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \sum_{b=b_0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b+2b_0}\mu^a/a!\overline{\alpha_a(J-G)},\overline{0}\right)}{\sum_{a=0}^{b+}\mu^a/a!}\right)^{w_b} \\
&\leq 3\sum_{b=0}^{b_0}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\|J-G\|},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \\
&\sum_{b=b_0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{2b_0-1}\mu^a/a!\overline{\alpha_a(J-G)}+\sum_{a=2b_0}^{b+2b_0}\mu^a/a!\overline{\alpha_a(J-G)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&\leq 3\sum_{b=0}^{b_0}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\|J-G\|},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + 2^{\aleph-1}\sum_{b=b_0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{2b_0-1}\mu^a/a!\overline{\alpha_a(J-G)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \\
&2^{\aleph-1}\sum_{b=b_0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=2b_0}^{b+2b_0}\mu^a/a!\overline{\alpha_a(J-G)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&\leq 3\sum_{b=0}^{b_0}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\|J-G\|},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + 2^{\aleph-1}\sum_{b=b_0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{2b_0-1}\mu^a/a!\overline{\alpha_a(J-G)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \\
&2^{\aleph-1}\sum_{b=b_0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^{a+2b_0}/(a+2b_0)!\overline{\alpha_{a+2b_0}(J-G)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&\leq 3\sum_{b=0}^{b_0}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\|J-G\|},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&\quad + 2^{2\aleph-1}\sup_{b=b_0}^{\infty}\overline{\tau}^{w_b}\left(\sum_{a=0}^{b_0}\mu^a/a!\overline{\|J-G\|},\overline{0}\right)\sum_{b=b_0}^{\infty}\left(\frac{q_b}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \\
&2^{\aleph-1}\sum_{b=b_0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\alpha_a(J)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} < \gamma.
\end{aligned}
\tag{22}
$$

Next, one concludes a negative example for the problem in [22] since $I_2 \in_A [\overline{\mathbb{B}^{\alpha}}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$, where $q = (1, 1/2, 1/3, 1/4, \ldots)$ and $w = (0, -1, 2, 2, 2, \ldots)$, but $(w_b) \notin \mathcal{I}$. □

Theorem 23. *The class* $(\overline{\mathbb{B}^{s}}_{(\mathbb{P}_F^{qw})_\varsigma}, \Upsilon)$ *is a* ***p-qB*** *ideal, where* $\Upsilon(B) = \varsigma((\overline{s_q(B)})_{q=0}^{\infty})$.

Proof. Obviously, Υ is a **p-qN** on $\overline{\mathbb{B}^{s}}_{(\mathbb{P}_F^{qw})_\varsigma}$ since ς is a **p-qN** on $(\mathbb{P}_F^{qw})_\varsigma$. Presume $(G_m)_{m\in\mathbb{N}_0}$ is a Cauchy sequence in $A[\overline{\mathbb{B}^{s}}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$. As $A\mathbb{B}_B \supseteq_A [\overline{\mathbb{B}^{s}}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$, we get

$$\Upsilon(G_j - G_m) = \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(G_j - G_m)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \geq \left(q_0\left\|G_j - G_m\right\|\right)^{w_0}, \tag{23}$$

so $(G_m)_{m\in\mathbb{N}_0}$ is a Cauchy sequence in $A\mathbb{B}_B$. As $A\mathbb{B}_B$ is a Banach space, then $G \in_A \mathbb{B}_B$ with $\lim_{m\to\infty}\|G_m - G\| = 0$. As $(\overline{s_b(G_m)})_{b=0}^{\infty} \in (\mathbb{P}_F^{qw})_\varsigma$, for all $m \in \mathbb{N}_0$. From Definition 12 parts (d2), (d3), and (d5), hence

$$\begin{aligned}\Upsilon(G) &= \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(G)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\ &\leq 2^{\aleph-1}\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_{[a/2]}(G - G_m)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + 2^{\aleph-1}\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_{[a/2]}(G_m)}, \overline{0}\right)}{\sum_{a=0}^{b}(a!/)}\right)^{w_b} \\ &\leq 2^{\aleph-1}\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\|G - G_m\|}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + 2^{\aleph-1}D_0\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(G_m)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} < \infty.\end{aligned} \tag{24}$$

So, $(\overline{s_b(G)})_{b=0}^{\infty} \in (\mathbb{P}_F^{qw})_\varsigma$, then $G \in_A [\overline{\mathbb{B}^{s}}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$. □

Theorem 24. *If* $w_b^{(1)} < w_b^{(2)}$ *and* $q_b^{(2)} \leq q_b^{(1)}$, *for each* $b \in \mathbb{N}_0$, *then*

$$_A\left[\overline{\mathbb{B}^{s}}_{\left(\mathbb{P}_F^{q^{(1)}w^{(1)}}\right)_\varsigma}\right]_B \underset{\neq}{\subset} {}_A\left[\overline{\mathbb{B}^{s}}_{\left(\mathbb{P}_F^{q^{(2)}w^{(2)}}\right)_\varsigma}\right]_B \underset{\neq}{\subset} {}_A\mathbb{B}_B. \tag{25}$$

Proof. Suppose $J \in_A [\overline{\mathbb{B}^{s}}_{(\mathbb{P}_F^{q^{(1)}w^{(1)}})_\varsigma}]_B$, hence $(\overline{s_b(J)}) \in (\mathbb{P}_F^{q^{(1)}w^{(1)}})_\varsigma$. One obtains

$$\sum_{b=0}^{\infty}\left(\frac{q_b^{(2)}\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(J)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b^{(2)}} < \sum_{b=0}^{\infty}\left(\frac{q_b^{(1)}\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(J)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b^{(1)}} < \infty, \tag{26}$$

hence $J \in_A [\overline{\mathbb{B}^{s}}_{(\mathbb{P}_F^{q^{(2)}w^{(2)}})_\varsigma}]_B$. Put $(\overline{s_b(J)})_{b=0}^{\infty}$ with $\overline{\tau}(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(J)}, \overline{0}) = \sum_{a=0}^{b}\mu^a/a!/q_b^{(1)}\sqrt{[w_b^{(1)}]b + 1}$, then $J \in_A \mathbb{B}_B$ with

$$\sum_{b=0}^{\infty}\left(\frac{q_b^{(1)}\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(J)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b^{(1)}}=\sum_{b=0}^{\infty}\frac{1}{b+1}=\infty,$$
$$\sum_{b=0}^{\infty}\left(\frac{q_b^{(2)}\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(J)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b^{(2)}}\leq\sum_{b=0}^{\infty}\left(\frac{q_b^{(1)}\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(J)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b^{(2)}}=\sum_{b=0}^{\infty}\left(\frac{1}{b+1}\right)^{w_b^{(2)}/w_b^{(1)}}<\infty. \tag{27}$$

Hence, $J\notin {}_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(1)}w^{(1)}})_\varsigma}]_B$ and $J\in{}_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(2)}w^{(2)}})_\varsigma}]_B$.

Clearly, ${}_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(2)}w^{(2)}})_\varsigma}]_B\subset{}_A\mathbb{B}_B$. Fix $(\overline{s_b(J)})_{b=0}^{\infty}$ with $\overline{\tau}(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(J)},\overline{0})=\sum_{a=0}^{b}\mu^a/a!/q_b^{(2)}\sqrt[b]{[w_b^{(2)}]b+1}$. So, $J\in{}_A\mathbb{B}_B$ and $J\notin{}_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(2)}w^{(2)}})_\varsigma}]_B$.

From Dvoretzky's theorem [23], one has A/W_b and $M_b\subseteq B$ mapped onto ℓ_2^b through isomorphisms V_b and Q_b such that $\|V_b\|\|V_b^{-1}\|\leq 2$ and $\|Q_b\|\|Q_b^{-1}\|\leq 2$, for each $b\in\mathbb{N}_0$. If T_b is the quotient operator from A onto A/W_b, I_b is the identity operator on ℓ_2^b and J_b is the natural embedding operator from M_b into B. Presume m_b is the Bernstein numbers [24]. □

Theorem 25. *Suppose* $(q_b)_{b=0}^{\infty}\notin\ell_{((w_b))}$, *then* $\overline{\mathbb{B}^{\alpha}}_{(\mathbb{P}_F^{qw})_\varsigma}$ *is minimum.*

Proof. Presuming ${}_A[\overline{\mathbb{B}^{\alpha}}_{\mathbb{P}_F^{qw}}]_B={}_A\mathbb{B}_B$, one obtains $\eta>0$ under $\Upsilon(U)\leq\eta\|U\|$, for all $U\in{}_A\mathbb{B}_B$ and $\Upsilon(U)=\sum_{b=0}^{\infty}(q_b\overline{\tau}(\sum_{a=0}^{b}\mu^a/a!\overline{\alpha_a(U)},\overline{0})/\sum_{a=0}^{b}\mu^a/a!)^{w_b}$. We obtain

$$\begin{aligned}1&=m_a(I_q)=m_a(Q_qQ_q^{-1}I_qV_qV_q^{-1})\leq\|Q_q\|m_a(Q_q^{-1}I_qV_q)\|V_q^{-1}\|=\|Q_q\|m_a(J_qQ_q^{-1}I_qV_q)\|V_q^{-1}\|\\&\leq\|Q_q\|d_a(J_qQ_q^{-1}I_qV_q)\|V_q^{-1}\|=\|Q_q\|d_a(J_qQ_q^{-1}I_qV_qT_q)\|V_q^{-1}\|\\&\leq\|Q_q\|\alpha_a(J_qQ_q^{-1}I_qV_qT_q)\|V_q^{-1}\|.\end{aligned}\tag{28}$$

Fixing $0\leq b\leq q$, one gets

$$\begin{aligned}&q_b\sum_{a=0}^{b}\mu^a/a!\leq q_b\overline{\tau}\left(\sum_{a=0}^{b}\|Q_q\|\mu^a/a!\overline{\alpha_a(J_qQ_q^{-1}I_qV_qT_q)}\|V_q^{-1}\|,\overline{0}\right)\Rightarrow,\\&(q_b)^{w_b}\leq\left(\|Q_q\|\|V_q^{-1}\|\right)^{w_b}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\alpha_a(J_qQ_q^{-1}I_qV_qT_q)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}.\end{aligned}\tag{29}$$

So, for some $\xi\geq 1$, then

$$\begin{aligned}&\sum_{b=0}^{q}(q_b)^{w_b}\leq\xi\|Q_q\|\|V_q^{-1}\|\sum_{b=0}^{q}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{\alpha_a(J_qQ_q^{-1}I_qV_qT_q)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\Rightarrow,\\&\sum_{b=0}^{q}\left(\frac{\sum_{a=0}^{b}\mu^a/a!q_a}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\leq\xi\|Q_q\|\|V_q^{-1}\|\Upsilon(J_qQ_q^{-1}I_qV_qT_q)\leq\xi\eta\|Q_q\|\|V_q^{-1}\|\|J_qQ_q^{-1}I_qV_qT_q\|\leq 4\xi\eta.\end{aligned}\tag{30}$$

As $q\longrightarrow\infty$, one has a contradiction. So, A and B both cannot be infinite-dimensional whenever ${}_A[\overline{\mathbb{B}^{\alpha}}_{\mathbb{P}_F^{qw}}]_B={}_A\mathbb{B}_B$. □

Theorem 26. *The class* $\overline{\mathbb{B}^{d}}_{\mathbb{P}_F^{qw}}$ *is minimum, whenever* $(q_b)_{b=0}^{\infty}\notin\ell_{((w_b))}$.

Lemma 27 (see [25]). *Presume $J\in {}_{A}\mathbb{B}_{B}$ and $J \notin {}_{A}\mathbb{R}_{B}$, then $X \in \mathbb{B}_{A}$ and $Y \in \mathbb{B}_{B}$ so that $YJXe_i = e_i$, for each $i \in \mathbb{N}_0$.*

Theorem 28 (see [25]). *If $\mathscr{H}_F$ is an infinite-dimensional Banach space, then*

$$\mathbb{I}_{\mathscr{H}_F} \underset{\neq}{\subset} \mathbb{R}_{\mathscr{H}_F} \underset{\neq}{\subset} \mathbb{K}_{\mathscr{H}_F} \underset{\neq}{\subset} \mathbb{B}_{\mathscr{H}_F}. \tag{31}$$

Theorem 29. *Presume $w_b^{(1)} < w_b^{(2)}$ and $q_b^{(2)} \leq q_b^{(1)}$, for each $b \in \mathbb{N}_0$, then*

$$\mathbb{B}\left({}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(2)}w^{(2)}}\right)_\varsigma}\right]_B, {}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(1)}w^{(1)}}\right)_\varsigma}\right]_B\right) = \mathbb{R}\left({}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(2)}w^{(2)}}\right)_\varsigma}\right]_B, {}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(1)}w^{(1)}}\right)_\varsigma}\right]_B\right). \tag{32}$$

Proof. If $X \in \mathbb{B}({}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(2)}w^{(2)}})_\varsigma}]_B, {}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(1)}w^{(1)}})_\varsigma}]_B)$ and $X \notin \mathbb{R}({}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(2)}w^{(2)}})_\varsigma}]_B, {}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(1)}w^{(1)}})_\varsigma}]_B)$. By Lemma 27, then $Y \in \mathbb{B}({}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(2)}w^{(2)}})_\varsigma}]_B)$ and $Z \in \mathbb{B}({}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(1)}w^{(1)}})_\varsigma}]_B)$ such that $\mathrm{ZXYI}_j = I_j$. Therefore, for every $j \in \mathbb{N}_0$, then

$$\begin{aligned}\left\|I_j\right\|_{{}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(1)}w^{(1)}}\right)_\varsigma}\right]_B} &= \sum_{b=0}^{\infty}\left(\frac{q_b^{(1)}\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a\left(I_j\right)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b^{(1)}}\\ &\leq \|\mathrm{ZXY}\|\left\|I_j\right\|_{{}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(2)}w^{(2)}}\right)_\varsigma}\right]_B} \leq \sum_{b=0}^{\infty}\left(\frac{q_b^{(2)}\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a\left(I_j\right)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b^{(2)}},\end{aligned} \tag{33}$$

which contradicts Theorem 24, hence $X \in \mathbb{R}({}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(2)}w^{(2)}})_\varsigma}]_B, {}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{q^{(1)}w^{(1)}})_\varsigma}]_B)$. □

Corollary 30. *Presume $w_b^{(1)} < w_b^{(2)}$ and $q_b^{(2)} \leq q_b^{(1)}$, for each $b \in \mathbb{N}_0$, then*

$$\mathbb{B}\left({}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(2)}w^{(2)}}\right)_\varsigma}\right]_B, {}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(1)}w^{(1)}}\right)_\varsigma}\right]_B\right) = \mathbb{K}\left({}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(2)}w^{(2)}}\right)_\varsigma}\right]_B, {}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{q^{(1)}w^{(1)}}\right)_\varsigma}\right]_B\right). \tag{34}$$

Proof. Obviously, since $\mathbb{R} \subset \mathbb{K}$. □

Definition 31 (see [25]). A Banach space $\mathscr{H}_F$ is defined as simple if $\mathbb{B}_{\mathscr{H}_F}$ is a unique nontrivial closed ideal.

Theorem 32. *The class $\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}$ is simple.*

Proof. Suppose $\mathbb{K}(A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}]_B)$ has $H \notin \mathbb{R}(A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}]_B)$. By Lemma 27, then there are $P, J \in \mathbb{B}H \notin \mathbb{R}(A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}]_B)$ such that $JHPI_j = I_j$. So, $I_{{}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}]_B} \in \mathbb{K}P, J \in \mathbb{B}H \notin \mathbb{R}(A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}]_B)$. Hence, $\mathbb{B}(A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}]_B) = \mathbb{K}(A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}]_B)$. Therefore, $\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}$ is a simple Banach space. □

Theorem 33. *Presume $\inf_b (q_b)^{w_b} > 0$, then ${}_{A}[(\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma})^\gamma]_B = {}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$.*

Proof. If $J \in {}_{A}[(\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma})^\gamma]_B$, then $(\overline{\gamma_b(J)})_{b=0}^{\infty} \in (\mathbb{P}_F^{\mathrm{qw}})_\varsigma$ and $\|J - \overline{\tau}(\overline{\gamma_b(J)}, \overline{0})I\| = 0$, for each $b \in \mathbb{N}_0$. We obtain $J = \overline{\tau}(\overline{\gamma_b(J)}, \overline{0})I$, for all $b \in \mathbb{N}_0$, so

$$\overline{\tau}\left(\overline{s_b(J)},\overline{0}\right) = \overline{\tau}\left(\overline{s_b\left(\overline{\tau}\left(\overline{\gamma_b(J)},\overline{0}\right)I\right)},\overline{0}\right) = \overline{\tau}\left(\overline{\gamma_b(J)},\overline{0}\right), \tag{35}$$

for all $b \in \mathbb{N}_0$. Hence, $(\overline{s_b(J)})_{b=0}^{\infty} \in (\mathbb{P}_F^{\mathrm{qw}})_\varsigma$, so $J \in {}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}]_B$.

After that, assume $J \in {}_{A}[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}]_B$. Then, $(\overline{s_b(J)})_{b=0}^{\infty} \in (\mathbb{P}_F^{\mathrm{qw}})_\varsigma$. So, one gets

$$\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(J)},\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \geq \inf_b (q_b)^{w_b}\sum_{b=0}^{\infty}\left[\overline{\tau}\left(\overline{s_b(J)},\overline{0}\right)\right]^{w_b}. \tag{36}$$

Hence, $\lim_{b\longrightarrow\infty}\overline{s_b(J)} = \overline{0}$. Assume $\|J - \overline{\tau}(\overline{s_b(J)}, \overline{0})I\|^{-1}$ exists, for all $b \in \mathbb{N}_0$. Hence, $\|J - \overline{\tau}(\overline{s_b(J)}, \overline{0})I\|^{-1}$ bounded and exists, for each $b \in \mathbb{N}_0$. Hence, $\lim_{b\longrightarrow\infty}\|J - \overline{\tau}(\overline{s_b(J)}, \overline{0})I\|^{-1} = \|J\|^{-1}$ bounded and exists. From $(\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{\mathrm{qw}})_\varsigma}, \Upsilon)$ which is a prequasi ideal, one has

$$I = JJ^{-1} \in {}_{A}\left[\overline{\mathbb{B}^s}_{\left(\mathbb{P}_F^{\mathrm{qw}}\right)_\varsigma}\right]_B \Rightarrow \left(\overline{s_b(I)}\right)_{b=0}^{\infty} \in \mathbb{P}_F^{\mathrm{qw}} \Rightarrow \lim_{b\longrightarrow\infty}\overline{s_b(I)} = \overline{0}. \tag{37}$$

One has a contradiction, as $\lim_{b\longrightarrow\infty}\overline{s_b(I)} = \overline{1}$. So, $\|J - \overline{\tau}(\overline{s_b(J)}, \overline{0})I\| = 0$, for every $b \in \mathbb{N}_0$. Hence,

$\|J - \overline{\tau}(\overline{\gamma_b(J)}, \overline{0})I\| = 0$, for all $b \in \mathbb{N}_0$. So, $J \in_A [(\overline{\mathbb{B}}^s{}_{(\mathbb{P}_F^{\text{qw}})_\varsigma})^\gamma]_B$. □

3. Kannan's Contraction Fixed Points

Supposing that the parts of Theorem 15 are established, the existence of a fixed point of the Kannan contraction operator acting on this new space and its associated **p-q** ideal are presented with some numerical examples to show our results.

The Banach fixed point theorem, as presented in reference [26], provided mathematicians with a means to extend the applicability of contraction operators by generalization, for instances, the Kannan contraction operator [27], Kannan operators in modular vector spaces [28], and Kannan **p-qN** contraction operator [29].

Definition 34 (see [13]). A **p-qN-**$\boldsymbol{\mathfrak{pssff}}$ ς on $\mathscr{H}_F$ holds the Fatou property (or in short **FPr**); presume for every $\left\{\overline{i^{(a)}}\right\} \subseteq [\mathscr{H}_F]_\varsigma$ such that $\lim_{a\longrightarrow\infty} \varsigma(\overline{i^{(a)}} - \overline{i}) = \overline{0}$ and $\overline{j} \in [\mathscr{H}_F]_\varsigma$, then $\varsigma(\overline{j} - \overline{i}) \leq \sup_q \inf_{a \geq q} \varsigma(\overline{j} - \overline{i^{(a)}})$.

We will use the following notations:

$$\begin{aligned}
\varsigma_1(\overline{j}) &= \left[\sum_{b=0}^{\infty}\left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{j_a}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph}, \\
\varsigma_2(\overline{j}) &= \sum_{b=0}^{\infty}\left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{j_a}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b},
\end{aligned} \tag{38}$$

for all $\overline{j} \in \mathbb{P}_F^{\text{qw}}$.

Theorem 35. *The function ς_1 holds the* ***FPr****.*

Proof. Presume $\left\{\overline{j^{(d)}}\right\} \subseteq (\mathbb{P}_F^{\text{qw}})_{\varsigma_1}$ so that $\lim_{d\longrightarrow\infty} \varsigma_1(\overline{j^{(d)}} - \overline{j}) = 0$. Clearly, $\overline{j} \in (\mathbb{P}_F^{\text{qw}})_{\varsigma_1}$. For each $\overline{i} \in (\mathbb{P}_F^{\text{qw}})_{\varsigma_1}$, hence

$$\begin{aligned}
\varsigma_1(\overline{i} - \overline{j}) &= \left[\sum_{b=0}^{\infty}\left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!(\overline{i_a} - \overline{j_a}), \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph} \\
&\leq \left[\sum_{b=0}^{\infty}\left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(\overline{i_a} - \overline{j_a^{(d)}}\right), \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph} + \left[\sum_{b=0}^{\infty}\left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(\overline{j_a^{(d)}} - \overline{j_a}\right), \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph} \\
&\leq \sup_q \inf_{d \geq q} \varsigma_1\left(\overline{i} - \overline{j^{(d)}}\right).
\end{aligned} \tag{39}$$

□

Theorem 36. *If $w_0 > 1$, then ς_2 does not hold the* ***FPr****.*

Proof. Suppose $\left\{\overline{j^{(d)}}\right\} \subseteq (\mathbb{P}_F^{\text{qw}})_{\varsigma_2}$ so that $\lim_{b\longrightarrow\infty} \varsigma_2(\overline{j^{(d)}} - \overline{j}) = 0$. Obviously, $\overline{j} \in (\mathbb{P}_F^{\text{qw}})_{\varsigma_2}$. For every $\overline{i} \in (\mathbb{P}_F^{\text{qw}})_{\varsigma_2}$, one obtains

$$\begin{aligned}
\varsigma_2(\overline{i} - \overline{j}) &= \sum_{b=0}^{\infty}\left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!(\overline{i_a} - \overline{j_a}), \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} \\
&\leq 2^{\aleph - 1}\left[\sum_{b=0}^{\infty}\left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(\overline{i_a} - \overline{j_a^{(d)}}\right), \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b} + \sum_{b=0}^{\infty}\left(\frac{q_b \overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(\overline{j_a^{(d)}} - \overline{j_a}\right), \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right] \\
&\leq 2^{\aleph - 1} \sup_q \inf_{d \geq q} \varsigma_2\left(\overline{i} - \overline{j^{(d)}}\right).
\end{aligned} \tag{40}$$

So, ς_2 does not verify the **FPr**. □

Definition 37 (see [29]). A mapping $G\colon [\mathscr{H}_F]_\varsigma \longrightarrow [\mathscr{H}_F]_\varsigma$ is said to be a Kannan ς-contraction, if there is $\varepsilon \in 0, [1/2)$ with $\varsigma(G\overline{j} - G\overline{k}) \leq \varepsilon(\varsigma(G\overline{j} - \overline{j}) + \varsigma(G\overline{k} - \overline{k}))$, for every $\overline{j}, \overline{k} \in [\mathscr{H}_F]_\varsigma$. Presume $G(\overline{j}) = \overline{j}$, then $\overline{j} \in [\mathscr{H}_F]_\varsigma$ is named a fixed point of G.

Theorem 38. *The space G has a unique fixed point, whenever $G\colon (\mathbb{P}_F^{qw})_{\varsigma_1} \longrightarrow (\mathbb{P}_F^{qw})_{\varsigma_1}$ is the Kannan ς_1-contraction operator.*

Proof. Presume $\overline{k} \in \mathbb{P}_F^{qw}$, hence $G^b\overline{k} \in \mathbb{P}_F^{qw}$. As G is a Kannan ς_1-contraction, one gets

$$\begin{aligned}
&\varsigma_1\left(G^{b+1}\overline{k} - G^b\overline{k}\right) \leq \varepsilon\left(\varsigma_1\left(G^{b+1}\overline{k} - G^b\overline{k}\right) + \varsigma_1\left(G^b\overline{k} - G^{b-1}\overline{k}\right)\right) \Rightarrow, \\
&\varsigma_1\left(G^{b+1}\overline{k} - G^b\overline{k}\right) \leq \frac{\varepsilon}{1-\varepsilon}\varsigma_1\left(G^b\overline{k} - G^{b-1}\overline{k}\right) \leq \left(\frac{\varepsilon}{1-\varepsilon}\right)^2 \varsigma_1\left(G^{b-1}\overline{k} - G^{b-2}\overline{k}\right) \leq, \ldots,, \\
&\leq \left(\frac{\varepsilon}{1-\varepsilon}\right)^b \varsigma_1(G\overline{k} - \overline{k}).
\end{aligned} \tag{41}$$

Hence, for all $a, b \in \mathbb{N}_0$ so that $a > b$, then

$$\begin{aligned}
\varsigma_1\left(G^b\overline{k} - G^a\overline{k}\right) &\leq \varepsilon\left(\varsigma_1\left(G^b\overline{k} - G^{b-1}\overline{k}\right) + \varsigma_1\left(G^a\overline{k} - G^{a-1}\overline{k}\right)\right) \\
&\leq \varepsilon\left(\left(\frac{\varepsilon}{1-\varepsilon}\right)^{b-1} + \left(\frac{\varepsilon}{1-\varepsilon}\right)^{a-1}\right)\varsigma_1(G\overline{k} - \overline{k}).
\end{aligned} \tag{42}$$

So, $\left\{G^b\overline{k}\right\}$ is a Cauchy sequence in $(\mathbb{P}_F^{qw})_{\varsigma_1}$, as $(\mathbb{P}_F^{qw})_{\varsigma_1}$ is a **p-qB** space. Hence, $\overline{q} \in (\mathbb{P}_F^{qw})_{\varsigma_1}$ with $\lim_{b\longrightarrow\infty} G^b\overline{k} = \overline{q}$ to show that $G(\overline{q}) = \overline{q}$. Since ς_1 verifies the **FPr**, one obtains

$$\varsigma_1(G\overline{q} - \overline{q}) \leq \sup \varsigma_1\left(G^{b+1}\overline{k} - G^b\overline{k}\right) \leq \sup_i \left(\frac{\varepsilon}{1-\varepsilon}\right)^b \varsigma_1(G\overline{k} - \overline{k}) = 0, \tag{43}$$

hence $G(\overline{q}) = \overline{q}$. So, $\overline{q}$ is a fixed point of G. To show the uniqueness of the fixed point, let one has two different fixed points $\overline{i}, \overline{q} \in (\mathbb{P}_F^{qw})_{\varsigma_1}$ of G. So,

$$\varsigma_1(\overline{i} - \overline{q}) \leq \varsigma_1(G\overline{i} - G\overline{q}) \leq \varepsilon(\varsigma_1(G\overline{i} - \overline{i}) + \varsigma_1(G\overline{q} - \overline{q})) = 0. \tag{44}$$

Therefore, $\overline{i} = \overline{q}$. □

Corollary 39. *If $G\colon (\mathbb{P}_F^{qw})_{\varsigma_1} \longrightarrow (\mathbb{P}_F^{qw})_{\varsigma_1}$ is Kannan ς_1-contraction, then G has a unique fixed point $\overline{q}$ such that $\varsigma_1(G^b\overline{k} - \overline{q}) \leq \varepsilon(\varepsilon/1-\varepsilon)^{b-1}\varsigma_1(G\overline{k} - \overline{k})$.*

Proof. Given Theorem 38, we obtain a unique fixed point $\overline{q}$ of G. Hence,

$$\varsigma_1\left(G^b\overline{k} - \overline{q}\right) = \varsigma_1\left(G^b\overline{k} - G\overline{q}\right) \leq \varepsilon\left(\varsigma_1\left(G^b\overline{k} - G^{b-1}\overline{k}\right) + \varsigma_1(G\overline{q} - \overline{q})\right) = \varepsilon\left(\frac{\varepsilon}{1-\varepsilon}\right)^{b-1} \varsigma_1(G\overline{k} - \overline{k}). \tag{45}$$

□

Definition 40 (see [13]). Presume $[\mathscr{H}_F]_\varsigma$ is a **p-qN-þꞩꞩſſ**, $G\colon [\mathscr{H}_F]_\varsigma \longrightarrow [\mathscr{H}_F]_\varsigma$, and $\overline{j} \in [\mathscr{H}_F]_\varsigma$. The operator G is called ς-sequentially continuous (or in short ς-**s.c**) at $\overline{j}$, if and only if, when $\lim_{i\longrightarrow\infty} \varsigma(\overline{g_i} - \overline{j}) = 0$, then $\lim_{i\longrightarrow\infty} \varsigma(G\overline{g_i} - G\overline{j}) = 0$.

Theorem 41. *Presume $w_0 > 1$ and $G\colon (\mathbb{P}_F^{qw})_{\varsigma_2} \longrightarrow (\mathbb{P}_F^{qw})_{\varsigma_2}$. The vector $\overline{k} \in (\mathbb{P}_F^{qw})_{\varsigma_2}$ is the unique fixed point of G, whenever the following parts are established:*

(g1) *G is Kannan ς_2-contraction,*

(g2) *G is ς_2- **s.c** at $\overline{k} \in (\mathbb{P}_F^{qw})_{\varsigma_2}$,*

(g3) *one has $\overline{u} \in (\mathbb{P}_F^{qw})_{\varsigma_2}$ such that $\{G^q\overline{u}\}$ has $\{G^{q_i}\overline{u}\}$ which converges to $\overline{k}$.*

Proof. Let $\overline{k}$ be not a fixed point of G, then $G\overline{k} \neq \overline{k}$. From parts (g2) and (g3), we get

$$\begin{aligned}
\lim_{q_i\longrightarrow\infty} \varsigma_2\left(G^{q_i}\overline{u} - \overline{k}\right) &= 0, \\
\lim_{q_i\longrightarrow\infty} \varsigma_2\left(G^{q_i+1}\overline{u} - \mathrm{G}\overline{\mathrm{k}}\right) &= 0.
\end{aligned} \tag{46}$$

As G is Kannan ς_2-contraction, then

$$\begin{aligned}
0 < \varsigma_2(\mathrm{G}\overline{\mathrm{k}} - \overline{k}) &= \varsigma_2\left(\left(\mathrm{G}\overline{\mathrm{k}} - G^{q_i+1}\overline{u}\right) + \left(G^{q_i}\overline{u} - \overline{k}\right) + \left(G^{q_i+1}\overline{u} - G^{q_i}\overline{u}\right)\right) \\
&\leq 2^{2\aleph-2}\varsigma_2\left(G^{q_i+1}\overline{u} - \mathrm{G}\overline{\mathrm{k}}\right) + 2^{2\aleph-2}\varsigma_2\left(G^{q_i}\overline{u} - \overline{k}\right) + 2^{\aleph-1}\varepsilon\left(\frac{\varepsilon}{1-\varepsilon}\right)^{q_i-1} \varsigma_2(\mathrm{G}\overline{\mathrm{u}} - \overline{u}).
\end{aligned} \tag{47}$$

By $q_i \longrightarrow \infty$, there is a contradiction. Then, $\overline{k}$ is a fixed point of G. For the uniqueness of $\overline{k}$, if one gets two different fixed points $\overline{k},\overline{r}\in (\mathbb{P}_F^{qw})_{\varsigma_2}$ of G. So,

$$\varsigma_2(\overline{k}-\overline{r}) \le \varsigma_2(G\overline{k}-G\overline{r}) \le \varepsilon\left(\varsigma_2(G\overline{k}-\overline{k})+\varsigma_2(G\overline{r}-\overline{r})\right)=0. \tag{48}$$

Hence, $\overline{k}=\overline{r}$. □

Example 1. Consider $A: (\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_1} \longrightarrow (\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_1}$ and

$$A(\overline{h})=\left\{\frac{\overline{h}}{4},\varsigma_1(\overline{h})\in[0,1),\frac{\overline{h}}{5},\varsigma_1(\overline{h})\in[1,\infty).\right. \tag{49}$$

If $\overline{h},\overline{r}\in (\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_1}$. Assume $\varsigma_1(\overline{h}),\varsigma_1(\overline{r})\in[0,1)$, then

$$\varsigma_1(A\overline{h}-A\overline{r})=\varsigma_1\left(\frac{\overline{h}}{4}-\frac{\overline{r}}{4}\right)\le\frac{1}{\sqrt[4]{27}}\left(\varsigma_1\left(\frac{3\overline{h}}{4}\right)+\varsigma_1\left(\frac{3\overline{r}}{4}\right)\right)=\frac{1}{\sqrt[4]{27}}\left(\varsigma_1(A\overline{h}-\overline{h})+\varsigma_1(A\overline{r}-\overline{r})\right). \tag{50}$$

Presume $\varsigma_1(\overline{h}),\varsigma_1(\overline{r})\in[1,\infty)$, then

$$\varsigma_1(A\overline{h}-A\overline{r})=\varsigma_1\left(\frac{\overline{h}}{5}-\frac{\overline{r}}{5}\right)\le\frac{1}{\sqrt[4]{64}}\left(\varsigma_1\left(\frac{4\overline{h}}{5}\right)+\varsigma_1\left(\frac{4\overline{r}}{5}\right)\right)=\frac{1}{\sqrt[4]{64}}\left(\varsigma_1(A\overline{h}-\overline{h})+\varsigma_1(A\overline{r}-\overline{r})\right). \tag{51}$$

Suppose $\varsigma_1(\overline{h})\in[0,1)$ and $\varsigma_1(\overline{r})\in[1,\infty)$, hence

$$\varsigma_1(A\overline{h}-A\overline{r})=\varsigma_1\left(\frac{\overline{h}}{4}-\frac{\overline{r}}{5}\right)\le\frac{1}{\sqrt[4]{27}}\varsigma_1\left(\frac{3\overline{h}}{4}\right)+\frac{1}{\sqrt[4]{64}}\varsigma_1\left(\frac{4\overline{r}}{5}\right)\le\frac{1}{\sqrt[4]{27}}\left(\varsigma_1\left(\frac{3\overline{h}}{4}\right)+\varsigma_1\left(\frac{4\overline{r}}{5}\right)\right)=\frac{1}{\sqrt[4]{27}}\left(\varsigma_1(A\overline{h}-\overline{h})+\varsigma_1(A\overline{r}-\overline{r})\right). \tag{52}$$

So, A is Kannan ς_1-contraction, as ς_1 verifies the **FPr**. By Theorem 38, A has a unique fixed point $\overline{\theta}$. Presume $\{\overline{h^{(b)}}\}\subseteq(\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_1}$ such that $\lim_{b\longrightarrow\infty}\varsigma_1(\overline{h^{(b)}}-\overline{h^{(0)}})=0$, where $\overline{h^{(0)}}\in(\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_1}$ so that $\varsigma_1(\overline{h^{(0)}})=1$. As ς_1 is continuous, one has

$$\lim_{b\longrightarrow\infty}\varsigma_1\left(A\overline{h^{(b)}}-A\overline{h^{(0)}}\right)=\lim_{b\longrightarrow\infty}\varsigma_1\left(\frac{\overline{h^{(b)}}}{4}-\frac{\overline{h^{(0)}}}{5}\right)=\varsigma_1\left(\frac{\overline{h^{(0)}}}{20}\right)>0. \tag{53}$$

So, A is not ς_1-**s.c** at $\overline{h^{(0)}}$, which explains that A is not continuous at $\overline{h^{(0)}}$.

Consider $\overline{h},\overline{r}\in(\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_2}$. If $\varsigma_2(\overline{h}),\varsigma_2(\overline{r})\in[0,1)$, then

$$\varsigma_2(A\overline{h}-A\overline{r})=\varsigma_2\left(\frac{\overline{h}}{4}-\frac{\overline{r}}{4}\right)\le\frac{2}{\sqrt{27}}\left(\varsigma_2\left(\frac{3\overline{h}}{4}\right)+\varsigma_2\left(\frac{3\overline{r}}{4}\right)\right)=\frac{2}{\sqrt{27}}\left(\varsigma_2(A\overline{h}-\overline{h})+\varsigma_2(A\overline{r}-\overline{r})\right). \tag{54}$$

If $\varsigma_2(\overline{h}),\varsigma_2(\overline{r})\in[1,\infty)$, hence

$$\varsigma_2(A\overline{h}-A\overline{r})=\varsigma_2\left(\frac{\overline{h}}{5}-\frac{\overline{r}}{5}\right)\le\frac{1}{4}\left(\varsigma_2\left(\frac{4\overline{h}}{5}\right)+\varsigma_2\left(\frac{4\overline{r}}{5}\right)\right)=\frac{1}{4}\left(\varsigma_2(A\overline{h}-\overline{h})+\varsigma_2(A\overline{r}-\overline{r})\right). \tag{55}$$

Suppose $\varsigma_2(\overline{h}) \in [0,1)$ and $\varsigma_2(\overline{r}) \in [1,\infty)$, then

$$\begin{aligned}\varsigma_2(A\overline{h} - A\overline{r}) &= \varsigma_2\left(\frac{\overline{h}}{4} - \frac{\overline{r}}{5}\right) \le \frac{2}{\sqrt{27}}\varsigma_2\left(\frac{3\overline{h}}{4}\right) + \frac{1}{4}\varsigma_2\left(\frac{4\overline{r}}{5}\right) \le \frac{2}{\sqrt{27}}\left(\varsigma_2\left(\frac{3\overline{h}}{4}\right) + \varsigma_2\left(\frac{4\overline{r}}{5}\right)\right)\\ &= \frac{2}{\sqrt{27}}\left(\varsigma_2(A\overline{h} - \overline{h}) + \varsigma_2(A\overline{r} - \overline{r})\right).\end{aligned} \tag{56}$$

So, A is Kannan ς_2-contraction and $T^q(\overline{h}) = \begin{cases} \overline{h}/4^q, & \varsigma_2(\overline{h}) \in [0,1),\\ \overline{h}/5^q, & \varsigma_2(\overline{h}) \in [1,\infty).\end{cases}$

Evidently, A is ς_2-s.c at $\overline{\theta}$ and $\{T^q\overline{h}\}$ has a $\{T^{q_j}\overline{h}\}$ which converges to $\overline{\theta}$. From Theorem 41, the vector $\overline{\theta}$ is the only fixed point of A.

Example 2. Consider $A: (\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_2} \longrightarrow (\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_2}$ and

$$A(\overline{h}) = \begin{cases} \frac{1}{4}(\overline{e_1} + \overline{h}), & \overline{h_0}(t) \in \left[0, \frac{1}{3}\right),\\ \frac{1}{3}\overline{e_1}, & \overline{h_0}(t) = \frac{1}{3},\\ \frac{1}{4}\overline{e_1}, & \overline{h_0}(t) \in \left(\frac{1}{3}, 1\right].\end{cases} \tag{57}$$

As $\overline{h_0}, \overline{r_0} \in [0, 1/3)$, then

$$\begin{aligned}\varsigma_2(A\overline{h} - A\overline{r}) &= \varsigma_2\left(\frac{1}{4}\left(\overline{h_0} - \overline{r_0}, \overline{h_1} - \overline{r_1}, \overline{h_2} - \overline{r_2}, \ldots\right)\right) \le \frac{2}{\sqrt{27}}\left(\varsigma_2\left(\frac{3\overline{h}}{4}\right) + \varsigma_2\left(\frac{3\overline{r}}{4}\right)\right)\\ &\le \frac{2}{\sqrt{27}}\left(\varsigma_2(A\overline{h} - \overline{h}) + \varsigma_2(A\overline{r} - \overline{r})\right).\end{aligned} \tag{58}$$

If $\overline{h_0}, \overline{r_0} \in (1/3, 1]$, then for every $\varepsilon > 0$, we obtain

$$\varsigma_2(A\overline{h} - A\overline{r}) = 0 \le \varepsilon\left(\varsigma_2(A\overline{h} - \overline{h}) + \varsigma_2(A\overline{r} - \overline{r})\right). \tag{59}$$

Suppose $\overline{h_0} \in [0, 1/3)$ and $\overline{r_0} \in (1/3, 1]$, then

$$\varsigma_2(A\overline{h} - A\overline{r}) = \varsigma_2\left(\frac{\overline{h}}{4}\right) \le \frac{1}{\sqrt{27}}\varsigma_2\left(\frac{3\overline{h}}{4}\right) = \frac{1}{\sqrt{27}}\varsigma_2(A\overline{h} - \overline{h}) \le \frac{1}{\sqrt{27}}\left(\varsigma_2(A\overline{h} - \overline{h}) + \varsigma_2(A\overline{r} - \overline{r})\right). \tag{60}$$

Hence, A is Kannan ς_2-contraction. Obviously, A is ς_2-**s.c** at $1/3\overline{e_1}$ and one has $\overline{h} \in (\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_2}$ such that $\overline{h_0} \in [0, 1/3)$ with $\{T^q\overline{h}\} = \left\{\sum_{a=1}^{q} 1/4^a\overline{e_1} + 1/4^q\overline{h}\right\}$ has a $\{T^{q_j}\overline{h}\} = \left\{\sum_{a=1}^{q_j} 1/4^a\overline{e_1} + 1/4^{q_j}\overline{h}\right\}$ which converges to $1/3\overline{e_1}$. According to Theorem 41, A has a unique fixed point $1/3\overline{e_1}$. Note that A is not continuous at $1/3\overline{e_1}$.

Presume $\overline{h}, \overline{r} \in (\mathbb{P}_F((1/a+5)_{a=0}^{\infty}, (2a+3/a+2)_{a=0}^{\infty}))_{\varsigma_1}$. When $\overline{h_0}, \overline{r_0} \in [0, 1/3)$, then

$$\begin{aligned}\varsigma_1(A\overline{h} - A\overline{r}) &= \varsigma_1\left(\frac{1}{4}\left(\overline{h_0} - \overline{r_0}, \overline{h_1} - \overline{r_1}, \overline{h_2} - \overline{r_2}, \ldots\right)\right) \le \frac{1}{\sqrt[4]{27}}\left(\varsigma_1\left(\frac{3\overline{h}}{4}\right) + \varsigma_1\left(\frac{3\overline{r}}{4}\right)\right)\\ &\le \frac{1}{\sqrt[4]{27}}\left(\varsigma_1(A\overline{h} - \overline{h}) + \varsigma_1(A\overline{r} - \overline{r})\right).\end{aligned} \tag{61}$$

Suppose $\overline{h_0}, \overline{r_0} \in (1/3, 1]$, then for all $\varepsilon > 0$, one can see

$$\varsigma_1(A\overline{h} - A\overline{r}) = 0 \le \varepsilon\left(\varsigma_1(A\overline{h} - \overline{h}) + \varsigma_1(A\overline{r} - \overline{r})\right). \tag{62}$$

If $\overline{h_0} \in [0, 1/3)$ and $\overline{r_0} \in (1/3, 1]$, then

$$\varsigma_1(A\overline{h} - A\overline{r}) = \varsigma_1\left(\frac{\overline{h}}{4}\right) \leq \frac{1}{\sqrt[4]{27}}\varsigma_1\left(\frac{3\overline{h}}{4}\right) = \frac{1}{\sqrt[4]{27}}\varsigma_1(A\overline{h} - \overline{h}) \leq \frac{1}{\sqrt[4]{27}}\left(\varsigma_1(A\overline{h} - \overline{h}) + \varsigma_1(A\overline{r} - \overline{r})\right). \tag{63}$$

So, A is Kannan ς_1-contraction. Since ς_1 satisfies the **FPr**. In view of Theorem 38, A has a unique fixed point $1/3\overline{e_1}$.

We will use in this part $\Upsilon(W) = \varsigma(\overline{(s_b(W))}_{b=0}^{\infty}) = [\sum_{b=0}^{\infty}(q_b\overline{\tau}(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(W)}, \overline{0})/\sum_{a=0}^{b}\mu^a/a!)^{w_b}]^{1/\aleph}$, for all $W \in_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$.

Definition 42 (see [10]). A function Υ on $\overline{\mathbb{B}^s}_{\mathscr{H}_F}$ verifies the **FPr** if for every $\{W_q\}_{q\in\mathbb{N}_0} \subseteq_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ such that $\lim_{q\longrightarrow\infty}\Upsilon(W_q - W) = 0$ and all $J \in_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$, then $\Upsilon(J - W) \leq \sup_n \inf_{q\geq n}\Upsilon(J - W_q)$.

Theorem 43. *The function Υ does not hold the **FPr**.*

Proof. Presume $\{J_q\}_{q\in\mathbb{N}_0} \subseteq_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ such that $\lim_{q\longrightarrow\infty}\Upsilon(J_q - J) = 0$. Evidently, $J \in_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$. Hence, for each $G \in_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$, then

$$\begin{aligned}
\Upsilon(G - J) &= \left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(G - J)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph} \\
&\leq \left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_{[a/2]}(G - J_i)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph} + \left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_{[a/2]}(J - J_i)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph} \\
&\leq \left(2^{2\aleph-1} + 2^{\aleph-1} + 2^{\aleph}\right)^{1/\aleph}\sup_q\inf_{i\geq q}\left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(G - J_i)}, \overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph}.
\end{aligned} \tag{64}$$

Therefore, Υ does not hold the **FPr**. □

Definition 44 (see [29]). A mapping L: $_A[[\overline{\mathbb{B}^s}]_{\mathscr{H}_F}]_B \longrightarrow {}_A[[\overline{\mathbb{B}^s}]_{\mathscr{H}_F}]_B$ is named a Kannan Υ-contraction; assume we have $\varepsilon \in [0,1/2)$ with $\Upsilon(LJ - LG) \leq \varepsilon(\Upsilon(LJ - J) + \Upsilon(LG - G))$, for all $J, G \in_A [[\overline{\mathbb{B}^s}]_{\mathscr{H}_F}]_B$.

Definition 45 (see [10]). Presume L: $_A[[\overline{\mathbb{B}^s}]_{\mathscr{H}_F}]_B \longrightarrow {}_A[[\overline{\mathbb{B}^s}]_{\mathscr{H}_F}]_B$ and $T \in_A [[\overline{\mathbb{B}^s}]_{\mathscr{H}_F}]_B$. The operator L is said to be Υ-**s.c** at T, if and only if, when $\lim_{q\longrightarrow\infty}\Upsilon(U_q - T) = 0$, then $\lim_{q\longrightarrow\infty}\Upsilon(LU_q - LT) = 0$.

Theorem 46. *If L: $_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B \longrightarrow {}_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$. The operator $U \in_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ is the unique fixed point of L, if the following parts are confirmed:*

(h1) *L is Kannan Υ-contraction,*

(h2) *L is Υ-**s.c** at $A \in_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$,*

(h2) *there is $T \in_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ such that $\{L^mT\}$ has $\{L^{m_i}T\}$ which converges to U.*

Proof. Let U be not a fixed point of L, then one gets $LU \neq U$. From the parts (h2) and (h3), one gets

$$\lim_{q_b\longrightarrow\infty}\Upsilon(L^{q_b}T - U) = 0 \text{ and } \lim_{q_b\longrightarrow\infty}\Upsilon\left(L^{q_b+1}T - GU\right) = 0. \tag{65}$$

As L is Kannan Υ-contraction operator, then

$$\begin{aligned}
0 < \Upsilon(LU - U) &= \Upsilon\left(\left(LU - G^{q_b+1}T\right) + \left(L^{q_b}T - U\right) + \left(L^{q_b+1}T - G^{q_b}T\right)\right) \\
&\leq \left(2^{2\aleph-1} + 2^{\aleph-1} + 2^{\aleph}\right)^{1/\aleph}\Upsilon\left(L^{q_b+1}T - GU\right) + \left(2^{2\aleph-1} + 2^{\aleph-1} + 2^{\aleph}\right)^{2/\aleph}\Upsilon\left(L^{q_b}T - U\right) \\
&\quad + \left(2^{2\aleph-1} + 2^{\aleph-1} + 2^{\aleph}\right)^{2/\aleph}\varepsilon\left(\frac{\varepsilon}{1-\varepsilon}\right)^{q_b-1}\Upsilon(LT - T).
\end{aligned} \tag{66}$$

We get a contradiction as $q_b \longrightarrow \infty$. Then, U is a fixed point of L. For the uniqueness of the fixed point U, assume one gets two different fixed points $U, V \in_A [\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ of L. So,

$$\Upsilon(U - V) \le \Upsilon(\mathrm{LU} - \mathrm{GV}) \le \varepsilon(\Upsilon(\mathrm{LU} - U) + \Upsilon(\mathrm{LV} - V)) = 0. \tag{67}$$

Therefore, $U = V$. □

Example 3. Consider

$$L: {}_A\left[\overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/b+4)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_\varsigma}\right]_B \longrightarrow {}_A\left[\overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/b+4)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_\varsigma}\right]_B,$$
$$L(K) = \begin{cases} \dfrac{K}{6}, & \Upsilon(K) \in [0,1), \\ \dfrac{K}{7}, & \Upsilon(K) \in [1,\infty). \end{cases} \tag{68}$$

Presume $K_1, K_2 \in \overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/b+4)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_\varsigma}$. If $\Upsilon(K_1), \Upsilon(K_2) \in [0,1)$, then

$$\begin{aligned}\Upsilon(\mathrm{LK}_1 - \mathrm{LK}_2) &= \Upsilon\left(\frac{K_1}{6} - \frac{K_2}{6}\right) \le \frac{\sqrt{2}}{\sqrt[4]{125}}\left(\Upsilon\left(\frac{5K_1}{6}\right) + \Upsilon\left(\frac{5K_2}{6}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{125}}(\Upsilon(\mathrm{LK}_1 - K_1) + \Upsilon(\mathrm{LK}_2 - K_2)).\end{aligned} \tag{69}$$

If $\Upsilon(K_1), \Upsilon(K_2) \in [1,\infty)$, then

$$\begin{aligned}\Upsilon(\mathrm{LK}_1 - \mathrm{LK}_2) &= \Upsilon\left(\frac{K_1}{7} - \frac{K_2}{7}\right) \le \frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Upsilon\left(\frac{6K_1}{7}\right) + \Upsilon\left(\frac{6K_2}{7}\right)\right) \\ &= \frac{\sqrt{2}}{\sqrt[4]{216}}(\Upsilon(\mathrm{LK}_1 - K_1) + \Upsilon(\mathrm{LK}_2 - K_2)).\end{aligned} \tag{70}$$

Suppose $\Upsilon(K_1) \in [0,1)$ and $\Upsilon(K_2) \in [1,\infty)$, one can see

$$\begin{aligned}\Upsilon(\mathrm{LK}_1 - \mathrm{LK}_2) &= \Upsilon\left(\frac{K_1}{6} - \frac{K_2}{7}\right) \le \frac{\sqrt{2}}{\sqrt[4]{125}}\Upsilon\left(\frac{5K_1}{6}\right) + \frac{\sqrt{2}}{\sqrt[4]{216}}\Upsilon\left(\frac{6K_2}{7}\right) \\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}}(\Upsilon(\mathrm{LK}_1 - K_1) + \Upsilon(\mathrm{LK}_2 - K_2)).\end{aligned} \tag{71}$$

Hence, L is Kannan Υ-contraction and $L^q(K) = \begin{cases} K/6^q, & \Upsilon(K) \in [0,1), \\ K/7^q, & \Upsilon(K) \in [1,\infty). \end{cases}$

Clearly, L is Υ-s.c at the zero operator Θ and $\{M^q K\}$ has a $\{M^{q_j} K\}$ which converges to Θ. From Theorem 46, Θ is the unique fixed point of L.

Presume $\{K^{(d)}\} \subseteq \overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/b+4)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_\varsigma}$ with $K^{(0)} \in \overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/b+4)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_\varsigma}$ with $\Upsilon(K^{(0)}) = 1$. As $\lim_{d\longrightarrow\infty}\Upsilon(K^{(d)} - K^{(0)}) = 0$, where Υ is continuous, then

$$\lim_{d\longrightarrow\infty}\Upsilon\left(LK^{(d)} - LK^{(0)}\right) = \lim_{d\longrightarrow\infty}\Upsilon\left(\frac{K^{(0)}}{6} - \frac{K^{(0)}}{7}\right) = \Upsilon\left(\frac{K^{(0)}}{42}\right) > 0. \tag{72}$$

So, L is not Υ-**s.c** at $K^{(0)}$. Hence, L is not continuous at $K^{(0)}$.

4. Applications

In this section, a solution in $(\mathbb{P}_F^{qw})_{\varsigma_1}$ to (1) is examined such that the parts of Theorem 15 are established.

Theorem 47. *System (1) holds one and only solution in* $(\mathbb{P}_F^{qw})_{\varsigma_1}$, *if* $\overline{\eta}$: $\mathbb{N}_0 \longrightarrow \mathscr{R}([0,1])$, *there is* $\varepsilon \in \mathscr{R}$ *such that* $\sup_b |\varepsilon|^{w_b/\aleph} \in [0,1/2)$ *and for each* $b \in \mathbb{N}_0$, *then*

$$\begin{aligned}&\left|\sum_{a=0}^{b}\left(\sum_{p\in\mathbb{N}_0}\Pi(a,p)\left[g\left(p,\overline{J_p}\right) - g\left(p,\overline{\eta_p}\right)\right]\right)\frac{\mu^a}{a!}\right|,\\ &\overline{\leq}|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{r_a} - \overline{J_a} + \sum_{p=0}^{\infty}\Pi(a,p)g\left(p,\overline{J_p}\right)\right)\frac{\mu^a}{a!}\right| + |\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{r_a} - \overline{\eta_a} + \sum_{p=0}^{\infty}\Pi(a,p)g\left(p,\overline{\eta_p}\right)\right)\frac{\mu^a}{a!}\right|.\end{aligned} \tag{73}$$

Proof. Suppose L: $(\mathbb{P}_F^{qw})_{\varsigma_1} \longrightarrow (\mathbb{P}_F^{qw})_{\varsigma_1}$ is defined by equation (2). By Theorem 38 and

$$\begin{aligned}\varsigma_1(L\overline{J} - L\overline{\eta}) &= \left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(L\overline{J_a} - L\overline{\eta_a}\right),\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph}\\ &= \left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\left(\sum_{p\in\mathbb{N}_0}\Pi(a,p)\left[g\left(p,\overline{J_p}\right) - g\left(p,\overline{\eta_p}\right)\right]\right)\mu^a/a!,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph}\\ &\leq \sup_b|\varepsilon|^{w_b/\aleph}\left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\left(\overline{r_a} - \overline{J_a} + \sum_{p=0}^{\infty}\Pi(a,p)g\left(p,\overline{J_p}\right)\right)\mu^a/a!,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph} +\\ &\quad\sup_b|\varepsilon|^{w_b/\aleph}\left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\left(\overline{r_a} - \overline{\eta_a} + \sum_{p=0}^{\infty}\Pi(a,p)g\left(p,\overline{\eta_p}\right)\right)\mu^a/a!,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph}\\ &= \sup_b|\varepsilon|^{w_b/\aleph}\left(\varsigma_1(L\overline{J} - \overline{J}) + \varsigma_1(L\overline{\eta} - \overline{\eta})\right).\end{aligned} \tag{74}$$

This completes the proofs. □

Example 4. Consider $(\mathbb{P}_F((1/b+1)_{b=0}^{\infty}, (2b+3/b+2)_{b=0}^{\infty}))_{\varsigma_1}$.

Suppose the following nonlinear uncertainty equation of fuzzy functions:

$$\overline{j_a} = \overline{\cos(2a+1)} + \sum_{q=0}^{\infty}2^{a+q}\frac{\overline{j_{a-2}^{r}}}{\overline{j_{a-1}^{d}} + \overline{q^2} + \overline{1}}, \tag{75}$$

with $r, d > 0$ and $\overline{j_{-2}}(t), \overline{j_{-1}}(t) > 0$, for all $t \in \mathscr{R}$, and if

$$L:\ \left(\mathbb{P}_F\left((1/b+1)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}\right)\right)_{\varsigma_1}\longrightarrow\left(\mathbb{P}_F\left((1/b+1)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}\right)\right)_{\varsigma_1} \tag{76}$$

is defined as

$$L\left(\overline{j_a}\right)_{a=0}^{\infty}=\left(\overline{\cos(2a+1)}+\sum_{q=0}^{\infty}2^{a+q}\frac{\overline{j_{a-2}^{r}}}{\overline{j_{a-1}^{d}}+\overline{q^2+1}}\right)_{a=0}^{\infty}. \tag{77}$$

Obviously, one obtains $\varepsilon\in\mathscr{R}$ such that $\sup_b|\varepsilon|^{2b+3/2b+4}\in[0,1/2)$ and for every $b\in\mathbb{N}_0$, then

$$\begin{aligned}&\left|\sum_{a=0}^{b}\left(\sum_{q=0}^{\infty}2^{a}\frac{\overline{j_{a-2}^{r}}}{\overline{j_{a-1}^{d}}+\overline{q^{2}+1}}\left(2^{q}-2^{q}\right)\right)\frac{\mu^{a}}{a!}\right|,\\&\overline{\le}|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{\cos(2a+1)}-\overline{j_a}+\sum_{q=0}^{\infty}2^{a+q}\frac{\overline{j_{a-2}^{r}}}{\overline{j_{a-1}^{d}}+\overline{q^{2}+1}}\right)\frac{\mu^{a}}{a!}\right|+,\\&|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{\cos(2a+1)}-\overline{\eta_a}+\sum_{q=0}^{\infty}2^{a+q}\frac{\overline{\eta_{a-2}^{r}}}{\overline{\eta_{a-1}^{d}}+\overline{q^{2}+1}}\right)\frac{\mu^{a}}{a!}\right|.\end{aligned} \tag{78}$$

From Theorem 47, system (8) has a unique solution in $\left(\mathbb{P}_F\left((1/b+1)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}\right)\right)_{\varsigma_1}$.

Theorem 48. *Consider $L:\ (\mathbb{P}_F^{qw})_{\varsigma_2}\longrightarrow(\mathbb{P}_F^{qw})_{\varsigma_2}$ is defined by (2) and $w_0>1$. System (1) holds one and only solution $\overline{l}\in(\mathbb{P}_F^{qw})_{\varsigma_2}$, if the following parts are established:*

(1) Suppose $\overline{v}:\mathbb{N}_0\longrightarrow\mathscr{R}([0,1])$ and $\overline{k}:\mathbb{N}_0\longrightarrow\mathscr{R}([0,1])$, if there is $\varepsilon\in\mathscr{R}$ with $2^{\aleph-1}\sup_b|\varepsilon|^{w_b}\in[0,1/2)$ and for each $b\in\mathbb{N}_0$, then

$$\begin{aligned}&\left|\sum_{a=0}^{b}\left(\sum_{p\in\mathbb{N}_0}\Pi(a,p)\left[g\left(p,\overline{j_p}\right)-g\left(p,\overline{k_p}\right)\right]\right)\frac{\mu^{a}}{a!}\right|,\\&\overline{\le}|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{v_a}-\overline{j_a}+\sum_{p=0}^{\infty}\Pi(a,p)g\left(p,\overline{j_p}\right)\right)\frac{\mu^{a}}{a!}\right|+|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{v_a}-\overline{k_a}+\sum_{p=0}^{\infty}\Pi(a,p)g\left(p,\overline{k_p}\right)\right)\frac{\mu^{a}}{a!}\right|.\end{aligned} \tag{79}$$

*(2) L is ς_2- **s.c** at $\overline{l}\in(\mathbb{P}_F^{qw})_{\varsigma_2}$,*

(3) one has $\overline{i}\in(\mathbb{P}_F^{qw})_{\varsigma_2}$ such that $\{W^{q}\overline{i}\}$ has $\{W^{q_j}\overline{i}\}$ converging to $\overline{l}$.

Proof. From Theorem 41 and

$$\begin{aligned}
\varsigma_2(\mathrm{L}\bar{\mathrm{j}}-\mathrm{L}\bar{\mathrm{k}}) &= \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(L\overline{j_a}-L\overline{k_a}\right),\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\\
&= \sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\left(\sum_{p\in\mathbb{N}_0}\Pi(a,p)\left[g\left(p,\overline{j_p}\right)-g\left(p,\overline{k_p}\right)\right]\right)\mu^a/a!,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\\
&\leq 2^{\aleph-1}\sup_b|\varepsilon|^{w_b}\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\left(\overline{v_a}-\overline{j_a}+\sum_{p=0}^{\infty}\Pi(a,p)g\left(p,\overline{j_p}\right)\right)\mu^a/a!,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}+\\
&\quad 2^{\aleph-1}\sup_b|\varepsilon|^{w_b}\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\left(\overline{v_a}-\overline{k_a}+\sum_{p=0}^{\infty}\Pi(a,p)g\left(p,\overline{k_p}\right)\right)\mu^a/a!,\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\\
&= 2^{\aleph-1}\sup_b|\varepsilon|^{w_b}\left(\varsigma_2(\mathrm{L}\bar{\mathrm{j}}-\bar{j})+\varsigma_2(\mathrm{L}\bar{\mathrm{k}}-\bar{k})\right).
\end{aligned}\tag{80}$$

This completes the proofs. ◻

Example 5. Let $(\mathbb{P}_F((1/b+1)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_{\varsigma_2}$.

Consider the Volterra-type summable equations (8).

Suppose L: $(\mathbb{P}_F((1/b+1)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_{\varsigma_2}\longrightarrow(\mathbb{P}_F((1/b+1)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_{\varsigma_2}$ is defined by (9).

When L is ς_2-**s.c** at $\bar{l}\in(\mathbb{P}_F((1/b+1)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_{\varsigma_2}$ and one has

$\bar{i}\in(\mathbb{P}_F((1/b+1)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_{\varsigma_2}$ such that $\{L^q\bar{i}\}$ has $\{L^{q_j}\bar{i}\}$ converging to $\bar{l}$. Clearly, there is $\varepsilon\in\mathscr{R}$ such that $2^{\aleph-1}\sup_b|\varepsilon|^{2b+3/b+2}\in[0,1/2)$ and for all $b\in\mathbb{N}_0$, then

$$\begin{aligned}
&\left|\sum_{a=0}^{b}\left(\sum_{q=0}^{\infty}2^a\frac{\overline{j_{a-2}^{b}}}{\overline{j_{a-1}^{d}}+\overline{q^2+1}}\left(2^q-2^q\right)\right)\frac{\mu^a}{a!}\right|,\\
&\leq|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{\cos(2a+1)}-\overline{j_a}+\sum_{q=0}^{\infty}2^{a+m}\frac{\overline{j_{a-2}^{b}}}{\overline{j_{a-1}^{d}}+\overline{q^2+1}}\right)\frac{\mu^a}{a!}\right|+,\\
&|\varepsilon|\left|\sum_{a=0}^{b}\left(\overline{\cos(2a+1)}-\overline{k_a}+\sum_{q=0}^{\infty}2^{a+m}\frac{\overline{k_{a-2}^{b}}}{\overline{k_{a-1}^{d}}+\overline{q^2+1}}\right)\frac{\mu^a}{a!}\right|.
\end{aligned}\tag{81}$$

By Theorem 49, the nonlinear uncertainty equation of fuzzy functions (8) has a unique solution $\bar{l}\in(\mathbb{P}_F((1/b+1)_{b=0}^{\infty},(2b+3/b+2)_{b=0}^{\infty}))_{\varsigma_2}$.

In this section, we propose a solution to system (10) at $K\in_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$. Supposing that the parts of Theorem 15 are established and $\Upsilon(K)=[\sum_{b=0}^{\infty}(q_b\overline{\tau}(\sum_{a=0}^{b}\mu^a/a!\overline{s_a(K)},\overline{0})/\sum_{a=0}^{b}\mu^a/a!)^{w_b}]^{1/\aleph}$, for each $K\in_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$. Consider the following nonlinear uncertainty equation of fuzzy functions:

$$\overline{s_a(K)}=\overline{s_a(J)}+\sum_{q=0}^{\infty}\Pi(a,q)f\left(q,\overline{s_q(K)}\right),\tag{82}$$

and if L: ${}_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B\longrightarrow{}_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ is defined as

$$L(K)=\left(\overline{s_a(J)}+\sum_{q=0}^{\infty}\Pi(a,q)f\left(q,\overline{s_q(K)}\right)\right)I.\tag{83}$$

Theorem 49. *System (10) holds one and only one solution $G\in_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$, if the following parts are confirmed:*

(1) Π: $\mathbb{N}_0^2\longrightarrow\mathscr{R}$, f: $\mathbb{N}_0\times\mathscr{R}([0,1])\longrightarrow\mathscr{R}([0,1])$, $J\in_A\mathbb{B}_B$, $U\in_A\mathbb{B}_B$, and for every $b\in\mathbb{N}_0$, one has ε such that $\sup_b\varepsilon^{w_b/\aleph}\in[0,0.5)$ and

$$\left|\sum_{q\in\mathbb{N}_0}\Pi(a,q)\left(f\left(q,\overline{s_q(K)}\right)-f\left(q,\overline{s_q(U)}\right)\right)\right|,$$
$$\underline{\leq}\varepsilon\left[\left|\overline{s_a(J)}-\overline{s_a(K)}+\sum_{q\in\mathbb{N}_0}\Pi(a,q)f\left(q,\overline{s_q(K)}\right)\right|+\left|\overline{s_a(J)}-\overline{s_a(U)}+\sum_{q\in\mathbb{N}_0}\Pi(a,q)f\left(q,\overline{s_q(U)}\right)\right|\right]. \tag{84}$$

(2) *L is* Υ-**s.c** *at an element* $G\in_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$.

(3) *There is* $Q\in_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ *such that* $\{L^aQ\}$ *has a* $\{L^{a_i}Q\}$ *converging to G.*

Proof. Consider L: $_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B\longrightarrow {}_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}]_B$ as defined by (11). From Theorem 46 and

$$\begin{aligned}\Upsilon(\mathrm{LK}-\mathrm{LU})&=\left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(\overline{s_a(K)}-\overline{s_a(U)}\right),\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph}\\
&=\left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\sum_{q\in\mathbb{N}_0}\Pi(a,q)\left(f\left(q,\overline{s_q(K)}\right)-f\left(q,\overline{s_q(U)}\right)\right),\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph}\\
&\leq\sup_a\varepsilon^{t_a/\aleph}\left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(\overline{s_a(J)}-\overline{s_a(K)}+\sum_{q\in\mathbb{N}_0}\Pi(a,q)f\left(q,\overline{s_q(K)}\right)\right),\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph}+\\
&\sup_b\varepsilon^{w_b/\aleph}\left[\sum_{b=0}^{\infty}\left(\frac{q_b\overline{\tau}\left(\sum_{a=0}^{b}\mu^a/a!\left(\overline{s_a(J)}-\overline{s_a(U)}+\sum_{q\in\mathbb{N}_0}\Pi(a,q)f\left(q,\overline{s_q(U)}\right)\right),\overline{0}\right)}{\sum_{a=0}^{b}\mu^a/a!}\right)^{w_b}\right]^{1/\aleph}\\
&=\sup_b\varepsilon^{w_b/\aleph}(\Upsilon(\mathrm{LK}-K)+\Upsilon(\mathrm{LU}-U)).\end{aligned}\tag{85}$$

This finishes the proof. □

Example 6. Let $_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/l!),(2l+3/l+2)))_\varsigma}]_B$, where $\Upsilon(G)=\sqrt{\sum_{b=0}^{\infty}((\overline{\tau}(\sum_{a=0}^{b}\mu^a/a!\overline{s_z(G)},\overline{0}))/(b!\sum_{a=0}^{b}\mu^a/a!))^{2b+3/b+2}}$, for all $G\in_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/l!),(2l+3/l+2)))_\varsigma}]_B$.

Consider the following nonlinear uncertainty equation of fuzzy functions:

$$\overline{s_z(G)}=\overline{\sin z}+\sum_{m=0}^{\infty}\frac{\cos\left(3m^2\right)\tanh(mz)\sinh^b\left|\overline{s_{z-2}(G)}\right|}{\sec^d\left|\overline{s_{z-1}(G)}\right|+\overline{\ln(m+z+1)}+\overline{1}}, \tag{86}$$

where $b,d>0$ and $z\geq 2$ and if W: $_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/l!),(2l+3/l+2)))_\varsigma}]_B\longrightarrow {}_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/l!),(2l+3/l+2)))_\varsigma}]_B$ is defined as

$$W(G)=\left(\overline{\sin z}+\sum_{m=0}^{\infty}\frac{\cos\left(3m^2\right)\tanh(mz)\sinh^b\left|\overline{s_{z-2}(G)}\right|}{\sec^d\left|\overline{s_{z-1}(G)}\right|+\overline{\ln(m+z+1)}+\overline{1}}\right)I. \tag{87}$$

Supposing that W is Υ-**s.c** at a point $D\in_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/l!),(2l+3/l+2)))_\varsigma}]_B$ and one has $B\in_A[\overline{\mathbb{B}^s}_{(\mathbb{P}_F((1/l!),(2l+3/l+2)))_\varsigma}]_B$ with $\{W^aB\}$ has a $\{W^{a_i}B\}$ converging to D. By Theorem 49 and

$$\left|\sum_{m=0}^{\infty}\frac{\tanh(mz)\sinh^b\left|\overline{s_{z-2}(G)}\right|}{\sec^d\left|\overline{s_{z-1}(G)}\right|+\overline{\ln(m+z+1)}+\overline{1}}\left(\cos\left(3m^2\right)-\cos\left(3m^2\right)\right)\right|,$$
$$\underline{\leq}\frac{1}{25}\left|\overline{\sin z}-\overline{s_z(G)}+\sum_{m=0}^{\infty}\frac{\cos\left(3m^2\right)\tanh(mz)\sinh^b\left|\overline{s_{z-2}(G)}\right|}{\sec^d\left|\overline{s_{z-1}(G)}\right|+\overline{\ln(m+z+1)}+\overline{1}}\right|,+\frac{1}{25}\left|\overline{\sin z}-\overline{s_z(T)}+\sum_{m=0}^{\infty}\frac{\cos\left(3m^2\right)\tanh(mz)\sinh^b\left|\overline{s_{z-2}(T)}\right|}{\sec^d\left|\overline{s_{z-1}(T)}\right|+\overline{\ln(m+z+1)}+\overline{1}}\right|. \tag{88}$$

This completes the proof.

5. Conclusion

We explained a few topological and geometric properties of $(\mathbb{P}_F^{qw})_\varsigma$ of the class $\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma}$ and of the class $(\overline{\mathbb{B}^s}_{(\mathbb{P}_F^{qw})_\varsigma})^\lambda$ in this article. The Kannan contraction operator on these spaces is analyzed, and the possibility of a fixed point is considered. We ran many numerical experiments to ensure our theories were correct. Fuzzy functions with nonlinear uncertainty equation implementations are also investigated. This novel fuzzy function space is used to investigate the fixed points of all contraction operators, providing a new universal solution space for a wide variety of stochastic nonlinear dynamical systems.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All the authors have read and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant no. UJ-22-DR-89. The authors, therefore, acknowledge with thanks the University of Jeddah for its technical and financial support.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
[2] D. Dubois and H. Prade, *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Springer Science & Business Media, New York, NY, USA, 1998.
[3] H. Ahmad, M. Younis, and M. E. Koksal, "Double controlled partial metric type spaces and convergence results," *Journal of Mathematics*, vol. 2021, Article ID 7008737, 11 pages, 2021.
[4] W. Mao, Q. Zhu, and X. Mao, "Existence, uniqueness and almost surely asymptotic estimations of the solutions to neutral stochastic functional differential equations driven by pure jumps," *Applied Mathematics and Computation*, vol. 254, pp. 252–265, 2015.
[5] L. Guo and Q. Zhu, "Stability analysis for stochastic Volterra–Levin equations with Poisson jumps: fixed point approach," *Journal of Mathematical Physics*, vol. 52, no. 4, Article ID 042702, 2011.
[6] I. Beg, M. Abbas, and M. W. Asghar, "Polytopic fuzzy sets and their applications to multiple-attribute decision-making problems," *International Journal of Fuzzy Systems*, vol. 24, no. 6, pp. 2969–2981, 2022.
[7] M. Mursaleen and F. Başar, "Sequence spaces: topics in modern summability theory," *Mathematics and Its Applications*, CRC Press/Taylor & Francis Group, London, UK, 2020.
[8] M. Mursaleen and A. K. Noman, "On some new sequence spaces of non-absolute type related to the spaces ℓp and ℓ∞ Ilpand l∞I," *Filomat*, vol. 25, no. 2, pp. 33–51, 2011.
[9] M. Mursaleen and F. Başar, "Domain of Cesàro mean of order one in some spaces of double sequences," *Studia Scientiarum Mathematicarum Hungarica*, vol. 51, no. 3, pp. 335–356, 2014.
[10] A. O. Mustafa and A. A. Bakery, "Decision making on the mappings' ideal solution of a fuzzy non-linear matrix system of Kannan-type," *Journal of Mathematics and Computer Science*, vol. 30, no. 1, pp. 48–66, 2022.
[11] M. Matloka, "Sequences of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 28, pp. 28–37, 1986.
[12] P. Salimi, Abdul Latif, and N. Hussain, "Modified α-ψ-contractive mappings with applications," *Fixed Point Theory and Applications*, vol. 2013, p. 151, 2013.
[13] A. A. Bakery and M. M. Mohammed, "Kannan nonexpansive operators on variable exponent Cesàro sequence space of fuzzy functions," *Journal of Function Spaces*, vol. 2022, Article ID 1992684, 18 pages, 2022.
[14] K. Abuasbeh, A. U. K. Niazi, H. M. Arshad, M. Awadalla, and S. Trabelsi, "Approximate controllability of fractional stochastic evolution inclusions with non-local conditions," *Fractal and Fractional*, vol. 7, no. 6, p. 462, 2023.
[15] A. U. K. Niazi, N. Iqbal, R. Shah, F. Wannalookkhee, and K. Nonlaopon, "Controllability for fuzzy fractional evolution equations in credibility space," *Fractal and Fractional*, vol. 5, no. 3, p. 112, 2021.
[16] N. Iqbal, A. U. K. Niazi, R. Shafqat, and S. Zaland, "Existence and uniqueness of mild solution for fractional-order controlled fuzzy evolution equation," *Journal of Function Spaces*, vol. 2021, Article ID 5795065, 8 pages, 2021.
[17] B. Altay and F. Başar, "Generalization of the sequence space lpderived by weighted means," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 147–185, 2007.
[18] A. Pietsch, *Eigenvalues And s-numbers*, Cambridge University Press, New York, NY, USA, 1986.
[19] M. M. Alsolmia and A. A. Bakery, "Multiplication mappings on a new stochastic space of a sequence of fuzzy function," *Journal of Mathematics and Computer Science*, vol. 29, no. 4, pp. 306–316, 2022.
[20] A. A. Bakery and O. K. S. K. Mohamed, "Orlicz generalized difference sequence space and the linked pre-quasi operator ideal," *Journal of Mathematics*, vol. 2020, Article ID 6664996, 9 pages, 2020.
[21] N. Faried and A. A. Bakery, "Small operator ideals formed by s numbers on generalized Cesáro and Orlicz sequence spacess*numbers on generalized Cesàro and Orlicz sequence spaces*," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 357, 2018.
[22] B. E. Rhoades, "Operators of A-p type," *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Serie IX. Matematica e Applicazioni*, vol. 59, no. 3-4, pp. 238–241, 1975.
[23] A. Pietsch, *Operator Ideals*, VEB Deutscher Verlag derWissenschaften, Berlin, Germany, 1978.
[24] A. Pietsch, "S-numbers of operators in banach spaces," *Studia Mathematica*, vol. 51, pp. 201–223, 1974.

[25] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam, Netherlands, 1980.
[26] S. Banach, “Sur les opérations dans les ensembles abstraits et leurs applications,” *Fundamenta Mathematicae*, vol. 3, pp. 133-–181, 1922.
[27] R. Kannan, “Some results on fixed points- II,” *The American Mathematical Monthly*, vol. 76, no. 4, pp. 405–408, 1969.
[28] S. J. H. Ghoncheh, “Some fixed point theorems for kannan mapping in the modular spaces,” *Ciĕncia e Natura*, vol. 37, pp. 462–466, 2015.
[29] A. A. Bakery and O. M. K. S. K. Mohamed, “Kannan prequasi contraction maps on Nakano sequence spaces,” *Journal of Function Spaces*, vol. 2020, Article ID 8871563, 10 pages, 2020.