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Research Article

Limit Directions of Julia Sets of Entire Functions and Complex Differential Equations

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The limiting directions of Julia sets of infinite order entire functions are studied by combining the theory of complex dynamic system and the theory of complex differential equations, in which the lower bound of the measure of limiting direction of Julia set of entire solutions of complex differential equations is obtained.

1. Introduction and Main Results

Let $\mathbb{C}$ be a complex plane and $f\colon \mathbb{C} \longrightarrow \overline{\mathbb{C}}$ be a transcendental meromorphic function, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. $f^n\,(n \in N)$ is denoted by the n-th iteration of f. The Fatou set $\mathcal{F}(f)$ of transcendental meromorphic function f is the subset of $\mathbb{C}$ and satisfies $\{f^n\}$ of f is a normal family. The Julia set $\mathcal{J}(f)$ of f is the complement of $\mathcal{F}(f)$ in $\mathbb{C}$. We all know that $\mathcal{F}(f)$ is completely invariant under f and open set. $\mathcal{J}(f)$ is closed and nonempty. Some fundamental knowledge about the complex dynamics of meromorphic functions can be found in[1, 2]. We assume that the reader is familiar with the basic results and standard notation of Nevanlinna's value distribution theory in $\mathbb{C}$. It plays a substantial role in our studying, such as $T(r, f), m(r, f)$, $N(r, f)$, which can be found in [3, 4]. For a function f meromorphic in $\mathbb{C}$, the order, lower order, and exponent of convergence of zeros of f are given by

$$\begin{aligned}\rho(f) &= \limsup_{r\longrightarrow+\infty} \frac{\log^+ T(r, f)}{\log r},\\ \mu(f) &= \liminf_{r\longrightarrow+\infty} \frac{\log^+ T(r, f)}{\log r},\\ \lambda(f) &= \limsup_{r\longrightarrow+\infty} \frac{\log^+ n(r, 1/f)}{\log r},\end{aligned} \tag{1}$$

respectively, where $\log^+ x = \max\{\log x, 0\}$, $T(r, f)$ is the Nevanlinna characteristic of f and $n(r, 1/f)$ denotes the number of f in $\{z\colon |z| < r\}$, counting multiplicities. If f is an entire function, then the Nevanlinna characteristic $T(r, f)$ can be replaced with $\log M(r, f)$, where $M(r, f) = \max_{|z|=r} |f(z)|$.

For any $a \in \mathbb{C}$, the deficiency of a with meromorphic function f is defined by

$$\begin{aligned}\delta(a, f) &= \liminf_{r\longrightarrow+\infty} \frac{m(r, 1/f-a)}{T(r, f)}\\ &= 1 - \limsup_{r\longrightarrow+\infty} \frac{N(r, 1/f-a)}{T(r, f)}.\end{aligned} \tag{2}$$

If $\delta(a, f) > 0$, then a is a deficient value of f.

The Lebesgue linear measure of a set $E \subset [0, +\infty]$ is denoted by $\mathrm{mes}(E) = \int_E dt$, and the logarithmic measure of a set $F \subset [1, +\infty]$ is defined by $\mathrm{m}_l(F) = \int_F dt/t$.

In the following, we introduce some related results on the limiting directions of Julia set of meromorphic functions. Baker [5] first observed that the Julia set cannot be contained in any finite set of straight lines for a transcendental entire function f. However, this is not true for transcendental meromorphic functions. For example, $\mathcal{J}(\tan z) = R$. Qiao [6] introduced the concept of limiting directions of $\mathcal{J}(f)$ and demonstrated that the Julia set of a transcendental entire function of finite order possesses an infinite number of limiting directions. The limiting direction of $\mathcal{J}(f)$ is defined as a ray $\arg z = \theta$ for which there exists an unbounded sequence $\{z_n\} \subseteq \mathcal{J}(f)$ such that $\lim_{n\longrightarrow+\infty}\arg z_n = \theta$. For further information, please refer to [6, 7]. Denoting

$$\Delta(f) = \{\theta \in [0,2\pi]\colon \arg z = \theta \text{ a limiting direction of } \mathcal{J}(\mathrm{f})\}. \tag{3}$$

It is evident that $\Delta(f)$ is a closed set. We denote its linear measure by $\operatorname{mes}(\Delta(f))$.

In [8], Qiao proved that if f is a transcendental entire function of finite lower order, then $\operatorname{mes}(\Delta(f)) = 2\pi$ if $\mu(f) < 1/2$; and $\operatorname{mes}(\Delta(f)) \geq \min\{2\pi, \pi/\mu(f)\}$ if $\mu(f) \geq 1/2$. Later, some observations were made for a transcendental meromorphic function f by [9, 10], respectively. They obtained that if $\mu(f) < +\infty$ and $\delta(\infty, f) > 0$, then

$$\operatorname{mes}(\Delta(f)) \geq \min\left\{2\pi, \frac{4}{\mu(A)}\arcsin\sqrt{\frac{\delta(\infty, A)}{2}}\right\}. \tag{4}$$

Many results have been obtained involving the limiting directions of the Julia set of transcendental meromorphic functions of finite order, for example [1, 7, 11–13] and therein in references.

However, the limiting direction of the Julia set of entire functions of infinite order remains an open problem. Recently, many scholars have studied the limiting direction of Julia set of entire solutions of infinite order by using the theory of differential equations, such as [11, 12, 14–16].

Furthermore, we define the common Julia limiting directions of derivatives and primitives of an entire function f by

$$L(f) \coloneqq \bigcap_{n\in\mathbb{Z}} \Delta\left(f^{(n)}\right), \tag{5}$$

where $f^{(n)}$ denotes the n-th derivative or n-th integral primitive of f for $n \geq 0$ or $n < 0$, respectively.

In order to introduce our results, some notations are needed.

Let α, β be two constants such that $0 < \alpha < \beta \leq 2\pi$,

$$\begin{aligned} \Omega(\alpha,\beta) &= \{z \in \mathbb{C}\colon \arg z \in (\alpha,\beta)\},\\ \Omega(r;\alpha,\beta) &= \Omega(\alpha,\beta) \cap \{z \in \mathbb{C}\colon |z| > r\},\\ \Omega(\alpha,\beta;r) &= \Omega(\alpha,\beta) \cap \{z \in \mathbb{C}\colon |z| \leq r\}, \end{aligned} \tag{6}$$

$\overline{\Omega}(\alpha,\beta)$ is the closed set of $\Omega(\alpha,\beta)$ and $\arg z = \theta \in [0,2\pi]$ is a ray emanating from the origin.

We recall the concept of an accumulation line of the zero sequence of a transcendental meromorphic function f in an angular domain $\Omega(\alpha,\beta;r)$, which can be found in [13, 17]. The radial convergence exponent of the zero sequence of f at the ray $\arg z = \theta$ is defined by

$$\lambda_\theta(f) = \lim_{\varepsilon\longrightarrow 0}\limsup_{r\longrightarrow+\infty}\frac{\log^+ n(\Omega(\theta-\varepsilon,\theta+\varepsilon;r),0,f)}{\log r}, \tag{7}$$

where $n(\Omega(\theta-\varepsilon,\theta+\varepsilon;r),0,f)$ denotes the number of zeros of f in $\Omega(\theta-\varepsilon,\theta+\varepsilon;r)$, counting multiplicities. If $\lambda_\theta(f) = \rho(f)$, then the ray $\arg z = \theta$ is considered to be an accumulation line of the zero sequence of f. This concept can be used to analyze the growth of solutions of differential equations, as described in [18]. The properties of solutions of the following equation (8) are needed in our results.

$$\omega'' + p(z)\omega = 0, \tag{8}$$

where $p(z) = a_n z^n + \cdots + a_0, a_n \neq 0$, which can be found in [19] [Chapter 7.4], see also Lemma 10 in Section 2 below. The properties of solutions of (8) are used to study the growth of solutions of complex differential equations

$$f'' + A(z)f' + B(z)f = 0. \tag{9}$$

See [18, 20] for more details.

In [21], the relationship between $T(r,A)$ and $\log M(r,A)$ is used to study the growth of solutions of (9), in which infinite order solutions of (9) are characterised by the following condition:

$$T(r,A) \sim \log M(r,A), \tag{10}$$

as $r \longrightarrow +\infty$ outside a set D of finite logarithmic measure. Later, in [22], Long et al. changed the condition to

$$T(r,A) \sim \alpha \log M(r,A), \tag{11}$$

as $r \longrightarrow +\infty$ outside a set of U zero upper logarithmic density, and got some results, where $\alpha \in [0,1]$.

The asymptotic relationships (10) and (11) can also be used to analyze the limiting direction of the Julia set of solutions of complex differential equations. Wang and Chen [16] studied the common limiting directions of the Julia set of solutions of (9).

Theorem 1 (see [16]). *Assume that A and B are entire functions, where B is transcendental and $T(r,B) \sim \log M(r,B)$ as $r \longrightarrow +\infty$ outside a set of finite logarithmic measure, A has a finite deficient value a, i.e., $\delta(a,A) > 0$. For every nontrivial solution f of (9), we have*

$$\operatorname{mes}(L(f)) \geq \min\left\{2\pi, \frac{4}{\mu(A)}\arcsin\sqrt{\frac{\delta(a, A)}{2}}\right\}. \tag{12}$$

Moreover, if $\lambda(A) < \infty$, then $\mu(f) = \infty$.

Inspired by Theorem 1 and the idea of asymptotic relations (11) in [22], which the asymptotical relationship (11), we investigate the limiting direction of the Julia set of entire solutions of (9) and derive a lower bound estimate for $\operatorname{mes}(L(f))$.

Theorem 2. *Let A be a nontrivial solution of (8), and the number of accumulation lines of zero sequence of A is strictly less than $n+2$. Let $\alpha \in [0,1]$, and let B be a transcendental entire function that satisfies*

$$T(r,B) \sim \alpha \log M(r,B), \tag{13}$$

as $r \longrightarrow +\infty$ outside a set G of zero upper logarithmic density, where $\alpha \in [0,1]$. Then, every nontrivial solution of (9) satisfies

$$\operatorname{mes}(L(f)) \geq \max\{0, \delta\}, \tag{14}$$

where $\delta = 2\pi/n + 2 - 2\pi(1-\alpha)$.

Next, the condition of B of Theorem 1 is replaced with the condition (13) and obtains the following result.

Theorem 3. *Let A be the entire function having a finite deficient value a, i.e., $\delta(a,A)>0$, and let B be given as Theorem 2. Then, every nontrivial solution of (9) satisfies*

$$\operatorname{mes}(L(f)) \geq \max\{0,\beta\}, \tag{15}$$

where $\beta = \min\left\{2\pi\alpha, 4/\mu(A)\arcsin\sqrt{\delta(a,A)/2} - 2\pi(1-\alpha)\right\}$.

The next result is related to Borel exceptional value.

Theorem 4. *Let A is an entire function of finite order having a finite Borel exceptional value, and let B be given as Theorem 2. Then, every nontrivial solution of (9) satisfies*

$$\operatorname{mes}(L(f)) \geq \max\{0,\gamma\}, \tag{16}$$

where $\gamma = 2\pi\alpha - \pi - 2\rho(A)\varepsilon_0$ *and* $\varepsilon_0 \in (0, \pi/4\rho(A))$.

Remark 5. It is well-known that Borel exception value is a deficient value and the contrary is not true. So Theorems 3 and 4 both are valuable.

2. Preliminary Lemma

In order to prove our results, we need some preliminary results. First of all, we recall Nevanlinna's characteristic in an angular domain $\overline{\Omega}(\alpha,\beta)$, which can be found in [23] [Chapter 2]. Let g be a meromorphic function on $\overline{\Omega}(\alpha,\beta)$. We define

$$\begin{aligned} A_{\alpha,\beta}(r,g) &= \frac{\omega}{\pi}\int_1^r\left(\frac{1}{t^{\omega}} - \frac{t^{\omega}}{r^{2\omega}}\right) \\ &\quad \cdot \left\{\log^+\left|g\left(te^{i\alpha}\right)\right| + \log^+\left|g\left(te^{i\beta}\right)\right|\right\}; \\ B_{\alpha,\beta}(r,g) &= \frac{2\omega}{\pi r^{\omega}}\int_{\alpha}^{\beta}\log^+\left|g\left(te^{i\theta}\right)\right|\sin\omega(\theta_{\alpha})d\theta; \\ C_{\alpha,\beta}(r,g) &= 2\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^{\omega}} - \frac{|b_n|^{\omega}}{r^{2\omega}}\right)\sin\omega(\theta_{\alpha})d\theta, \end{aligned} \tag{17}$$

where $\omega = \pi/\beta-\alpha$, and $b_n = |b_n|e^{i\theta_n}$ are the poles of g on $\overline{\Omega}(\alpha,\beta)$, appearing with their respective multiplicities. The Nevanlinna angular characteristic of g in $\Omega(\alpha,\beta)$ is defined as follows:

$$S_{\alpha,\beta}(r,g) = A_{\alpha,\beta}(r,g) + B_{\alpha,\beta}(r,g) + C_{\alpha,\beta}(r,g), \tag{18}$$

and the order of g in $\overline{\Omega}(\alpha,\beta)$ is defined by

$$\rho_{\alpha,\beta}(g) = \limsup_{r\longrightarrow+\infty}\frac{\log S_{\alpha,\beta}(r,g)}{\log r}. \tag{19}$$

Lemma 6 (see [23]). *Let f be a meromorphic function on $\Omega(\alpha-\varepsilon,\beta+\varepsilon)$ for $\varepsilon>0$ and $0<\alpha<\beta\leq 2\pi$. Then,*

$$A_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + B_{\alpha,\beta}\left(r,\frac{f'}{f}\right) \leq K\left(\log^+ S_{\alpha-\varepsilon,\beta+\varepsilon}(r,f) + \log r + 1\right), \tag{20}$$

possibly except a set with finite linear measure.

Lemma 7 (see [10]). *Let $f\colon \Omega(r_0;\theta_1,\theta_2) \longrightarrow U$ be holomorphic function, where U is a hyperbolic domain. If there exists a point $a \in \partial U/\{\infty\}$ such that $C_U(a)>0$, then there exists a constant $d>0$ such that for sufficiently small $\varepsilon>0$,*

$$|f(z)| = O\left(|z|^d\right), z \in \Omega(r_0;\theta_1+\varepsilon,\theta_2-\varepsilon), |z| \longrightarrow +\infty. \tag{21}$$

Remark 8. A domain W in $\overline{\mathbb{C}}\backslash\{\infty\}$ is called hyperbolic if $\overline{\mathbb{C}}\backslash W$ contains at least three points. For $a \in \mathbb{C}\backslash W$, the hyperbolic distance from a to W is defined as $C_W(a) = \inf\{\lambda_W(z)|z-a|\colon z \in W\}$, where $\lambda_W(a)$ is the hyperbolic density on W. It is well-known that if every component of W is simply connected, then $C_W(a) \geq 1/2$, see [10] for more details.

The following lemma provides estimates for the logarithmic derivatives of functions that are analytic in an angular domain. First, let us take a look at the definition of R-set, see [24] for detail. Set $B(z_n,r_n) = \{z\colon |z-z_n|<r_n\}$. Then, $\bigcup_{n=1}^{\infty}B(z_n,r_n)$ is called an R-set if $\sum_{n=1}^{\infty}r_n<\infty$ and $z_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$. It is clear that $\left\{|z|\colon z \in \bigcup_{n=1}^{\infty}B(z_n,r_n)\right\}$ is a set of finite linear measure.

Lemma 9 (see [25]). *Let $z = re^{i\theta}, r > r_0 + 1$ for some $r_0>0$ and $\alpha\leq\theta\leq\beta$, where $0<\beta-\alpha\leq 2\pi$. Assuming that $n\geq 2$ is an integer, g is analytic in $\Omega(r_0;\alpha,\beta)$ with $\rho_{\alpha,\beta}(g)<+\infty$. Choose $\alpha<\alpha_1<\beta_1<\beta$. Then, for every $\varepsilon_j \in (0,(\beta_j-\alpha_j)/2)$ $(j=1,2,\ldots,n-1)$ outside a set of linear measure zero with*

$$\alpha_j = \alpha + \sum_{s=1}^{j-1}\varepsilon_s, \beta_j = \beta + \sum_{s=1}^{j-1}\varepsilon_s, \quad j = 2,3,\ldots,n-1. \tag{22}$$

There exist $K>0$ and $M>0$ that only depending on $g, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$ and $\Omega(\alpha_{n-1},\beta_{n-1})$, not depending on z, such that

$$\begin{aligned} \left|\frac{g'(z)}{g(z)}\right| &\leq Kr^M(\sin k(\theta-\alpha))^{-2}, \\ \left|\frac{g^{(n)}(z)}{g(z)}\right| &\leq Kr^M\left(\sin k(\theta-\alpha)\prod_{j=1}^{n-1}\sin k_{\varepsilon_j}(\theta-\alpha_j)\right)^{-2}, \end{aligned} \tag{23}$$

for all $z \in \Omega(\alpha_{n-1},\beta_{n-1})$ outside an R-set, where $k = \pi/\beta-\alpha$ and $k_{\varepsilon_j} = \pi/\beta_j-\alpha_j$, $j = 1,2,\ldots,n-1$.

In order to recall the properties of solutions of (8), the following definitions are needed. Let A be an entire function with finite positive order $\rho(A)$. If for any $\theta \in (\alpha,\beta)$, we have

$$\lim_{r\longrightarrow+\infty}\frac{\log\log\left|A\left(re^{i\theta}\right)\right|}{\log r}=\rho(A), \tag{24}$$

then A blows up exponentially in $\overline{\Omega}(\alpha,\beta)$. If for any $\theta\in(\alpha,\beta)$, one has that

$$\lim_{r\longrightarrow+\infty}\frac{\log\log\left|A\left(re^{i\theta}\right)\right|^{-1}}{\log r}=\rho(A). \tag{25}$$

Then, in $\overline{\Omega}(\alpha,\beta)$, A decays to zero exponentially.

The following lemma will be employed to establish Theorem 2.

Lemma 10 (see [19]). *Let A be a nontrivial solution of equation (8). Set $\theta_j=2j\pi-\arg(a_n)/n+2$ and $S_j=\Omega(\theta_j,\theta_{j+1})$, where $j=0,1,\ldots,n+1$ and $\theta_{n+2}=\theta_0+2\pi$. Then, A has the following properties:*

(i) In each sector S_j, A either blows up or decays exponentially to zero;

(ii) If A decays exponentially to zero in S_j, then it must blow up exponentially in S_{j-1} and S_{j+1}, but it is possible for A to blow up exponentially in many adjacent sectors;

(iii) If A decays exponentially to zero in S_j, then A has at most finitely many zeros in any closed subsector within $S_{j-1}\cup\overline{S_j}\cup S_{j+1}$;

(iv) If A blows up exponentially in S_{j-1} and S_j, then, for each $\varepsilon>0$, A has infinitely many zeros in each sector $\overline{\Omega}(\theta_j-\varepsilon,\theta_j+\varepsilon)$. Moreover, as $r\longrightarrow+\infty$,

$$n\left(\overline{\Omega}\left(\theta_j-\varepsilon,\theta_j+\varepsilon;r\right),0,A\right)=(1+o(1))\frac{2\sqrt{|a_n|}}{\pi(n+2)}r^{n+2/2}, \tag{26}$$

where $n(\overline{\Omega}(\theta_j-\varepsilon,\theta_j+\varepsilon;r),0,A)$ is the number of zeros of A counting multiplicity in $\overline{\Omega}(\theta_j-\varepsilon,\theta_j+\varepsilon;r)$.

Remark 11. It is well-known that the set of accumulation rays of the zero sequence of every nontrivial solution of (8) is a subset of $\{\theta_j\colon j=0,1,\ldots,n+1\}$, and the number of accumulation rays is less than or equal to $n+2$.

Next, we will introduce some information on the correlation between Pólya peak and deficient value. Edrei [26] demonstrated that if f is an entire function with $\mu(f)<+\infty$, then, for any finite ρ where $\mu(f)\le\rho\le\rho(f)$, there is a series of Pólya peaks of order ρ. As proven in [4], [Theorem 1.1.3] and [23], there is a positive, increasing, and unbounded sequence $\{r_n\}$ that is outside of an exceptional set F with finite logarithmic measure. The main result of [27] is the spread relation on the Pólya peak, which is stated in the following lemma.

Lemma 12 (see [27]). *Let f be a transcendental meromorphic function with finite lower order μ and one deficient value a. Let $\Lambda(r)$ be a positive function such that $\Lambda(r)=o(T(r,f))$ as $r\longrightarrow+\infty$. Then, for any fixed sequence of Pólya peaks $\{r_n\}$ of order μ, we have*

$$\liminf_{r\longrightarrow+\infty}\operatorname{mes}D_\Lambda(r_n,a)\ge\min\left\{2\pi,\frac{4}{\mu}\arcsin\sqrt{\frac{\delta(a,f)}{2}}\right\}, \tag{27}$$

where $D_\Lambda(r,a)$ is defined by $D_\Lambda(r,\infty)=\{\theta\in[-\pi,\pi]\colon|f(z)|>e^{\Lambda(r)}\}$, and for finite a, $D_\Lambda(r,a)=\{\theta\in[-\pi,\pi]\colon|f(z)-a|<e^{-\Lambda(r)}\}$.

The following two lemmas are related to Borel exceptional value.

Lemma 13 (see [15]). *Let f be an entire function of finite order with a finite Borel exceptional value c. Then, there exist an entire function $h(z)$ with $\rho(h)<\rho(f)$ and a polynomial $P(z)$ of degree $\deg(P)=\rho(f)$ such that*

$$f(z)=h(z)e^{P(z)}+c. \tag{28}$$

Lemma 14 (see [28]). *Let $Q(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$, where $n\in\mathbb{N}^+$, $a_n=b_ne^{i\theta_n}$ with $b_n>0$, and $\theta_n\in[0,2\pi]$. For any given $\varepsilon\in(0,\pi/4n)$, we introduce $2n$ open angles*

$$S_j=\left\{z\in\mathbb{C}\colon -\frac{\theta_n}{n}+(2j-1)\frac{\pi}{2n}+\varepsilon<\arg z<-\frac{\theta_n}{n}+(2j+1)\frac{\pi}{2n}-\varepsilon\right\}, \tag{29}$$

where $j=0,1,\ldots,2n-1$. Then there exists a positive number $R=R(\varepsilon)$ such that, for $|z|=r>R$,

$$\begin{cases}\operatorname{Re}\{Q(z)\}>b_n(1-\varepsilon)\sin(n\varepsilon)r^n, & \text{if } z\in S_j \text{ and } j \text{ is even},\\ \operatorname{Re}\{Q(z)\}<-b_n(1-\varepsilon)\sin(n\varepsilon)r^n, & \text{if } z\in S_j \text{ and } j \text{ is odd}.\end{cases} \tag{30}$$

3. Proofs of Main Theorems

Proof of Theorem 15. Since $\delta = 2\pi/n+2 - 2\pi(1-\alpha)$, if $\delta \le 0$, then it is clear that $\mathrm{mes}(L(f)) \ge 0$. We will prove $\mathrm{mes}(L(f)) \ge \delta$ for $\delta > 0$. Since the number of accumulation lines of zero sequence of A is strictly less than $n+2$, there exists at least one ray $\arg z = \theta$ is not the accumulation line of the zero sequence of A. Without loss of generally, let $i_0 \in \{0, 1, \ldots, n+1\}$ such that the ray $\arg z = \theta_{i_0}$ is not an accumulation line of the zero sequence of A, which implies that A decays to zero exponentially in either S_{i_0-1} or S_{i_0}. In fact, if A blows up in both S_{i_0-1} and S_{i_0}, then by condition (iv) of Lemma 10, on the one hand

$$\begin{aligned}\lambda_{\theta_{i_0}}(A) &= \lim_{\varepsilon\longrightarrow 0}\limsup_{r\longrightarrow+\infty}\frac{\log^+ n\left(\Omega\left(\theta_{i_0}-\varepsilon, \theta_{i_0}+\varepsilon; r\right), 0, A\right)}{\log r}\\ &= \frac{n+2}{2} = \rho(A),\end{aligned} \tag{31}$$

which is impossible. Without loss of generality, assume that A decays to zero exponentially in sector $S_{i_0} = \Omega(\theta_{i_0}, \theta_{i_0+1})$, $0 \le i_0 \le n+1$. That is, for any $\theta \in D_{i_0} = \{\arg z: z \in S_{i_0}\}$, we have that

$$\lim_{r\longrightarrow+\infty}\frac{\log\log\left|A\left(re^{i\theta}\right)\right|^{-1}}{\log r} = \rho(A), \tag{32}$$

and $\mathrm{mes}(D_{i_0}) = 2\pi/n+2$. By simple calculations, for any given $\varepsilon > 0$, there exists $r_0 > 1$, such that for all $z \in S_{i_0}$ and $|z| = r > r_0$, we have

$$\left|A\left(re^{i\theta}\right)\right| \le \exp\left(-r^{(\rho(A)-\varepsilon)}\right). \tag{33}$$

On the other hand, for any sufficiently large positive constants M_1, we define $I = \{z \in \mathbb{C}: |B(z)| > |z|^{M_1}\}$ and $E(r) = \{\theta \in [0, 2\pi]: z = re^{i\theta} \in I\}$. Then for some $r_1 > 0$, if $r > r_1$, we have

$$2\pi T(r,B) = \int_{E(r)}\log^+\left|B\left(re^{i\theta}\right)\right|\mathrm{d}\theta + \int_{[0,2\pi]/E(r)}\log^+\left|B\left(re^{i\theta}\right)\right|\mathrm{d}\theta \le \mathrm{mes}(E(r))\log M(r,B) + M_1\log r(2\pi - \mathrm{mes}(E(r))), \tag{34}$$

which gives

$$2\pi \le \mathrm{mes}(E(r))\frac{\log M(r,B)}{T(r,B)} + \frac{M_1\log r}{T(r,B)}(2\pi - \mathrm{mes}(E(r))). \tag{35}$$

Since B is transcendental and satisfies (13) outside of G, it yields that, for $r \notin G$,

$$\liminf_{r\longrightarrow+\infty}\mathrm{mes}(E(r)) \ge 2\pi\alpha. \tag{36}$$

By the Heine theorem, there exists an infinite sequence $\{r_n\} \subset (r_1, +\infty)\backslash G$ satisfies $\lim_{n\longrightarrow+\infty} r_n = +\infty$, such that

$$\mathrm{mes}(E(r_n)) \ge 2\pi\alpha. \tag{37}$$

Set $E_n = \bigcup_{i=n}^{\infty} E(r_i)$. It is easy to see that E_n is a monotone-decreasing measurable set. Moreover, set $\widetilde{E} = \cap_{n=1}^{\infty} E_n$. Then $\widetilde{E}$ is independent of r, thus, by the Monotone Convergence theorem and (37), we get

$$\begin{aligned}&\mathrm{mes}(\widetilde{E}) = \lim_{n\longrightarrow+\infty}\mathrm{mes}(E_n) = \lim_{n\longrightarrow+\infty}\mathrm{mes}\left(\bigcup_{i=n}^{\infty}(E_i)\right) \ge 2\pi\alpha,\\ &\mathrm{mes}\left(D_{i_0}\cap\widetilde{E}\right) = \mathrm{mes}\left(D_{i_0}/\widetilde{E}^C\right) \ge \mathrm{mes}\left(D_{i_0}\right) - \mathrm{mes}\left(\widetilde{E}^C\right) \ge \frac{2\pi}{n+2} - 2\pi(1-\alpha) = \delta > 0.\end{aligned} \tag{38}$$

Suppose that $\mathrm{mes}(L(f)) < \delta$. Then there exists an interval (α, β) such that

$$(\alpha,\beta) \subset D_{i_0}\cap\widetilde{E},\ (\alpha,\beta)\cap L(f) = \varnothing. \tag{39}$$

Thus, every ray $\arg z = \theta \in (\alpha, \beta)$ is not a limiting direction of Julia set of $f^{(n_\theta)}$ for some integer n_θ depending on θ. Then there exists an angular domain $\Omega(\theta-\xi_\theta, \theta+\xi_\theta)$ such that

$$(\theta-\xi_\theta, \theta+\xi_\theta) \subset (\alpha,\beta), \Omega(r; \theta-\xi_\theta, \theta+\xi_\theta)\cap\mathcal{J}\left(f^{(n_\theta)}\right) = \varnothing, \tag{40}$$

for sufficiently large $r \notin G$, ξ_θ is a constant depending on θ. This means that there exist the corresponding r_θ and an unbounded Fatou component U_θ of $\mathcal{F}(f^{(n_\theta)})$ (see [29]), such that $\Omega(r_\theta; \theta-\xi_\theta, \theta+\xi_\theta) \subset U_\theta$. Take an unbounded and connected set $\Gamma \subset \partial U_\theta$, then

$$f: \Omega(r_\theta; \theta-\xi_\theta, \theta+\xi_\theta) \longrightarrow \frac{\mathbb{C}}{\Gamma}, \tag{41}$$

is analytic. Because $\mathbb{C}\backslash\Gamma$ is simply connected, so that $\mathbb{C}\backslash\Gamma$ is hyperbolic and open, then by Remark 8, for any $a \in \Gamma\{\infty\}$, we have $C_{\mathbb{C}\backslash\Gamma}(a) \ge 1/2$.

Applying Lemma 7 to mapping f mentioned above, there exists a positive constant d_1 such that

$$\left|f^{(n_\theta)}(z)\right| = O\left(|z|^{d_1}\right), \tag{42}$$

holds for all $\arg z \in \Omega(r_\theta; \theta - \xi_\theta + \varepsilon, \theta + \xi_\theta - \varepsilon)$.

If $n_\theta > 0$, noting the fact that

$$f^{(n_\theta - 1)}(z) = \int_0^z f^{(n_\theta)}(\zeta)\mathrm{d}\zeta + C, \tag{43}$$

where C is a constant, and the integral path can be chosen as the segment of a straight line from 0 to z because this integral is independent of the path. It follows from (42) and (43) that

$$\left|f^{(n_\theta - 1)}(z)\right| = O\left(|z|^{d_1+1}\right), \tag{44}$$

holds for all $z \in \Omega(r_\theta; \theta - \xi_\theta + \varepsilon, \theta + \xi_\theta - \varepsilon)$.

Repeating the above discussion n_θ times, we can deduce that for $z \in \Omega(r_\theta; \theta - \xi_\theta + \varepsilon, \theta + \xi_\theta - \varepsilon)$, we have

$$|f(z)| = O\left(|z|^{(d_1+n_\theta)}\right). \tag{45}$$

Therefore, from the definition of Nevanlinna angular characteristic, we know that

$$S_{\theta-\xi_\theta+\varepsilon,\theta+\xi_\theta-\varepsilon}(r, f) = O(\log r). \tag{46}$$

If $n_\theta < 0$, in view of [23], [Lemma 2.2.1] and Lemma 6, we conclude that

$$\begin{aligned} &S_{\theta-\xi_\theta+\varepsilon+\varepsilon',\theta+\xi_\theta-\varepsilon-\varepsilon'}\left(r, f^{(n_\theta+1)}\right) \\ &\le S_{\theta-\xi_\theta+\varepsilon+\varepsilon',\theta+\xi_\theta-\varepsilon-\varepsilon'}\left(r, \frac{f^{(n_\theta+1)}}{f^{(n_\theta)}}\right) \\ &\quad + S_{\theta-\xi_\theta+\varepsilon+\varepsilon',\theta+\xi_\theta-\varepsilon-\varepsilon'}\left(r, f^{(n_\theta)}\right) \\ &\le O\left(\log^+ S_{\theta-\xi_\theta+\varepsilon,\theta+\xi_\theta-\varepsilon}\left(r, f^{(n_\theta)}\right) + \log r\right) \\ &\quad + S_{\theta-\xi_\theta+\varepsilon+\varepsilon',\theta+\xi_\theta-\varepsilon-\varepsilon'}\left(r, f^{(n_\theta)}\right) \\ &= O\left(\log^+ S_{\theta-\xi_\theta+\varepsilon,\theta+\xi_\theta-\varepsilon}\left(r, f^{(n_\theta)}\right) + \log r\right), \end{aligned} \tag{47}$$

for $|n_\theta|\varepsilon' = \varepsilon$. Then,

$$S_{\theta-\xi_\theta+\varepsilon+\varepsilon',\theta+\xi_\theta-\varepsilon-\varepsilon'}\left(r, f^{(n_\theta+1)}\right) = O(\log r). \tag{48}$$

By (42), we get $S_{\theta-\xi_\theta+\varepsilon,\theta+\xi_\theta-\varepsilon}(r, f^{(n_\theta)}) = O(\log r)$. By using the similar argument $|n_\theta|$ times, it yields that

$$S_{\theta-\xi_\theta+2\varepsilon,\theta+\xi_\theta-2\varepsilon}(r, f) = O(\log r). \tag{49}$$

It follows from (46) and (49), whenever n_θ is positive or not,

$$S_{\alpha',\beta'}(r, f) = O(\log r), \tag{50}$$

where

$$\begin{aligned} &\alpha' = \theta - \xi_\theta + \varepsilon, \beta' = \theta + \xi_\theta - \varepsilon, n_\theta \ge 0, \\ &\alpha' = \theta - \xi_\theta + 2\varepsilon, \beta' = \theta + \xi_\theta - 2\varepsilon, n_\theta < 0. \end{aligned} \tag{51}$$

This implies that $\rho_{\alpha',\beta'}(r, f) = 0$. By Lemma 9, there exist two constants $M_2 > 0$ and $K > 0$ such that

$$\left|\frac{f^{(n)}(z)}{f(z)}\right| \le K r^{M_2}, \tag{52}$$

holds for $z \in \Omega(r_\theta; \theta - \xi_\theta + 2\varepsilon, \theta + \xi_\theta - 2\varepsilon)$ outside a R-set.

Combining (9), (33), and (52), we have

$$\begin{aligned} r_n^{M_1} &\le \left|B\left(r_n e^{i\theta}\right)\right| \le \left|\frac{f''\left(r_n e^{i\theta}\right)}{f\left(r_n e^{i\theta}\right)}\right| + \left|A\left(r_n e^{i\theta}\right)\right|\left|\frac{f'\left(r_n e^{i\theta}\right)}{f\left(r_n e^{i\theta}\right)}\right| \\ &\le C\exp\left(-r^{(\rho(A)-\varepsilon)}\right) r_n^{M_2}, \end{aligned} \tag{53}$$

holds for $z_n = r_n e^{i\theta} \in \Omega(r_\theta; \theta - \xi_\theta + 2\varepsilon, \theta + \xi_\theta - 2\varepsilon)$ outside an R-set and sufficiently large $|z_n| = r_n \notin G$, where C is a positive constant. This is impossible, since M_1 can be taken sufficiently large and M_2 is a finite positive constant. Hence, we obtain

$$\operatorname{mes}(L(f)) \ge \delta. \tag{54}$$

Hence, Theorem 2 is completely proved. □ □

Proof of Theorem 16. Since $\beta = \min\{2\pi\alpha, 4/\mu(A)\arcsin\sqrt{\delta(a, A)/2} - 2\pi(1-\alpha)\}$, if $\beta \le 0$, then it is clear that $\operatorname{mes}(L(f)) \ge 0$, Let us prove that $\operatorname{mes}(L(f)) \ge \beta$ for $\beta > 0$, we first assume that

$$\operatorname{mes}(L(f)) < \beta. \tag{55}$$

Using Lemma 12 to A, we can take an increasing and unbounded sequence $\{r_k\}$ such that

$$\liminf_{k\longrightarrow+\infty} \operatorname{mes}(D(r_k)) \ge \min\left\{2\pi, \frac{\pi}{\mu(A)}\arcsin\sqrt{\frac{\delta(a, A)}{2}}\right\}, \tag{56}$$

where $D(r_k) = \{\theta \in [0, 2\pi]: \log|A(r_k e^{i\theta}) - a| < 1\}$, $r_k \notin \{|z| : z \in H\} \cup G$, with H being the R-set. Clearly, for $\theta \in D(r_k)$, $|A(r_k e^{i\theta})| \le e + |a|$. Let $\widetilde{E}$ is defined as in the Proof of Theorem 15. Then, $\operatorname{mes}(D(r_k) \cap \widetilde{E}) \ge \beta$. Similarly as in the Proof of Theorem 15, then there exists an interval (α, β) such that

$$(\alpha, \beta) \subset (\widetilde{E} \cap D(r_k)), (\alpha, \beta) \cap L(f) = \varnothing. \tag{57}$$

Clearly, (52) holds. By (52) and $|B(z)| > |z^{M_1}|$, where M_1 is defined as in the Proof of Theorem 15, we have

$$\begin{aligned} \left|r_k e^{i\theta}\right|^{M_1} &< \left|B\left(r_k e^{i\theta}\right)\right| \le \left|\frac{f''\left(r_k e^{i\theta}\right)}{f\left(r_k e^{i\theta}\right)}\right| \\ &\quad + \left|A\left(r_k e^{i\theta}\right)\right|\left|\frac{f'\left(r_k e^{i\theta}\right)}{f\left(r_k e^{i\theta}\right)}\right| \le C r_k^{M_2}, \end{aligned} \tag{58}$$

holds for $z_k \in \Omega(r_\theta; \theta - \xi_\theta + 2\varepsilon, \theta + \xi_\theta - 2\varepsilon)$ outside an R-set and sufficiently large $|z_k| = r_k$. This is impossible, since M_1 can be taken sufficiently large and M_2 is a finite positive constant. Hence, we have

$$\operatorname{mes}(L(f)) \geq \beta. \tag{59}$$

Theorem 3 is completely proved. □

Proof of Theorem 17. Since $\gamma = 2\pi\alpha - \pi - 2\rho(A)\varepsilon_0$, if $\gamma \leq 0$, then it is clear that $\operatorname{mes}(L(f)) \geq 0$, we will prove $\operatorname{mes}(L(f)) \geq \gamma$ for $\gamma > 0$. Let a be a Borel exceptional value of A. According to Lemma 13, there exists an entire function g with $\rho(g) < \rho(A)$ and a polynomial P of $\deg P = \rho(A)$ such that

$$A(z) = g(z)e^{P(z)} + a, \tag{60}$$

where $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, $a_n \neq 0, n \in \mathbb{N}^+$.

Let

$$S_j = \left\{ z \in \mathbb{C}\colon \arg z \in \left(-\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon_0, -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon_0 \right) \right\}, \tag{61}$$

where $j = 0, 1, \ldots, 2n-1$ and $\varepsilon_0 \in (0, \pi/4\rho(A))$. Set

$$E_j(\theta) = \left\{ \arg z\colon z \in S_j \right\}, j = 0, 1, \ldots, 2n-1. \tag{62}$$

By (61), we have

$$\operatorname{mes}\left(E_j(\theta)\right) = \frac{\pi}{n} - 2\varepsilon_0. \tag{63}$$

For any $j_1 \neq j_2$ and $j_1, j_2 \in \{0, 1, \ldots, 2n-1\}$, we have

$$E_{j_1}(\theta) \cap E_{j_2}(\theta) = \varnothing. \tag{64}$$

Since $\rho(g) < \rho(A) = n$, it follows from (60) and Lemma 14, for any sufficiently small $\eta \in (0, \rho(A) - \rho(g)/3)$,

$$|A(z) - a| \leq |g(z)| \exp(\operatorname{Re}\{P(z)\}) \leq \exp\left(r^{\rho(g)+\eta} - C_0 r^n\right) < \exp(-Cr^n), \tag{65}$$

as $|z| = r \longrightarrow +\infty$ and $z \in \bigcup_{i=1}^n S_{2i-1}$, where C_0 and C are positive constants.

It is clear that

$$\operatorname{mes}\left(\bigcup_{i=1}^n S_{2i-1}\right) = n\operatorname{mes}\left(E_j(\theta)\right) = \pi - 2n\varepsilon_0 = \pi - 2\rho(A)\varepsilon_0. \tag{66}$$

So $\operatorname{mes}((\bigcup_{i=1}^n E_{2i-1}) \cap \widetilde{E}) \geq \pi - 2\rho(A)\varepsilon_0 - 2\pi(1-\alpha) = \gamma > 0$, where $\widetilde{E}$ is defined as in the Proof of Theorem 15.

Suppose that $\operatorname{mes}(L(f)) < \gamma$. By using a similar reason as in the Proof of Theorem 15, there exists an interval (α, β) such that

$$(\alpha, \beta) \subset \left(\bigcup_{i=1}^n E_{2i-1}\right) \cap \widetilde{E}, (\alpha, \beta) \cap L(f) = \varnothing. \tag{67}$$

It is clear that (52) holds. By (9), (52), and (65), we have

$$\left|re^{i\theta}\right|^{M_1} < \left|B\left(re^{i\theta}\right)\right| \leq \left|\frac{f''\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| + \left|A\left(re^{i\theta}\right)\right| \left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| \leq C\exp(-Cr^n)r^{M_2}, \tag{68}$$

holds for $z \in \Omega(r_\theta; \theta - \xi_\theta + 2\varepsilon, \theta + \xi_\theta - 2\varepsilon)$ outside an R-set and sufficiently large $|z| = r$. This is impossible, since M_1 can be taken sufficiently large and M_2 is a finite positive constant. So, we obtain

$$\operatorname{mes}(L(f)) \geq \gamma. \tag{69}$$

Hence, Theorem 4 is completely proved. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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