Hindawi
Journal of Mathematics
Volume 2023, Article ID 4723233, 14 pages
https://doi.org/10.1155/2023/4723233

Research Article

Economic Decision-Making Using Rough Topological Structures

M. A. El-Gayar [1] **R. Abu-Gdairi** [2] **M. K. El-Bably** [3] **and D. I. Taher** [4]

[1] *Department of Mathematics, Faculty of Science, Helwan University, Helwan 11795, Egypt*
[2] *Department of Mathematics, Faculty of Science, Zarqa University, Zarqa 13132, Jordan*
[3] *Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt*
[4] *New Cairo Technology University, Cairo, Egypt*

Correspondence should be addressed to D. I. Taher; dtaher@nctu.edu.eg

Received 29 September 2022; Revised 22 January 2023; Accepted 1 April 2023; Published 5 May 2023

Academic Editor: Costas Siriopoulos

This paper suggests new approximations that are inspired by topological structures. The primary goal of this work is to define four neighborhoods resulting from a binary relation. Thus, we have four distinct techniques for approximating rough sets. The suggested approaches represent topological generalizations of the previous works. The characteristics and connections of these approaches are investigated. For the sake of the application, we provide some useful examples to compare our techniques to those in the published literature. The merit of the current technique is to obtain a more accurate decision for the problems in which these cases are the appropriate frame to describe them; for instance, machine learning (ML, for short) applications of finance, etc. To demonstrate this fact, an economic application is proposed. We employ the proposed technique in defining accurate decisions to identify the growth of countries. An algorithm for decision-making problems is proposed and tested on fictitious data to compare our methods with the previous approaches.

1. Introduction

Machine learning has gradually permeated the financial sector in recent years, having a significant impact on reshaping the landscape of quantitative finance. Many financial institutions, including banks, insurance companies, and even regulators, are already using this technology to address complex financial decision problems, analyze large financial datasets, price complex financial instruments, manage operational risk, and forecast future price paths. Furthermore, the development of free and easy-to-use programming languages, such as **R** and **Python**), has broadened the applicability and investigation of ML applications outside of finance.

As ML models use big data to learn and improve predictability and performance, algorithmic trading and block chain-based finance are gaining traction. However, the success of any ML approach is heavily dependent on collecting and using the appropriate data, as well as applying the correct algorithm. In this context, a purely mathematical approach with no theoretical foundations can produce erroneous results, creating or exacerbating both financial and nonfinancial risk. As financial complexity rises, so do transactional and operational costs, and ML enables analysts to handle a greater volume of data and mine information previously unattainable through automated transaction processes. Although predicting stock price direction has been studied for years by individuals and financial firms, there is a large body of the literature on the subject (for instance see [1–7]) that did not use mathematical methods for predicting stock price direction, and thus identifying economic growth. However, empirical research focusing on fixed-income market direction prediction, particularly using machine learning methodologies, is scarce, and such literature is rarely repeatable. There are earlier studies of the use of R/S analysis and Hurst exponent in the stock market (see [1–3]) and the mutual fund industry (see [4]), while applications in air pollution are discussed in paper [5]. On the other hand, Pavlidis, et al. [6] introduced and discussed some methods for financial forecasting. Sfetsos and Siriopoulos suggested and studied time series forecasting with a hybrid clustering scheme and pattern recognition in [7].

A lot of real-world situations require some formulas of approximations to appropriate mathematical structures. The magnificence of applying topological structures in rough approximations allows for an approximation of qualitative concepts (i.e., subsets) with no coding or assumption. Topologists used relations to build a general topology which is the applicable mathematical structure for any group linked by relations. We conclude that the relations have been entered to build topological structures in a variety of fields such as in rough sets and their extensions [8–11], rough multisets [12], decision-making problems [13–16], medical applications [14, 17–19], bipolar soft ordered topology [20], economic fields [21], topological reductions of attributes for predicting of a lung cancer disease [22] and heart failure [23], biochemistry [24–26], computer sciences [27–30], structure analysis [31], fuzzy soft approaches [32–34], topological study of zeolite socony mobil-5 [35], near sets theory [36], and covering rough sets [37–40]. In 2022, Dalkılıç [41] introduced some topological structures of virtual fuzzy parameterized fuzzy soft sets and proposed some applications of his methods. It is important to notice that rough sets approach have many applications and interests in several fields such as bipolar soft set theory [42] and virtual intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets and their applications in decision-making which were examined and studied by Dalkılıç and Demirtaş in [43, 44].

Rough set theory is a mathematical strategy for dealing with ambiguity in which exact lower and upper approximation sets are used. These approximations relate to the strict set contained in the rough set's minimal (resp. maximum) rough set (resp. containing). The equivalence relation represents a central idea in Pawlak's original rough set theory, and it appears to be a stringent requirement that restricts theory's applicability. So, many authors introduce several generalizations to Pawlak's models. Some of them used reflexive relations [10], similarity relations [45–47], general binary relations [46–52], topological structures [6, 53–55], and coverings [37–40]. Marei proposed some different methods based on topological structures and neighborhoods to generalize Pawlak rough sets in [56–58]. On the other hand, Rafat [54] introduced and studied some methods based on the ideal concept and topological structures to generalize the previous methods such as [59–61]. Some relationships between the rough set approach [62, 63] and the other branches studied in [64–67].

Monsef et al. [8] proposed a new concept known as "the j-neighborhood space" (abbreviated j-**NS**), which represents a generalized type of neighborhood space. In fact, they presented a framework for generalizing Pawlak's classical rough set theory, as well as some other generalizations through different topologies induced by a binary relation. As a result, they devised some of rough set approximations to satisfy all of the axioms of Pawlak's principle without any constraints. These procedures paved the way for more topological applications in a rough context, while also assisting in the formalization of many real-world applications.

The main motivations of the current work are as follows: first is to propose a generalization to the idea of "basic-neighborhood" given by Abu-Gdairi et al. [17]. Therefore, we initiate four different topologies induced from these neighborhoods, and hence we study the relationships between these topologies and the previous one [8]. Based on the suggested topologies, four different methodologies to approximate rough sets in the generalized approximation space are given. Comparisons among accuracy measures of these kinds of approximations are achieved and the best one is well-defined. Furthermore, we compare the proposed techniques to the previous ones (namely, Monsef et al. [8], Abu-Gdairi et al. [17], Dai et al. [45], and Yao [50, 51]). As a result, we demonstrate that the proposed methods are more accurate than the alternatives.

Finally, we employ the proposed technique in defining accurate decisions of an economic application. In fact, we explain the meaning of j-basic approximations in making-decision of growth rates of countries. The main aim of applying the proposed approaches here is to confirm between the experimental data and its mathematical analysis. The mathematical study depends on the classification of data. Thus, we use a decision table (an information system with decision attributes) of five countries and a set of attributes that measure the national product in these countries. The decision attribute in that table is to decide the growth of the country. Consequently, we conclude that the suggested approximations are more precise than other approaches and are very useful in determining data ambiguity and assisting in decision-making in real-life problems such as medical diagnosis and ML applications of finance, which requires precise decisions.

Therefore, the fundamental goals of the current manuscript are as follows:

(i) Suggesting some extensions to "basic-neighborhoods" [17]

(ii) Constructing different topologies from relations and studying their properties

(iii) Generating some new types of generalized rough sets as mathematical tools for decision-making

(iv) Investigating some comparisons with the other approaches in the literature

(v) Applying the proposed methods in economic applications in order to broaden the applicability and investigation of ML applications outside finance

(vi) Planning an algorithm and framework for the introduced methods using MATLAB for decision-making problems

2. j-Neighborhood Space

The present section is dedicated to outlining the fundamental concepts of the j-**NS**.

Definition 1 (see [8]). Suppose that $\mathbb{R}$ is a binary relation on a nonempty finite set $\mathbb{U}$. Then, the j-neighborhood of

$e \in \mathbb{U}$, de noted by $\Omega_{j}(e)$, $j \in \{r, \ell, i, u\}$ can defined as follows:

(i) r-neighborhood: $\Omega_{r}(e) = \{q \in \mathbb{U}: e\mathbb{R}q\}$
(ii) ℓ-neighborhood : $\Omega_{\ell}(e) = \{q \in \mathbb{U}: q\mathbb{R}e\}$
(iii) i-neighborhood: $\Omega_{i}(e) = \Omega_{r}(e) \cap \Omega_{\ell}(e)$
(iv) u-neighborhood: $\Omega_{u}(e) = \Omega_{r}(e) \cup \Omega_{\ell}(e)$

Definition 2 (see [8]). If $\mathbb{R}$ is a binary relation on $\mathbb{U}$ and $\chi_{j}: \mathbb{U} \longrightarrow P(\mathbb{U})$ is a function which gives $\forall e \in \mathbb{U}$ its j-neighborhood in $P(\mathbb{U})$. Then, the triple $(\mathbb{U}, \mathbb{R}, \chi_{j})$ is called a j-neighborhood space (abbreviated j-**NS**).

Theorem 1 (see [8]). *Let $(\mathbb{U}, \mathbb{R}, \chi_{j})$ be a j-**NS**, then for each $j \in \{r, \ell, i, u\}$, the collection $T_{j} = \{Q \subseteq \mathbb{U}: \forall q \in Q, \Omega_{j}(q) \subseteq Q\}$ represents a topology on $\mathbb{U}$.*

Definition 3 (see [8]). Consider $(\mathbb{U}, \mathbb{R}, \chi_{j})$ is a j-**NS**. The subset $Q \subseteq \mathbb{U}$ is supposed to be "*j- open set*" if $Q \in T_{j}$, and its complement is named "*j- closed set.*" The family K_{j} of all j-closed sets of a j-**NS** is defined by $K_{j} = \{Q \subseteq \mathbb{U}: Q^{c} \in T_{j}\}$.

Definition 4 (see [8]). If $(\mathbb{U}, \mathbb{R}, \chi_{j})$ is a j-**NS** and $Q \subseteq \mathbb{U}$. Then, the "*j-lower*" and "*j-upper*" approximations of Q are well-defined, respectively, as follows:

$$\begin{aligned} \underline{\mathbb{R}}_{j}(Q) &= \cup\{\mathcal{B} \in T_{j}: \mathcal{B} \subseteq Q\} \\ &= \mathbf{Int}_{j}(Q), \\ \overline{\mathbb{R}}_{j}(Q) &= \cap\{\mathcal{M} \in K_{j}: Q \subseteq \mathcal{M}\} \\ &= \mathbf{Cl}_{j}(Q), \end{aligned} \tag{1}$$

where $\mathbf{Int}_{j}(Q)$ (resp. $\mathbf{Cl}_{j}(Q)$) represents j-interior (resp. j-closure) of Q.

Definition 5 (see [8]). Let $(\mathbb{U}, \mathbb{R}, \chi_{j})$ be a j-**NS** and $Q \subseteq \mathbb{U}$. Then, the subset Q is called a "*j-exact*" set if $\underline{\mathbb{R}}_{j}(Q) = \overline{\mathbb{R}}_{j}(Q) = Q$. If not, it is "*$j$-rough.*"

Definition 6 (see [8]). Consider $(\mathbb{U}, \mathbb{R}, \chi_{j})$ is a j-**NS** and $Q \subseteq \mathbb{U}$. The "*j-boundary,*" "*j-positive,*" and "*j-negative*" regions of Q are given, respectively, as follows:

$$\begin{aligned} \mathcal{B}nd_{j}(Q) &= \overline{\mathbb{R}}_{j}(Q) - \underline{\mathbb{R}}_{j}(Q), \\ \mathcal{P}os_{j}(Q) &= \underline{\mathbb{R}}_{j}(Q), \\ \mathcal{N}eg_{j}(Q) &= \mathbb{U} - \overline{\mathbb{R}}_{j}(Q). \end{aligned} \tag{2}$$

The "*j-accuracy*" of the approximations is given as follows:

$$\psi_{j}(Q) = \frac{\left|\underline{\mathbb{R}}_{j}(Q)\right|}{\left|\overline{\mathbb{R}}_{j}(Q)\right|}, \text{where} \left|\overline{\mathbb{R}}_{j}(Q)\right| \neq 0. \tag{3}$$

Clearly, $0 \leq \psi_{j}(Q) \leq 1$ and if $\psi_{j}(Q) = 1$, then Q is a j-exact set. Else, it is j-rough.

3. Topologies Generated from Neighborhoods

The present section is devoted to generalizing the concept of "basic-neighborhood [17]" into new types, and thus we generate four different topologies from these neighborhoods.

Definition 7. Suppose that $\mathbb{R}$ is a binary relation on a nonempty finite set $\mathbb{U}$. Then, we define the following neighborhoods of $e \in \mathbb{U}$:

(i) r-basic neighborhood: $\Omega^{\mathfrak{b}}_{r}(e) = \{q \in \mathbb{U}: \Omega_{r}(q) \subseteq \Omega_{r}(e)\}$
(ii) ℓ-basic neighborhood: $\Omega^{\mathfrak{b}}_{\ell}(e) = \{q \in \mathbb{U}: \Omega_{\ell}(q) \subseteq \Omega_{\ell}(e)\}$
(iii) i-basic neighborhood: $\Omega^{\mathfrak{b}}_{i}(e) = \Omega^{\mathfrak{b}}_{r}(e) \cap \Omega^{\mathfrak{b}}_{\ell}(e)$
(iv) u-basic neighborhood: $\Omega^{\mathfrak{b}}_{u}(e) = \Omega^{\mathfrak{b}}_{r}(e) \cup \Omega^{\mathfrak{b}}_{\ell}(e)$

The next consequences state the foremost characteristics of the previous neighborhoods.

Lemma 1. *Suppose that $\mathbb{R}$ represents a binary relation on $\mathbb{U}$. For each $j \in \{r, \ell, i, u\}$:*

(i) $e \in \Omega^{\mathfrak{b}}_{j}(e)$
(ii) $\Omega^{\mathfrak{b}}_{j}(e) \neq \varphi$
(iii) If $q \in \Omega^{\mathfrak{b}}_{j}(e)$, then $\Omega^{\mathfrak{b}}_{j}(q) \subseteq \Omega^{\mathfrak{b}}_{j}(e)$, for each $j \in \{r, \ell, i\}$

Proof. Firstly, the proof of (i) and (ii) is obvious by Definition 7. (iii) We prove the item in a case of $j = r$ only, and the other cases in a similar way.

By using Definition 7, if $q \in \Omega^{\mathfrak{b}}_{r}(e)$, then $\Omega_{r}(q) \subseteq \Omega_{r}(e) \ldots (1)$

Now, let $k \in \Omega^{\mathfrak{b}}_{r}(q)$. Then, $\Omega_{r}(k) \subseteq \Omega_{r}(q)$. Thus, by (1), $\Omega_{r}(k) \subseteq \Omega_{r}(e)$ and this implies $k \in \Omega^{\mathfrak{b}}_{r}(e)$. Accordingly, $\Omega^{\mathfrak{b}}_{j}(q) \subseteq \Omega^{\mathfrak{b}}_{j}(e)$. □

Lemma 2. *Let $\mathbb{R}$ be a binary relation on $\mathbb{U}$. Then, for every $e \in \mathbb{U}$:*

(i) $\Omega^{\mathfrak{b}}_{i}(e) \subseteq \Omega^{\mathfrak{b}}_{r}(e) \subseteq \Omega^{\mathfrak{b}}_{u}(e)$
(ii) $\Omega^{\mathfrak{b}}_{i}(e) \subseteq \Omega^{\mathfrak{b}}_{\ell}(e) \subseteq \Omega^{\mathfrak{b}}_{u}(e)$

Proof. Forthright □

Remark 1. In Example 1, we will illustrate that:

(i) Item (iii) of Lemma 1 is not true in the case of $j = u$
(ii) The j-basic neighborhoods and j-neighborhoods are independent (noncomparable) in a general case, for each $j \in \{r, \ell, i, u\}$, and $\mathbb{R}$ be a binary relation on $\mathbb{U}$

The following lemma illustrates the relationships between the j-basic neighborhoods and j-neighborhoods.

Lemma 3. *If $(\mathbb{U}, \mathbb{R}, \chi_{j})$ is a j-**NS** and $\mathbb{R}$ is a reflexive relation. Then, $\forall j \in \{r, \ell, i, u\}$: $\Omega^{\mathfrak{b}}_{j}(e) \subseteq \Omega_{j}(e)$ and $\forall e \in \mathbb{U}$.*

Proof. Firstly, by using Definition 7, if $q \in \Omega^{b}_{j}(e)$, then $\Omega_{j}(q) \subseteq \Omega_{j}(e) \ldots (1)$

Since $\mathbb{R}$ is a reflexive relation, then $q \in \Omega_{j}(q)$. Therefore, by (1), $q \in \Omega_{j}(e)$ and thus, $\Omega^{b}_{j}(e) \subseteq \Omega_{j}(e)$ and $\forall e \in \mathbb{U}$. □

Lemma 4. *If* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *is a* j-**NS** *and* $\mathbb{R}$ *is a transitive relation. Then, for every* $j \in \{r, \ell, i, u\}$: $\Omega_{j}(e) \subseteq \Omega^{b}_{j}(e)$ *and* $\forall e \in \mathbb{U}$.

Proof. In what follows, we demonstrate the lemma in the case of $j = r$ only, and similarly, we can prove the other cases.

Firstly, let $g \in \Omega_{r}(e)$, then $e\mathbb{R}g \ldots (1)$

Now, we necessity to show that $\Omega_{r}(g) \subseteq \Omega_{r}(e)$ as follows:

Let $h \in \Omega_{r}(g)$, then $g\mathbb{R}h$. Therefore, by transitivity of $\mathbb{R}$ and using (1), we obtain $e\mathbb{R}h$. Hence, $h \in \Omega_{r}(e)$ which implies $\Omega_{r}(g) \subseteq \Omega_{r}(e)$. Consequently, $g \in \Omega^{b}_{j}(e)$. □

Corollary 1. *If* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *is a* j-**NS** *and* $\mathbb{R}$ *is a preorder (reflexive and transitive) relation. Then, for every* $j \in \{r, \ell, i, u\}$: $\Omega_{j}(e) = \Omega^{b}_{j}(e)$ *and* $\forall e \in \mathbb{U}$.

Corollary 2. *If* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *is a* j-**NS** *and* $\mathbb{R}$ *is an equivalence relation. Then, for each* $j \in \{r, \ell, i, u\}$: $\Omega^{b}_{j}(e) = \Omega_{j}(e) = [e]_{\mathbb{R}}$, *where* $[e]_{\mathbb{R}}$ *represents the equivalence class of* $e \in \mathbb{U}$.

The following result (depending on Theorem 1) discusses an interesting technique to construct dissimilar topologies depending on the previous neighborhoods.

Theorem 2. *If* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *is a* j-**NS**. *Then, for each* $j \in \{r, \ell, i, u\}$, *the collection* $T^{b}_{j} = \{Q \subseteq \mathbb{U}: \forall q \in Q, \Omega^{b}_{j}(q) \subseteq Q\}$ *is a topology on* $\mathbb{U}$.

Proof

(T1) Clearly, $\mathbb{U}$ and φ belong to T^{b}_{j}.

(T2) If $\{Q_{k}: k \in \mathcal{K}\}$ is a class of members in T^{b}_{j} and $q \in \cup_{k} Q_{k}$, then, $\exists k_{o} \in \mathcal{K}$ such that $q \in Q_{k_{o}}$. Thus, $\Omega^{b}_{j}(q) \subseteq Q_{k_{o}}$ and this implies $\Omega^{b}_{j}(q) \subseteq \cup_{k} Q_{k}$. Therefore, $\cup_{k} Q_{k} \in T^{b}_{j}$.

(T3) Let $Q_{1}, Q_{2} \in T^{b}_{j}$ and $q \in Q_{1} \cap Q_{2}$. Then, $q \in Q_{1}$ and $q \in Q_{2}$ which implies $\Omega^{b}_{j}(q) \subseteq Q_{1}$ and $\Omega^{b}_{j}(q) \subseteq Q_{2}$. Thus, $\Omega^{b}_{j}(q) \subseteq Q_{1} \cap Q_{2}$ and hence, $Q_{1} \cap Q_{2} \in T^{b}_{j}$.

From (T1), (T2), and (T3), T^{b}_{j} forms a topology on $\mathbb{U}$.

By using Lemma 2, it is easy to verify the following result which gives the relationships among different topologies T^{b}_{j}. □

Proposition 1. *If* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *is a* j-**NS**, *then*

(i) $T^{b}_{u} \subseteq T^{b}_{r} \subseteq T^{b}_{i}$

(ii) $T^{b}_{u} \subseteq T^{b}_{\ell} \subseteq T^{b}_{i}$

The opposite of Proposition 1 is incorrect as illustrated in Example 1.

Example 1. Let $\mathbb{U} = \{q_{1}, q_{2}, q_{3}, q_{4}\}$ and $\mathbb{R} = \{(q_{1}, q_{1}), (q_{1}, q_{4}), (q_{2}, q_{1}), (q_{2}, q_{3}), (q_{3}, q_{3}), (q_{3}, q_{4}), (q_{4}, q_{1})\}$. Thus, we get

$\Omega_{r}(q_{1}) = \{q_{1}, q_{4}\}, \Omega_{\ell}(q_{1}) = \{q_{1}, q_{2}, q_{4}\}, \Omega_{i}(q_{1}) = \{q_{1}, q_{4}\}, \Omega_{u}(q_{1}) = \{q_{1}, q_{2}, q_{4}\}, \Omega_{r}(q_{2}) = \{q_{1}, q_{3}\}, \Omega_{\ell}(q_{2}) = \varphi, \Omega_{i}(q_{2}) = \varphi, \Omega_{u}(q_{2}) = \{q_{1}, q_{3}\}, \Omega_{r}(q_{3}) = \{q_{3}, q_{4}\}, \Omega_{\ell}(q_{3}) = \{q_{2}, q_{3}\}, \Omega_{i}(q_{3}) = \{q_{3}\}, \Omega_{u}(q_{3}) = \{q_{2}, q_{3}, q_{4}\}, \Omega_{r}(q_{4}) = \{q_{1}\}, \Omega_{\ell}(q_{4}) = \{q_{1}, q_{3}\}, \Omega_{i}(q_{4}) = \{q_{1}\}$, and $\Omega_{u}(q_{4}) = \{q_{1}, q_{3}\}$

Consequently, we obtain

$\Omega^{b}_{r}(q_{1}) = \{q_{1}, q_{4}\}, \Omega^{b}_{\ell}(q_{1}) = \{q_{1}, q_{2}\}, \Omega^{b}_{i}(q_{1}) = \{q_{1}\}, \Omega^{b}_{u}(q_{1}) = \{q_{1}, q_{2}, q_{4}\}, \Omega^{b}_{r}(q_{2}) = \{q_{2}, q_{4}\}, \Omega^{b}_{\ell}(q_{2}) = \{q_{2}\}, \Omega^{b}_{i}(q_{2}) = \{q_{2}\}, \Omega^{b}_{u}(q_{2}) = \{q_{2}, q_{4}\}, \Omega^{b}_{r}(q_{3}) = \{q_{3}\}, \Omega^{b}_{\ell}(q_{3}) = \{q_{2}, q_{3}\}, \Omega^{b}_{i}(q_{3}) = \{q_{3}\}, \Omega^{b}_{u}(q_{3}) = \{q_{2}, q_{3}\}, \Omega^{b}_{r}(q_{4}) = \{q_{4}\}, \Omega^{b}_{\ell}(q_{4}) = \{q_{2}, q_{4}\}, \Omega^{b}_{i}(q_{4}) = \{q_{4}\}$, and $\Omega^{b}_{u}(q_{4}) = \{q_{2}, q_{4}\}$

Accordingly, we generate the following topologies:

$$
\begin{aligned}
T^{b}_{r} &= \{\mathbb{U}, \varphi, \{q_{3}\}, \{q_{4}\}, \{q_{1}, q_{4}\}, \{q_{2}, q_{4}\}, \{q_{3}, q_{4}\}, \\
&\quad \{q_{1}, q_{2}, q_{4}\}, \{q_{2}, q_{3}, q_{4}\}, \{q_{1}, q_{2}, q_{4}\}\{q_{1}, q_{3}, q_{4}\}\}, \\
T^{b}_{\ell} &= \{\mathbb{U}, \varphi, \{q_{2}\}, \{q_{1}, q_{2}\}, \{q_{2}, q_{3}\}, \{q_{2}, q_{4}\}, \\
&\quad \{q_{1}, q_{2}, q_{3}\}, \{q_{1}, q_{2}, q_{4}\}, \{q_{2}, q_{3}, q_{4}\}\}, \\
T^{b}_{i} &= \mathcal{P}(\mathbb{U}), \\
T^{b}_{u} &= \{\mathbb{U}, \varphi, \{q_{2}, q_{4}\}, \{q_{1}, q_{2}, q_{4}\}, \{q_{2}, q_{3}, q_{4}\}\}.
\end{aligned}
\tag{4}
$$

The consequent proposition gives the relationships between the topologies T_{j} and T^{b}_{j}.

Remark 2. For any a j-**NS** $(\mathbb{U}, \mathbb{R}, \chi_{j})$, and by using Example 1, we notice that

(i) The topologies T_{j} and T^{b}_{j} are independent in general case

(ii) The topologies T^{b}_{r} and T^{b}_{ℓ} are independent in general case

Proposition 2. *If* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *is a* j-**NS** *and* $\mathbb{R}$ *is a reflexive relation, then for each* $j \in \{r, \ell, i, u\}$: $T_{j} \subseteq T^{b}_{j}$.

Proof. By Lemma 3, the proof is clear.

The following example illustrates that the converse of Proposition 2 is not true in general. □

Example 2. Let $\mathbb{U} = \{q_{1}, q_{2}, q_{3}, q_{4}\}$ and $\mathbb{R} = \{(q_{1}, q_{1}), (q_{1}, q_{2}), (q_{2}, q_{1}), (q_{2}, q_{2}), (q_{2}, q_{3}), (q_{3}, q_{3}), (q_{4}, q_{4})\}$ be a reflexive relation on $\mathbb{U}$. Thus, we compute the topologies T_{j} and T^{b}_{j} in the case of $j = r$ and the others similarly.

$$
\begin{aligned}
T_{r} &= \{\mathbb{U}, \varphi, \{q_{3}\}, \{q_{4}\}, \{q_{3}, q_{4}\}, \{q_{1}, q_{2}, q_{3}\}\}, \\
T^{b}_{r} &= \{\mathbb{U}, \varphi, \{q_{1}\}, \{q_{3}\}, \{q_{4}\}, \{q_{1}, q_{3}\}, \{q_{1}, q_{4}\}, \\
&\quad \{q_{3}, q_{4}\}, \{q_{1}, q_{2}, q_{3}\}, \{q_{1}, q_{3}, q_{4}\}\}.
\end{aligned}
\tag{5}
$$

Figure 1 summarizes the relationships among different topologies in the case of $\mathcal{R}$ is a reflexive relation.

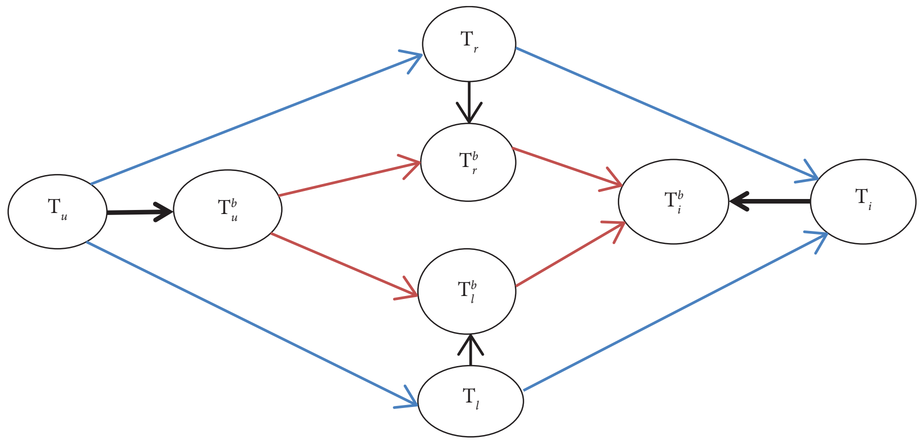

Figure 1: The connections between different topologies of T_{j} and $T_{j}^{\mathfrak{b}}$ in the case of reflexivity relation.

4. Rough Approximations Generated by Different Topologies

We present four methods for approximating rough sets using the interior and closure generated by topology $T_{j}^{\mathfrak{b}}$, for each $j \in \{r, \ell, i, u\}$ in this section.

Definition 8. If $(\mathbb{U}, \mathbb{R}, \chi_{j})$ is a j-**NS**, then $\mathcal{Q} \subseteq \mathbb{U}$ is supposed to be an *j-basic open* set and if $\mathcal{Q} \in T_{j}^{\mathfrak{b}}$, its complement called an *j-basic closed* set. The family $K_{j}^{\mathfrak{i}}$ of all j-basic closed sets is given by $K_{j}^{\mathfrak{b}} = \{K \subseteq \mathbb{U}\colon K^c \in T_{j}^{\mathfrak{b}}\}$. Moreover, we define the following points:

(i) The *j-basic interior* of $\mathcal{Q} \subseteq \mathbb{U}$ is: $\mathrm{Int}_{j}^{\mathfrak{b}}(\mathcal{Q}) = \cup\{\mathcal{G} \in T_{j}^{\mathfrak{b}}\colon \mathcal{G} \subseteq \mathcal{Q}\}$

(ii) The *j-basic closure* of $\mathcal{Q} \subseteq \mathbb{U}$ is: $Cl_{j}^{\mathfrak{b}}(\mathcal{Q}) = \cap\{\mathcal{H} \in K_{j}^{\mathfrak{b}}\colon \mathcal{Q} \subseteq \mathcal{H}\}$

Definition 9. Let $(\mathbb{U}, \mathbb{R}, \chi_{j})$ be a j-**NS** and $\mathcal{Q} \subseteq \mathbb{U}$. Then, the j-basic *lower* and *upper* approximations of $\mathcal{Q}$ are proposed, respectively, as follows:

$$\underline{\mathbb{R}}_{j}^{\mathfrak{b}}(\mathcal{Q}) = \mathrm{Int}_{j}^{\mathfrak{b}}(\mathcal{Q}),$$
$$\overline{\mathbb{R}}_{j}^{\mathfrak{b}}(\mathcal{Q}) = \mathrm{Cl}_{j}^{\mathfrak{b}}(\mathcal{Q}). \tag{6}$$

Definition 10. Let $(\mathbb{U}, \mathbb{R}, \chi_{j})$ be a j-**NS** and $\mathcal{Q} \subseteq \mathbb{U}$. The *$j$-basic boundary*, *$j$-basic positive*, and *$j$-basic negative* regions of $\mathcal{Q}$ are proposed, respectively, as follows:

$$\mathcal{B}nd_{j}^{\mathfrak{b}}(\mathcal{Q}) = \overline{\mathbb{R}}_{j}^{\mathfrak{b}}(\mathcal{Q}) - \underline{\mathbb{R}}_{j}^{\mathfrak{b}}(\mathcal{Q}),$$
$$\mathcal{P}os_{j}^{\mathfrak{b}}(\mathcal{Q}) = \underline{\mathbb{R}}_{j}^{\mathfrak{b}}(\mathcal{Q}), \tag{7}$$
$$\mathcal{N}eg_{j}^{\mathfrak{b}}(\mathcal{Q}) = \mathbb{U} - \overline{\mathbb{R}}_{j}^{\mathfrak{b}}(\mathcal{Q}).$$

Moreover, the *j-basic accuracy* of the j-basic approximations of $\mathcal{Q} \subseteq \mathbb{U}$ is suggested as follows:

$$\psi_{j}^{\mathfrak{b}}(\mathcal{Q}) = \frac{\left|\underline{\mathbb{R}}_{j}^{\mathfrak{b}}(\mathcal{Q})\right|}{\left|\overline{\mathbb{R}}_{j}^{\mathfrak{b}}(\mathcal{Q})\right|}, \text{where} \left|\overline{\mathbb{R}}_{j}^{\mathfrak{b}}(\mathcal{Q})\right| \neq 0. \tag{8}$$

It is obvious that, $0 \leq \psi_{j}^{\mathfrak{b}}(\mathcal{Q}) \leq 1$ and if $\psi_{j}^{\mathfrak{b}}(\mathcal{Q}) = 1$, then $\mathcal{Q}$ is called an j-basic definable (j-basic exact) set. If not, it is an j-basic rough set.

Example 3. By using Example 1, we obtain the following equation:

$$\begin{aligned}
T_{r}^{\mathfrak{b}} &= \{\mathbb{U}, \varphi, \{q_3\}, \{q_4\}, \{q_1, q_4\}, \{q_2, q_4\},\\
&\quad \{q_3, q_4\}\{q_1, q_2, q_4\}, \{q_2, q_3, q_4\}, \{q_1, q_3, q_4\}\},\\
K_{r}^{\mathfrak{b}} &= \{\mathbb{U}, \varphi, \{q_1\}, \{q_2\}, \{q_3\}\{q_1, q_2\}, \{q_1, q_3\},\\
&\quad \{q_2, q_3\}\{q_1, q_2, q_4\}, \{q_1, q_2, q_3\}\},\\
T_{\ell}^{\mathfrak{b}} &= \{\mathbb{U}, \varphi, \{q_2\}, \{q_1, q_2\}, \{q_2, q_3\},\\
&\quad \{q_2, q_4\}\{q_1, q_2, q_3\}, \{q_1, q_2, q_4\}, \{q_2, q_3, q_4\}\},\\
K_{\ell}^{\mathfrak{b}} &= \{\mathbb{U}, \varphi, \{q_1\}, \{q_3\}, \{q_4\}, \{q_1, q_3\}, \{q_1, q_4\},\\
&\quad \{q_3, q_4\}, \{q_1, q_3, q_4\}\},\\
T_{i}^{\mathfrak{b}} &= K_{i}^{\mathfrak{b}} = \mathcal{P}(\mathbb{U}),\\
T_{u}^{\mathfrak{b}} &= \{\mathbb{U}, \varphi, \{q_2, q_4\}, \{q_1, q_2, q_4\}, \{q_2, q_3, q_4\}\},\\
K_{u}^{\mathfrak{b}} &= \{\mathbb{U}, \varphi, \{q_1\}, \{q_3\}, \{q_1, q_3\}\}.
\end{aligned} \tag{9}$$

Thus, we can get Tables 1 and 2 that give the j-basic lower, j-basic upper approximations, and the j-basic accuracy of approximations for all subsets of $\mathbb{U}$.

Remark 3. According to Tables 1 and 2 of Example 3, we conclude that by using different types of $T_{j}^{\mathfrak{b}}$ in building approximations of the sets, the finest of them is that assumed by $T_{i}^{\mathfrak{b}}$ because, for every $\mathcal{Q} \subseteq \mathbb{U}$, $\psi_{u}^{\mathfrak{b}}(\mathcal{Q}) \leq \psi_{r}^{\mathfrak{b}}(\mathcal{Q}) \leq \psi_{i}^{\mathfrak{b}}(\mathcal{Q})$, and $\psi_{u}^{\mathfrak{b}}(\mathcal{Q}) \leq \psi_{\ell}^{\mathfrak{b}}(\mathcal{Q}) \leq \psi_{i}^{\mathfrak{b}}(\mathcal{Q})$. In addition, these approaches are more accurate than the other in [1, 6].

The following proposition imposes some properties of the j-basic approximations:

Table 1: Comparison among different types of j-basic approximations.

$\mathscr{P}(\mathbb{U})$	$T^{\mathfrak{b}}_{r}$		$T^{\mathfrak{b}}_{\ell}$		$T^{\mathfrak{b}}_{i}$		$T^{\mathfrak{b}}_{u}$	
	$\underline{\mathbb{R}}^{\mathfrak{b}}_{r}(\mathcal{Q})$	$\overline{\mathbb{R}}^{\mathfrak{b}}_{r}(\mathcal{Q})$	$\underline{\mathbb{R}}^{\mathfrak{b}}_{\ell}(\mathcal{Q})$	$\overline{\mathbb{R}}^{\mathfrak{b}}_{\ell}(\mathcal{Q})$	$\underline{\mathbb{R}}^{\mathfrak{b}}_{i}(\mathcal{Q})$	$\overline{\mathbb{R}}^{\mathfrak{b}}_{i}(\mathcal{Q})$	$\underline{\mathbb{R}}^{\mathfrak{b}}_{u}(\mathcal{Q})$	$\overline{\mathbb{R}}^{\mathfrak{b}}_{u}(\mathcal{Q})$
$\{q_1\}$	φ	$\{q_1\}$	φ	$\{q_1\}$	$\{q_1\}$	$\{q_1\}$	φ	$\{q_1\}$
$\{q_2\}$	φ	$\{q_2\}$	$\{q_2\}$	$\mathbb{U}$	$\{q_2\}$	$\{q_2\}$	φ	$\mathbb{U}$
$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	φ	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$	φ	$\{q_3\}$
$\{q_4\}$	$\{q_4\}$	$\mathbb{U}$	φ	$\{q_4\}$	$\{q_4\}$	$\{q_4\}$	φ	$\mathbb{U}$
$\{q_1, q_2\}$	φ	$\{q_1, q_2\}$	$\{q_1, q_2\}$	$\mathbb{U}$	$\{q_1, q_2\}$	$\{q_1, q_2\}$	φ	$\mathbb{U}$
$\{q_1, q_3\}$	$\{q_3\}$	$\{q_1, q_3\}$	φ	$\{q_1, q_3\}$	$\{q_1, q_3\}$	$\{q_1, q_3\}$	φ	$\{q_1, q_3\}$
$\{q_1, q_4\}$	$\{q_1, q_4\}$	$\{q_1, q_2, q_4\}$	φ	$\{q_1, q_4\}$	$\{q_1, q_4\}$	$\{q_1, q_4\}$	φ	$\{q_1\}$
$\{q_2, q_3\}$	$\{q_3\}$	$\{q_1, q_3\}$	$\{q_2, q_3\}$	$\mathbb{U}$	$\{q_2, q_3\}$	$\{q_2, q_3\}$	φ	$\{q_3\}$
$\{q_2, q_4\}$	$\{q_2, q_4\}$	$\{q_1, q_2, q_4\}$	$\{q_2, q_4\}$	$\mathbb{U}$	$\{q_2, q_4\}$	$\{q_2, q_4\}$	$\{q_2, q_4\}$	$\mathbb{U}$
$\{q_3, q_4\}$	$\{q_3, q_4\}$	$\mathbb{U}$	φ	$\{q_3, q_4\}$	$\{q_3, q_4\}$	$\{q_3, q_4\}$	φ	$\mathbb{U}$
$\{q_1, q_2, q_3\}$	$\{q_3\}$	$\{q_1, q_2, q_3\}$	$\{q_1, q_2, q_3\}$	$\mathbb{U}$	$\{q_1, q_2, q_3\}$	$\{q_1, q_2, q_3\}$	φ	$\mathbb{U}$
$\{q_1, q_2, q_4\}$	$\{q_1, q_2, q_4\}$	$\{q_1, q_2, q_4\}$	$\{q_1, q_2, q_4\}$	$\mathbb{U}$	$\{q_1, q_2, q_4\}$	$\{q_1, q_2, q_4\}$	$\{q_1, q_2, q_4\}$	$\mathbb{U}$
$\{q_2, q_3, q_4\}$	$\{q_2, q_3, q_4\}$	$\mathbb{U}$	$\{q_2, q_3, q_4\}$	$\mathbb{U}$	$\{q_1, q_3, q_4\}$	$\{q_1, q_3, q_4\}$	$\{q_2, q_3, q_4\}$	$\mathbb{U}$
$\{q_1, q_3, q_4\}$	$\{q_1, q_3, q_4\}$	$\mathbb{U}$	φ	$\{q_1, q_3, q_4\}$	$\{q_2, q_3, q_4\}$	$\{q_2, q_3, q_4\}$	φ	$\mathbb{U}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	φ	$\mathbb{U}$

Table 2: Comparison among different types of j-basic accuracies.

$\mathscr{P}(\mathbb{U})$	$\psi^{\mathfrak{b}}_{r}(\mathcal{Q})$	$\psi^{\mathfrak{b}}_{\ell}(\mathcal{Q})$	$\psi^{\mathfrak{b}}_{i}(\mathcal{Q})$	$\psi^{\mathfrak{b}}_{u}(\mathcal{Q})$
$\{q_1\}$	0	*0*	1	0
$\{q_2\}$	0	1/4	1	0
$\{q_3\}$	1	*0*	1	0
$\{q_4\}$	1/4	*0*	1	0
$\{q_1, q_2\}$	*0*	1/2	1	0
$\{q_1, q_3\}$	1/2	*0*	1	0
$\{q_1, q_4\}$	2/3	*0*	1	0
$\{q_2, q_3\}$	1/2	1/2	1	0
$\{q_2, q_4\}$	2/3	1/2	1	1/2
$\{q_3, q_4\}$	1/2	*0*	1	0
$\{q_1, q_2, q_3\}$	1/3	3/4	1	0
$\{q_1, q_2, q_4\}$	1	3/4	1	3/4
$\{q_1, q_3, q_4\}$	3/4	3/4	1	3/4
$\{q_2, q_3, q_4\}$	3/4	0	1	0
$\mathbb{U}$	1	1	1	1

Proposition 3. *Let* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *be a* j-**NS** *and* $\mathcal{Q}, \mathcal{S} \subseteq \mathbb{U}$. *Thus,*

(1) $\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}) \subseteq \mathcal{Q} \subseteq \overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q})$

(2) $\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathbb{U}) = \overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathbb{U}) = \mathbb{U}$ *and* $\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\varphi) = \overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\varphi) = \varphi$

(3) $\overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q} \cup \mathcal{S}) = \overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}) \cup \overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{S})$

(4) $\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q} \cap \mathcal{S}) = \underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}) \cap \underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{S})$

(5) If $\mathcal{Q} \subseteq \mathcal{S}$ then $\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}) \subseteq \underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{S})$

(6) If $\mathcal{Q} \subseteq \mathcal{S}$ then $\overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}) \subseteq \overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{S})$

(7) $\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q} \cup \mathcal{S}) \supseteq \underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}) \cup \underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{S})$

(8) $\overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q} \cap \mathcal{S}) \subseteq \overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}) \cap \overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{S})$

(9) $\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}) = [\overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}^{c})]^{c}$, *where* $\mathcal{Q}^{c}$ *represents a complement of* $\mathcal{Q}$

(10) $\overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}) = [\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q}^{c})]^{c}$

(11) $\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q})) = \underline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q})$

(12) $\overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\overline{\mathbb{R}}^{\mathfrak{b}}_{j}(\mathcal{Q})) = \overline{R}^{\mathfrak{b}}_{j}(\mathcal{Q})$

Proof. Though using properties of the j-basic interior and j-basic closure, we can prove these properties.

The preceding proposition is one of the distinctions between our approaches and those of other proposals such as [6, 15, 43, 48, 49].

The following results, which illustrate the relationships among the suggested approximations (j-basic approximations), are simple to prove using Proposition 1, so the proof is omitted. □

Proposition 4. *Suppose that* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *is a* j-**NS** *and* $\mathcal{Q} \subseteq \mathbb{U}$. *Then,*

(1) $\underline{\mathbb{R}}^{\mathfrak{b}}_{u}(\mathcal{Q}) \subseteq \underline{\mathbb{R}}^{\mathfrak{b}}_{r}(\mathcal{Q}) \subseteq \underline{\mathbb{R}}^{\mathfrak{b}}_{i}(\mathcal{Q})$

(2) $\underline{\mathbb{R}}^{\mathfrak{b}}_{u}(\mathcal{Q}) \subseteq \underline{\mathbb{R}}^{\mathfrak{b}}_{\ell}(\mathcal{Q}) \subseteq \underline{\mathbb{R}}^{\mathfrak{b}}_{i}(\mathcal{Q})$

(3) $\overline{\mathbb{R}}^{\mathfrak{b}}_{i}(\mathcal{Q}) \subseteq \overline{\mathbb{R}}^{\mathfrak{b}}_{r}(\mathcal{Q}) \subseteq \overline{\mathbb{R}}^{\mathfrak{b}}_{u}(\mathcal{Q})$

(4) $\overline{\mathbb{R}}^{\mathfrak{b}}_{i}(\mathcal{Q}) \subseteq \overline{\mathbb{R}}^{\mathfrak{b}}_{\ell}(\mathcal{Q}) \subseteq \overline{\mathbb{R}}^{\mathfrak{b}}_{u}(\mathcal{Q})$

Corollary 3. *If* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *is a* j-**NS** *and* $\mathcal{Q} \subseteq \mathbb{U}$. *Then,*

(1) $\mathcal{B}nd^{\mathfrak{b}}_{i}(\mathcal{Q}) \subseteq \mathcal{B}nd^{\mathfrak{b}}_{r}(\mathcal{Q}) \subseteq \mathcal{B}nd^{\mathfrak{b}}_{u}(\mathcal{Q})$

(2) $\mathcal{B}nd^{\mathfrak{b}}_{i}(\mathcal{Q}) \subseteq \mathcal{B}nd^{\mathfrak{b}}_{\ell}(\mathcal{Q}) \subseteq \mathcal{B}nd^{\mathfrak{b}}_{u}(\mathcal{Q})$

(3) $\psi^{\mathfrak{b}}_{u}(\mathcal{Q}) \leq \psi^{\mathfrak{b}}_{r}(\mathcal{Q}) \leq \psi^{\mathfrak{b}}_{i}(\mathcal{Q})$

(4) $\psi^{\mathfrak{b}}_{u}(\mathcal{Q}) \leq \psi^{\mathfrak{b}}_{\ell}(\mathcal{Q}) \leq \psi^{\mathfrak{b}}_{i}(\mathcal{Q})$

(5) *The subset* $\mathcal{Q}$ *is an* u-*basic exact set* $\Rightarrow$ $\mathcal{Q}$ *is* r-*basic exact* $\Rightarrow$ $\mathcal{Q}$ *is* i-*basic exact*

(6) *The subset* $\mathcal{Q}$ *is an* u-*basic exact set* $\Rightarrow$ $\mathcal{Q}$ *is* ℓ-*basic exact* $\Rightarrow$ $\mathcal{Q}$ *is* i-*basic exact*

Remark 4. Example 3 demonstrates that the opposition of the preceding results is not true in general.

The following results show comparisons between the proposed approximations (j-basic approximations) and the previous approximations (j-approximations [8]).

Theorem 3. *Let* $(\mathbb{U}, \mathbb{R}, \chi_{j})$ *be a* j-**NS** *and* $\mathcal{Q} \subseteq \mathbb{U}$. *If* $\mathbb{R}$ *is a reflexive relation on* $\mathbb{U}$, *then for each* $j \in \{r, \ell, i, u\}$:

(1) $\underline{\mathbb{R}}_{j}(\mathcal{Q}) \subseteq \underline{\mathbb{R}}^{b}_{j}(\mathcal{Q})$

(2) $\overline{\mathbb{R}}^{b}_{j}(\mathcal{Q}) \subseteq \overline{\mathbb{R}}_{j}(\mathcal{Q})$

Proof. The first point will be proved, and the others in a similar way.

Suppose that $q \in \underline{\mathbb{R}}_{j}(\mathcal{Q})$. Then, using Definition 4, $\mathcal{B} \in T_{j}$ such that $q \in \mathcal{B} \subseteq \mathcal{Q}$. However, from Proposition 2, $T_{j} \subseteq T^{b}_{j}$. Hence, $\mathcal{B} \in T^{b}_{j}$ such that $q \in \mathcal{B} \subseteq \mathcal{Q}$ and this means that $q \in \underline{\mathbb{R}}^{b}_{j}(\mathcal{Q})$. Consequently, $\underline{\mathbb{R}}_{j}(\mathcal{Q}) \subseteq \underline{\mathbb{R}}^{b}_{j}(\mathcal{Q})$. □

Corollary 4. *If $(\mathbb{U}, \mathbb{R}, \chi_{j})$ is a j-***NS** *and $\mathcal{Q} \subseteq \mathbb{U}$. Then,*

(1) $\mathcal{B}nd^{b}_{j}(\mathcal{Q}) \subseteq \mathcal{B}nd_{j}(\mathcal{Q})$

(2) $\psi_{j}(\mathcal{Q}) \leq \psi^{b}_{j}(\mathcal{Q})$

(3) If $\mathcal{Q}$ is an j-exact set, then it is j-basic exact

Remark 5. The contrary of previous results is not true as shown in Example 4.

Example 4. Consider Example 2, and then we will compare the j-basic approximations with the j-approximations in the case of $j = r$ and the others similarly.

Firstly, we evaluate the topologies T^{b}_{j} and T_{j} (resp. the class of all closed sets K^{b}_{j} and K_{j}) in the case of $j = r$ as follows:

$$\begin{aligned}
T_{r} &= \{\mathbb{U}, \varphi, \{q_3\}, \{q_4\}, \{q_3, q_4\}, \{q_1, q_2, q_3\}\},\\
K_{r} &= \{\mathbb{U}, \varphi, \{q_4\}, \{q_1, q_2\}, \{q_1 q_2, q_4\}, \{q_1, q_2, q_3\}\},\\
T^{b}_{r} &= \{\mathbb{U}, \varphi, \{q_1\}, \{q_3\}, \{q_4\}, \{q_1, q_3\}, \{q_1, q_4\}, \{q_3, q_4\},\\
&\quad \{q_1, q_2, q_3\}, \{q_1, q_3, q_4\}\},\\
K^{b}_{r} &= \{\mathbb{U}, \varphi, \{q_2\}, \{q_4\}, \{q_1, q_2\}, \{q_2, q_3\}, \{q_2, q_4\}, \{q_1, q_2, q_3\},\\
&\quad \{q_1, q_2, q_4\}\{q_2, q_{3,} q_4\}\}.
\end{aligned} \tag{10}$$

Hence, we obtain Table 3 which exemplifies a comparison among the r-accuracy of r-approximations and r-basic accuracy of r-basic approximations of all subsets of $\mathbb{U}$.

Remark 6. According to Table 3 of Example 4, we notice that r-basic approximations are more accurate than r-approximations of sets since $\psi_{r}(\mathcal{Q}) \leq \psi^{b}_{r}(\mathcal{Q})$. Therefore, it may say that the recommended approximations "j-basic approximations" represent golden tools in eliminating the ambiguity of sets. For example, in Table 3, the subset $\mathcal{Q} = \{q_1, q_2\}$ and its r-approximations are $\underline{\mathbb{R}}_{r}(\mathcal{Q}) = \varphi$ and $\overline{\mathbb{R}}_{r}(\mathcal{Q}) = \{q_1, q_2\}$ which implies $\mathcal{B}nd_{r}(\mathcal{Q}) = \{q_1, q_2\}$ and $\psi_{r}(\mathcal{Q}) = 0$ and this means that $\mathcal{Q}$ is a r-rough set. Moreover, the r-positive region of $\mathcal{Q}$ is $\mathcal{P}os_{r}(\mathcal{Q}) = \varphi$ although $\mathcal{Q}$ consists of two elements which is a contradiction to the knowledge of Example 4. On the other hand, we find r-basic approximations of $\mathcal{Q}$ are $\underline{\mathbb{R}}^{b}_{r}(\mathcal{Q}) = \{q_1\}$ and $\overline{\mathbb{R}}^{b}_{r}(\mathcal{Q}) = \{q_1, q_2\}$, that is, the r-basic positive region of $\mathcal{Q}$ is $\mathcal{P}os^{b}_{r}(\mathcal{Q}) = \{q_1\}$ and $\psi^{b}_{r}(\mathcal{Q}) = 1/2$.

Table 3: Comparison between r-accuracies and r-basic accuracies.

$\mathcal{P}(\mathbb{U})$	$\psi_{r}(\mathcal{Q})$	$\psi^{b}_{r}(\mathcal{Q})$
$\{q_1\}$	0	1/2
$\{q_2\}$	0	0
$\{q_3\}$	1/3	1/2
$\{q_4\}$	1	1
$\{q_1, q_2\}$	0	1/2
$\{q_1, q_3\}$	1/3	2/3
$\{q_1, q_4\}$	1/3	2/3
$\{q_2, q_3\}$	1/3	1/2
$\{q_2, q_4\}$	1/4	1/2
$\{q_3, q_4\}$	1/2	2/3
$\{q_1, q_2, q_3\}$	1/3	1
$\{q_1, q_2, q_4\}$	1/2	2/3
$\{q_1, q_3, q_4\}$	1/2	3/4
$\{q_2, q_3, q_4\}$	1/2	2/3
$\mathbb{U}$	1	1

5. Economic Application

Since the 1950s, most Western countries' official policy goal has been economic development. In general, growth rates have been slightly slower since the 1970s than in the two preceding years. Furthermore, most countries' economic growth has yet to recover from the 2008 recession. A growing number of economists and commentators have challenged the (still primary) expectation that GDP growth will continue to rise at a regular rate of 2.5 percent in the coming period (see [21]). Mainstream economists propose a new standard yearly growth ratio of 1% or less, owing to a lower predictable rate of industrial progress, and thus a lower rate of productivity growth. Because of the decline in oil production, other resource constraints, and the negative consequences of environmental degradation and climate change, zero growth or even disastrous trends are considered abnormal. Therefore, many countries (for example, USA, England, and Middle East) tended to pay attention to the economy and study the growth rate to contribute to overcoming this crisis.

In the current section, we provide an applied example of how approximations can be used as tools with a coefficient of precision for making accurate decisions in discussing the growth rate of countries. For this, we use a data table with a decision attribute [21], and then apply our own and other methods to show which methods are accurate in making the right decision. National production can be designed using three methods, as shown in the example below. Because this system depends on a reflexive relation, Pawlak's rough sets are inapplicable. As a result, we implement the proposed methods, as well as the previous methods in this country's decision-making system, and then we compare these decision-making methods.

Firstly, we recall some other methodologies for approximating rough sets that given in [45, 50].

5.1. Yao's Method

Definition 11 (see [50]). Let $\mathbb{R}$ be a binary relation on $\mathbb{U}$. Then, **Yao**-lower and **Yao**-upper approximations of $\mathcal{Q} \subseteq \mathbb{U}$ are given, respectively, as follows:

$$\begin{aligned} \mathcal{L}(\mathcal{Q}) &= \{q \in \mathbb{U}\colon \Omega_r(q) \subseteq \mathcal{Q}\}, \\ \mathcal{U}(\mathcal{Q}) &= \{q \in \mathbb{U}\colon \Omega_r(q) \cap \mathcal{Q} \neq \varphi\}. \end{aligned} \tag{11}$$

Therefore, **Yao**-accuracy of approximations of $\mathcal{Q} \subseteq \mathbb{U}$ is given as follows:

$$\delta(\mathcal{Q}) = \frac{|\mathcal{L}(\mathcal{Q})|}{|\mathcal{U}(\mathcal{Q})|}, \tag{12}$$

such that $\mathcal{U}(\mathcal{Q}) \neq \varphi$

5.2. Dai et al.'s Method

Definition 12 (see [45]). Let $\mathbb{R}$ be a binary relation on $\mathbb{U}$. Then, for each $q \in \mathbb{U}$, we define its maximal-neighborhood as follows:

$$\Omega_{\sqcup}(q) = \begin{cases} \bigcup_{q \in \Omega_r(e)} \Omega_r(e), & if\, \exists q \text{ such that } q \in \Omega_r(e), \\ \varphi, & \text{Otherwise.} \end{cases} \tag{13}$$

Definition 13 (see [45]). If $\mathbb{R}$ is a binary relation on $\mathbb{U}$. Then, the maximal-lower and maximal-upper approximations of $\mathcal{Q} \subseteq \mathbb{U}$ are provided, respectively, as follows:

$$\begin{aligned} \mathcal{L}_{\sqcup}(\mathcal{Q}) &= \{q \in \mathbb{U}\colon \Omega_{\sqcup}(q) \subseteq \mathcal{Q}\}, \\ \mathcal{U}_{\sqcup}(\mathcal{Q}) &= \{q \in \mathbb{U}\colon \Omega_{\sqcup}(q) \cap \mathcal{Q} \neq \varphi\}. \end{aligned} \tag{14}$$

Therefore, the maximal-accuracy of approximations of $\mathcal{Q} \subseteq \mathbb{U}$ is defined as follows:

$$\mu(\mathcal{Q}) = \frac{|\mathcal{L}_{\sqcup}(\mathcal{Q})|}{|\mathcal{U}_{\sqcup}(\mathcal{Q})|}, \tag{15}$$

such that $\mathcal{U}_{\sqcup}(\mathcal{Q}) \neq \varphi$.

Now, we present an economic application which was constructed from [21].

Example 5 (see [21]). We consider $\mathbb{U} = \{q_1, q_2, q_3, q_4, q_5\}$ is a world of five countries, and $\mathbb{A} = \{a_1, a_2, a_3\}$ the set of attributes which measure the national product in these countries, where a_1 means a product method, a_2 means a spending method, and a_3 means an income method and decision attribute $\mathbb{D}$ ={growth and not growth}.

Now, assume that the assessment sets of the attributes are specified as follows:$\mathcal{V}_{a_1} = \{\mathcal{F}, \mathcal{T}, \mathcal{V}\}$ where $\mathcal{F}$, $\mathcal{T}$, and $\mathcal{V}$ denote to finishing product style, taxes, and value-added style, respectively.$\mathcal{V}_{a_2} = \{\mathcal{C}, \mathcal{I}, \mathcal{G}\}$ where $\mathcal{C}$, $\mathcal{I}$, and $\mathcal{G}$ denote to consumption, investment, and government, respectively.$\mathcal{V}_{a_3} = \{\mathcal{S}, \mathcal{P}, \mathcal{R}\}$ where $\mathcal{S}$, $\mathcal{P}$, and $\mathcal{R}$ denote to salaries, profits, and rents, respectively.

The next procedure is proposing a relation; it is given according to the requirements of the standpoint of system's experts. In this example, we propose the following relation: $q_m \mathbb{R}_{a_k} q_n \Longleftrightarrow \mathcal{V}_{a_m}(q_m) \subseteq \mathcal{V}_{a_n}(q_n)$, for each $m, n \in \{1, 2, 3, 4, 5\}$ and $k \in \{1, 2, 3\}$.

It should be noted that the stated relation can be replaced based on the opinions of the system's specialists.

Therefore, from Table 4, we obtain the following point:

(i) For the attribute a_1, we obtain the reflexive relation as follows:

$$\begin{aligned} \mathbb{R}_{a_1} = \{&(q_1, q_1), (q_1, q_2), (q_2, q_2), (q_3, q_2), \\ &(q_3, q_3), (q_3, q_4), (q_4, q_4), \\ &(q_5, q_4), (q_5, q_5)\}. \end{aligned} \tag{16}$$

Thus, $q_1\mathbb{R}_{a_1} = \{q_1, q_2\}$, $q_2\mathbb{R}_{a_1} = \{q_2\}$, $q_3\mathbb{R}_{a_1} = \{q_2, q_3, q_4\}$, $q_4\mathbb{R}_{a_1} = \{q_4\}$, and $q_5\mathbb{R}_{a_1} = \{q_4, q_5\}$. Similarly, $q_1\mathbb{R}_{a_2} = q_2\mathbb{R}_{a_2} = \{q_1, q_2, q_3, q_4\}$, $q_3\mathbb{R}_{a_2} = \{q_3\}$, $q_4\mathbb{R}_{a_2} = \{q_4\}$, $q_5\mathbb{R}_{a_2} = \{q_4, q_5\}$, $q_1\mathbb{R}_{a_3} = \{q_1, q_2, q_4, q_5\}$, $q_2\mathbb{R}_{a_3} = \{q_2, q_4\}$, $q_3\mathbb{R}_{a_3} = \{q_3, q_4, q_5\}$, $q_4\mathbb{R}_{a_3} = \{q_4\}$, and $q_5\mathbb{R}_{a_3} = \{q_4, q_5\}$.

It is clear that the suggested relation is reflexive; this means that the Pawlak approximations space fails to describe this system.

Therefore, from all previous relations, we construct the following j-neighborhoods, for each $j \in \{r, \ell, i\}$, to describe the set of all condition attributes of Table 4 as follows: $\Omega_r(q_m) = \cup_k q_m \mathbb{R}_{a_k}$, for each $k \in \{1, 2, 3\}$ and $m \in \{1, 2, 3, 4, 5\}$.

(i) j-neighborhoods of all $q_m \in \mathbb{U}$ as follows:

$\Omega_r(q_1) = \mathbb{U}, \Omega_\ell(q_1) = \{q_1, q_2\}, \Omega_i(q_1) = \{q_1, q_2\}$, $\Omega_r(q_2) = \{q_1, q_2, q_3, q_4\}$, $(q_2) = \{q_1, q_2, q_3\}$, $\Omega_i(q_2) = \{q_1, q_2, q_3\}$, $\Omega_r(q_3) = \{q_2, q_3, q_4, q_5\}$, $\Omega_\ell(q_3) = \{q_1, q_2, q_3\}$, $\Omega_i(q_3) = \{q_2, q_3\}$, $\Omega_r(q_4) = \{q_4\}, \Omega_\ell(q_4) = \mathbb{U}, \Omega_i(q_4) = \{q_4\}, \Omega_r(q_5) = \{q_4, q_5\}$, $\Omega_\ell(q_5) = \{q_3, q_5\}$, and $\Omega_i(q_5) = \{q_5\}$

(ii) j-basic neighborhoods of all $q_m \in \mathbb{U}$ as follows: $\Omega^{\mathfrak{b}}_r(q_1) = \mathbb{U}, \Omega^{\mathfrak{b}}_\ell(q_1) = \{q_1\}$, $\Omega^{\mathfrak{b}}_i(q_1) = \{q_1\}$, $\Omega^{\mathfrak{b}}_r(q_2) = \{q_2, q_4\}, \Omega^{\mathfrak{b}}_\ell(q_2) = \{q_1, q_2, q_3\}$, $\Omega^{\mathfrak{b}}_i(q_2) = \{q_2\}$, $\Omega^{\mathfrak{b}}_r(q_3) = \{q_3, q_4, q_5\}$, $\Omega^{\mathfrak{b}}_\ell(q_3) = \{q_1, q_2, q_3\}$, $\Omega^{\mathfrak{b}}_i(q_3) = \{q_3\}$, $\Omega^{\mathfrak{b}}_r(q_4) = \{q_4\}$, $\Omega^{\mathfrak{b}}_\ell(q_4) = \mathbb{U}$, $\Omega^{\mathfrak{b}}_i(q_4) = \{q_4\}$, $\Omega^{\mathfrak{b}}_r(q_5) = \{q_4, q_5\}$, $\Omega^{\mathfrak{b}}_\ell(q_4) = \{q_5\}$, and $\Omega^{\mathfrak{b}}_i(q_4) = \{q_5\}$

So, the j-topologies (resp. j-closed sets) and j-basic topologies (resp. j-basic closed sets), for each $j \in \{r, \ell, i\}$ generated by these neighborhoods are given, respectively, as follows:

Table 4: Information system [21].

	a_1	a_2	a_3	$\mathbb{D}$
q_1	$\{\mathcal{F}\}$	$\{\mathcal{G}\}$	$\{\mathcal{S}\}$	Growth
q_2	$\{\mathcal{F},\mathcal{T}\}$	$\{\mathcal{G}\}$	$\{\mathcal{P},\mathcal{S}\}$	Growth
q_3	$\{\mathcal{T}\}$	$\{\mathcal{I},\mathcal{G}\}$	$\{\mathcal{R}\}$	Not growth
q_4	$\{\mathcal{T},\mathcal{V}\}$	$\{\mathcal{G},\mathcal{C}\}$	$\{\mathcal{R},\mathcal{P},\mathcal{S}\}$	Growth
q_5	$\{\mathcal{V}\}$	$\{\mathcal{C}\}$	$\{\mathcal{R},\mathcal{S}\}$	Not growth

Table 5: Comparison between the suggested approaches and previous methods, namely, Monsef et al. [8], Abu-Gdairi et al. [17], Dai et al. [45], and Yao [50].

$\mathcal{P}(\mathbb{U})$	Current methods			Abd El-Monsef et al.			Yao method	Dai et al. method
	$\psi^{\mathfrak{b}}_{r}(\mathcal{Q})$	$\psi^{\mathfrak{b}}_{\ell}(\mathcal{Q})$	$\psi^{\mathfrak{b}}_{i}(\mathcal{Q})$	$\psi_{r}(\mathcal{Q})$	$\psi_{\ell}(\mathcal{Q})$	$\psi_{i}(\mathcal{Q})$	$\delta(\mathcal{Q})$	$\mu(\mathcal{Q})$
$\{q_1\}$	0	1/5	1	0	0	0	0	0
$\{q_2\}$	0	0	1	0	0	0	0	0
$\{q_3\}$	0	0	1	0	0	0	0	0
$\{q_4\}$	1/5	0	1	1/5	0	1	1/5	0
$\{q_5\}$	0	1/5	1	0	0	1	0	0
$\{q_1, q_2\}$	0	1/4	1	0	0	0	0	0
$\{q_1, q_3\}$	0	1/4	1	0	0	0	0	0
$\{q_1, q_4\}$	1/5	1/4	1	1/5	0	1/4	1/5	0
$\{q_1, q_5\}$	0	1/5	1	0	0	1/4	0	0
$\{q_2, q_3\}$	0	0	1	0	0	0	0	0
$\{q_2, q_4\}$	2/5	0	1	1/5	0	1/4	1/5	0
$\{q_2, q_5\}$	0	1/4	1	0	0	1/4	0	0
$\{q_3, q_4\}$	1/5	0	1	1/5	0	1/4	1/5	0
$\{q_3, q_5\}$	0	1/4	1	0	0	1/4	0	0
$\{q_4, q_5\}$	2/5	1/2	1	2/5	0	1	2/5	0
$\{q_1, q_2, q_3\}$	0	3/4	1	0	3/5	1	0	0
$\{q_1, q_2, q_4\}$	2/5	1/4	1	1/5	0	1/4	1/5	0
$\{q_1, q_2, q_5\}$	0	2/5	1	0	0	1/4	0	0
$\{q_1, q_3, q_4\}$	1/5	1/4	1	1/5	0	1/4	1/5	0
$\{q_1, q_3, q_5\}$	0	2/5	1	0	0	1/4	0	0
$\{q_1, q_4, q_5\}$	2/5	2/5	1	2/5	0	2/5	2/5	0
$\{q_2, q_3, q_4\}$	2/5	0	1	1/5	0	1/4	1/5	0
$\{q_2, q_3, q_5\}$	0	1/4	1	0	0	1/4	0	0
$\{q_2, q_4, q_5\}$	3/5	0	1	2/5	0	2/5	2/5	0
$\{q_3, q_4, q_5\}$	3/5	1/4	1	2/5	0	2/5	2/5	0
$\{q_1, q_2, q_3, q_4\}$	2/5	3/4	1	1/5	3/5	1	2/5	0
$\{q_1, q_2, q_3, q_5\}$	0	4/5	1	0	4/5	1	0	0
$\{q_1, q_2, q_4, q_5\}$	2/5	2/5	1	2/5	0	2/5	2/5	0
$\{q_1, q_3, q_4, q_5\}$	3/5	2/5	1	2/5	0	2/5	2/5	0
$\{q_2, q_3, q_4, q_5\}$	4/5	1/4	1	2/5	0	2/5	3/5	0
$\mathbb{U}$	1	1	1	1	1	1	1	1

$$
\begin{aligned}
&T_{r} = K_{\ell} = \{\mathbb{U}, \varphi, \{q_4\}, \{q_4, q_5\}\},\\
&T_{\ell} = K_{r} = \{\mathbb{U}, \varphi, \{q_1, q_2, q_3\}, \{q_1, q_2, q_3, q_5\}\},\\
&T_{i} = K_{i} = \{\mathbb{U}, \varphi, \{q_4\}, \{q_5\}, \{q_4, q_5\}, \{q_1, q_2, q_3\}, \{q_1, q_2, q_3, q_4\}, \{q_1, q_2, q_3, q_5\}\},\\
&T^{\mathfrak{b}}_{r} = \{\mathbb{U}, \varphi, \{q_4\}, \{q_2, q_4\}, \{q_4, q_5\}, \{q_2, q_4, q_5\}, \{q_3, q_4, q_5\}, \{q_2, q_3, q_4, q_5\}\},\\
&K^{\mathfrak{b}}_{r} = \{\mathbb{U}, \varphi, \{q_1\}, \{q_1, q_2\}, \{q_1, q_3\}, \{q_1, q_2, q_3\}, \{q_1, q_3, q_5\}, \{q_1, q_2, q_3, q_5\}\},\\
&T^{\mathfrak{b}}_{\ell} = \{\mathbb{U}, \varphi, \{q_1\}, \{q_5\}, \{q_1, q_5\}, \{q_1, q_2, q_3\}, \{q_1, q_2, q_3, q_5\}\},\\
&K^{\mathfrak{b}}_{\ell} = \{\mathbb{U}, \varphi, \{q_4\}, \{q_4, q_5\}, \{q_2, q_3, q_4\}, \{q_1, q_2, q_3, q_4\}, \{q_2, q_3, q_4, q_5\}\},\\
&T^{\mathfrak{b}}_{i} = K^{\mathfrak{b}}_{i} = \mathcal{P}(\mathbb{U}).
\end{aligned}
\tag{17}
$$

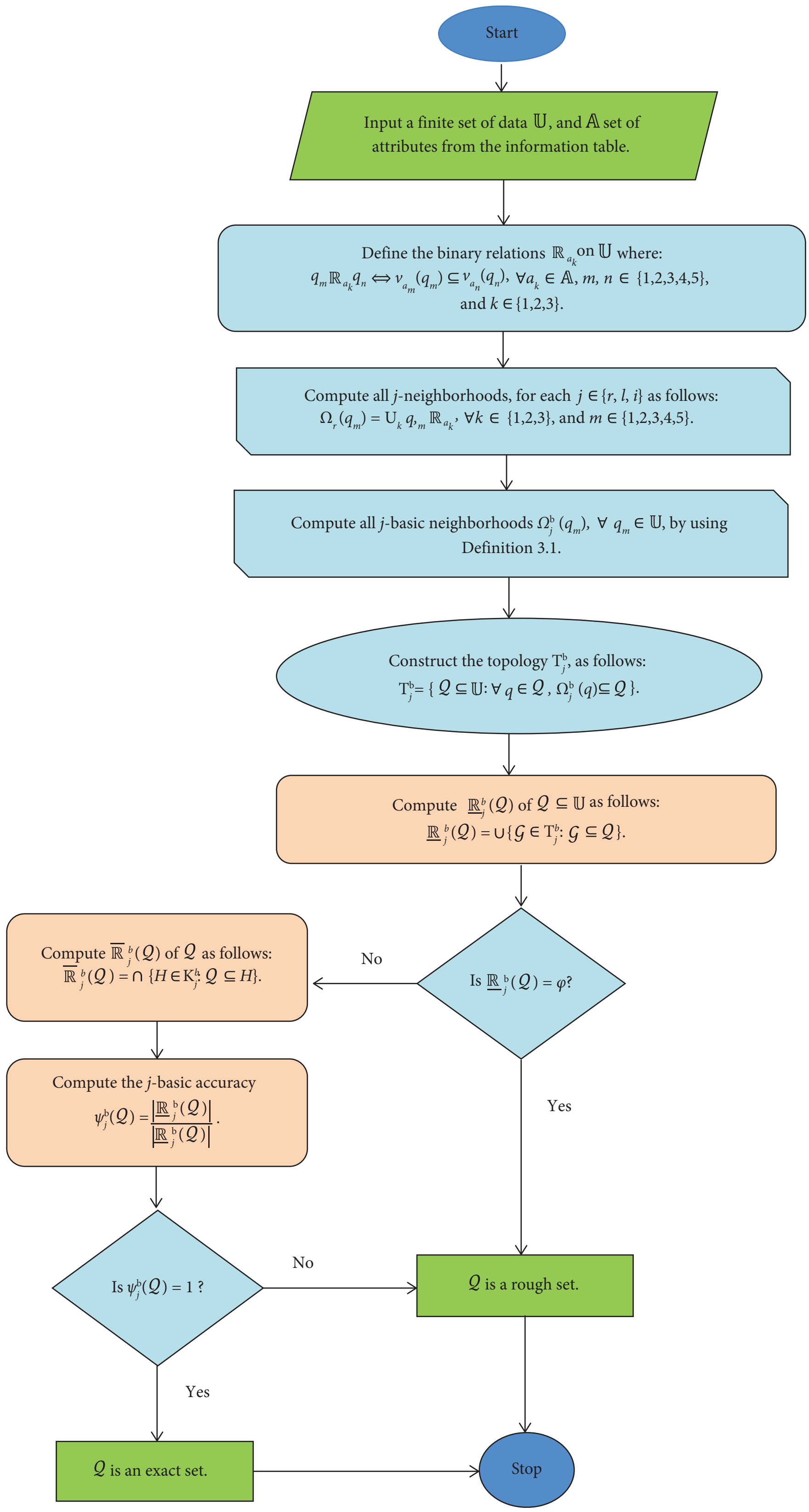

Figure 2: Flowchart for decision-making using the j-basic accuracy measure.

Input: initiate an information table generating from the given data such that the first column contains the set of objects $\mathbb{U}$ and the set of attributes $\mathbb{A}$ as a first row.
Output: An accurate decision for exact and rough sets.
(1) Define the binary relations $q_m \mathbb{R}_{a_k} q_n \Longleftrightarrow \mathcal{V}_{a_m}(q_m) \subseteq \mathcal{V}_{a_n}(q_n)$, for each $a_k \in \mathbb{A}$, $m, n \in \{1,2,3,4,5\}$, and $k \in \{1,2,3\}$.
(2) **for** each $j \in \{r, \ell, i\}$, **do**
(3) Compute all j-neighborhoods, as follows:
$\Omega_r(q_m) = \cup_k q_m \mathbb{R}_{a_k}$, for each $k \in \{1,2,3\}$ and $m \in \{1,2,3,4,5\}$.
(4) Compute all j-basic neighborhoods $\Omega_j^{\mathfrak{b}}(q_m)$, using **Definition 7**.
(5) Construct the topology $T_j^{\mathfrak{b}}$, using **Step 4**, as follows:
$T_j^{\mathfrak{b}} = \{Q \subseteq \mathbb{U} \colon \forall q \in Q, \Omega_j^{\mathfrak{b}}(q) \subseteq Q\}$.
end
(6) **for** each $Q \subseteq \mathbb{U}$, **do**
(7) Compute the j-basic lower approximation $\underline{\mathbb{R}}_j^{\mathfrak{b}}(Q) = \cup\{\mathcal{G} \in T_j^{\mathfrak{b}} \colon \mathcal{G} \subseteq Q\}$.
(8) **if** $\underline{\mathbb{R}}^{\mathfrak{b}}(Q) = \varphi$, **then**
(9) **return** Q is a rough set.
(10) **else**
(11) Compute the j-basic upper approximation $\overline{\mathbb{R}}_j^{\mathfrak{b}}(Q) = \cap\{\mathcal{H} \in K_j^{\mathfrak{b}} \colon Q \subseteq \mathcal{H}\}$.
(12) Compute the j-basic accuracy $\psi_j^{\mathfrak{b}}(Q) = |\underline{\mathbb{R}}_j^{\mathfrak{b}}(Q)|/|\overline{\mathbb{R}}_j^{\mathfrak{b}}(Q)|$.
(13) **if** $\psi_j^{\mathfrak{b}}(Q) = 1$, **then**
(14) **return** Q is an exact set.
(15) **else**
(16) **return** Q is a rough set.
(17) **end**
(18) **end**
(19) **end**

ALGORITHM 1: An algorithm for using the j-approximations in decision-making problems.

Firstly, we present comparisons amongst the suggested approaches and the previous methods in ([8, 17, 45, 50]) as shown in Table 5.

Now, we illustrate the importance of proposed techniques in decision-making.

Firstly, Table 4 represents a decision table with decision attributes that is given by expert. So, we apply the proposed techniques for two subsets $\mathcal{A} = \{q_1, q_2, q_4\}$ and $\mathcal{B} = \{q_3, q_5\}$ that characterize the set of growth and not growth countries, respectively. Consequently, we compute the approximations using Monsef et al. [8], Abu-Gdairi et al. [17], Dai et al. [45], and Yao [50] and the suggested methods, using Table 5, as follows:

5.3. Current Methods. $\underline{\mathbb{R}}^{\mathfrak{b}}(\mathcal{A}) = \overline{\mathbb{R}}_i^{\mathfrak{b}}(\mathcal{A}) = \mathcal{A}$, and hence $\psi_i^{\mathfrak{b}}(\mathcal{A}) = 1$. Accordingly, $\mathcal{A}$ is an i-basic exact set, and thus the countries q_1, q_2, and q_4 represent a growth countries, and then this result conforms to Table 4. Similarly, the other set $\mathcal{B} = \{q_3, q_5\}$.

5.4. Yao Method. $\mathcal{L}(\mathcal{A}) = \{q_4\}$ and $\mathcal{U}(\mathcal{A}) = \mathbb{U}$, and hence $\delta_i^{\mathfrak{b}}(\mathcal{A}) = 1/5$. Accordingly, $\mathcal{A}$ is an Yao-rough set, and thus the countries q_1 and q_2 represent not growth countries, and then this result contradicts to Table 4. Similarly, the other set $\mathcal{B} = \{q_3, q_5\}$.

5.5. Abd El-Monsef et al. Method. $\underline{\mathbb{R}}_i(\mathcal{A}) = \{q_4\}$ and $\overline{\mathbb{R}}_i(\mathcal{A}) = \{q_1, q_2, q_3, q_4\}$, and hence $\psi_i(\mathcal{A}) = 1/4$. Accordingly, $\mathcal{A}$ is an i-rough set, and thus the countries q_1 and q_2 represent not growth countries, and then this result contradicts to Table 4. Similarly, the other set $\mathcal{B} = \{q_3, q_5\}$.

5.6. Dai et al. Method. $\mathcal{L}_{\sqcup}(\mathcal{A}) = \varphi$ and $\mathcal{U}_{\sqcup}(\mathcal{A}) = \mathbb{U}$, and hence $\mu(\mathcal{A}) = 0$. Accordingly, $\mathcal{A}$ is a totally maximal-rough set, and thus the countries q_1, q_2, and q_4 represent not growth countries, and then this result contradicts to Table 4. Similarly, the other set $\mathcal{B} = \{q_3, q_5\}$.

5.7. Concluding Notes. According to the previous comparisons, we conclude the following points:

(1) Table 4 represents a decision table with decision attributes given by the expert. Therefore, the countries q_1, q_2, and q_4 (resp. q_3 and q_5) are surely growth (resp. not growth) countries.
(2) According to our approaches, the countries q_1, q_2, and q_4 (resp. q_3 and q_5) are surely growth (resp. not growth) countries which conform to Table 4. So, we can surely select the countries which are growth countries or not. Accordingly, the suggested approximations may be useful in making decisions and help in extracting knowledge in the information system of other real-life problems.
(3) According to the previous methods ([8, 17, 45, 50]), the countries q_1 and q_2 represent not growth countries which is a contradiction to Table 4.

Therefore, we cannot decide whether the country is a growth country or not.

(4) As a result, we can conclude that the proposed approximations are more precise than the other approaches and are very useful in determining data ambiguity and assisting in decision-making in real-life problems such as medical diagnosis, which requires precise decisions.

(5) This method also allows us to deal with various real problems under any arbitrary relation, whereas Pawlak method requires an equivalent relation to simulate the problems under consideration. Since the best approximations and accuracy measures are produced, the current approach is more suited to dealing with big samples. Thus, improvement of these operators and increasing their values of accuracy leads to an accurate prediction.

6. An Algorithm and Flowchart

This section provides an algorithm and flowchart (see Figure 2) for decision-making problems. The proposed algorithm is tested on fictitious data and compared to existing methods. This technique is a simple tool that can be used in MATLAB. (See Algorithm 1).

The following figure (Figure 2) represents a simple flowchart of the accuracy measures induced from Algorithm 1.

7. Conclusion and Discussion

The appropriated mathematical structural for any collection linked by relations is the general topology. So, the study of neighborhood system that generated via relations and their relationships with topology represents very interesting topic of rough set approaches and many applications such as machine learning (ML) and decision-making problems. In recent years, ML has increasingly entered the financial sector, having a substantial impact on redefining the landscape of quantitative finance. Many financial institutions, including banks, insurance companies, and even regulators, are already putting this technology to use to solve complex financial decision problems, analyze large financial datasets, price complex financial instruments, manage operational risk, and forecast future price paths. In the current manuscript, we provided new mathematical approaches to confirm between the experimental data and its mathematical analysis. The mathematical study depends on the classification of data.

Firstly, we introduced and studied new extensions to the notion "basic-neighborhood," and hence we developed a theory of rough sets based on four different basic-neighborhood systems. Furthermore, using Theorem 1 in [6], a new method to produce four different topologies made via these neighborhoods is suggested. Comparisons of these topologies with the previous one were conducted. Consequently, we used these new topologies to generate and investigate new generalizations to Pawlak rough sets. We have compared the proposed methods to the previous ones (Monsef et al. [8], Abu-Gdairi et al. [17], Dai et al. [45], and Yao [50]) and found that these methods are more accurate than others. Many examples and comparisons to clarify the significance of the suggested techniques were superimposed.

The merits of these methods are as follows:

(1) It enlarges the space of practical problems that we can deal with them

(2) It keeps the main properties of original approximation spaces under fewer conditions, which are evaporated in some previous approaches

(3) Improve the approximation operators and accuracy measures compared to the previous approaches, and hence we obtain a more accurate decision for the problems in which these cases are the appropriate frame to describe them; for instance, infectious diseases such as COVID-19, ML applications, and decision-making problems.

Finally, we used the proposed approaches to decide on an economic decision table. In fact, we used a decision table (an information system with decision attributes) of five countries and a set of attributes that measure the national product in these countries. The decision attribute in that table is to decide the growth of the country. Consequently, we conclude that the suggested approximations are more precise than other approaches and are very useful in determining data ambiguity and assisting in decision-making in real-life problems such as medical diagnosis and ML applications of finance which requires precise decisions. Therefore, we believe that using these techniques will be easier in application fields and beneficial for applying many topological concepts in future studies. Furthermore, an algorithm with a flowchart was provided and tested on fictitious data in order to compare it to existing methods, and hence it can be a simple technique to use in MATLAB.

In the forthcoming works, we will study the following points:

(i) Using the suggested methods in more applications such as in medical fields [14, 19] and economic applications [1–5]

(ii) Emphasis the basic-approximations concept in some other frames such as soft sets [19], the soft rough sets [23], and fuzzy rough sets [15]

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] K. Sirlantzis and C. Siriopoulos, "Deterministic chaos in stock markets: empirical results from monthly returns," *Neural Network World*, vol. 3, pp. 855–864, 1993.

[2] C. Siriopoulos, "Investigating the behaviour of mature and emerging capital markets," *Indian Journal of Quantitative Economics*, vol. 11, pp. 76–98, 1996.

[3] C. Siriopoulos, A. Samitas, V. Dimitropoulos, A. Boura, and D. M. AlBlooshi, "Health economics of air pollution," in *Pollution Assessment for Sustainable Practices in Applied Sciences and Engineering*, A. Mohamed, V. G. Rodrigues, and K. E. Paleologos, Eds., Elsevier, Amsterdam, Netherlands, 2022.

[4] C. Siriopoulos and M. Skaperda, "Investing in mutual funds: are you paying for performance or for the ties of the manager?" *Bulletin of Applied Economics*, vol. 7, pp. 153–164, 2020.

[5] M. A. Bal, L. M. Batrancea, L. Gaban, M.-I. Rus, and H. Tulai, "Fractality of borsa istanbul during the COVID-19 pandemic," *Mathematics*, vol. 10, no. 14, 2022.

[6] N. G. Pavlidis, D. K. Tasoulis, V. P. Plagianakos, C. Siriopoulos, and M. N. Vrahatis, "Computational intelligence methods for financial forecasting," *Lecture Series on Computer and Computational Sciences*, vol. 1, pp. 1–3, 2005.

[7] A. Sfetsos and C. Siriopoulos, "Time series forecasting with a hybrid clustering scheme and pattern recognition," *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, vol. 34, no. 3, pp. 399–405, 2004.

[8] M. A. E. Monsef, O. A. Embaby, and M. E. Bably, "Comparison between rough set approximations based on different topologies," *International Journal of Granular Computing, Rough Sets and Intelligent Systems*, vol. 3, no. 4, pp. 292–305, 2014.

[9] M. E. Abd El-Monsef, M. A. El-Gayar, and R. M. Aqeel, "A comparison of three types of rough fuzzy sets based on two universal sets," *International Journal of Machine Learning and Cybernetics*, vol. 8, no. 1, pp. 343–353, 2017.

[10] E. A. Abo-Tabl and M. K. El-Bably, "Rough topological structure based on reflexivity with some applications," *AIMS Mathematics*, vol. 7, no. 6, pp. 9911–9999, 2022.

[11] A. A. Allam, M. Y. Bakeir, and E. A. Abo-Tabl, "New approach for basic rough set concepts," in *Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing, Lecture Notes in Artificial Intelligence 3641*, D. Slezak, G. Wang, M. Szczuka, I. Dntsch, and Y. Yao, Eds., pp. 64–73, Springer, Berlin, Germany, 2005.

[12] E. A. Abo-Tabl, "On topological properties of generalized rough multisets," *Annals of Fuzzy Mathematics and Informatics*, vol. 19, no. 1, pp. 95–107, 2021.

[13] R. Abu-Gdairi, M. A. El-Gayar, T. M. Al-shami, A. S. Nawar, M. K. El-Bably, and M. K. El-Bably, "Some topological approaches for generalized rough sets and their decision-making applications," *Symmetry*, vol. 14, no. 1, 2022.

[14] M. A. El-Gayar and A. E. F. El Atik, "Topological models of rough sets and decision making of COVID-19," *Complexity*, vol. 2022, Article ID 2989236, 10 pages, 2022.

[15] H. Lu, A. M. Khalil, W. Alharbi, and M. A. El-Gayar, "A new type of generalized picture fuzzy soft set and its application in decision making," *Journal of Intelligent and Fuzzy Systems*, vol. 40, no. 6, pp. 12459–12475, 2021.

[16] O. Dalkılıç and N. Demirtaş, "Virtual intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets and their application in decision-making," *Gümüşhane Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, vol. 12, no. 3, pp. 769–780, 2022.

[17] R. Abu-Gdairi, M. A. El-Gayar, M. K. El-Bably, and K. K. Fleifel, "Two different views for generalized rough sets with applications," *Mathematics*, vol. 9, 2021.

[18] M. K. El-Bably, R. Abu-Gdairi, and M. A. El-Gayar, "Medical diagnosis for the problem of Chikungunya disease using soft rough sets," *AIMS Mathematics*, vol. 8, no. 4, pp. 9082–9105, 2023.

[19] M. K. El-Bably and A. E. F. A. El Atik, "Soft β-rough sets and their application to determine COVID-19," *Turkish Journal of Mathematics*, vol. 45, no. 3, pp. 1133–1148, 2021.

[20] N. Demirtaş and O. Dalkılıç, "Bipolar soft ordered topology and a new definition for bipolar soft topology," *Journal of Universal Mathematics*, vol. 4, no. 2, pp. 259–270, 2021.

[21] M. K. El-Bably and M. El-Sayed, "Three methods to generalize Pawlak approximations via simply open concepts with economic applications," *Soft Computing*, vol. 26, no. 10, pp. 4685–4700, 2022.

[22] M. K. El-Bably and E. A. Abo-Tabl, "A topological reduction for predicting of a lung cancer disease based on generalized rough sets," *Journal of Intelligent and Fuzzy Systems*, vol. 41, pp. 335–346, 2021.

[23] M. K. El-Bably, M. I. Ali, and E. S. A. Abo-Tabl, "New topological approaches to generalized soft rough approximations with medical applications," *Journal of Mathematics*, vol. 2021, Article ID 2559495, 16 pages, 2021.

[24] R. A. Hosny, R. Abu-Gdairi, and M. K. El-Bably, "Approximations by ideal minimal structure with chemical application," *Intelligent Automation and Soft Computing*, vol. 36, no. 3, pp. 3073–3085, 2023.

[25] B. M. R. Stadler and P. F. Stadler, "The topology of evolutionary Biology," in *Modeling in Molecular Biology*, G. Ciobanu and G. Rozenberg, Eds., Springer, Berlin, Germany, 2004.

[26] M. M. El-Sharkasy, "Topological model for recombination of DNA and RNA," *International Journal of Biomathematics*, vol. 11, no. 8, Article ID 1850097, 2018.

[27] A. Kandil, S. A. El-Sheikh, M. Hosny, and M. Raafat, "Generalization of nano topological spaces induced by different neighborhoods based on ideals and its applications," *Tbilisi Mathematical Journal*, vol. 14, no. 1, pp. 135–148, 2021.

[28] M. A. E. Monsef, M. E. Gayar, and R. M. Aqeel, "On relationships between revised rough fuzzy approximation operators and fuzzy topological spaces," *International Journal of Granular Computing, Rough Sets and Intelligent Systems*, vol. 3, no. 4, pp. 257–271, 2014.

[29] E. Khalimsky, "Topological structures in computer science," *Journal of Applied Mathematics and Simulation*, vol. 1, Article ID 921206, pp. 25–40, 1987.

[30] J. Kortelainen, "On relationship between modified sets, topological spaces and rough sets," *Fuzzy Sets and Systems*, vol. 61, no. 1, pp. 91–95, 1994.

[31] A. Galton, "A generalized topological view of motion in discrete space," *Theoretical Computer Science*, vol. 305, no. 1-3, pp. 111–134, 2003.

[32] S. Mishra and R. Srivastava, "Fuzzy soft compact topological spaces," *Journal of Mathematics*, vol. 2016, Article ID 2480842, 7 pages, 2016.

[33] S. Y. Musa and B. A. Asaad, "Topological structures via bipolar hypersoft sets," *Journal of Mathematics*, vol. 2022, Article ID 2896053, 14 pages, 2022.

[34] S. Y. Musa and B. A. Asaad, "Connectedness on bipolar hypersoft topological spaces," *Journal of Intelligent and Fuzzy Systems*, vol. 43, no. 4, pp. 4095–4105, 2022.

[35] N. Saeed, K. Long, T. U. Islam, Z. S. Mufti, and A. Abbas, "Topological study of zeolite socony mobil-5 via degree-based topological indices," *Journal of Chemistry*, vol. 2021, Article ID 5522800, 13 pages, 2021.

[36] E. A. Marei, M. E. Abd El-Monsef, and H. M. Abu-Donia, "Modification of near sets theory," *Fundamenta Informaticae*, vol. 137, no. 3, pp. 387–402, 2015.

[37] W. Zhu, "Topological approaches to covering rough sets," *Information Sciences*, vol. 177, no. 6, pp. 1499–1508, 2007.

[38] W. Zhu, "Generalized rough sets based on relations," *Information Sciences*, vol. 177, no. 22, pp. 4997–5011, 2007.

[39] W. Zhu and F.-Y. Wang, "On three types of covering-based rough sets," *IEEE Transactions on Knowledge and Data Engineering*, vol. 19, no. 8, pp. 1131–1144, 2007.
[40] W. Zhu and F.-Y. Wang, "Reduction and axiomization of covering generalized rough sets," *Information Sciences*, vol. 152, pp. 217–230, 2003.
[41] O. Dalkılıç, "On topological structures of virtual fuzzy parametrized fuzzy soft sets," *Complex and Intelligent Systems*, vol. 8, no. 1, pp. 337–348, 2022.
[42] H. Y. Saleh, B. A. Asaad, and R. A. Mohammed, "Bipolar soft generalized topological structures and their application in decision making," *European Journal of Pure and Applied Mathematics*, vol. 15, no. 2, pp. 646–671, 2022.
[43] O. Dalkılıç and N. Demirtaş, "Decision analysis review on the concept of class for bipolar soft set theory," *Computational and Applied Mathematics*, vol. 41, no. 5, 2022.
[44] O. Dalkılıç and N. Demirtaş, "A mathematical analysis of the relationship between the vaccination rate and COVID-19 pandemic in Turkey," *Turkish Journal of Forecasting*, vol. 6, 2022.
[45] J. H. Dai, S. C. Gao, and G. J. Zheng, "Generalized rough set models determined by multiple neighborhoods generated from a similarity relation," *Soft Computing*, vol. 22, no. 7, pp. 2081–2094, 2018.
[46] R. Slowinski and D. Vanderpooten, "A generalized definition of rough approximations based on similarity," *IEEE Transactions on Knowledge and Data Engineering*, vol. 12, no. 2, pp. 331–336, 2000.
[47] A. Skowron and J. Stepaniuk, "Tolerance approximation spaces," *Fundamenta Informaticae*, vol. 27, no. 2,3, pp. 245–253, 1996.
[48] T. Y. Lin, "Granular computing on binary relations I: data mining and neighborhood systems, II: rough set representations and belief functions," in *Rough Sets in Knowledge Discovery 1*, L. Polkowski and A. Skowron, Eds., PhysicaVerlag, Heidelberg, Germany, 1998.
[49] Z. Pawlak and A. Skowron, "Rough sets: some extensions," *Information Sciences*, vol. 177, no. 1, pp. 28–40, 2007.
[50] Y. Y. Yao, "Two views of the theory of rough sets in finite universes," *International Journal of Approximate Reasoning*, vol. 15, no. 4, pp. 291–317, 1996.
[51] Y. Y. Yao, "Relational interpretations of neighborhood operators and rough set approximation operators," *Information Sciences*, vol. 111, no. 1–4, pp. 239–259, 1998.
[52] W. Zhu, "Relationship between generalized rough sets based on binary relation and covering," *Information Sciences*, vol. 179, no. 3, pp. 210–225, 2009.
[53] L. Polkowski, "Metric spaces of topological rough sets from countable knowledge bases," *Foundations of Computing and Decision Sciences*, vol. 18, no. 3–4, 1993.
[54] M. Raafat, *A Study of Some Topological Structures and Some of Their Applications*, Ph.D. thesis, Faculty of Education, Ain Shams University, Cairo, Egypt, 2020.
[55] H. Zhang, Y. Ouyang, and Z. Wang, "Note on "Generalized rough sets based on reflexive and transitive relations," *Information Sciences*, vol. 179, no. 4, pp. 471–473, 2009.
[56] E. A. Marei, *Neighborhood System and Decision Making, Master's Thesis*, Zagazig University, Zagazig, Egypt, 2007.
[57] E. A. Marei, "Generalized soft rough approach with a medical decision making problem," *European Journal of Scientific Research*, vol. 133, no. 1, pp. 49–65, 2015.
[58] E. A. Marei, "Rough set approximations on A semi bitopological view," *International Journal of Scientific and Innovative Mathematical Research*, vol. 3, no. 12, pp. 59–70, 2015.
[59] C. Largeron and S. Bonnevay, "A pretopological approach for structural analysis," *Information Sciences*, vol. 144, no. 1-4, pp. 169–185, 2002.
[60] G. L. Liu and W. Zhu, "The algebraic structures of generalized rough set theory," *Information Sciences*, vol. 178, no. 21, pp. 4105–4113, 2008.
[61] Y. Y. Yao, "Rough sets, neighborhood systems and granular computing," in *Proceedings of the IEEE Canadian Conference on Electrical and Computer Engineering*, pp. 1553–1559, Edmonton, AB, Canada, May 1999.
[62] Z. Pawlak, "Rough sets," *International Journal of Computer and Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.
[63] Z. Pawlak, *Rough Sets Theoretical Aspects of Reasoning about Data*, Kluwer- Academic Publishers, Dordrecht, Netherlands, 1991.
[64] Y. Y. Yao and T. Y. Lin, "Generalization of rough sets using modal logics," *Intelligent Automation and Soft Computing*, vol. 2, pp. 103–119, 1996.
[65] F. Tufail, M. Shabir, and E. S. A. Abo-Tabl, "A comparison of promethee and topsis techniques based on bipolar soft covering-based rough sets," *IEEE Access*, vol. 10, pp. 37586–37602, 2022.
[66] C. Wang, C. Wu, and D. Chen, "A systematic study on attribute reduction with rough sets based on general binary relations," *Information Sciences*, vol. 178, no. 9, pp. 2237–2261, 2008.
[67] W. Z. Wu and W. X. Zhang, "Rough set approximations vs. measurable spaces," in *Proceedings of the IEEE International Conference on Granular Computing*, pp. 329–332, Atlanta, GA, USA, May 2006.