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Research Article

Numerical Solutions of Time-Fractional Whitham–Broer–Kaup Equations via Sumudu Decomposition Method

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In this paper, the coupled system of Whitham–Broer–Kaup equations of the Caputo fractional derivative (CFD) is studied using the Sumudu decomposition method (SDM). Using different dispersion relations, these equations are needed to describe the properties of waves in shallow water. The current investigation for the future scheme includes convergence and error analysis. We use two examples to demonstrate the leverage and effectiveness of the proposed scheme, and the error analysis is discussed to ensure its accuracy. The numerical simulation is carried out to ensure the accuracy of the future technique. The obtained numerical and graphical results are presented, and the proposed scheme is computationally very accurate and simple to study and solve fractionally coupled nonlinear complex phenomena encountered in science and technology.

1. Introduction

Partial differential equations with nonlinearities are used to describe a wide range of phenomena in applied sciences, engineering, and technology, ranging from gravitation to dynamics [1–3]. Indeed, nonlinear PDEs are important tools for modeling nonlinear dynamical phenomena in a range of areas, including mathematical biology, plasma physics, solid-state physics, and fluid dynamics, as shown in [4]. A suitable set of partial differential equations can represent the majority of dynamical systems. Partial differential equations are also well known for their application in the solution of mathematical problems such as the Poincare conjecture and the Calabi conjecture.

It has been noticed that the nonlinear evolution of shallow water waves in fluid dynamics is represented by a coupled system of Whitham–Broer–Kaup (WBK) equations [5]. Whitham [6], Broer [7], and Kaup [8] proposed the coupled system of the aforementioned equations. The aforementioned equations describe the propagation of shallow water waves with different spreading relations; see [9].

Given below are the governing equations in classical order for the aforementioned phenomenon

$$
\begin{aligned}
&\frac{\partial \xi}{\partial \tau} + \xi \frac{\partial \xi}{\partial \eta} + \frac{\partial \varphi}{\partial \eta} + c_1 \frac{\partial^2 \xi}{\partial \eta^2} = 0, \\
&\frac{\partial \varphi}{\partial \tau} + \xi \frac{\partial \varphi}{\partial \eta} + \varphi \frac{\partial \xi}{\partial \eta} + c_2 \frac{\partial^3 \xi}{\partial \eta^3} - c_1 \frac{\partial^2 \phi}{\partial \eta^2} = 0,
\end{aligned}
\tag{1}
$$

where $\xi = \xi(\eta, \tau)$ is horizontal velocity, $\varphi = \varphi(\eta, \tau)$ is height that deviates from equilibrium position of the liquid, and c_1, c_2 are constants which are represented in different diffusion powers.

Investigating the findings of nonlinear PDEs has been a major area of research over the last few decades. Numerous writers have developed a variety of mathematical methods for examining the approximate results of nonlinear PDEs. Aminikhah and Biazar [10] solved the coupled model of the Brusselator and Burger equations using the homotopy perturbation method (HPM). Noor and Mohyud-Din [11] used HPM to investigate the solutions of various classical orders of PDEs. For approximate solutions to other classically ordered partial differential equations by using other methods, see [12–25]. Various methods have been used to investigate the solution to the given nonlinear coupled system (1) of partial differential equations. Using the hyperbolic function method, Xie et al. [1] found some new solitary wave solutions. El-Sayed and Kaya [26] used the Adomian decomposition method (ADM) to obtain approximate solutions. Rafei and Daniali [27] used the variational iteration method (VIM) to create the analytical solutions. Ahmad et al. [4] used the ADM (Adomian decomposition method) in conjunction with He's polynomial to investigate the coupled scheme result of coupled system (1). Recently, Haq and Ishaq [28] used the optimal homotopy asymptotic method (OHAM) to solve the WBK equations and obtain numerical solutions.

Partial differential equations of fractional order have been getting a lot of attention lately [29–34] because they are used in a wide range of applied sciences, such as control theory, pattern reorganization, signal processing, system identification, and image processing. Because fractional-order differential equations describe many physical phenomena more accurately than classical order differential equations in many sciences, there is a strong reason to find good numerical solutions to fractional-order differential equations.

Now, we present a coupled (WBK) equation of fractional order of the form:

$$\begin{aligned} &\frac{\partial^{\nu}\xi}{\partial\tau^{\nu}} + \xi\frac{\partial^{\lambda}\xi}{\partial\eta^{\lambda}} + \frac{\partial^{\lambda}\varphi}{\partial\eta^{\lambda}} + c_1\frac{\partial^{2\lambda}\xi}{\partial\eta^{2\lambda}} = 0,\\ &\frac{\partial^{\nu}\varphi}{\partial\tau^{\nu}} + \xi\frac{\partial^{\lambda}\varphi}{\partial\eta^{\lambda}} + \varphi\frac{\partial^{\lambda}\xi}{\partial\eta^{\lambda}} + c_2\frac{\partial^{3\lambda}\xi}{\partial\eta^{3\lambda}} - c_1\frac{\partial^{2\lambda}\phi}{\partial\eta^{2\lambda}} = 0, \eta, \tau \geq 0, 0 < \nu, \lambda \leq 1, \end{aligned} \tag{2}$$

with the initial conditions

$$\xi(\eta, 0) = \alpha(\eta), \varphi(\eta, 0) = \beta(\eta). \tag{3}$$

It should be noted that if $\nu = \lambda = 1$, the system equation (2) becomes the standard WBK equations.

When $c_2 = 1$ and $c_1 = 0$, Eq. (2) transforms into a modified Boussinesq (MB) equation, whereas when $c_2 = 1$ and $c_1 \neq 0$, the system means an approximate long wave (ALW) equation. Similarly, the solution for the fractional order of WBK partial differential equations was investigated using various analytical and numerical methods, such as the residual power series method [2], the Riccati subequation method [35], the exponential function method [36], the coupled fractional reduced differential transform method (CFRDTM) [37], the q-homotopy analysis transform method (q-HATM) [38], the Laplace Adomian decomposition method (LADM) [3], the Yang decomposition method (YDM) [39], and the new iterative method (NIM) [40]. In 1980, Adomian presented the Adomian decomposition method (ADM), which is an efficient method for finding numerical and explicit solutions to a large class of differential equations representing physical problems. It works effectively for initial value problems and boundary value problems for partial, fractional, and ordinary differential equations, including linear and nonlinear equations. The Adomian decomposition method combined with the Sumudu transformation leads to a powerful method called the Sumudu decomposition method (SDM), which was introduced by Devendra et al. [41]. The SDM has also been used in a number of papers to obtain numerical solutions to nonlinear partial differential equations of fractional order, as shown in [42–44].

In this paper, we use the Sumudu decomposition method (SDM) to look into both the general and numerical solutions of the coupled system of fractional-order Whitham–Broer–Kaup equations. SDM is a simple and very effective method that doesn't need to be perturbation or liberalization. We compare the results of our proposed method with those of other well-known methods, such as ADM, VIM, OHAM, NIM, and LADM. We can see that the proposed method outperforms the mentioned method for solving nonlinear fractional-order PDEs.

2. Preliminaries

In this section, we explain the basic definitions and concepts that will be used in this work.

Definition 1 (see [33]). Suppose the function $\varphi \in \mathbb{C}_{\mu}, \mu \geq -1$, then the Riemann-Liouville integral operator of order $\nu \geq 0$, is known as follows:

$$J^{\nu}\varphi(\tau) = \begin{cases} \dfrac{1}{\Gamma(\nu)}\displaystyle\int_0^{\tau}(\tau-\delta)^{\nu-1}\varphi(\delta)d\delta, \nu > 0, & \delta > 0,\\ \varphi(\tau), & \nu = 0. \end{cases} \tag{4}$$

Definition 2 (see [34]). Suppose $\varphi \in \mathbb{C}_{-1}^{n}$, then the Caputo fractional derivative of order ν if $n-1 < \nu < n, n \in \mathbb{N}$, is defined as follows:

$$D_\tau^\nu \varphi(\tau) = \begin{cases} \dfrac{1}{\Gamma(n-\nu)} \displaystyle\int_0^\tau (\tau-\delta)^{n-\nu-1} \varphi^n(\delta) d\delta, & n-1<\nu<n, n \in \mathbb{N}, \\ \dfrac{\partial^n \varphi(\tau)}{\partial \tau^n}, & \nu = n. \end{cases} \quad (5)$$

Definition 3. The Sumudu transform (ST) of the function $\varphi(\tau)$ is defined by

$$S[\varphi(\tau)] = \phi(\rho) = \int_0^\infty (1/\rho)\varphi(\tau) e^{-(\tau/\rho)} d\tau, \tau > 0. \quad (6)$$

Definition 4. The Sumudu transform of derivative of fractional order in Caputo' form is

$$S[D^\nu \varphi(\tau)] = \frac{\phi(\rho)}{\rho^\nu} - \sum_{j=0}^{r-1} \frac{\varphi^{(j)}(0)}{\rho^{\nu-j}}, r-1<\nu\le r, \quad (7)$$

where $\phi(\rho)$ represents the ST of $\varphi(\tau)$.

3. Sumudu Decomposition Method (SDM)

This section illustrates the new approach of using ST combined with the decomposition method to find the solution to the general form of the nonlinear FPDE.

$$D_\tau^\nu \varphi(\eta,\tau) + R\varphi(\eta,\tau) + N\varphi(\eta,\tau) = \psi(\eta,\tau), \tau>0, 0<\nu\le 1, \quad (8)$$

with the initial condition:

$$\varphi(\eta,0) = \beta(\eta), \quad (9)$$

where $D_\tau^\nu = \partial^\nu \varphi(\eta,\tau)/\partial\tau^\nu$, while R is a linear operator, N is a nonlinear term, and ψ is the source function.

Taking ST to equation (8), we get

$$S[D_\tau^\nu \varphi(\eta,\tau) + R[\varphi(\eta,\tau] + N\varphi(\eta,\tau)] = S[\psi(\eta,\tau)], \tau>0, 0<\nu\le 1, \quad (10)$$

or

$$S[\varphi(\eta,\tau)] = \varphi(\eta,0) + \rho^\nu S[\psi(\eta,\tau)] - \rho^\nu S[R[\varphi(\eta,\tau)] + N\varphi(\eta,\tau)]. \quad (11)$$

Taking S^{-1} of equation (11), we get

$$\varphi(\eta,\tau) = \Omega(\eta,\tau) - S^{-1}[\rho^\upsilon S[R[\varphi(\eta,\tau)] + N[\varphi(\eta,\tau)]]], \quad (12)$$

such that: $\Omega(\eta,\tau) = S^{-1}[\varphi(\eta,0) + \rho^\upsilon S[\psi(\eta,\tau)]]$. (13)

Now, use the decomposition approach by assuming

$$\varphi(\eta,\tau) = \sum_{i=0}^\infty \varphi_i(\eta,\tau). \quad (14)$$

The nonlinear term can be decomposed as follows:

$$N\varphi(\eta,\tau) = \sum_{i=0}^\infty A_i, \quad (15)$$

for some Adomian polynomial A_i that is given by

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N\left(\sum_{i=0}^\infty \lambda^i \varphi_i\right)\right]_{\lambda=0}, \quad i = 1,2,\cdots. \quad (16)$$

Substituting equations (14) and (15) in equation (12), we get

$$\sum_{i=0}^\infty \varphi_i(\eta,\tau) = \Omega(\eta,\tau) - S^{-1}\left[\rho^\nu S\left[R[\varphi(\eta,\tau)] + \sum_{i=0}^\infty A_i\right]\right]. \quad (17)$$

As result, the following recurrence relation is obtained:

$$\begin{aligned} &\varphi_0(\eta,\tau) = \Omega(\eta,\tau), \\ &\varphi_{k+1}(\eta,\tau) = -S^{-1}[\rho^\upsilon S[R[\varphi_k(\eta,\tau)] + A_k]], \quad k = 0,1,2,\ldots. \end{aligned} \quad (18)$$

4. Convergence Analysis of SDM Solution

In this section, we establish SDM convergence and uniqueness.

Theorem 1 (see [45–47]). *The SDM of equation (8) has a unique solution whenever* $0<\delta<1$, *such that* $\delta = (\delta_1+\delta_2)\tau^{(\upsilon)}/\Gamma(\upsilon+1)$.

Proof. Let $I = (C[E], \|.\|)$ the Banach space of all continuous functions on $E = [0,\tau]$ with the norm $\|.\|$, we define a mapping $F: I \longrightarrow I$ where

$$\varphi_{k+1}(\eta,\tau)=\Omega(\eta,\tau)-S^{-1}\left[\rho^{v}S\left[R\left[\varphi_{k}(\eta,\tau)\right]+N\left[\varphi_{k}(\eta,\tau)\right]\right]\right],\quad k=0,1,2,\ldots. \tag{19}$$

Now, suppose $R[\varphi(\eta,\tau)]$ and $N[\varphi(\eta,\tau)]$ are also Lipschitzian with $|R\varphi(\eta,\tau)-R\varphi(\eta,\tau)|<\delta_1|\varphi(\eta,\tau)-\varphi(\eta,\tau)|$, and $|N\varphi(\eta,\tau)-N\varphi(\eta,\tau)|<\delta_2|\varphi(\eta,\tau)-\varphi(\eta,\tau)|$, where δ_1, and δ_2 are Lipschitz constant, and $\varphi,\widehat{\varphi}$ is two distinct solutions of the equation (8).

$$\begin{aligned}\|F\varphi-F\varphi\|&=\max_{\tau\in E}\left|\begin{array}{l}S^{-1}\left[\rho^{\nu}S[R[\varphi(\eta,\tau)]+N[\varphi(\eta,\tau)]]\right]\\-S^{-1}\left[\rho^{\nu}S[Q\varphi(\eta,\tau)+N[\varphi(\eta,\tau)]]\right]\end{array}\right|\\&\leq\max_{\tau\in E}\left|\begin{array}{l}S^{-1}\left[\rho^{\nu}S[R\,\varphi(\eta,\tau)-R\varphi(\eta,\tau)]\right]\\+S^{-1}\left[\rho^{\nu}S[N[\varphi(\eta,\tau)]-N[\varphi(\eta,\tau)]]\right]\end{array}\right|\\&\leq\max_{\tau\in E}\left[\delta_1S^{-1}\left[\rho^{\nu}S|\varphi(\eta,\tau)-\varphi(\eta,\tau)|\right]+\delta_2S^{-1}\left[\rho^{\nu}S|\varphi(\eta,\tau)-\varphi(\eta,\tau)|\right]\right]\\&\leq\max_{\tau\in E}(\delta_1+\delta_2)\left[S^{-1}\left[\rho^{\nu}S|\varphi(\eta,\tau)-\widehat{\varphi}(\eta,\tau)|\right]\right]\\&\leq(\delta_1+\delta_2)\left[S^{-1}\left[\rho^{\nu}S\|\varphi(\eta,\tau)-\widehat{\varphi}(\eta,\tau)\|\right]\right]\\&=\frac{(\delta_1+\delta_2)\tau^{(\nu)}}{\Gamma(\nu+1)}\|\varphi(\eta,\tau)-\widehat{\varphi}(\eta,\tau)\|.\end{aligned} \tag{20}$$

Since $0<\delta<1$, the mapping is contraction. Then, there would be a unique solution to (8). □

Theorem 2 (see [45–47]). *The SDM solution of equation (8) is convergent.*

Proof. Suppose $G_n=\sum_{i=0}^{n}\varphi_i(\eta,\tau)$. We are going to prove that $\{G_n\}$ is a Cauchy sequence in Banach space I:

$$\begin{aligned}\left\|G_n-G_m\right\|&=\max_{\tau\in E}\left|G_n-G_m\right|=\max_{\tau\in E}\left|\sum_{i=m+1}^{n}\varphi_i(\eta,\tau)\right|\\&\leq\max_{\tau\in E}\left|S^{-1}\left[\rho^{\nu}S\left[\sum_{i=m+1}^{n}\left(R\left[\varphi_{i-1}(\eta,\tau)\right]+N\left[\varphi_{i-1}(\eta,\tau)\right]\right)\right]\right]\right|\\&=\max_{\tau\in E}\left|S^{-1}\left[\rho^{\nu}S\left[\sum_{i=m}^{n-1}\left(R\left[\varphi_j(\eta,\tau)\right]+N\left[\varphi_i(\eta,\tau)\right]\right)\right]\right]\right|\\&\leq\max_{\tau\in E}\left|S^{-1}\left[\rho^{\nu}S\left[\left(R(\varphi_{n-1})-R(\varphi_{m-1})+N(\varphi_{n-1})-N(\varphi_{m-1})\right)\right]\right]\right|\\&\leq\delta_1\max_{\tau\in E}\left|S^{-1}\left[\rho^{\nu}S\left[(\varphi_{n-1})-(\varphi_{m-1})\right]\right]\right|\\&\quad+\delta_2\max_{\tau\in E}\left|S^{-1}\left[\rho^{\nu}S\left[(\varphi_{n-1})-(\varphi_{m-1})\right]\right]\right|\\&=\frac{(\delta_1+\delta_2)\tau^{(\nu)}}{\Gamma(\nu+1)}\left\|\varphi_{n-1}(\eta,\tau)-\varphi_{m-1}(\eta,\tau)\right\|.\end{aligned} \tag{21}$$

Let $n=m+1$, and then,

$$\left\|G_{m+1}-G_m\right\| \leq \delta\left\|G_m-G_{m-1}\right\| \leq \delta^2\left\|G_{m-1}-G_{m-2}\right\| \leq \cdots \leq \delta^m\left\|G_1-G_0\right\|. \tag{22}$$

By using the triangle inequality, we have

$$\begin{aligned}\left\|G_n-G_m\right\| &\leq \left\|G_{m+1}-G_m\right\|+\left\|G_{m+2}-G_{m+1}\right\|+\cdots+\left\|G_n-G_{n-1}\right\|\\ &\leq \left[\delta^m+\delta^{m+1}+\cdots+\delta^{n-1}\right]\left\|G_1-G_0\right\|\\ &\leq \delta^m\left[1+\delta+\delta^2+\cdots+\delta^{n-m-1}\right]\left\|G_1-G_0\right\|\\ &\leq \delta^m\left(\frac{1-\delta^{n-m}}{1-\delta}\right)\left\|\varphi_1\right\|.\end{aligned} \tag{23}$$

Since $0<\delta<1$, we have $(1-\delta^{n-m})<1$, then

$$\left\|G_n-G_m\right\| \leq \frac{\delta^m}{1-\delta}\max_{\tau\in E}\left\|\varphi_1\right\|. \tag{24}$$

However, $\|\varphi_1\|<\infty$, then $\|G_n-G_m\| \longrightarrow 0$ as $m\longrightarrow\infty$, and hence, $\{G_n\}$ is a Cauchy sequence in I, and so the series $\sum_{j=0}^{n}\varphi_j(\eta,\tau)$ converges. □

Theorem 3. *The maximum absolute truncation error of equation (14) to equation (8) is estimated to be*

$$\max_{\tau\in I}\left|\varphi(\eta,\tau)-\sum_{i=0}^{m}\varphi_i(\eta,\tau)\right| \leq \frac{\delta^m}{1-\delta}\max_{\tau\in I}\left\|\varphi_1\right\|. \tag{25}$$

Proof. From Theorem 2, we have

$$\left\|G_n-G_m\right\| \leq \frac{\delta^m}{1-\delta}\max_{\tau\in E}\left\|\varphi_1\right\|, \tag{26}$$

as $n\longrightarrow\infty$, then $G_n\longrightarrow\varphi(\eta,\tau)$, so we have

$$\left\|\varphi(\eta,\tau)-G_m\right\| \leq \frac{\delta^m}{1-\delta}\max_{\tau\in E}\left\|\varphi_1\right\|. \tag{27}$$

Then,

$$\max_{\tau\in I}\left|\varphi(\eta,\tau)-\sum_{j=0}^{m}\varphi_j(\eta,\tau)\right| \leq \frac{\delta^m}{1-\delta}\max_{\tau\in I}\left\|\varphi_1\right\|. \tag{28}$$

□

5. Application

This part talks about the general steps for solving equation (2) numerically with given initial conditions. Taking ST of equation (2) as

$$\begin{aligned}&S\left[\frac{\partial^\nu\xi}{\partial\tau^\nu}+\xi\frac{\partial^\lambda\xi}{\partial\eta^\lambda}+\frac{\partial^\lambda\varphi}{\partial\eta^\lambda}+c_1\frac{\partial^{2\lambda}\xi}{\partial\eta^{2\lambda}}=0\right],\\ &S\left[\frac{\partial^\nu\varphi}{\partial\tau^\nu}+\xi\frac{\partial^\lambda\varphi}{\partial\eta^\lambda}+\varphi\frac{\partial^\lambda\xi}{\partial\eta^\lambda}+c_2\frac{\partial^{3\lambda}\xi}{\partial\eta^{3\lambda}}-c_1\frac{\partial^{2\lambda}\phi}{\partial\eta^{2\lambda}}=0\right], \eta,\tau\geq 0, 0<\nu,\lambda\leq 1.\end{aligned} \tag{29}$$

Applying ST, we get

$$\begin{aligned}&\rho^{-\nu}S[\xi(\eta,\tau)]-\rho^{-\nu}\xi(\eta,0)=-S\left[\xi\frac{\partial^\lambda\xi}{\partial\eta^\lambda}+\frac{\partial^\lambda\varphi}{\partial\eta^\lambda}+c_1\frac{\partial^{2\lambda}\xi}{\partial\eta^{2\lambda}}\right],\\ &\rho^{-\nu}S[\varphi(\eta,\tau)]-\rho^{-\nu}\varphi(\eta,0)=-S\left[\xi\frac{\partial^\lambda\varphi}{\partial\eta^\lambda}+\varphi\frac{\partial^\lambda\xi}{\partial\eta^\lambda}+c_2\frac{\partial^{3\lambda}\xi}{\partial\eta^{3\lambda}}-c_1\frac{\partial^{2\lambda}\phi}{\partial\eta^{2\lambda}}\right].\end{aligned} \tag{30}$$

By using initial conditions equation (3), we obtain

$$S[\xi(\eta,\tau)] = \alpha(\eta) - \rho^{\gamma} S\left[\xi\frac{\partial^{\lambda}\xi}{\partial\eta^{\lambda}} + \frac{\partial^{\lambda}\varphi}{\partial\eta^{\lambda}} + c_1\frac{\partial^{2\lambda}\xi}{\partial\eta^{2\lambda}}\right],$$
$$S[\varphi(\eta,\tau)] = \beta(\eta) - \rho^{\gamma} S\left[\xi\frac{\partial^{\lambda}\varphi}{\partial\eta^{\lambda}} + \varphi\frac{\partial^{\lambda}\xi}{\partial\eta^{\lambda}} + c_2\frac{\partial^{3\lambda}\xi}{\partial\eta^{3\lambda}} - c_1\frac{\partial^{2\lambda}\phi}{\partial\eta^{2\lambda}}\right]. \tag{31}$$

Taking S^{-1} of equation (31), we get

$$\xi(\eta,\tau) = \alpha(\eta) - S^{-1}\left[\rho^{\gamma} S\left[\xi\frac{\partial^{\lambda}\xi}{\partial\eta^{\lambda}} + \frac{\partial^{\lambda}\varphi}{\partial\eta^{\lambda}} + c_1\frac{\partial^{2\lambda}\xi}{\partial\eta^{2\lambda}}\right]\right],$$
$$\varphi(\eta,\tau) = \beta(\eta) - S^{-1}\left[\rho^{\gamma} S\left[\xi\frac{\partial^{\lambda}\varphi}{\partial\eta^{\lambda}} + \varphi\frac{\partial^{\lambda}\xi}{\partial\eta^{\lambda}} + c_2\frac{\partial^{3\lambda}\xi}{\partial\eta^{3\lambda}} - c_1\frac{\partial^{2\lambda}\phi}{\partial\eta^{2\lambda}}\right]\right]. \tag{32}$$

Now, use the decomposition approach by assuming

$$\xi(\eta,\tau) = \sum_{i=0}^{\infty}\xi_i(\eta,\tau), \varphi(\eta,\tau) = \sum_{i=0}^{\infty}\varphi_i(\eta,\tau). \tag{33}$$

The nonlinear terms can be decomposed as follows:

$$\xi\frac{\partial^{\lambda}\xi}{\partial\eta^{\lambda}} = \sum_{i=0}^{\infty}A_i, \xi\frac{\partial^{\lambda}\varphi}{\partial\eta^{\lambda}} = \sum_{i=0}^{\infty}Q_i, \varphi\frac{\partial^{\lambda}\xi}{\partial\eta^{\lambda}} = \sum_{i=0}^{\infty}P_i, \tag{34}$$

for some Adomian polynomials $A_i, Q_i,$ and P_i that are given by

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[\sum_{i=0}^{\infty}\lambda^i\xi_i\sum_{i=0}^{\infty}\lambda^i\frac{\partial^{\lambda}\xi_i}{\partial\eta^{\lambda}}\right]_{\lambda=0},$$
$$Q_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[\sum_{i=0}^{\infty}\lambda^i\xi_i\sum_{i=0}^{\infty}\lambda^i\frac{\partial^{\lambda}\varphi_i}{\partial\eta^{\lambda}}\right]_{\lambda=0},$$
$$P_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[\sum_{i=0}^{\infty}\lambda^i\varphi_i\sum_{i=0}^{\infty}\lambda^i\frac{\partial^{\lambda}\xi_i}{\partial\eta^{\lambda}}\right]_{\lambda=0}. \tag{35}$$

Substituting equations (33) and (34) in equation (32), we get

$$\sum_{i=0}^{\infty}\xi_i(\eta,\tau) = \alpha(\eta) - S^{-1}\left[\rho^{\gamma} S\left[\sum_{i=0}^{\infty}A_i + \frac{\partial^{\lambda}\varphi}{\partial\eta^{\lambda}} + c_1\frac{\partial^{2\lambda}\xi}{\partial\eta^{2\lambda}}\right]\right],$$
$$\sum_{i=0}^{\infty}\varphi_i(\eta,\tau) = \beta(\eta) - S^{-1}\left[\rho^{\gamma} S\left[\sum_{i=0}^{\infty}Q_i + \sum_{i=0}^{\infty}P_i + c_2\frac{\partial^{3\lambda}\xi}{\partial\eta^{3\lambda}} - c_1\frac{\partial^{2\lambda}\phi}{\partial\eta^{2\lambda}}\right]\right]. \tag{36}$$

As result, the following recurrence relations are obtained

$$
\begin{aligned}
&\xi_0(\eta,\tau) = \xi_0(\eta,0) = \alpha(\eta),\\
&\varphi_0(\eta,\tau) = \varphi_0(\eta,0) = \beta(\eta),\\
&\xi_1(\eta,\tau) = -S^{-1}\left[\rho^{\gamma}S\left[I_0 + \frac{\partial^{\lambda}\varphi_0}{\partial\eta^{\lambda}} + c_1\frac{\partial^{2\lambda}\xi_0}{\partial\eta^{2\lambda}}\right]\right],\\
&\varphi_1(\eta,\tau) = -S^{-1}\left[\rho^{\gamma}S\left[Q_0 + P_0 + c_2\frac{\partial^{3\lambda}\xi_0}{\partial\eta^{3\lambda}} - c_1\frac{\partial^{2\lambda}\phi_0}{\partial\eta^{2\lambda}}\right]\right],\\
&\xi_2(\eta,\tau) = -S^{-1}\left[\rho^{\gamma}S\left[I_1 + \frac{\partial^{\lambda}\varphi_1}{\partial\eta^{\lambda}} + c_1\frac{\partial^{2\lambda}\xi_1}{\partial\eta^{2\lambda}}\right]\right],\\
&\varphi_2(\eta,\tau) = -S^{-1}\left[\rho^{\gamma}S\left[Q_1 + P_1 + c_2\frac{\partial^{3\lambda}\xi_1}{\partial\eta^{3\lambda}} - c_1\frac{\partial^{2\lambda}\phi_1}{\partial\eta^{2\lambda}}\right]\right],\\
&\xi_3(\eta,\tau) = -S^{-1}\left[\rho^{\gamma}S\left[I_2 + \frac{\partial^{\lambda}\varphi_2}{\partial\eta^{\lambda}} + c_1\frac{\partial^{2\lambda}\xi_2}{\partial\eta^{2\lambda}}\right]\right],\\
&\varphi_3(\eta,\tau) = -S^{-1}\left[\rho^{\gamma}S\left[Q_2 + P_2 + c_2\frac{\partial^{3\lambda}\xi_2}{\partial\eta^{3\lambda}} - c_1\frac{\partial^{2\lambda}\phi_2}{\partial\eta^{2\lambda}}\right]\right],\\
&\quad\vdots\\
&\xi_k(\eta,\tau) = -S^{-1}\left[\rho^{\gamma}S\left[A_{k-1} + \frac{\partial^{\lambda}\varphi_{k-1}}{\partial\eta^{\lambda}} + c_1\frac{\partial^{2\lambda}\xi_{k-1}}{\partial\eta^{2\lambda}}\right]\right], k\geq 1,\\
&\varphi_k(\eta,\tau) = -S^{-1}\left[\rho^{\gamma}S\left[Q_{k-1} + P_{k-1} + c_2\frac{\partial^{3\lambda}\xi_{k-1}}{\partial\eta^{3\lambda}} - c_1\frac{\partial^{2\lambda}\phi_{k-1}}{\partial\eta^{2\lambda}}\right]\right], k\geq 1.
\end{aligned}
\tag{37}
$$

Following that, one can get the solution of equation (2)

$$
\begin{aligned}
\xi(\eta,\tau) &= \xi_0(\eta,\tau) + \xi_1(\eta,\tau) + \xi_2(\eta,\tau) + \xi_3(\eta,\tau) + \cdots = \sum_{i=0}^{\infty}\xi_i(\eta,\tau),\\
\varphi(\eta,\tau) &= \varphi_0(\eta,\tau) + \varphi_1(\eta,\tau) + \varphi_2(\eta,\tau) + \varphi_3(\eta,\tau) + \cdots = \sum_{i=0}^{\infty}\varphi_i(\eta,\tau).
\end{aligned}
\tag{38}
$$

6. Elucidative Examples

In this section, two examples of nonlinear coupled WBK equations are solved to demonstrate the performance and efficiency of the SDM.

Example 1. Consider the coupled system of WBK with $c_1 = 0, c_2 = 1, \lambda = 1$, we obtain

$$\frac{\partial^{\nu}\xi}{\partial\tau^{\nu}}+\xi\frac{\partial\xi}{\partial\eta}+\frac{\partial\varphi}{\partial\eta}=0,$$
$$\frac{\partial^{\nu}\varphi}{\partial\tau^{\nu}}+\xi\frac{\partial\varphi}{\partial\eta}+\varphi\frac{\partial\xi}{\partial\eta}+\frac{\partial^{3\lambda}\xi}{\partial\eta^{3\lambda}}=0, 0<\nu\leq 1, 0<\tau\leq 1, -100\leq\eta\leq 100, \tag{39}$$

with the ICs:

$$\xi(\eta,0)=\varepsilon-2h\coth[h(\eta+c)], \varphi(\eta,0)=-2h^{2}\operatorname{csch}^{2}[h(\eta+c)], \tag{40}$$

where ε, h, and n are arbitrary constants.

The exact solutions of equation (39) at classical order $\nu=1$ are given by

$$\xi(\eta,\tau)=\varepsilon-2h\quad\coth[h(\eta+c-\varepsilon\tau)], \varphi(\eta,\tau)=-2h^{2}\quad\operatorname{csch}^{2}[h(\eta+c-\varepsilon\tau)]. \tag{41}$$

Using the procedure equation (36), we have

$$\sum_{i=0}^{\infty}\xi_i(\eta,\tau)=\varepsilon-2h\coth[h(\eta+c)]-S^{-1}\left[\rho^{\nu}S\left[\sum_{i=0}^{\infty}A_i+\sum_{i=0}^{\infty}\frac{\partial\varphi_i}{\partial\eta}\right]\right],$$
$$\sum_{i=0}^{\infty}\varphi_i(\eta,\tau)=-2h^{2}\operatorname{csch}^{2}[h(\eta+c)]-S^{-1}\left[\rho^{\nu}S\left[\sum_{i=0}^{\infty}Q_i+\sum_{i=0}^{\infty}P_i+\sum_{i=0}^{\infty}\frac{\partial^{3}\xi_i}{\partial\eta^{3}}\right]\right]. \tag{42}$$

Using the procedure equation (37), we have

$$\xi_0(\eta,\tau)=\varepsilon-2h\coth[h(\eta+c)], \varphi_0(\eta,\tau)=-2h^{2}\operatorname{csch}^{2}[h(\eta+c)],$$
$$\xi_1(\eta,\tau)=-\frac{2\tau^{\nu}\varepsilon h^{2}\operatorname{csch}^{2}[h(\eta+c)]}{\Gamma(1+\nu)},$$
$$\varphi_1(\eta,\tau)=-\frac{4\tau^{\nu}h^{3}\varepsilon\coth[(\eta+c)h]\operatorname{csch}^{2}[h(\eta+c)]}{\Gamma(1+\nu)},$$
$$\xi_2(\eta,\tau)=-\frac{4\tau^{2\nu}h^{3}\varepsilon^{2}\coth[(\eta+c)h]\operatorname{csch}^{2}[h(\eta+c)]}{\Gamma(1+2\nu)},$$
$$\varphi_2(\eta,\tau)=-\frac{4\tau^{2\nu}h^{4}\varepsilon^{2}(2+\cosh[2(\eta+c)h])\operatorname{csch}^{4}[h(\eta+c)]}{\Gamma(1+2\nu)},$$
$$\xi_3(\eta,\tau)=-\frac{2\tau^{3\nu}h^{4}\varepsilon^{2}\operatorname{csch}^{5}[(\eta+c)h]}{\Gamma[1+\nu]^{2}\Gamma[1+3\nu]}\left(\begin{array}{c}-4h\cosh[(\eta+c)h]\Gamma[1+2\nu]\\+\Gamma[1+\nu]^{2}\left(\begin{array}{c}8h\cosh[(\eta+c)h]\\+\varepsilon\left(\begin{array}{c}3\sinh[(\eta+c)h]\\+\sinh[3(\eta+c)h]\end{array}\right)\end{array}\right)\end{array}\right),$$
$$\varphi_3(\eta,\tau)=-\frac{2\tau^{3\nu}h^{5}\varepsilon^{2}\operatorname{csch}^{6}[(\eta+c)h]}{\Gamma[1+\nu]^{2}\Gamma[1+3\nu]}\left(\begin{array}{c}-4h(3+2\cosh[2(\eta+c)h])\Gamma[1+2\nu]\\+\Gamma[1+\nu]^{2}\left(\begin{array}{c}24h+16h\quad\cosh[2(\eta+c)h]\\+10\varepsilon\sinh[2(\eta+c)h]\\+\varepsilon\sinh[4(\eta+c)h]\end{array}\right)\end{array}\right). \tag{43}$$

In the same way, we can find the rest of the terms and then

$$\xi(\eta,\tau)=\xi_0(\eta,\tau)+\xi_1(\eta,\tau)+\xi_2(\eta,\tau)+\xi_3(\eta,\tau)+\cdots,$$
$$\varphi(\eta,\tau)=\varphi_0(\eta,\tau)+\varphi_1(\eta,\tau)+\varphi_2(\eta,\tau)+\varphi_3(\eta,\tau)+\cdots. \tag{44}$$

After four terms, we have the solution of equation (39) as

$$\begin{aligned}\xi(\eta,\tau)&=\varepsilon-2h\coth[h(\eta+c)]-\frac{2\tau^{\nu}\varepsilon h^2\operatorname{csch}^2[h(\eta+c)]}{\Gamma(1+\nu)}\\&-\frac{4\tau^{2\nu}h^3\varepsilon^2\coth[(\eta+c)h]\operatorname{csch}^2[h(\eta+c)]}{\Gamma(1+2\nu)}\\&-\frac{2\tau^{3\nu}h^4\varepsilon^2\operatorname{csch}^5[(\eta+c)h]}{\Gamma[1+\nu]^2\Gamma[1+3\nu]}\left(\begin{array}{c}-4h\cosh[(\eta+c)h]\Gamma[1+2\nu]\\+\Gamma[1+\nu]^2\left(\begin{array}{c}8h\cosh[(\eta+c)h]\\+\varepsilon\left(\begin{array}{c}3\sinh[(\eta+c)h]\\+\sinh[3(\eta+c)h]\end{array}\right)\end{array}\right)\end{array}\right),\\ \varphi(\eta,\tau)&=-2h^2\operatorname{csch}^2[h(\eta+c)]-\frac{4\tau^{\nu}h^3\varepsilon\coth[(\eta+c)h]\operatorname{csch}^2[h(\eta+c)]}{\Gamma(1+\nu)}\\&-\frac{4\tau^{2\nu}h^4\varepsilon^2(2+\cosh[2(\eta+c)h])\operatorname{csch}^4[h(\eta+c)]}{\Gamma(1+2\nu)}\\&-\frac{2\tau^{3\nu}h^5\varepsilon^2\operatorname{csch}^6[(\eta+c)h]}{\Gamma[1+\nu]^2\Gamma[1+3\nu]}\left(\begin{array}{c}-4h(3+2\cosh[2(\eta+c)h])\Gamma[1+2\nu]\\+\Gamma[1+\nu]^2\left(\begin{array}{c}24h+16h\cosh[2(\eta+c)h]\\+10\varepsilon\sinh[2(\eta+c)h]\\+\varepsilon\sinh[4(\eta+c)h]\end{array}\right)\end{array}\right).\end{aligned} \tag{45}$$

Example 2. Consider the coupled system of WBK with $c_1 = (1/2), c_2 = 0, \lambda = 1,$ we obtain

$$\begin{aligned}&\frac{\partial^{\nu}\xi}{\partial\tau^{\nu}}+\xi\frac{\partial\xi}{\partial\eta}+\frac{\partial\varphi}{\partial\eta}+\frac{1}{2}\frac{\partial^2\xi}{\partial\eta^2}=0,\\&\frac{\partial^{\nu}\varphi}{\partial\tau^{\nu}}+\xi\frac{\partial\varphi}{\partial\eta}+\varphi\frac{\partial\xi}{\partial\eta}-\frac{1}{2}\frac{\partial^2\phi}{\partial\eta^2}=0, 0<\nu\leq 1, 0<\tau\leq 1, -100\leq\eta\leq 100,\end{aligned} \tag{46}$$

with the ICs:

$$\xi(\eta,0)=\varepsilon-h\coth[h(\eta+c)],\varphi(\eta,0)=-h^2\operatorname{csch}^2[h(\eta+c)]. \tag{47}$$

The exact solutions of equation (46) at classical order $\nu = 1$ are given by

$$\xi(\eta,\tau)=\varepsilon-h\coth[h(\eta+c-\varepsilon\tau)],\ \varphi(\eta,\tau)=-h^2\operatorname{csch}^2[h(\eta+c-\varepsilon\tau)]. \tag{48}$$

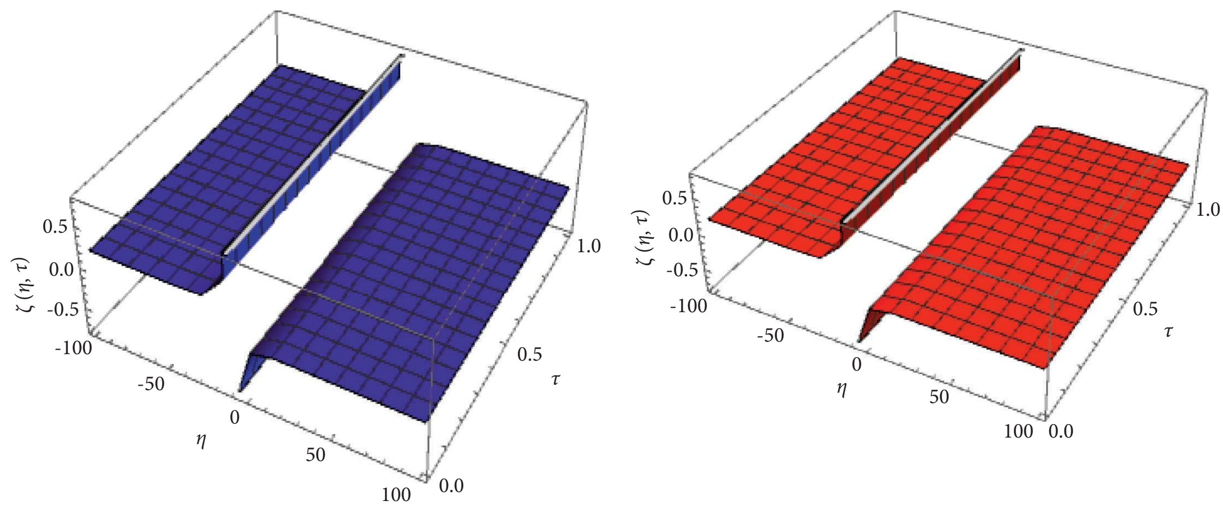
As a result of following the same procedure as in Example 1, we have

$$\begin{aligned}
&\xi_0(\eta,\tau)=\varepsilon-h\coth[h(\eta+c)],\ \varphi_0(\eta,\tau)=-h^2\operatorname{csch}^2[h(\eta+c)],\\
&\xi_1(\eta,\tau)=-\frac{\tau^{\nu}\varepsilon h^2\operatorname{csch}^2[h(\eta+c)]}{\Gamma(1+\nu)},\\
&\varphi_1(\eta,\tau)=-\frac{2\tau^{\nu}h^3\varepsilon\coth[(\eta+c)h]\operatorname{csch}^2[h(\eta+c)]}{\Gamma(1+\nu)},\\
&\xi_2(\eta,\tau)=-\frac{2\tau^{2\nu}h^3\varepsilon^2\coth[(\eta+c)h]\operatorname{csch}^2[h(\eta+c)]}{\Gamma(1+2\nu)},\\
&\varphi_2(\eta,\tau)=-\frac{2\tau^{2\nu}h^4\varepsilon^2(2+\cosh[2(\eta+c)h])\operatorname{csch}^4[h(\eta+c)]}{\Gamma(1+2\nu)},\\
&\xi_3(\eta,\tau)=-\frac{\tau^{3\nu}h^4\varepsilon^2\operatorname{csch}^5[(\eta+c)h]}{\Gamma[1+\nu]^2\Gamma[1+3\nu]}\left(\begin{array}{c}-2h\cosh[(\eta+c)h]\Gamma[1+2\nu]\\ +\Gamma[1+\nu]^2\left(\begin{array}{c}4h\cosh[(\eta+c)h]\\ +\varepsilon\left(\begin{array}{c}3\sinh[(\eta+c)h]\\ +\sinh[3(\eta+c)h]\end{array}\right)\end{array}\right)\end{array}\right),\\
&\varphi_3(\eta,\tau)=-\frac{\tau^{3\nu}h^5\varepsilon^2\operatorname{csch}^6[(\eta+c)h]}{\Gamma[1+\nu]^2\Gamma[1+3\nu]}\left(\begin{array}{c}-2h(3+2\cosh[2(\eta+c)h])\Gamma[1+2\nu]\\ +\Gamma[1+\nu]^2\left(\begin{array}{c}12h+8h\cosh[2(\eta+c)h]\\ +10\varepsilon\sinh[2(\eta+c)h]\\ +\varepsilon\sinh[4(\eta+c)h]\end{array}\right)\end{array}\right).
\end{aligned} \tag{49}$$

In the same way, we can find the rest of the terms and then

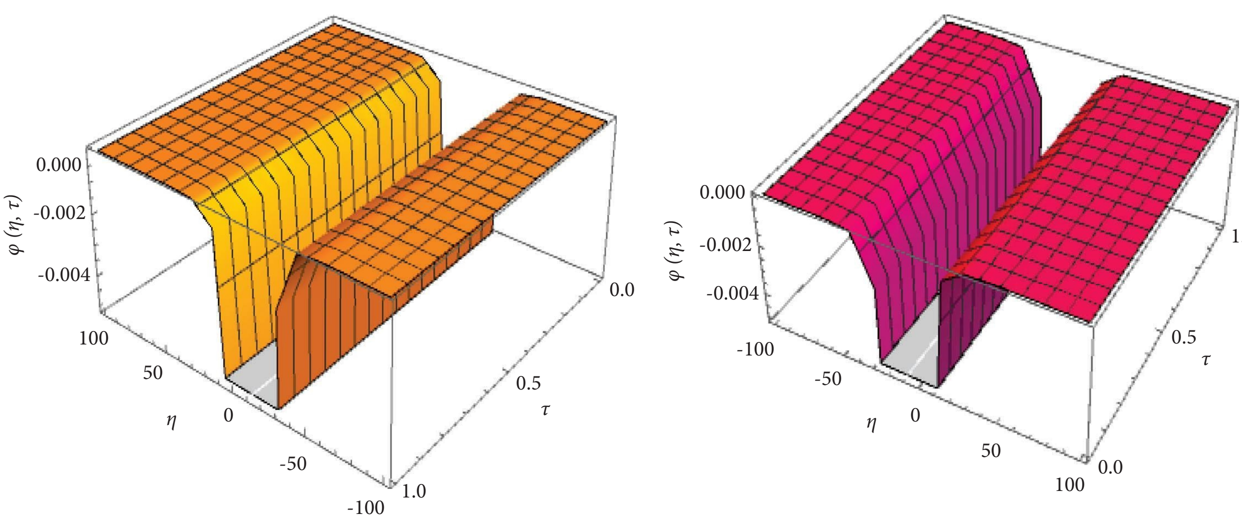
$$\begin{aligned}
\xi(\eta,\tau)&=\xi_0(\eta,\tau)+\xi_1(\eta,\tau)+\xi_2(\eta,\tau)+\xi_3(\eta,\tau)+\cdots,\\
\varphi(\eta,\tau)&=\varphi_0(\eta,\tau)+\varphi_1(\eta,\tau)+\varphi_2(\eta,\tau)+\varphi_3(\eta,\tau)+\cdots.
\end{aligned} \tag{50}$$

After four terms, we have the solution of equation (46) as



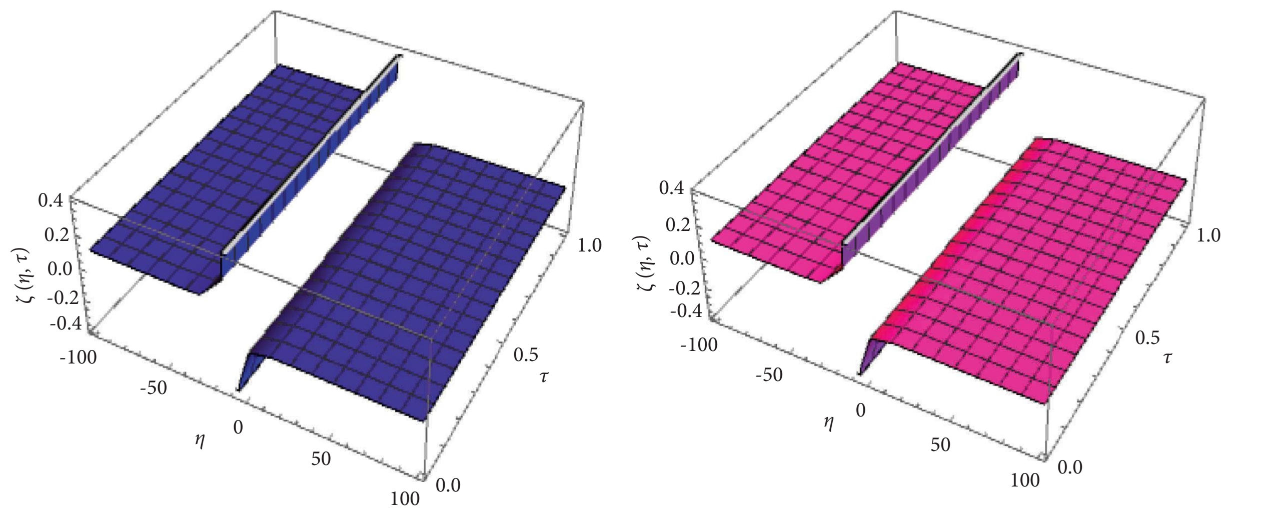
(a) (b)

Figure 1: Behavior of (a) approximate solution of $\xi(\eta, \tau)$ and (b) exact solution for Example 1 at $\varepsilon = 0.005, h = 0.1, c = 10$, and $\nu = 1$.



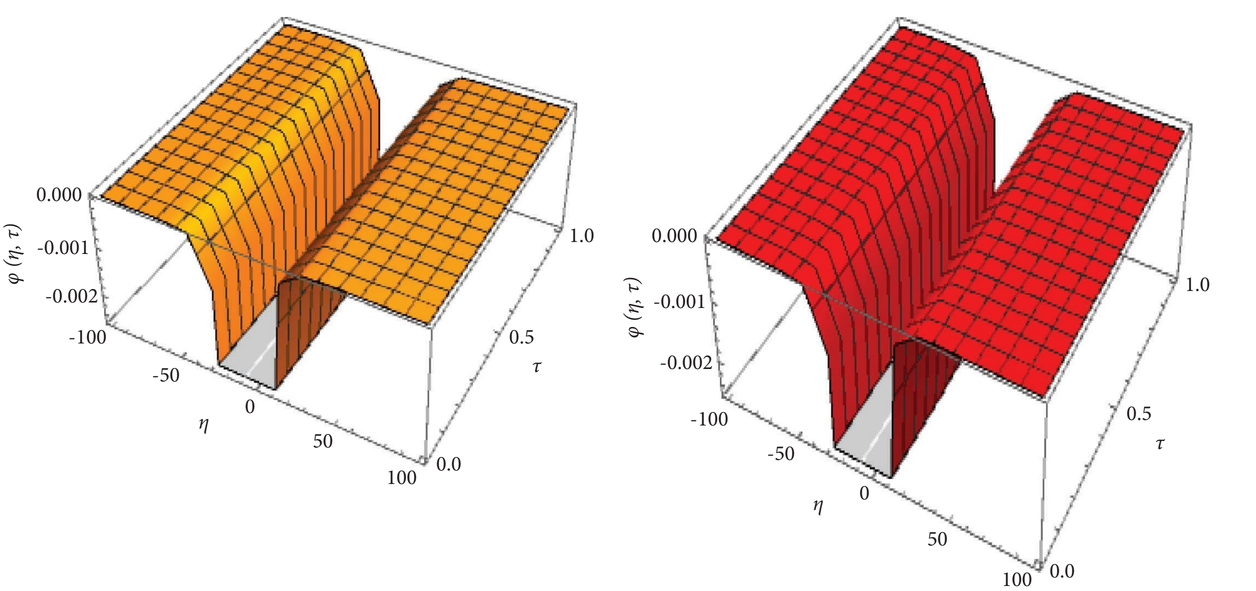
(a) (b)

Figure 2: Behavior of (a) approximate solution of $\varphi(\eta, \tau)$ and (b) exact solution for Example 1 at $\varepsilon = 0.005, h = 0.1, c = 10$, and $\nu = 1$.



(a) (b)

Figure 3: Behavior of (a) approximate solution of $\xi(\eta,\tau)$ and (b) exact solution for Example 2 at $\varepsilon = 0.005, h = 0.1, c = 10$, and $\nu = 1$.



(a) (b)

Figure 4: Behavior of (a) approximate solution of $\varphi(\eta,\tau)$ and (b) exact solution for Example 2 at $\varepsilon = 0.005, h = 0.1, c = 10$, and $\nu = 1$.

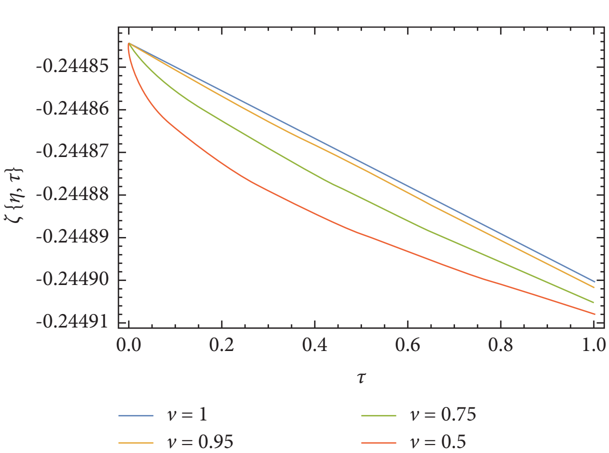


Figure 5: Exact and approximate SDM solution of $\xi(\eta,\tau)$ at $\nu = 1, 0.95, 0.75$, and 0.5 for Example 1, when $\varepsilon = 0.005$, $h = 0.1$, $c = 10$, and $\eta = 1$.

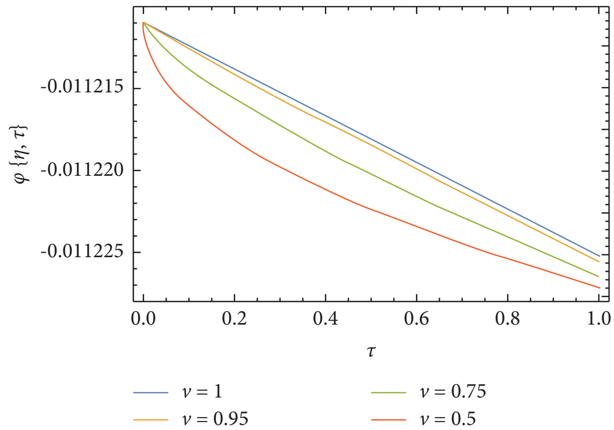


Figure 6: Exact and approximate SDM solution of $\varphi(\eta, \tau)$ at $\nu = 1, 0.95, 0.75$, and 0.5 for Example 1, when $\varepsilon = 0.005$, $h = 0.1$, $c = 10$, and $\eta = 1$.

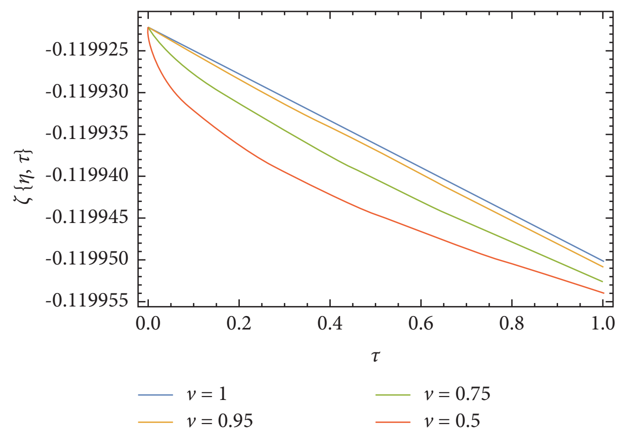


Figure 7: Exact and approximate SDM solution of $\xi(\eta, \tau)$ at $\nu = 1, 0.95, 0.75$, and 0.5 for Example 2, when $\varepsilon = 0.005$, $h = 0.1$, $c = 10$, and $\eta = 1$.

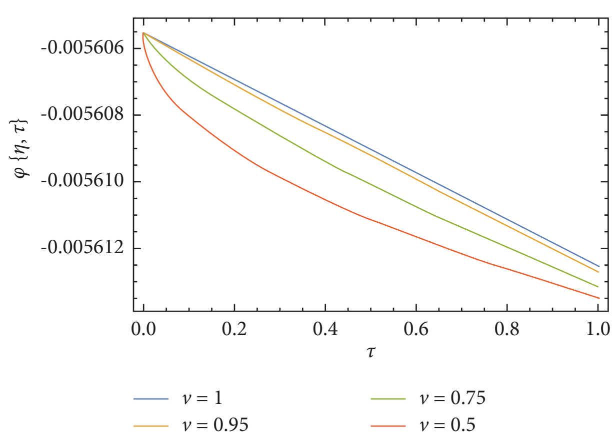


Figure 8: Exact and approximate SDM solution of $\varphi(\eta, \tau)$ at $\nu = 1, 0.95, 0.75$, and 0.5 for Example 2, when $\varepsilon = 0.005$, $h = 0.1$, $c = 10$, and $\eta = 1$.

Table 1: Comparison of SDM solution of $\xi(\eta,\tau)$ with VIM [26], ADM [27], and OHAM [28] of Example 1 with $\varepsilon = 0.005$, $h = 0.1$, $c = 10$, and $\nu = 1$.

(η,τ)	$\lvert\xi_{\text{Exact}} - \xi_{\text{VIM}}\rvert$	$\lvert\xi_{\text{Exact}} - \xi_{\text{ADM}}\rvert$	$\lvert\xi_{\text{Exact}} - \xi_{\text{OHAM}}\rvert$	$\lvert\xi_{\text{Exact}} - \xi^{(4)}_{\text{SDM}}\rvert$
(0.1, 0.1)	6.35269×10^{-5}	8.16297×10^{-7}	6.35267×10^{-5}	5.55112×10^{-17}
(0.1, 0.3)	1.90854×10^{-4}	7.64245×10^{-7}	1.90854×10^{-4}	5.55112×10^{-17}
(0.1, 0.5)	3.18549×10^{-4}	7.16083×10^{-7}	3.18548×10^{-4}	6.66134×10^{-16}
(0.2, 0.1)	6.18930×10^{-5}	3.26243×10^{-6}	6.18931×10^{-5}	1.11022×10^{-16}
(0.2, 0.3)	1.85945×10^{-4}	3.05458×10^{-6}	1.85945×10^{-4}	1.11022×10^{-16}
(0.2, 0.5)	3.10352×10^{-4}	2.86226×10^{-6}	3.10352×10^{-4}	7.77156×10^{-16}
(0.3, 0.1)	6.03095×10^{-5}	7.33445×10^{-6}	6.03098×10^{-5}	0
(0.3, 0.3)	1.81187×10^{-4}	6.86758×10^{-6}	1.81187×10^{-4}	1.11023×10^{-16}
(0.3, 0.5)	3.02408×10^{-4}	6.43557×10^{-6}	3.02408×10^{-4}	6.66133×10^{-16}
(0.4, 0.1)	5.87746×10^{-5}	1.30286×10^{-5}	5.87749×10^{-5}	0
(0.4, 0.3)	1.76574×10^{-4}	1.22000×10^{-5}	1.76574×10^{-4}	5.55112×10^{-17}
(0.4, 0.5)	2.94707×10^{-4}	1.14333×10^{-5}	2.94708×10^{-4}	6.10623×10^{-16}
(0.5, 0.1)	5.72867×10^{-5}	2.03415×10^{-5}	5.72865×10^{-4}	0
(0.5, 0.3)	1.72102×10^{-4}	1.90489×10^{-5}	1.72102×10^{-4}	1.11020×10^{-16}
(0.5, 0.5)	2.87241×10^{-4}	1.78528×10^{-5}	2.87240×10^{-4}	6.10623×10^{-16}

Table 2: Comparison of SDM solution of $\varphi(\eta,\tau)$ with VIM [26], ADM [27], and OHAM [28] of Example 1 with $\varepsilon = 0.005$, $h = 0.1$, $c = 10$, and $\nu = 1$.

(η,τ)	$\lvert\varphi_{\text{Exact}} - \varphi_{\text{VIM}}\rvert$	$\lvert\varphi_{\text{Exact}} - \varphi_{\text{ADM}}\rvert$	$\lvert\varphi_{\text{Exact}} - \varphi_{\text{OHAM}}\rvert$	$\lvert\varphi_{\text{Exact}} - \varphi^{(4)}_{\text{SDM}}\rvert$
(0.1, 0.1)	1.65942×10^{-5}	5.88676×10^{-5}	1.65942×10^{-5}	3.46945×10^{-18}
(0.1, 0.3)	4.98691×10^{-5}	5.56914×10^{-5}	4.98691×10^{-5}	5.20417×10^{-17}
(0.1, 0.5)	8.32598×10^{-5}	5.27169×10^{-5}	8.26491×10^{-4}	3.60822×10^{-16}
(0.2, 0.1)	1.06813×10^{-5}	1.18213×10^{-4}	1.06812×10^{-5}	6.93889×10^{-18}
(0.2, 0.3)	4.83269×10^{-5}	1.11833×10^{-4}	4.83269×10^{-5}	4.68375×10^{-17}
(0.2, 0.5)	8.06837×10^{-5}	1.05858×10^{-4}	7.94290×10^{-4}	3.46945×10^{-16}
(0.3, 0.1)	1.55880×10^{-5}	1.78041×10^{-4}	1.55880×10^{-5}	6.93889×10^{-18}
(0.3, 0.3)	4.68440×10^{-5}	1.68429×10^{-4}	4.68439×10^{-5}	3.98986×10^{-17}
(0.3, 0.5)	7.82068×10^{-5}	1.59428×10^{-4}	7.63646×10^{-4}	3.22659×10^{-16}
(0.4, 0.1)	1.51135×10^{-5}	2.38356×10^{-4}	1.51135×10^{-5}	5.20417×10^{-18}
(0.4, 0.3)	4.54174×10^{-5}	2.25483×10^{-4}	4.54174×10^{-5}	3.46945×10^{-17}
(0.4, 0.5)	7.58243×10^{-5}	2.13430×10^{-4}	7.34471×10^{-4}	3.05311×10^{-16}
(0.5, 0.1)	1.46569×10^{-5}	2.99162×10^{-4}	1.46569×10^{-5}	1.73472×10^{-18}
(0.5, 0.3)	4.40448×10^{-5}	2.83001×10^{-4}	4.40448×10^{-5}	4.16334×10^{-17}
(0.5, 0.5)	7.35317×10^{-5}	2.67868×10^{-4}	7.06678×10^{-4}	2.87964×10^{-16}

Table 3: Comparison of SDM solution of $\xi(\eta,\tau)$ with LADM [3] and NIM [40] of Example 2 with $\varepsilon = 0.005$, $h = 0.1$, $c = 10$, and $\nu = 1$.

(η,τ)	$\lvert\xi_{\text{Exact}} - \xi^{(3)}_{\text{LADM}}\rvert$	$\lvert\xi_{\text{Exact}} - \xi^{(3)}_{\text{NIM}}\rvert$	$\lvert\xi_{\text{Exact}} - \xi^{(3)}_{\text{SDM}}\rvert$	$\lvert\xi_{\text{Exact}} - \xi^{(4)}_{\text{SDM}}\rvert$
(0.1, 0.1)	7.1000×10^{-9}	1.20348×10^{-13}	1.20737×10^{-14}	2.77556×10^{-17}
(0.1, 0.3)	6.5000×10^{-9}	3.25026×10^{-12}	3.26655×10^{-13}	2.77556×10^{-17}
(0.1, 0.5)	5.9000×10^{-9}	1.50478×10^{-11}	1.51246×10^{-12}	3.33067×10^{-16}
(0.2, 0.1)	2.8200×10^{-8}	1.13895×10^{-13}	1.16573×10^{-14}	2.77556×10^{-17}
(0.2, 0.3)	2.5900×10^{-8}	3.07447×10^{-12}	3.14054×10^{-13}	4.16334×10^{-17}
(0.2, 0.5)	2.4100×10^{-8}	1.42339×10^{-11}	1.45411×10^{-12}	3.60822×10^{-16}
(0.3, 0.1)	6.3367×10^{-8}	1.07747×10^{-13}	1.11855×10^{-14}	1.38778×10^{-17}
(0.3, 0.3)	5.8500×10^{-8}	2.90939×10^{-12}	3.02064×10^{-13}	5.55112×10^{-17}
(0.3, 0.5)	5.4000×10^{-8}	1.34695×10^{-11}	1.39849×10^{-12}	3.19190×10^{-16}
(0.4, 0.1)	1.124×10^{-7}	1.02029×10^{-13}	1.0783×10^{-14}	1.38778×10^{-17}
(0.4, 0.3)	1.0390×10^{-7}	2.75424×10^{-12}	2.90587×10^{-13}	2.77556×10^{-17}
(0.4, 0.5)	9.6100×10^{-8}	1.27514×10^{-11}	1.34555×10^{-12}	3.19190×10^{-16}
(0.5, 0.1)	1.7550×10^{-7}	9.66033×10^{-14}	1.03528×10^{-14}	0
(0.5, 0.3)	1.6220×10^{-7}	2.60846×10^{-12}	2.79707×10^{-13}	5.55512×10^{-17}
(0.5, 0.5)	1.5010×10^{-7}	1.20763×10^{-11}	1.29505×10^{-12}	3.19190×10^{-16}

Table 4: Comparison of SDM solution of $\varphi(\eta,\tau)$ with LADM [3] and NIM [40] of Example 2 with $\varepsilon = 0.005$, $h = 0.1$, $c = 10$, and $\nu = 1$.

(η,τ)	$\lvert\varphi_{\text{Exact}} - \varphi^{(3)}_{\text{LADM}}\rvert$	$\lvert\varphi_{\text{Exact}} - \varphi^{(3)}_{\text{NIM}}\rvert$	$\lvert\varphi_{\text{Exact}} - \varphi^{(3)}_{\text{SDM}}\rvert$	$\lvert\varphi_{\text{Exact}} - \varphi^{(4)}_{\text{SDM}}\rvert$
(0.1, 0.1)	9.5512×10^{-10}	6.71962×10^{-14}	4.78523×10^{-15}	1.73472×10^{-18}
(0.1, 0.3)	8.0600×10^{-10}	1.81427×10^{-12}	1.29184×10^{-13}	2.60209×10^{-17}
(0.1, 0.5)	6.7700×10^{-10}	8.39947×10^{-12}	5.98136×10^{-13}	1.80411×10^{-16}
(0.2, 0.1)	3.8210×10^{-9}	6.30876×10^{-14}	4.55885×10^{-15}	3.46945×10^{-18}
(0.2, 0.3)	3.224×10^{-9}	1.70328×10^{-12}	1.23005×10^{-13}	2.34188×10^{-17}
(0.2, 0.5)	2.7060×10^{-9}	7.88563×10^{-12}	5.69531×10^{-13}	1.73472×10^{-16}
(0.3, 0.1)	8.597×10^{-9}	5.92521×10^{-14}	4.33594×10^{-15}	3.46945×10^{-18}
(0.3, 0.3)	7.252×10^{-9}	1.59992×10^{-12}	1.17176×10^{-13}	1.99493×10^{-17}
(0.3, 0.5)	6.0910×10^{-9}	7.40708×10^{-12}	5.42553×10^{-13}	1.61329×10^{-16}
(0.4, 0.1)	1.5284×10^{-8}	5.56907×10^{-14}	4.13818×10^{-15}	2.60209×10^{-18}
(0.4, 0.3)	1.2893×10^{-8}	1.50359×10^{-12}	1.11679×10^{-13}	1.73472×10^{-17}
(0.4, 0.5)	1.0827×10^{-8}	6.96112×10^{-12}	5.17104×10^{-13}	1.52656×10^{-16}
(0.5, 0.1)	2.3880×10^{-8}	5.23618×10^{-14}	3.94476×10^{-15}	8.67362×10^{-19}
(0.5, 0.3)	2.0144×10^{-8}	1.41377×10^{-12}	1.06495×10^{-13}	2.08167×10^{-17}
(0.5, 0.5)	1.6916×10^{-8}	6.54527×10^{-12}	4.93078×10^{-13}	1.43982×10^{-16}

$$
\begin{aligned}
\xi(\eta,\tau) &= \varepsilon - h\coth[h(\eta+c)] - \frac{\tau^{\nu}\varepsilon h^2\operatorname{csch}^2[h(\eta+c)]}{\Gamma(1+\nu)} \\
&- \frac{2\tau^{2\nu}h^3\varepsilon^2\coth[(\eta+c)h]\operatorname{csch}^2[h(\eta+c)]}{\Gamma(1+2\nu)} \\
&- \frac{\tau^{3\nu}h^4\varepsilon^2\operatorname{csch}^5[(\eta+c)h]}{\Gamma[1+\nu]^2\Gamma[1+3\nu]}\left(\begin{array}{c} -2h\cosh[(\eta+c)h]\Gamma[1+2\nu] \\ +\Gamma[1+\nu]^2\left(\begin{array}{c} 4h\cosh[(\eta+c)h] \\ +\varepsilon\left(\begin{array}{c} 3\sinh[(\eta+c)h] \\ +\sinh[3(\eta+c)h]\end{array}\right)\end{array}\right)\end{array}\right), \\
\varphi(\eta,\tau) &= -h^2\operatorname{csch}^2[h(\eta+c)] - \frac{2\tau^{\nu}h^3\varepsilon\coth[(\eta+c)h]\operatorname{csch}^2[h(\eta+c)]}{\Gamma(1+\nu)} \\
&- \frac{2\tau^{2\nu}h^4\varepsilon^2(2+\cosh[2(\eta+c)h])\operatorname{csch}^4[h(\eta+c)]}{\Gamma(1+2\nu)} \\
&- \frac{\tau^{3\nu}h^5\varepsilon^2\operatorname{csch}^6[(\eta+c)h]}{\Gamma[1+\nu]^2\Gamma[1+3\nu]}\left(\begin{array}{c} -2h(3+2\cosh[2(\eta+c)h])\Gamma[1+2\nu] \\ +\Gamma[1+\nu]^2\left(\begin{array}{c} 12h+8h\cosh[2(\eta+c)h] \\ +10\varepsilon\sinh[2(\eta+c)h] \\ +\varepsilon\sinh[4(\eta+c)h]\end{array}\right)\end{array}\right).
\end{aligned}
\tag{51}
$$

7. Numerical Results and Discussion

This section uses graphs and tables to compare the approximate and exact solutions and talk about how accurate and useful the proposed method is. Figures 1–4 show the 3D plot solutions of Examples 1 and 2 obtained by the present method in comparison to the exact solutions at $\nu = 1$. These figures show that the SDM's "approximate" solutions are almost the same as the "exact" solutions. For different fractional values of ν, Figures 5–8 show line plots of the approximate solutions from the proposed method and the exact solutions from Examples 1 and 2. As we can see from the figures, when $\nu \longrightarrow 1$, the numerical solutions approach the exact solutions. In Tables 1–4, the SDM results of this study are compared to methods that have been used in the past for Examples 1 and 2. These tables show that the simulation results using the methods described in [3, 26–28, 40] are less accurate than the simulation results using the method under consideration. This shows that the method under consideration is efficient and reliable.

8. Conclusion

In this paper, the SDM is used to find the solution to a coupled system of WBK equations of fractional order. To demonstrate consistency and applicability, the current framework includes convergence and error analysis. Two examples are used to demonstrate and validate the efficiency of the algorithm under consideration. We can see from the tables and plots that the proposed technique is more effective and precise than other methods. As a result, we can conclude that the proposed algorithm is extremely powerful and well-organized for studying coupled systems arising from physical phenomena; both fractional- and integer-ordered derivatives analytically and numerically describe real-world problems in a systematic and improved manner.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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