Hindawi
Journal of Mathematics
Volume 2023, Article ID 4599585, 13 pages
https://doi.org/10.1155/2023/4599585

Research Article

Algebraic Techniques for Canonical Forms and Applications in Split Quaternionic Mechanics

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Received 3 March 2023; Revised 4 October 2023; Accepted 12 October 2023; Published 7 November 2023

Academic Editor: Zafar Ullah

The algebra of split quaternions is a recently increasing topic in the study of theory and numerical computation in split quaternionic mechanics. This paper, by means of a real representation of a split quaternion matrix, studies the problem of canonical forms of a split quaternion matrix and derives algebraic techniques for finding the canonical forms of a split quaternion matrix. This paper also gives two applications for the right eigenvalue and diagonalization in split quaternionic mechanics.

1. Introduction

A split quaternion (or coquaternion), which was found in 1849 by James Cockle, is in the form of

$$q = q_1 + q_2 i + q_3 j + q_4 k, \tag{1a}$$

$$i^2 = -1, j^2 = k^2 = 1, \text{ijk} = 1, \tag{1b}$$

where q_1, q_2, q_3, q_4 are real numbers. One can easily get that $\text{ij} = -\text{ji} = k, \text{jk} = -\text{kj} = -i, \text{ki} = -\text{ik} = j$. A quaternion, which was found in 1843 by William Rowan Hamilton, is in the form of $q = q_1 + q_2 i + q_3 j + q_4 k$, $i^2 = j^2 = k^2 = -1, \text{ijk} = -1$, where q_1, q_2, q_3, q_4 are real numbers, and $\text{ij} = -\text{ji} = k, \text{jk} = -\text{kj} = i, \text{ki} = -\text{ik} = j$. Let $\mathbf{H}_s$ and $\mathbf{H}$ denote, respectively, the split quaternion ring and the quaternion ring. The ring $\mathbf{H}_s$ and $\mathbf{H}$ are two different associative and noncommutative Clifford algebras, and the former ring, which contains zero-divisors, nilpotent elements, and nontrivial idempotents, is a noncommutative ring [1–7].

The split quaternions are used to express Lorentzian rotations, and works on the geometric and mechanical meaning of split quaternions can be found in [8–17]. Moreover, in the study of the relation between complexified classical and non-Hermitian quantum mechanics, there are findings that complexified mechanical systems can alternatively be thought of as certain quaternionic and split quaternionic extensions of the underlying real mechanical systems [12, 18–24]. This identification gives rise to the possibility of using algebraic techniques of split quaternions for dealing with problems in complexified classical and quantum mechanics.

It is known that the theory and algebraic technique for Jordan forms of a matrix play essential roles in the study of matrix theory and its applications. In [25, 26], the authors studied the problems of Jordan forms of complex and quaternion matrices, respectively; for instance, for any $n \times n$ quaternion matrices A, A are uniquely similar to the Jordan form [26], and the authors gave many applications not only in the theory study but also in the numerical computation. This paper, by means of a real representation of a split quaternion matrix, studies the problems of canonical forms of a split quaternion matrix and derives algebraic techniques for finding the canonical forms of a split quaternion matrix. This paper also gives two applications for the right eigenvalue and diagonalization of a split quaternion matrix in split quaternionic mechanics.

Denote the real number field and complex number field, respectively, by $\mathbf{R}$ and $\mathbf{C}$ and the split quaternion ring by $\mathbf{H}_s = \mathbf{R} \oplus \mathbf{R}i \oplus \mathbf{R}\text{j} \oplus \mathbf{R}k$. A split quaternion λ is called to be a right (left) eigenvalue of a split quaternion matrix $A \in \mathbf{H}_s^{n\times n}$ if $A\alpha = \alpha\lambda\,(A\alpha = \lambda\alpha)$ for nonzero vector α, and α is called to be an eigenvector corresponding to the eigenvalue λ. For $A \in \mathbf{H}_s^{n\times n}$, A is said to be nonsingular if $AB = BA = I_n$ for a matrix $B \in \mathbf{H}_s^{n\times n}$. Two matrices $A, B \in \mathbf{H}_s^{n\times n}$ are said to be similar if $T^{-1}AT = B$ for a nonsingular matrix $T \in \mathbf{H}_s^{n\times n}$. If A is similar to B, write $A \sim B$, and if A is permutation similar to B, write $A \overset{ps}{\sim} B$. A matrix $A \in \mathbf{H}_s^{n\times n}$ is called to be diagonalizable if $T^{-1}AT = J$ is a diagonal matrix for a nonsingular matrix $T \in \mathbf{H}_s^{n\times n}$.

2. Real Representations of a Split Quaternion

Given $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbf{H}_s^{m\times n}$, $A_1, A_2, A_3, A_4 \in \mathbf{R}^{m\times n}$, a real representation of A was given as [13]

$$A^R = \begin{bmatrix} A_1 + A_3 & -A_2 + A_4 \\ A_2 + A_4 & A_1 - A_3 \end{bmatrix} \in \mathbf{R}^{2m\times 2n}, \tag{2}$$

and the mapping $f\colon A \longrightarrow A^R$ is an isomorphism of ring $\mathbf{H}_s^{m\times n}$ onto ring $\mathbf{R}^{2m\times 2n}$.

For $A, B \in \mathbf{H}_s^{m\times n}$, $C \in \mathbf{H}_s^{n\times p}$, $r \in \mathbf{R}$, we have the following results:

$$(A + B)^R = A^R + B^R,\ (AC)^R = A^R C^R,\ (rA)^R = rA^R. \tag{3}$$

Therefore, the statements above imply that $\mathbf{H}_s^{m\times n}$ is isomorphic to $\mathbf{R}^{2m\times 2n}$, that is, $\mathbf{H}_s^{m\times n} \cong \mathbf{R}^{2m\times 2n}$. Therefore, $A \sim B$ if and only if $A^R \sim B^R$ for two matrices $A, B \in \mathbf{H}_s^{n\times n}$. Moreover, $A \in \mathbf{H}_s^{n\times n}$ is nonsingular if and only if A^R is nonsingular with $(A^R)^{-1} = (A^{-1})^R$.

Remark 1. For $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbf{H}_s^{m\times n}$, $A_1, A_2, A_3, A_4 \in \mathbf{R}^{m\times n}$, the real representation of A is not unique. The real representation of A can also be of the form

$$A^R = \begin{bmatrix} A_1 + A_4 & A_2 + A_3 \\ -A_2 + A_3 & A_1 - A_4 \end{bmatrix} \in \mathbf{R}^{2m\times 2n}. \tag{4}$$

Similarly, this real representation also gives an isomorphism of ring $\mathbf{H}_s^{m\times n}$ onto ring $\mathbf{R}^{2m\times 2n}$.

Proposition 2 (see [26], chapter 6.7). *Let $A \in \mathbf{H}_s^{n\times n}$. Then the complex eigenvalues of the real representation A^R given in (2) appear in conjugate pairs, and A^R is similar to the real block-diagonal matrix, that is,*

$$A^R \sim \sum_{u=1}^{t} \oplus J_{2l_u}\left(\lambda_u, \overline{\lambda}_u\right) \oplus \sum_{v=1}^{s} \oplus J_{m_v}(\mu_v), \tag{5}$$

where $\lambda_u = a_u + b_u\mathrm{i}$, $a_u, b_u \in \mathbf{R}$, are imaginary eigenvalues and μ_v are real eigenvalues of real representation A^R, and $2\sum_{u=1}^{t} l_u + \sum_{v=1}^{s} m_v = 2n$. Moreover, $J_{2l_u}(\lambda_u, \overline{\lambda}_u)$ and $J_{m_v}(\mu_v)$ are the matrices of $2l_u \times 2l_u$ and $m_v \times m_v$, respectively, as follows.

$$J_{2l_u}\left(\lambda_u, \overline{\lambda}_u\right) = \begin{bmatrix} F_u & I_2 & & \\ & F_u & \ddots & \\ & & \ddots & I_2 \\ & & & F_u \end{bmatrix}, J_{m_v}(\mu_v) = \begin{bmatrix} \mu_v & 1 & & \\ & \mu_v & \ddots & \\ & & \ddots & 1 \\ & & & \mu_v \end{bmatrix}, \tag{6}$$

and $F_u = \begin{bmatrix} a_u & -b_u \\ b_u & a_u \end{bmatrix}$, $J_{m_v}(\mu_v)$ is the Jordan block of the eigenvalue μ_v.

3. A Canonical Form of a Split Quaternion Matrix

Given $A \in \mathbf{H}_s^{n\times n}$, let all different complex eigenvalues of the real representation A^R be as follows:

$$\lambda_1, \overline{\lambda}_1, \ldots, \lambda_t, \overline{\lambda}_t, \mu_1, \mu_2, \ldots, \mu_s, \tag{7}$$

in which $\lambda_u = a_u + b_u\mathrm{i}$ are imaginary, $a_u, b_u\,(>0)$, and μ_v are real; by Proposition 2, there exists a nonsingular matrix $P \in \mathbf{R}^{2n\times 2n}$ such that $P^{-1}A^R P$ is a real block-diagonal matrix in (5), in which $J_{2l_u}(\lambda_u, \overline{\lambda}_u)$ and $J_{m_v}(\mu_v)$ are given in (6), and $2\sum_{u=1}^{t} l_u + \sum_{v=1}^{s} m_v = 2n$.

For the cases that the real representation A^R has imaginary eigenvalues, let all $J_{2l_u}(\lambda_u, \overline{\lambda}_u)$ matrices in (6) be as follows:

$$J_{2l_1}\left(\lambda_1, \overline{\lambda}_1\right), J_{2l_2}\left(\lambda_2, \overline{\lambda}_2\right), \ldots, J_{2l_t}\left(\lambda_t, \overline{\lambda}_t\right). \tag{8}$$

Clearly, for the Jordan blocks $J_{l_u}(\lambda_u)$, by definition (2), we have

$$J_{l_u}^R(\lambda_u) = \begin{bmatrix} a_u & 1 & & & -b_u & & & \\ & a_u & \ddots & & & -b_u & & \\ & & \ddots & 1 & & & \ddots & \\ & & & a_u & & & & -b_u \\ b_u & & & & a_u & 1 & & \\ & b_u & & & & a_u & \ddots & \\ & & \ddots & & & & & 1 \\ & & & b_u & & & & a_u \end{bmatrix}_{2l_u\times 2l_u}$$

$$\overset{\text{ps}}{\sim}\begin{bmatrix} F_u & I_2 & & & & & & \\ & F_u & \ddots & & & & & \\ & & \ddots & I_2 & & & & \\ & & & F_u & I_2 & & & \\ & & & & F_u & I_2 & & \\ & & & & & F_u & \ddots & \\ & & & & & & \ddots & I_2 \\ & & & & & & & F_u \end{bmatrix}_{2l_u\times 2l_u} = J_{2l_u}\left(\lambda_u, \overline{\lambda}_u\right). \quad (9)$$

For the case that the real representation A^R has real eigenvalues, let all the Jordan blocks $J_{m_v}(\mu_v)$ in (10) be as follows:

$$J_{m_1}(\mu_1), J_{m_2}(\mu_2), \ldots, J_{m_s}(\mu_s). \quad (10)$$

Suppose that an integer m_t for some $1 \le t \le s$ is even; then we have

$$L_{m_t/2}\left(\mu_t + \frac{k-\mathrm{i}}{2}\right) \equiv \begin{bmatrix} \mu_t + \frac{k-\mathrm{i}}{2} & \frac{i+k}{2} & & \\ & \mu_t + \frac{k-i}{2} & \ddots & \\ & & \ddots & \frac{i+k}{2} \\ & & & \mu_t + \frac{k-i}{2} \end{bmatrix}_{m_t/2\times m_t/2}. \quad (11)$$

Clearly, by (2), we have

$$L^R_{m_t/2}\left(\mu_t + \frac{k-\mathrm{i}}{2}\right) = \begin{bmatrix} \mu_t & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \mu_t & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_t & 0 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 0 & \mu_t & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 & \mu_t & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \mu_t \end{bmatrix}_{m_t\times m_t} \overset{\text{ps}}{\sim} J_{m_t}(\mu_t). \quad (12)$$

Suppose that an integer m_v for some $1 \le v \le s$ is odd; since $\sum_{v=1}^{s} m_v = 2(n - \sum_{u=1}^{t} l_u)$, there exists a Jordan block $J_{m_u}(\mu_u)$ such that m_u is also odd; set

$$J(\mu_u, \mu_v) = \begin{bmatrix} J_{m_u}(\mu_u) & 0 \\ 0 & J_{m_v}(\mu_v) \end{bmatrix}_{m_u+m_v}. \quad (13)$$

There are three cases for two odds m_u and m_v as follows:

Case 1. $m_u = m_v = 1$. In this case, we have

$$J(\mu_u, \mu_v) = \begin{bmatrix} \mu_u & \\ & \mu_v \end{bmatrix} = \begin{cases} \mu_u^R, & \mu_u = \mu_v, \\ \left(\dfrac{\mu_u + \mu_v}{2} + \dfrac{\mu_u - \mu_v}{2} j\right)^R, & \mu_u \neq \mu_v. \end{cases} \tag{14}$$

Case 2. $m_u = 1, m_v \geq 3$. In this case, construct the following split quaternion matrix:

$$M_{(1+m_v)/2\times(1+m_v)/2}\left(\mu_v + \frac{k-i}{2}\right) \equiv \begin{bmatrix} \dfrac{\mu_u + \mu_v}{2} + \dfrac{\mu_u - \mu_v}{2} j & \dfrac{i+k}{2} \\ 0 & L_{(m_v-1)/2}\left(\mu_v + \dfrac{k-i}{2}\right) \end{bmatrix}, \tag{15}$$

and it is clear to get the following equality:

$$\begin{aligned} M_{(1+m_v)/2\times(1+m_v)/2}\left(\mu_v + \frac{k-i}{2}\right) &= \begin{bmatrix} \mu_u & 0 & 0 & 0 \\ 0 & \mu_v I_{(m_v-1)/2} & 0 & I_{(m_v-1)/2} \\ 0 & 1 & \mu_v & 0 \\ 0 & J_{(m_v-1)/2}(0) & 0 & \mu_v I_{(m_v-1)/2} \end{bmatrix}_{(1+m_v)/2\times(1+m_v)/2} \\ \overset{ps}{\sim} \begin{bmatrix} \mu_u & 0 & 0 & 0 \\ 0 & \mu_v & 1 & 0 \\ 0 & 0 & \mu_v I_{(m_v-1)/2} & I_{(m_v-1)/2} \\ 0 & 0 & J_{(m_v-1)/2}(0) & \mu_v I_{(m_v-1)/2} \end{bmatrix} &\overset{ps}{\sim} \begin{bmatrix} \mu_u & 0 \\ 0 & J_{m_v}(\mu_v) \end{bmatrix}_{1+m_v} = J(\mu_u, \mu_v). \end{aligned} \tag{16}$$

Case 3. $m_u \geq 3, m_v \geq 3$. In this case, construct the $(m_u + m_v)/2 \times (m_u + m_v)/2$ split quaternion matrix:

$$\begin{aligned}
&M_{(m_u+m_v)/2\times(m_u+m_v)/2}\left(\mu_u+\frac{k-i}{2},\mu_v+\frac{k-i}{2}\right)\\
&\equiv\begin{bmatrix} L_{(m_u-1)/2}\left(\mu_u+\frac{k-i}{2}\right) & e_{(m_u-1)/2}\left(\frac{i+k}{2}\right) & 0\\ 0 & \frac{\mu_u+\mu_v}{2}+\frac{\mu_u-\mu_v}{2}\mathrm{j} & e_1^T\left(\frac{i+k}{2}\right)\\ 0 & 0 & L_{(m_v-1)/2}\left(\mu_v+\frac{k-i}{2}\right)\end{bmatrix}\\
&=\begin{bmatrix}
\mu_u+\frac{k-i}{2} & \frac{i+k}{2} & & & & & & & \\
 & \mu_u+\frac{k-i}{2} & \ddots & & & & & & \\
 & & \ddots & \frac{i+k}{2} & & & & & \\
 & & & \mu_u+\frac{k-i}{2} & \frac{i+k}{2} & & & & \\
 & & & & \frac{\mu_u+\mu_v}{2}+\frac{\mu_u-\mu_v}{2}j & \frac{i+k}{2} & & & \\
 & & & & & \mu_v+\frac{k-i}{2} & \frac{i+k}{2} & & \\
 & & & & & & \mu_v+\frac{k-i}{2} & \ddots & \\
 & & & & & & & \ddots & \frac{i+k}{2}\\
 & & & & & & & & \mu_v+\frac{k-i}{2}
\end{bmatrix},
\end{aligned}\tag{17}$$

where $e^T_{(m_u-1)/2}((i+k)/2)=(0,\ldots,0,(i+k)/2)\in\mathbf{R}^{1\times(m_u-1)/2}$, $e_1^T\,((i+k)/2)=((i+k)/2,0,\ldots,0)\in\mathbf{R}^{1\times(m_v-1)/2}$.

By the fact that $L_{(m_v-1)/2}(\mu_v+(k-i)/2)=\mu_v I_{(m_v-1)/2}+(1/2)J_{(m_v-1)/2}(-1)i+(1/2)J_{(m_v-1)/2}(1)k$, we easily get the following equality:

$$M^{\mathrm{R}}_{(m_u+m_v)/2\times(m_u+m_v)/2}\left(\mu_u+\frac{k-i}{2},\mu_v+\frac{k-i}{2}\right)$$
$$=\begin{bmatrix} \mu_u I_{(m_u-1)/2} & 0 & 0 & I_{(m_u-1)/2} & 0 & 0 \\ 0 & \mu_u & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_v I_{(m_v-1)/2} & 0 & 0 & I_{(m_v-1)/2} \\ J_{(m_u-1)/2}(0) & e_{(m_u-1)/2} & 0 & \mu_u I_{(m_u-1)/2} & 0 & 0 \\ 0 & 0 & e_1^T & 0 & \mu_v & 0 \\ 0 & 0 & J_{(m_v-1)/2}(0) & 0 & 0 & \mu_v I_{(m_v-1)/2} \end{bmatrix} \tag{18}$$
$$\overset{\mathrm{ps}}{\sim}\begin{bmatrix} \mu_u I_{(m_u-1)/2} & I_{(m_u-1)/2} & 0 & & & \\ J_{(m_u-1)/2}(0) & \mu_u I_{(m_u-1)/2} & e_{(m_u-1)/2} & & & \\ 0 & 0 & \mu_u & & & \\ & & & \mu_v I_{(m_v-1)/2} & I_{(m_v-1)/2} & 0 \\ & & & J_{(m_v-1)/2}(0) & \mu_v I_{(m_v-1)/2} & 0 \\ & & & e_1^T & 0 & \mu_v \end{bmatrix}.$$

It is easy to get by direct calculation

$$\begin{bmatrix} \mu_u I_{(m_u-1)/2} & I_{(m_u-1)/2} & 0 \\ J_{(m_u-1)/2}(0) & \mu_u I_{(m_u-1)/2} & e_{(m_u-1)/2} \\ 0 & 0 & \mu_u \end{bmatrix} \overset{\mathrm{ps}}{\sim} J_{m_u}(\mu_u),$$
$$\begin{bmatrix} \mu_v I_{(m_v-1)/2} & I_{(m_v-1)/2} & 0 \\ J_{(m_v-1)/2}(0) & \mu_v I_{(m_v-1)/2} & 0 \\ e_1^T & 0 & \mu_v \end{bmatrix} \overset{\mathrm{ps}}{\sim} J_{m_v}(\mu_v). \tag{19}$$

Therefore,

$$M^{R}_{(m_u+m_v)/2\times(m_u+m_v)/2}\left(\mu_u+\frac{k-i}{2},\mu_v+\frac{k-i}{2}\right)\overset{\mathrm{ps}}{\sim}\begin{bmatrix} J_{m_u}(\mu_u) & 0 \\ 0 & J_{m_v}(\mu_v) \end{bmatrix}=J(\mu_u,\mu_v). \tag{20}$$

From the aforementioned discussions, we can obtain the following results:

$$A^R \sim \left\{ \sum_u \oplus J_{l_u}(\lambda_u) \oplus \sum_{\mu_p \neq \mu_q} \oplus \left(\frac{\mu_p + \mu_q}{2} + \frac{\mu_p - \mu_q}{2} \mathrm{j} \right) \oplus \sum_v \oplus \mu_v \right.$$
$$\oplus \sum_{\substack{t \\ m_t \text{even}}} \oplus L_{m_t/2}\left(\mu_t + \frac{k-i}{2} \right) \oplus \sum_{\substack{h \\ m_h \text{odd} \\ m_h \geq 3}} \oplus M_{(1+m_h)/2\times(1+m_h)/2}\left(\mu_h + \frac{k-i}{2} \right) \quad (21)$$
$$\left. \oplus \sum_{\substack{r,s \\ m_r, m_s \text{odd} \\ m_r \geq 3, m_s \geq 3}} \oplus M_{(m_r+m_s)/2\times(m_r+m_s)/2}\left(\mu_r + \frac{k-i}{2}, \mu_s + \frac{k-i}{2} \right) \right\}^R.$$

Theorem 3. *Let $A \in \mathbf{H}_s^{n\times n}$. Then A is similar to the following canonical form:*

$$A \sim \sum_u \oplus J_{l_u}(\lambda_u) \oplus \sum_{\mu_p \neq \mu_q} \oplus \left(\frac{\mu_p + \mu_q}{2} + \frac{\mu_p - \mu_q}{2} \mathrm{j} \right) \oplus \sum_v \oplus \mu_v$$
$$\oplus \sum_{\substack{t \\ m_t \text{even}}} \oplus L_{m_t/2}\left(\mu_t + \frac{k-i}{2} \right) \oplus \sum_{\substack{h \\ m_h \text{odd} \\ m_h \geq 3}} \oplus M_{(1+m_h)/2\times(1+m_h)/2}\left(\mu_h + \frac{k-i}{2} \right) \quad (22)$$
$$\oplus \sum_{\substack{r,s \\ m_r, m_s \text{odd} \\ m_r \geq 3, m_s \geq 3}} \oplus M_{(m_r+m_s)/2\times(m_r+m_s)/2}\left(\mu_r + \frac{k-i}{2}, \mu_s + \frac{k-i}{2} \right),$$

in which the split quaternion matrices $L_{m_t/2}(\mu_t + (k-i)/2)$, $M_{(1+m_h)/2\times(1+m_h)/2}(\mu_h + (k-i)/2)$, and $M_{(m_r+m_s)/2\times(m_r+m_s)/2}(\mu_r + (k-i)/2, \mu_s + (k-i)/2)$ are defined in (11), (15), and (17), respectively.

Similar to Theorem 3, we can obtain Corollary 4.

Corollary 4. *Let $A \in \mathbf{H}_s^{n\times n}$. Then A is similar to the upper bidiagonal matrix, that is,*

$$A \sim \begin{bmatrix} \lambda_1 & \sigma_1 & & & \\ & \lambda_2 & \sigma_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \sigma_{n-1} \\ & & & & \lambda_n \end{bmatrix}_{n\times n}, \tag{23}$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_n \in \{a_u + b_u i, (\mu_p + \mu_q)/2 + (\mu_p - \mu_q)/2 j, \mu_v, \mu_t + (k-i)/2\}$ *are right split quaternion eigenvalues of the split quaternion matrix* A, *and* $\sigma_1, \sigma_2, \ldots, \sigma_{n-1} \in \{0, 1, (i+k)/2\}$.

Remark 5. For $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbf{H}_s^{m\times n}$, $A_1, A_2, A_3, A_4 \in \mathbf{R}^{m\times n}$, the canonical form of A is not unique. For example, if a real representation of A is defined in (4), then it is easy to get the following result:

$$\begin{aligned} A^R \sim \Bigg\{ & \sum_u \oplus J_{l_u}(a_u - b_u k) \oplus \sum_{\mu_p \neq \mu_q} \oplus \left(\frac{\mu_p + \mu_q}{2} + \frac{\mu_p - \mu_q}{2} i\right) \oplus \sum_v \oplus \mu_v \\ & \oplus \sum_{\substack{t \\ m_t \text{even}}} \oplus L_{m_t/2}\left(\mu_t + \frac{k-j}{2}\right) \oplus \sum_{\substack{h \\ m_h \text{odd} \\ m_h \geq 3}} \oplus M_{(1+m_h)/2\times(1+m_h)/2}\left(\mu_h + \frac{k-j}{2}\right) \\ & \oplus \sum_{\substack{r,s \\ m_r, m_s \text{odd} \\ m_r \geq 3, m_s \geq 3}} \oplus M_{(m_r+m_s)/2\times(m_r+m_s)/2}\left(\mu_r + \frac{k-j}{2}, \mu_s + \frac{k-j}{2}\right) \Bigg\}^R. \end{aligned} \tag{24}$$

By Remark 5, similar to Theorem 3, we can obtain Theorem 6.

Theorem 6. *Let* $A \in \mathbf{H}_s^{n\times n}$, *and*

$$A^R \sim \sum_u \oplus J_{2l_u}(\lambda_u, \bar{\lambda}_u) \oplus \sum_v \oplus J_{m_v}(\mu_v), \tag{25}$$

in which $\lambda_u = a_u + b_u i$ *are imaginary eigenvalues,* μ_v *are real eigenvalues of real representation* A^R, *and* $J_{2l_u}(\lambda_u, \bar{\lambda}_u)$ *and* $J_{m_v}(\mu_v)$ *are given in (6). Then* A *is similar to the following canonical form:*

$$\begin{aligned} A \sim & \sum_u \oplus J_{l_u}(a_u - b_u k) \oplus \sum_{\mu_p \neq \mu_q} \oplus \left(\frac{\mu_p + \mu_q}{2} + \frac{\mu_p - \mu_q}{2} i\right) \oplus \sum_v \oplus \mu_v \\ & \oplus \sum_{\substack{t \\ m_t \text{even}}} \oplus L_{m_t/2}\left(\mu_t + \frac{k-j}{2}\right) \oplus \sum_{\substack{h \\ m_h \text{odd} \\ m_h \geq 3}} \oplus M_{(1+m_h)/2\times(1+m_h)/2}\left(\mu_h + \frac{k-j}{2}\right) \\ & \oplus \sum_{\substack{r,s \\ m_r, m_s \text{odd} \\ m_r \geq 3, m_s \geq 3}} \oplus M_{(m_r+m_s)/2\times(m_r+m_s)/2}\left(\mu_r + \frac{k-j}{2}, \mu_s + \frac{k-j}{2}\right), \end{aligned} \tag{26}$$

in which the split quaternion matrices $L_{m_t/2}(\mu_t + (k-j)/2)$, $M_{(1+m_h)/2\times(1+m_h)/2}(\mu_h + (k-j)/2)$, *and* $M_{(m_r+m_s)/2\times(m_r+m_s)/2}(\mu_r + (k-j)/2, \mu_s + (k-j)/2)$ *are defined in (27)–(29), respectively.*

$$L_{m_t/2}\left(\mu_t + \frac{k-j}{2}\right) = \begin{bmatrix} \mu_t + \frac{k-j}{2} & \frac{j+k}{2} & & \\ & \mu_t + \frac{k-j}{2} & \ddots & \\ & & \ddots & \frac{j+k}{2} \\ & & & \mu_t + \frac{k-j}{2} \end{bmatrix}_{m_t/2\times m_t/2}, \tag{27}$$

$$M_{(1+m_h)/2\times(1+m_h)/2}\left(\mu_h + \frac{k-j}{2}\right) \equiv \begin{bmatrix} \frac{\mu_h+\mu_v}{2} + \frac{\mu_h-\mu_v}{2}i & \frac{j+k}{2} \\ 0 & L_{(m_v-1)/2}\left(\mu_v + \frac{k-j}{2}\right) \end{bmatrix}, \tag{28}$$

$$\cdot M_{(m_r+m_s)/2\times(m_r+m_s)/2}\left(\mu_r + \frac{k-j}{2}, \mu_s + \frac{k-j}{2}\right) \equiv \begin{bmatrix} L_{(m_r-1)/2}\left(\mu_r + \frac{k-j}{2}\right) & e_{(m_r-1)/2}\left(\frac{j+k}{2}\right) & 0 \\ 0 & \frac{\mu_r+\mu_s}{2} + \frac{\mu_r-\mu_s}{2}\mathrm{i} & e_1^T\left(\frac{j+k}{2}\right) \\ 0 & 0 & L_{(m_s-1)/2}\left(\mu_s + \frac{k-j}{2}\right) \end{bmatrix}. \tag{29}$$

Corollary 7. *Let* $A \in \mathbf{H}_s^{n\times n}$. *Then A is similar to the upper bidiagonal matrix, that is,*

$$A \sim \begin{bmatrix} \eta_1 & \tau_1 & & & \\ & \eta_2 & \tau_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \tau_{n-1} \\ & & & & \eta_n \end{bmatrix}_{n\times n}, \tag{30}$$

in which $\eta_1, \eta_2, \ldots, \eta_n \in \{a_u - b_u i, (\mu_p + \mu_q)/2 + (\mu_p - \mu_q)/2i, \mu_v, \mu_t + (k-j)/2\}$ *are right split quaternion eigenvalues of A, and* $\tau_1, \tau_2, \ldots, \tau_{n-1} \in \{0, 1, (j+k)/2\}$.

From the statements above, we know that the proofs of Theorems 3 and 6 are constructive; they give two algebraic techniques for finding canonical forms by means of real representation of a split quaternion matrix.

4. Applications

This section, by means of canonical forms of a split quaternion matrix, gives two applications for the eigenvalue and diagonalization of a split quaternion matrix.

If $x^{-1}px = q$ for a nonsingular split quaternion x, the two split quaternions p and q are called similar, which can be written as $p \sim q$. Denote by $[q]$ the equivalence class that contains q. The split quaternion q is called to be a principal element of $[q]$.

By Theorem 3, we get the following result.

Theorem 8. *Let* $A \in \mathbf{H}_s^{n\times n}$, *all different complex eigenvalues of the real representation* A^R *be given in (7). Then there exist four possible different kinds of principal eigenvalues of A as follows:*

$$\lambda_u = a_u + b_u i, \frac{\mu_p + \mu_q}{2} + \frac{\mu_p - \mu_q}{2} j, \mu_v, \mu_t + \frac{k-i}{2}, \tag{31}$$

in which $\mu_v, \mu_v, \mu_p, \mu_q, \mu_p \neq \mu_q$, and μ_t are real eigenvalues of A^R and $\lambda_u = a_u + b_u i$ are imaginary eigenvalues of A^R with real $a_u, b_u\,(>0)$.

Moreover, all possible right eigenvalues of A are the following equivalence classes with principal element $\lambda_u = a_u + b_u i$, $(\mu_p + \mu_q)/2 + (\mu_p - \mu_q)/2 j, \mu_v$ and $\mu_t + (k - i)/2$, respectively.

$$\bigcup_{\lambda_u}[\lambda_u], \bigcup_{\mu_p,\mu_q}\left[\frac{\mu_p + \mu_q}{2} + \frac{\mu_p - \mu_q}{2} j\right], \bigcup_{\mu_v}[\mu_v], \bigcup_{\mu_t}\left[\mu_t + \frac{k-i}{2}\right]. \tag{32}$$

Remark 9. For $A \in \mathbf{H}_s^{n\times n}$, Theorem 8 not only derives all possible right eigenvalues of A but also suggests a technique for finding the eigenvalues by means of real representation of A.

By the fact that $\mathbf{H}_s^{n\times n}$ is isomorphic to $\mathbf{R}^{2n\times 2n}$, for $A, B \in \mathbf{H}_s^{n\times n}$, we know that $A \sim B$ if and only if $A^R \sim B^R$. In addition, if $A \in \mathbf{H}_s^{n\times n}$ is diagonalizable, then by Theorem 3, we get the following result:

$$A \sim J = \sum_u \oplus \lambda_u \oplus \sum_{\mu_p \neq \mu_q} \oplus \left(\frac{\mu_p + \mu_q}{2} + \frac{\mu_p - \mu_q}{2} j\right) \oplus \sum_v \oplus \mu_v \oplus \sum_t \oplus \left(\mu_t - \frac{1}{2} i + \frac{1}{2} k\right), \tag{33}$$

and by (2) and (3), we have

$$\begin{aligned} A^R &\overset{\mathrm{ps}}{\sim} \sum_u \oplus \lambda_u^R \oplus \sum_{\mu_p \neq \mu_q} \oplus \left(\frac{\mu_p + \mu_q}{2} + \frac{\mu_p - \mu_q}{2} j\right)^R \oplus \sum_v \oplus \mu_v^R \oplus \sum_t \oplus \left(\mu_t - \frac{1}{2} i + \frac{1}{2} k\right)^R \\ &\overset{\mathrm{s}}{\sim} \sum_u \oplus \begin{bmatrix} a_u & -b_u \\ b_u & a_u \end{bmatrix} \oplus \sum_{\mu_p \neq \mu_q} \oplus \begin{bmatrix} \mu_p & 0 \\ 0 & \mu_q \end{bmatrix} \oplus \sum_v \oplus \begin{bmatrix} \mu_v & 0 \\ 0 & \mu_v \end{bmatrix} \oplus \sum_t \oplus \begin{bmatrix} \mu_t & 1 \\ 0 & \mu_t \end{bmatrix} \\ &\overset{\mathrm{s}}{\sim} \sum_u \oplus \begin{bmatrix} \lambda_u & 0 \\ 0 & \overline{\lambda}_u \end{bmatrix} \oplus \sum_{\mu_p \neq \mu_q} \oplus \begin{bmatrix} \mu_p & 0 \\ 0 & \mu_q \end{bmatrix} \oplus \sum_v \oplus \begin{bmatrix} \mu_v & 0 \\ 0 & \mu_v \end{bmatrix} \oplus \sum_t \oplus \begin{bmatrix} \mu_t & 1 \\ 0 & \mu_t \end{bmatrix}, \end{aligned} \tag{34}$$

and the characteristic polynomial of A^R is

$$f_{A^R}(x) = \prod_u \left(x^2 + p_u x + q_u\right)^{m_u} \prod_r \left(x^2 + c_r x + d_r\right)^{n_r} \prod_v (x - \mu_v)^{2h_v} \prod_t (x - \mu_t)^{2g_t}, \tag{35}$$

where $p_u^2 < 4q_u$, $c_r^2 \geq 4d_r$, and $p_u, q_u, c_r, d_r, \mu_v, \mu_t$ are real numbers. Therefore, $A \in \mathbf{H}_s^{n\times n}$ is diagonalizable if and only if A^R satisfies (34) and (35).

From the aforementioned discussions, we can obtain the following results.

Theorem 10. *Let $A \in \mathbf{H}_s^{n\times n}$. Then A is diagonalizable if and only if the characteristic polynomial of A^R has the following form:*

$$f_{A^R}(x) = \prod_u \left(x^2 + p_u x + q_u\right)^{m_u} \prod_r \left(x^2 + c_r x + d_r\right)^{n_r} \prod_v (x - \mu_v)^{2h_v} \prod_t (x - \mu_t)^{2g_t}, \tag{36}$$

where $p_u^2 < 4q_u$, $c_r^2 \geq 4d_r$, *and* $p_u, q_u, c_r, d_r, \mu_t$ *are real numbers, and for any eigenvalue* μ *of* A^R,

$$\operatorname{rank}\left(\mu I_{2n} - A^R\right) = 2n - s, \quad \text{or } 2n - \left(\frac{1}{2}\right)s, \tag{37}$$

where s is the multiplicity of the eigenvalue μ *in which case the diagonal matrix* J *is as follows:*

$$J = \sum_u \oplus \lambda_u \oplus \sum_{\mu_p \neq \mu_q} \oplus \left(\frac{\mu_p + \mu_q}{2} + \frac{\mu_p - \mu_q}{2} j\right) \oplus \sum_v \oplus \mu_v \oplus \sum_t \oplus \left(\mu_t - \frac{1}{2} i + \frac{1}{2} k\right). \tag{38}$$

Remark 11. Theorem 10 derives an algebraic technique for finding the diagonal split quaternion matrix J in (33) by means of real representation A^R of a split quaternion matrix A. Moreover, if A is diagonalizable, then A^R can be diagonalizable by statement (34).

Example 1. Let

$$A = \begin{bmatrix} -2+i-j-k & 1+i+3j-k & 2-i-4j+2k & -3+i+2j-2k \\ -1-2j & 3+j & -2+i & -i-j \\ 1-i+3j+k & -3-i-j+k & 5+2i-j-2k & -2-2i+2j+2k \\ 1-i+3j+k & -3-i-j+k & 3+i-j-2k & -i+2j+2k \end{bmatrix}$$
$$= \begin{bmatrix} -2 & 1 & 2 & -3 \\ -1 & 3 & -2 & 0 \\ 1 & -3 & 5 & -2 \\ 1 & -3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & 2 & -2 \\ -1 & -1 & 1 & -1 \end{bmatrix} i + \begin{bmatrix} -1 & 3 & -4 & 2 \\ -2 & 1 & 0 & -1 \\ 3 & -1 & -1 & 2 \\ 3 & -1 & -1 & 2 \end{bmatrix} j + \begin{bmatrix} -1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 2 \\ 1 & 1 & -2 & 2 \end{bmatrix} k. \tag{39}$$

By (2), we have

$$A^R = \begin{bmatrix} -3 & 4 & -2 & -1 & -2 & -2 & 3 & -3 \\ -3 & 4 & -2 & -1 & 0 & 0 & -1 & 1 \\ 4 & -4 & 4 & 0 & 2 & 2 & -4 & 4 \\ 4 & -4 & 2 & 2 & 2 & 2 & -3 & 3 \\ 0 & 0 & 1 & -1 & -1 & -2 & 6 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 & -2 & 6 & -4 \\ 0 & 0 & -1 & 1 & -2 & -2 & 4 & -2 \end{bmatrix}, \tag{40}$$

and by direct calculation, we get

$$f_{A^R}(x) = \left(x^2 - 4x + 5\right)\left(x^2 - 6x + 8\right)(x-1)^2 x^2, \tag{41}$$

and the eigenvalues of A^R are as follows:

$$\lambda_1 = 2+i, \lambda_2 = 2-i, \mu_1 = 4, \mu_2 = 2, \quad \mu_3 = 1, \mu_4 = 1, \mu_5 = 0, \mu_6 = 0. \tag{42}$$

(1) By Theorem 8, the principal eigenvalues of the split quaternion matrix A are

$$\nu_1 = 2+i, \nu_2 = 3+j, \nu_3 = 1, \nu_4 = \frac{k-i}{2}, \tag{43}$$

and all right eigenvalues of A are $p^{-1}\nu_s p$ for any nonsingular split quaternion p, $s = 1, 2, 3, 4$.

(2) For eigenvalues λ of real representation matrix A^R, it is easy to get that $\operatorname{rank}(\lambda I_8 - A^R) = 8 - s$ for $\lambda = \lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3$, and $\operatorname{rank}(\lambda I_8 - A^R) = 8 - (1/2)s$ for $\lambda = \mu_5$, in which s is the multiplicity of eigenvalue λ. Then by Theorem 10, A is diagonalizable, and by (38), A is similar to diagonal matrix as follows:

$$J = \begin{bmatrix} 2+i & 0 & 0 & 0 \\ 0 & 3+j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{k-i}{2} \end{bmatrix}. \tag{44}$$

(3) By (5)–(14), A^R is similar to the following canonical form:

$$A^R \sim (\lambda_u)^R \oplus \left(\frac{\mu_1+\mu_2}{2} + \frac{\mu_1-\mu_2}{2} j\right)^R \oplus (\mu_3)^R \oplus \left(\mu_5 + \frac{k-i}{2}\right)^R$$
$$\overset{\mathrm{ps}}{\sim} \left\{(\lambda_u) \oplus \left(\frac{\mu_1+\mu_2}{2} + \frac{\mu_1-\mu_2}{2} j\right) \oplus (\mu_3) \oplus \left(\mu_5 + \frac{k-i}{2}\right)\right\}^R$$
$$\overset{\mathrm{ps}}{\sim} \left\{(2+i) \oplus (3+j) \oplus (1) \oplus \left(\frac{k-i}{2}\right)\right\}^R, \tag{45}$$

and therefore, by Theorem 3, A has a canonical form J as follows:

$$J = (2+i) \oplus (3+j) \oplus (1) \oplus \left(\frac{k-i}{2}\right) = \begin{bmatrix} 2+i & 0 & 0 & 0 \\ 0 & 3+j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \dfrac{k-i}{2} \end{bmatrix}. \tag{46}$$

5. Conclusions

This paper, by means of the real representation of a split quaternion matrix, studies the problems of canonical forms of a split quaternion matrix and derives algebraic techniques for finding the canonical forms of a split quaternion matrix. This paper also gives two applications for the right eigenvalue and diagonalization of a split quaternion matrix in split quaternionic mechanics. This paper enriches the split quaternion matrix algebraic properties of split quaternionic mechanics and will certainly promote the development of split quaternionic mechanics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Tongsong Jiang was supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-02-2023-947, February 16, 2023). V. I. Vasil'ev was supported by the Ministry of Science and Higher Education of the Russian Federation (grant no. FSRG-2023-0025). Dong Zhang, Zhenwei Guo, and Gang Wang were supported by the Russian Science Foundation (grant no. 23-71-30013) and Chinese Government Scholarship (CSC nos. 202108370086, 202108370087, and 202008370340).

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