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Research Article

Novel Soliton Solutions for the (3 + 1)-Dimensional Sakovich Equation Using Different Analytical Methods

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In the work that we are doing now, our goal is to find a solution to the (3 + 1)-dimensional Sakovich equation, which is an equation that can be used to characterize the movement of nonlinear waves. This approach sees extensive use across the board in engineering's many subfields. This new equation can represent more dispersion and nonlinear effects, which allows it to accept more different application scenarios. As a result, it has a wider range of potential applications in the physical world. As a consequence of this, when the two components are combined, they can be used to easily counteract the effects of nonlinearity and dispersion on the integrability of partially differential equations. This equation represents a novel physical paradigm that should be looked into further. We use the following three analytical techniques: the $\exp(-\psi(\eta))$ expansion method is the first one, the second is the $(G'/kG' + G + r)$ expansion method, and the Bernoulli sub-ODE technique is the last one. Lastly, the soliton solutions obtained are presented by graphs in two and three dimensions, which present different kinds of solitons such as dark, periodic, exponential, bright, and singular soliton solutions.

1. Introduction

Nonlinear partial differential equations, also known as NLPDEs, are used to model a wide variety of natural phenomena, including those found in mathematical physics, fluid flow, plasma physics, mathematical modeling, optical fibers, optics, biology, heat, mechanics, engineering, fluid dynamics, Chemical Physics, Biomathematics, fluid dynamics, Biophysics, Neurophysics, Demography, and so on. Finding the precise solutions of the pertinent NLEEs is crucial for improving our comprehension of nonlinear phenomena and their application to practical problems. A lot of various strategies for obtaining exact solutions to NLEEs have been introduced such as Hirota's method [1–3], Bäcklund transformation method [4], He-Laplace variational iteration method [5], modified homotopy perturbation method [6], Lie symmetry analysis [7, 8], the extended tanh method [9], and numerous other techniques [10–12]. A very important model with wide applications in different fields is the (2 + 1)-dimensional second-order Sakovich equation, which can be used to interpret the dynamics of the water waves in an elongate, limited, hollow duct. Sakovich presented a new second-order three-dimensional nonlinear wave equation [13].

$$u_{xz} + u_{yy} + 2uu_{xy} + 6u^2u_{xx} + 2(u_{xx})^2 = 0. \quad (1)$$

Equation (1) possesses KdV-type multisoliton solutions and passes the Painlevé test for integrability. In 2020, Wazwaz [14] protracted Sakovich equation (1) to the following equation:

$$u_{xt} + u_{xx} + u_{yy} + u_{xy} + u_{xz} + u_{yz} + 2uu_{xy} + 6u^2 u_{xx} + 2u_{xx}^2 = 0. \tag{2}$$

Many researchers present investigated this equation using various techniques such as kumar et al. employ Lie symmetry analysis and extended Jacobian elliptic function expansion method [11], Saglam Ozkan and Yasar employed the logarithmic transformation with the ansatz function technique [15], the new modified extended direct algebraic (NMEDA) technique is used by Younis et al. [16], and Shailendra Singh et al. solved the equation with variable coefficients [17] employing the Painlevé analysis and auto-Bäcklund transformation methods.

Newly, Wazwaz et al. [18] present an extension to (2) by adding two terms, the first one is a linear term u_{zz} which characterizes the dispersion effect of the second order along the direction of z-axis and the second term is a nonlinear term uu_{xx} which characterizes the nonlinear effect reasoned by u and the dispersion of the second order along the direction of x-axis, (2) takes the following form:

$$u_{xt} + u_{xx} + u_{yy} + u_{zz} + u_{xy} + u_{xz} + u_{yz} + 2uu_{xy} + 6u^2 u_{xx} + 2u_{xx}^2 + uu_{xx} = 0. \tag{3}$$

Our goal in the current article is to scrutinize the soliton solution of (3) using the following three powerful methods: the former one is the $\exp(-\psi(\eta))$ expansion method [19–22], second, the new technique is $(G'/kG' + G + r)$ expansion method [23], and the last one is the Bernoulli sub-ODE technique [24, 25].

The paper is organized as follows: In Section 2, we demonstrate the intrinsic steps for the three methods. In Section 3, we apply the mentioned methods and exhibit the analytical solutions. Most of the solutions are presented by graphs in two and three dimensions in Section 4. Finally, Section 5 provides synopsis for the present work.

2. The Principle Outlines

2.1. The $\exp(-\psi(\eta))$ *Expansion Method.* Introducing the following nonlinear partial differential equation:

$$F\left(u, u_t, u_x, u_y, u_z, u_{xt}, u_{xx}, u_{yt}, u_{yy}, u_{xz}, u_{zz}, \ldots\right) = 0, \tag{4}$$

where the function $u = u(x, y, z, t)$ is unknown and F is a polynomial in u and its partial derivatives.

Step 1. Mixing all the independent variables x, y, z, and t to get one new variable η,

$$u(x, y, z, t) = u(\eta), \eta = ax + by + dz - ct, \tag{5}$$

where a, b, d are constants and c is the wave speed. Plugging the transformation of the traveling wave (5) in (4) diminishes (4) to the next ODE.

$$H(u', u'', u''', u'''', \ldots) = 0, \tag{6}$$

where H is a polynomial in $u(\eta)$ and its derivatives.

Step 2. To acquire the solution of (6), we postulate the soliton solution as finite series.

$$u(\eta) = \sum_{i=0}^{N} A_i (\exp(-\psi(\eta)))^i, \tag{7}$$

where the constants A_i $(0 \le i \le N)$ will be matured, such that $A_N \ne 0$, and $\psi = \psi(\eta)$ fulfill the next ODE.

$$\psi'(\eta) = \exp(-\psi(\eta)) + \mu \exp(\psi(\eta)) + \lambda. \tag{8}$$

The solutions of (8) are conferred as follows:

Family 1: when $\lambda^2 - 4\mu > 0, \mu \ne 0$,

$$\psi(\eta) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)}\tanh\left(\left(\sqrt{(\lambda^2 - 4\mu)}/2\right)(\eta + \varepsilon)\right) - \lambda}{2\mu}\right), \tag{9}$$

$$\psi(\eta) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)}\coth\left(\left(\sqrt{(\lambda^2 - 4\mu)}/2\right)(\eta + \varepsilon)\right) - \lambda}{2\mu}\right). \tag{10}$$

Family 2: when $\lambda^2 - 4\mu < 0, \mu \ne 0$,

$$\psi(\eta) = \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\left(\sqrt{(\lambda^2-4\mu)}/2\right)(\eta+\varepsilon)\right)-\lambda}{2\mu}\right), \quad (11)$$

$$\psi(\eta) = \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\cot\left(\left(\sqrt{(\lambda^2-4\mu)}/2\right)(\eta+\varepsilon)\right)-\lambda}{2\mu}\right). \quad (12)$$

Family 3: when $\lambda^2 - 4\mu > 0, \lambda \neq 0, \mu = 0$,

$$\psi(\eta) = -\ln\left(\frac{\lambda}{\exp(\lambda(\eta+\varepsilon))-1}\right). \quad (13)$$

Family 4: when $\lambda^2 - 4\mu = 0, \lambda \neq 0, \mu \neq 0$,

$$\psi(\eta) = \ln\left(-\frac{2(\lambda(\eta+E)+2)}{\lambda^2(\eta+\varepsilon)}\right). \quad (14)$$

Family 5: when $\lambda^2 - 4\mu = 0, \lambda = 0, \mu = 0$,

$$\psi(\eta) = \ln(\eta+\varepsilon). \quad (15)$$

Step 3. Through balancing the highest power nonlinear term with the highest order derivative term in (6), N is deliberated. We insert (7) into (6), a set of equations achieved as the coefficients of $\exp(-\psi(\eta))$ with the same powers vanished. The Mathematica program is employed to solve the previous set of equations.

2.2. The $(G'/kG'+G+r)$ Expansion Method. The substantial procedures of the $(G'/kG'+G+r)$ expansion method are introduced as follows:

Step 1: presuming the solution of (6) is given by

$$u(\eta) = \sum_{i=0}^{N} m_i F(\eta)^i, \quad (16)$$

where $F(\eta) = (G'/kG'+G+r), G(\eta)$ fulfill the second-order differential equation.

$$G''(\eta) = -\frac{\lambda}{k}G'(\eta) - \frac{\mu}{k^2}G(\eta) - \frac{\mu}{k^2}r, \quad (17)$$

where m_i are constants to be determined later and $k, r, \lambda,$ and μ are arbitrary constants. The following ordinary differential equation is satisfied by $F = F(\eta)$:

$$F'(\eta) = (\lambda-\mu-1)F(\eta)^2 + \frac{1}{k}(2\mu-\lambda)F(\eta) - \frac{1}{k^2}\mu. \quad (18)$$

Step 2: referring to the preceding section to determine N.

Step 3: we attain two families of solutions of (18).

Family 1: when $\Delta = \lambda^2 - 4\mu > 0$

The solution of (17) is as follows:

$$G = -r + p_1 e^{(1/2k)(-\lambda-\sqrt{\Delta})\eta} + p_2 e^{(1/2k)(-\lambda+\sqrt{\Delta})\eta}, \quad (19)$$

p_1 and p_2 are arbitrary constants, hold the relation $r^2 + p_1^2 + p_2^2 \neq 0$. Then,

$$F(\eta) = \frac{p_1(\lambda+\sqrt{\Delta}) + p_2(\lambda-\Delta)e^{(\sqrt{\Delta}\eta/k)}}{kp_1(\lambda-2+\sqrt{\Delta}) + kp_2(\lambda-2-\Delta)e^{(\sqrt{\Delta}\eta/k)}},$$

$$F(\eta) = \frac{[\lambda(p_2-p_1)-\sqrt{\Delta}(p_2+p_1)]\sinh(\sqrt{\Delta}\eta/2k) + [\lambda(p_2+p_1)-\sqrt{\Delta}(p_2-p_1)]\cosh(\sqrt{\Delta}\eta/2k)}{k[(\lambda-2)(p_2-p_1)-\sqrt{\Delta}(p_2+p_1)]\sinh(\sqrt{\Delta}\eta/2k) + k[(\lambda-2)(p_2+p_1)-\sqrt{\Delta}(p_2-p_1)]\cosh(\sqrt{\Delta}\eta/2k)}, \quad (20)$$

$$F(\eta) = \begin{cases} \dfrac{\lambda-2\mu}{2k(\lambda-\mu-1)} - \dfrac{\sqrt{\Delta}}{2k(\lambda-\mu-1)}\tanh\left(\dfrac{\sqrt{\Delta}\eta}{2k}\right), & (\lambda-2)(p_2-p_1)-\sqrt{\Delta}(p_2+p_1)=0, \\ \dfrac{\lambda-2\mu}{2k(\lambda-\mu-1)} - \dfrac{\sqrt{\Delta}}{2k(\lambda-\mu-1)}\coth\left(\dfrac{\sqrt{\Delta}\eta}{2k}\right), & (\lambda-2)(p_2+p_1)-\sqrt{\Delta}(p_2-p_1)=0. \end{cases} \quad (21)$$

Family 2: when $\Delta = \lambda^2 - 4\mu < 0$

The solution of (17) is as follows:

$$G = -r + e^{-\lambda\eta/2k}\left(p_1 \cos\left(\frac{\sqrt{-\Delta}\,\eta}{2k}\right) + p_2 \sin\left(\frac{\sqrt{-\Delta}\,\eta}{2k}\right)\right), \tag{22}$$

$$F(\eta) = \frac{\left(\lambda p_1 - \sqrt{-\Delta}\,p_2\right)\cos\left(\sqrt{-\Delta}\,\eta/2k\right) + \left(\lambda p_2 + \sqrt{-\Delta}\,p_1\right)\sin\left(\sqrt{-\Delta}\,\eta/2k\right)}{k\left((\lambda-2)p_1 - \sqrt{-\Delta}\,p_2\right)\cos\left(\sqrt{-\Delta}\,\eta/2k\right) + k\left((\lambda-2)p_2 + \sqrt{-\Delta}\,p_1\right)\sin\left(\sqrt{-\Delta}\,\eta/2k\right)}, \tag{23}$$

$$F(\eta) = \begin{cases} \dfrac{\lambda-2\mu}{2k(\lambda-\mu-1)} + \dfrac{\sqrt{-\Delta}}{2k(\lambda-\mu-1)}\tan\left(\dfrac{\sqrt{-\Delta}\,\eta}{2k}\right), & (\lambda-2)p_2 + \sqrt{-\Delta}\,p_1 = 0, \\ \dfrac{\lambda-2\mu}{2k(\lambda-\mu-1)} - \dfrac{\sqrt{-\Delta}}{2k(\lambda-\mu-1)}\cot\left(\dfrac{\sqrt{-\Delta}\,\eta}{2k}\right), & (\lambda-2)p_1 - \sqrt{-\Delta}\,p_2 = 0. \end{cases} \tag{24}$$

Step 4: inserting (16) and (18) into (6), a system of equations engender as the coefficients with identical powers of $F(\eta)$ vanished. Utilizing Mathematica program to solve this system of equations.

2.3. Bernoulli Sub-ODE Method. Interpreting the fundamental steps briefly, we get the following:

Step 1: deeming the solution of (6) is

$$u(\eta) = \sum_{i=0}^{N} f_i (G(\eta))^i, \tag{25}$$

where the constants $f_i\,(0 \le i \le N)$ will be determined, $f_N \ne 0$. $G(\eta)$ fulfills the ordinary differential equation,

$$G' + \lambda G = \mu G^2, \tag{26}$$

where $\lambda, \mu \ne 0$, the next equation offered the solution of $G(\eta)$,

$$G(\eta) = \frac{1}{(\mu/\lambda) + \alpha e^{\lambda\eta}}. \tag{27}$$

Step 2: calculating the non-negative integer N from (6) by the balance principle.

Step 3: inserting (25) and (26) into (6), a polynomial in $G(\eta)$ is achieved. Each coefficient in this polynomial is set equal zero. Mathematica program was used in solving the preceding set of equations.

3. Applications

Converting (3) into the next ordinary differential equation using the transformation (5), we acquire the following equation:

$$u''(\eta)\left(a^2 + ab + b^2 - ac + ad + bd + d^2 + a(a+2b)u(\eta) + 6a^2 u(\eta)^2 + 2a^4 u''(\eta)\right) = 0. \tag{28}$$

Balancing $u(\eta)^2 u''(\eta)$ with $u''(\eta)^2$ in (28) we get, $3N + 2 = 2N + 4$, yields $N = 2$.

3.1. Soliton Solutions Applying the $\exp(-\psi(\eta))$ *Expansion Method.* From (7), the solution of (28) can be expressed by

$$u(\eta) = A_0 + A_1(\exp(-\psi(\eta))) + A_2(\exp(-\psi(\eta)))^2. \tag{29}$$

Substituting (29) into (28) and using (8), then set the coefficients of the identical power of $\exp(-\psi(\eta))$ equal to zero, we get the next system of equations as follows:

$$\begin{aligned}
&\mu(\lambda A_1 + 2\mu A_2)\left(a^2 + b^2 + d^2 + bd + a(b-c+d) + a(a+2b)A_0 + 6a^2A_0^2 + 2a^4\mu(\lambda A_1 + 2\mu A_2)\right) = 0,\\
&a\lambda\mu\left(a + 2b + 4a^3\left(\lambda^2 + 2\mu\right) + 12aA_0\right)A_1^2 + 6\lambda\mu A_2\left(a^2 + b^2 + d^2 + bd + a(b-c+d) + aA_0(a+2b+6aA_0) + 8a^4\mu^2A_2\right)\\
&\quad + A_1\left(\left(\lambda^2 + 2\mu\right)\left(a^2 + b^2 + d^2 + bd + a(b-c+d) + aA_0(a+2b+6aA_0)\right) + 2a\mu^2\left(a + 2b + 8a^3\left(2\lambda^2 + \mu\right) + 12aA_0\right)A_2\right) = 0,\\
&6a^2\lambda\mu A_1^3 + 2A_2\left(2\left(\lambda^2 + 2\mu\right)\left(a^2 + b^2 + d^2 + bd + a(b-c+d) + aA_0(a+2b+6aA_0)\right) + a\mu^2\left(a + 2b + 4a^3\left(13\lambda^2 + 8\mu\right) + 12aA_0\right)A_2\right)\\
&\quad + \lambda A_1\left(3\left(a^2 + b^2 + d^2 + bd + a(b-c+d) + aA_0(a+2b+6aA_0)\right) + a\mu\left(7a + 14b + 8a^3\left(5\lambda^2 + 13\mu\right) + 84aA_0\right)A_2\right)\\
&\quad + aA_1^2\left(a\left(\lambda^2 + 2\mu\right) + 2b\left(\lambda^2 + 2\mu\right) + 2a^3\left(\lambda^4 + 10\lambda^2\mu + 4\mu^2\right) + 12a\left(\left(\lambda^2 + 2\mu\right)A_0 + \mu^2A_2\right)\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&6a^2A_1^3(\lambda^2+2\mu)+a\lambda A_1^2(3a+6b+4a^3(3\lambda^2+8\mu)+36aA_0+48a\mu A_2)+2\lambda A_2(5(a^2+b^2+d^2+bd+a(b-c+d)+aA_0(a+2b+6aA_0))\\
&\quad+a\mu(3a+6b+8a^3(6\lambda^2+17\mu)+36aA_0)A_2)+A_1(2(a^2+b^2+d^2+bd+a(b-c+d)+aA_0(a+2b+6aA_0))\\
&\quad+a(5a(\lambda^2+2\mu)+10b(\lambda^2+2\mu)+16a^3(\lambda^4+11\lambda^2\mu+5\mu^2)+60a(\lambda^2+2\mu)A_0)A_2+24a^2\mu^2A_2^2)=0,\\
&18a^2\lambda A_1^3+2A_2(3(a^2+b^2+d^2+bd+a(b-c+d)+aA_0(a+2b+6aA_0))\\
&\quad+2a(a(\lambda^2+2\mu)+2b(\lambda^2+2\mu)+4a^3(2\lambda^2+\mu)(\lambda^2+11\mu)+12a(\lambda^2+2\mu)A_0)A_2+6a^2\mu^2A_2^2)+a\lambda A_1A_2(13a+26b+8a^3(11\lambda^2+31\mu)\\
&\quad+78a(2A_0+\mu A_2))+2aA_1^2(a+2b+a^3(13\lambda^2+8\mu)+6a(2A_0+3A_2(\lambda^2+2\mu)))=0,\\
&2a(6aA_1^3+12a\lambda A_1^2(a^2+4A_2)+\lambda A_2^2(5a+10b+8a^3(10\lambda^2+29\mu)+60aA_0+18a\mu A_2))\\
&\quad+A_1A_2(4(a+2b+2a^3(11\lambda^2+7\mu))+48aA_0+27aA_2(\lambda^2+2\mu))=0,\\
&2a(a\lambda A_1A_2(76a^2+69A_2)+A_1^2(4a^3+30aA_2)+A_2^2(3a+6b+4a^3(37\lambda^2+24\mu)+12a(3A_0+A_2(\lambda^2+2\mu))))=0,\\
&12a^2A_2(5\lambda A_2(4a^2+A_2)+A_1(4a^2+7A_2))=0,\\
&36a^2A_2^2(2a^2+A_2)=0,
\end{aligned} \tag{30}$$

We achieve the following sets of solutions as we solve the preceding system:

Set 1:

$$\begin{aligned}
A_1&=-2a^2\lambda,\ A_2=-2a^2,\ d=\frac{1}{4}(-a+2a^3(\lambda^2+8\mu)+12aA_0\\
&\quad-((-a(11a-16c+4a^3(\lambda^2+8\mu)+4a^5(3\lambda^4+32\lambda^2\mu+160\mu^2)\\
&\quad+8aA_0(3+14a^2(\lambda^2+8\mu)+42A_0)))^{(1/2)}),\\
b&=-\frac{1}{2}a(1+2a^2(\lambda^2+8\mu)+12A_0).
\end{aligned} \tag{31}$$

Set 2:

$$\begin{aligned}
A_1&=-2a^2\lambda,\ A_2=-2a^2,\ d=\frac{1}{4}(-a+2a^3(\lambda^2+8\mu)+12aA_0\\
&\quad+((-a(11a-16c+4a^3(\lambda^2+8\mu)+4a^5(3\lambda^4+32\lambda^2\mu+160\mu^2)\\
&\quad+8aA_0(3+14a^2(\lambda^2+8\mu)+42A_0)))^{(1/2)}),\\
b&=-\frac{1}{2}a(1+2a^2(\lambda^2+8\mu)+12A_0).
\end{aligned} \tag{32}$$

Next, we present five families of traveling wave solutions.

Family 1. when $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$u(x,y,z,t) = A_0 + A_1\left(\frac{2\mu}{-\lambda - \sqrt{(\lambda^2 - 4\mu)}\tanh\left(\left(\sqrt{(\lambda^2 - 4\mu)}/2\right)(ax + by + dz - ct)\right)}\right) + A_2\left(\frac{2\mu}{-\lambda - \sqrt{(\lambda^2 - 4\mu)}\tanh\left(\left(\sqrt{(\lambda^2 - 4\mu)}/2\right)(ax + by + dz - ct)\right)}\right)^2, \tag{33}$$

$$u(x,y,z,t) = A_0 + A_1\left(\frac{2\mu}{-\lambda - \sqrt{(\lambda^2 - 4\mu)}\coth\left(\left(\sqrt{(\lambda^2 - 4\mu)}/2\right)(ax + by + dz - ct)\right)}\right) + A_2\left(\frac{2\mu}{-\lambda - \sqrt{(\lambda^2 - 4\mu)}\coth\left(\left(\sqrt{(\lambda^2 - 4\mu)}/2\right)(ax + by + dz - ct)\right)}\right)^2. \tag{34}$$

Family 2. when $\lambda^2 - 4\mu < 0, \mu \neq 0$,

$$u(x,y,z,t) = A_0 + A_1\left(\frac{2\mu}{-\lambda + \sqrt{(4\mu - \lambda^2)}\tan\left(\left(\sqrt{(4\mu - \lambda^2)}/2\right)(ax + by + dz - ct)\right)}\right) + A_2\left(\frac{2\mu}{-\lambda + \sqrt{(4\mu - \lambda^2)}\tan\left(\left(\sqrt{(4\mu - \lambda^2)}/2\right)(ax + by + dz - ct)\right)}\right)^2, \tag{35}$$

$$u(x,y,z,t) = A_0 + A_1\left(\frac{2\mu}{-\lambda + \sqrt{(4\mu - \lambda^2)}\cot\left(\left(\sqrt{(4\mu - \lambda^2)}/2\right)(ax + by + dz - ct)\right)}\right) + A_2\left(\frac{2\mu}{-\lambda + \sqrt{(4\mu - \lambda^2)}\cot\left(\left(\sqrt{(4\mu - \lambda^2)}/2\right)(ax + by + dz - ct)\right)}\right)^2. \tag{36}$$

Family 3. when $\lambda^2 - 4\mu > 0, \lambda \neq 0, \mu = 0$,

$$u(x,y,z,t) = A_0 + A_1\left(\frac{\lambda}{-1 + e^{\lambda(ax+by+dz-ct)}}\right) + A_2\left(\frac{\lambda}{-1 + e^{\lambda(ax+by+dz-ct)}}\right)^2. \tag{37}$$

Family 4. when $\lambda^2 - 4\mu = 0, \lambda \neq 0, \mu \neq 0$,

$$u(x, y, z, t) = A_0 - A_1\left(\frac{\lambda^2 (ax + by + dz - ct)}{2(2 + \lambda(ax + by + dz - ct))}\right) + A_2\left(\frac{\lambda^2 (ax + by + dz - ct)}{2(2 + \lambda(ax + by + dz - ct))}\right)^2. \tag{38}$$

Family 5. when $\lambda^2 - 4\mu = 0, \lambda = 0, \mu = 0$,

$$u(x, y, z, t) = A_0 + A_1\left(\frac{1}{ax + by + dz - ct}\right) + A_2\left(\frac{1}{ax + by + dz - ct}\right)^2. \tag{39}$$

3.2. Soliton Solutions Applying the $(G'/kG' + G + r)$ Expansion Method. Based on (16), the solution of (28) is presented by the following equation:

$$u(\eta) = m_0 + m_1 F(\eta) + m_2 F(\eta)^2. \tag{40}$$

Inserting (40) into (28) and placing the coefficients of $F(\eta)$ with equal power equal to zero, we achieve the system as follows:

$$\begin{aligned}
&\mu(k(\lambda - 2\mu)m_1 + 2\mu m_2)((a^2 + b^2 + d^2 + bd + a(b - c + d))k^4 \\
&\quad + a(a + 2b)k^4 m_0 + 6a^2 k^4 m_0^2 + 2a^4 \mu(k(\lambda - 2\mu)m_1 + 2\mu m_2)) = 0, \\
&k(a\mu k^2 (\lambda - 2\mu)(ak^2 + 2bk^2 + 4a^3(\lambda^2 - 6\lambda\mu + 2\mu(1 + 3\mu)) + 12ak^2 m_0) + m_1^2 \\
&\quad + 6\mu m_2 (\lambda - 2\mu)(k^4(a^2 + b^2 + d^2 + bd + a(b - c + d) + am_0(a + 2b + 6am_0)) \\
&\quad + 8a^4\mu^2 m_2) + km_1(k^4(\lambda^2 - 6\lambda\mu + 2\mu(1 + 3\mu))(a^2 + b^2 + d^2 + bd + a(b - c + d) + am_0(a + 2b + 6am_0)) \\
&\quad + 2a\mu^2(ak^2 + 2bk^2 + 8a^3(2\lambda^2 + \mu - 9\lambda\mu + 9\mu^2) + 12ak^2 m_0)m_2) = 0, \\
&k^2(6a^2k^3\mu m_1^3(\lambda - 2\mu) + 2m_2(2k^4(\lambda^2 - 6\lambda\mu + 2\mu(1 + 3\mu))(a^2 + b^2 + d^2 + bd + a(b - c + d) + am_0(a + 2b + 6am_0)) \\
&\quad + a\mu^2(ak^2 + 2bk^2 + 4a^3(13\lambda^2 - 60\lambda\mu + 4\mu(2 + 15\mu)) + 12ak^2 m_0)m_2) \\
&\quad + km_1(\lambda - 2\mu)(-3k^4(-1 + \lambda - \mu)(a^2 + b^2 + d^2 + bd + a(b - c + d) + am_0(a + 2b + 6am_0)) \\
&\quad + a\mu(7ak^2 + 14bk^2 + 8a^3(5\lambda^2 - 33\lambda\mu + \mu(13 + 33\mu)) + 84ak^2 m_0)m_2) \\
&\quad + ak^2 m_1^2(2a^3(\lambda^4 + 2\lambda^2\mu(5 - 9\lambda) + 2(2 + 3\lambda(-8 + 13\lambda))\mu^2 + 24\mu^3(2 - 5\lambda) + 60\mu^4) \\
&\quad + ak^2(\lambda^2 - 6\lambda\mu + 2\mu(1 + 3\mu)) + 2bk^2(\lambda^2 - 6\lambda\mu + 2\mu(1 + 3\mu)) + 12a(k^2 m_0(\lambda^2 + 2\mu - 6\lambda\mu + 6\mu^2) + \mu^2 m_2)))) = 0, \\
&k^3(6a^2k^3(\lambda^2 - 6\lambda\mu + 2\mu(1 + 3\mu))m_1^3 - ak^2(\lambda - 2\mu)m_1^2((-1 + \lambda - \mu)(3ak^2 + 6bk^2 + 4a^3(3\lambda^2 - 20\lambda\mu + 4\mu(2 + 5\mu)) \\
&\quad + 36ak^2 m_0) - 48a\mu m_2) \\
&\quad + 2m_2(\lambda - 2\mu)(-5k^4(-1 + \lambda - \mu)(a^2 + b^2 + d^2 + bd + a(b - c + d) + am_0(a + 2b + 6am_0)) \\
&\quad + a\mu(3ak^2 + 6bk^2 + 8a^3(6\lambda^2 - 41\lambda\mu + \mu(17 + 41\mu)) + 36ak^2 m_0)m_2) \\
&\quad + km_1(2(a^2 + b^2 + d^2 + bd + a(b - c + d))k^4(1 - \lambda + \mu)^2 + a(12ak^4(1 - \lambda + \mu)^2 m_0^2 \\
&\quad + m_2(16a^3(\lambda^4\lambda^2\mu(11 - 19\lambda) + (5 + 6\lambda(-9 + 14\lambda))\mu^2 + 2(27 - 65\lambda)\mu^3 + 65\mu^4) \\
&\quad + 5ak^2(\lambda^2 - 6\lambda\mu + 2\mu(1 + 3\mu)) + 10bk^2(\lambda^2 - 6\lambda\mu + 2\mu(1 + 3\mu)) + 24a\mu^2 m_2) \\
&\quad + 2k^2 m_0(k^2(a + 2b)(1 - \lambda + \mu^2 + 30am_2(\lambda^2 + 2\mu - 6\lambda\mu + 6\mu^2)))))) = 0,
\end{aligned}$$

$$
\begin{aligned}
&k^4\Big(-18a^2k^3m_1^3(\lambda-2\mu)(-1+\lambda-\mu)-akm_1m_2(\lambda-2\mu)((-1+\lambda-\mu)\big(13ak^2+26bk^2+8a^3\big(11\lambda^2-75\lambda\mu+\mu(31+75\mu)\big)\\
&\quad+156am_0k^2\big)-78am_2\mu\big)+2ak^2m_1^2\big((1-\lambda+\mu)^2\big(ak^2+2bk^2+a^3\big(13\lambda^2-60\lambda\mu+4\mu(2+15\mu)\big)+12am_0k^2\big)\\
&\quad+18a\big(\lambda^2-6\lambda\mu+2\mu(1+3\mu)\big)m_2\big)+2m_2\big(3k^4\big(a^2+b^2+d^2+bd+a(b-c+d)\big)(1-\lambda+\mu)^2\\
&\quad+a\big(18ak^4m_0^2(1-\lambda+\mu)^2+2m_2\big(ak^2\big(\lambda^2-6\lambda\mu+2\mu(1+3\mu)\big)\big)\big)\\
&\quad+2bk^2\big(\lambda^2-6\lambda\mu+2\mu(1+3\mu)\big)+4a^3\big(2\lambda^2+\mu-9\lambda\mu+9\mu^2\big)\big(\lambda^2-15\lambda\mu+\mu(11+15\mu)\big)+3am_2\mu^2\big)\\
&\quad+3m_0k^2\big((a+2b)k^2(1-\lambda+\mu)^2+8am_2\big(\lambda^2+2\mu-6\lambda\mu+6\mu^2\big)\big)\Big)=0,\\
&2ak^5\big(6ak^3m_1^3(1-\lambda+\mu)^2+12ak^2m_1^2(\lambda-2\mu)(-1+\lambda-\mu)\big(-a^2(1-\lambda+\mu)^2-4m_2\big)\\
&\quad-m_2^2(\lambda-2\mu)\big((-1+\lambda-\mu)\big(5ak^2+10bk^2+8a^3\big(10\lambda^2-69\lambda\mu\\
&\quad+\mu(29+69\mu)\big)+60am_0k^2\big)-18am_2\mu\big)+km_1m_2\big(4(1-\lambda+\mu)^2\big(ak^2+2bk^2+2a^3\big(11\lambda^2-51\lambda\mu+\mu(7+51\mu)\big)\\
&\quad+12ak^2m_0\big)+27a\big(\lambda^2-6\lambda\mu+2\mu(1+3\mu)\big)m_2\big)\big)=0,\\
&2ak^6\Big(akm_1m_2(\lambda-2\mu)(-1+\lambda-\mu)\big(-76a^2(1-\lambda+\mu)^2-69m_2\big)+2ak^2m_1^2\times(1-\lambda+\mu)^2\big(2a^2(1-\lambda+\mu)^2+15m_2\big)\\
&\quad+m_2^2\big((1-\lambda+\mu)^2\big(3ak^2+6bk^2+4a^3\big(37\lambda^2-172\lambda\mu+4\mu(6+43\mu)\big)+36am_0k^2\big)+12a\big(\lambda^2-6\lambda\mu+2\mu(1+3\mu)\big)m_2\big)\Big)=0,\\
&12a^2k^7m_2(1-\lambda+\mu)\Big(km_1(1-\lambda+\mu)\big(-4a^2(1-\lambda+\mu)^2-7m_2\big)+5m_2\big(\lambda-2\mu\big)\big(4a^2(1-\lambda+\mu)^2+m_2\big)\Big)=0,\\
&36a^2k^8m_2^2(1-\lambda+\mu)^2\big(2a^2(1-\lambda+\mu)^2+m_2\big)=0.
\end{aligned}
\tag{41}
$$

We obtain the following two sets of solutions when solving the anterior system of equations:

Set 1

$$
\begin{aligned}
m_0&=-\frac{ak^2+2bk^2+2a^3\big(\lambda^2-12\lambda\mu+4\mu(2+3\mu)\big)}{12ak^2},\\
m_1&=\frac{2a^2(\lambda-2\mu)(-1+\lambda-\mu)}{k},\\
m_2&=-2a^2(1-\lambda+\mu)^2,\\
d&=-\frac{1}{12k^4}\Big(6k^4(a+b)+\big(-6k^4\big(17a^2k^4+14b^2k^4+8ak^4(b-3c)+4a^6\big(\lambda^2-4\mu^2\big)\big)\big)^{(1/2)}\Big).
\end{aligned}
\tag{42}
$$

Set 2

$$\begin{aligned}
m_0 &= -\frac{ak^2 + 2bk^2 + 2a^3\left(\lambda^2 - 12\lambda\mu + 4\mu(2+3\mu)\right)}{12ak^2},\\
m_1 &= \frac{2a^2(\lambda - 2\mu)(-1+\lambda-\mu)}{k},\\
m_2 &= -2a^2(1-\lambda+\mu)^2,\\
d &= \frac{1}{12k^4}\left(-6k^4(a+b) + \left(-6k^4\left(17a^2k^4 + 14b^2k^4 + 8ak^4(b-3c) + 4a^6\left(\lambda^2 - 4\mu^2\right)\right)\right)^{(1/2)}\right).
\end{aligned} \tag{43}$$

In what follows, we introduce two families of traveling wave solutions.

Family 1: when $\Delta = \lambda^2 - 4\mu > 0$,

$$\begin{aligned}
u(x,y,z,t) = m_0 &+ m_1\left(\frac{\lambda - 2\mu}{2k(\lambda-\mu-1)} - \frac{\sqrt{\Delta}}{2k(\lambda-\mu-1)}\tanh\left(\frac{\sqrt{\Delta}(ax+by+dz-ct)}{2k}\right)\right)\\
&+ m_2\left(\frac{\lambda - 2\mu}{2k(\lambda-\mu-1)} - \frac{\sqrt{\Delta}}{2k(\lambda-\mu-1)}\tanh\left(\frac{\sqrt{\Delta}(ax+by+dz-ct)}{2k}\right)\right)^2.
\end{aligned} \tag{44}$$

Under the constrain, $(\lambda-2)(p_2-p_1) - \sqrt{\Delta}(p_2+p_1) = 0$,

$$\begin{aligned}
u(x,y,z,t) = m_0 &+ m_1\left(\frac{\lambda - 2\mu}{2k(\lambda-\mu-1)} - \frac{\sqrt{\Delta}}{2k(\lambda-\mu-1)}\coth\left(\frac{\sqrt{\Delta}(ax+by+dz-ct)}{2k}\right)\right)\\
&+ m_2\left(\frac{\lambda - 2\mu}{2k(\lambda-\mu-1)} - \frac{\sqrt{\Delta}}{2k(\lambda-\mu-1)}\coth\left(\frac{\sqrt{\Delta}(ax+by+dz-ct)}{2k}\right)\right)^2.
\end{aligned} \tag{45}$$

Under the constrain, $(\lambda-2)(p_2+p_1) - \sqrt{\Delta}(p_2-p_1) = 0$.

Family 2: when $\Delta = \lambda^2 - 4\mu < 0$,

$$\begin{aligned}
u(x,y,z,t) = m_0 &+ m_1\left(\frac{\lambda - 2\mu}{2k(\lambda-\mu-1)} + \frac{\sqrt{-\Delta}}{2k(\lambda-\mu-1)}\tan\left(\frac{\sqrt{-\Delta}(ax+by+dz-ct)}{2k}\right)\right)\\
&+ m_2\left(\frac{\lambda - 2\mu}{2k(\lambda-\mu-1)} + \frac{\sqrt{-\Delta}}{2k(\lambda-\mu-1)}\tan\left(\frac{\sqrt{-\Delta}(ax+by+dz-ct)}{2k}\right)\right)^2.
\end{aligned} \tag{46}$$

Under the constrain, $(\lambda-2)p_2 + \sqrt{-\Delta}p_1 = 0$,

$$\begin{aligned}
u(x,y,z,t) = m_0 &+ m_1\left(\frac{\lambda - 2\mu}{2k(\lambda-\mu-1)} + \frac{\sqrt{-\Delta}}{2k(\lambda-\mu-1)}\cot\left(\frac{\sqrt{-\Delta}(ax+by+dz-ct)}{2k}\right)\right)\\
&+ m_2\left(\frac{\lambda - 2\mu}{2k(\lambda-\mu-1)} + \frac{\sqrt{-\Delta}}{2k(\lambda-\mu-1)}\cot\left(\frac{\sqrt{-\Delta}(ax+by+dz-ct)}{2k}\right)\right)^2,
\end{aligned} \tag{47}$$

Under the constrain, $(\lambda-2)p_1-\sqrt{-\Delta}p_2=0$.

3.3. Soliton Solutions Applying the Bernoulli Sub-ODE Method. Based on (25), the solution of (28) is given by the following equation:

$$u(\eta)=f_0+f_1G(\eta)+f_2G(\eta)^2. \tag{48}$$

Inserting (48) into (28) and placing the coefficients with equal powers of $G(\eta)$ equal to zero, we achieve the next system as follows:

$$\begin{aligned}
&\lambda^2 f_1\left(a^2+b^2+d^2+bd+a(b-c+d)+af_0(a+2b+6af_0)\right)=0,\\
&\lambda\left(-3f_1\mu\left(a^2+b^2+d^2+bd+a(b-c+d)+af_0(a+2b+6af_0)\right)+a\lambda(a+2b+2a^3\lambda^2+12af_0)f_1^2\right.\\
&\quad\left.+4f_2\lambda\left(a^2+b^2+d^2+bd+a(b-c+d)+af_0(a+2b+6af_0)\right)\right)=0-3a\lambda\mu\left(a+2b+4a^3\lambda^2+12af_0\right)f_1^2\\
&\quad+6a^2\lambda^2f_1^3-10\lambda\mu\left(a^2+b^2+d^2+bd+a(b-c+d)+af_0(a+2b+6af_0)\right)f_2\\
&\quad+f_1\left(2\mu^2\left(a^2+b^2+d^2+bd+a(b-c+d)+af_0(a+2b+6af_0)\right)\right.\\
&\quad\left.+a\lambda^2f_2\left(5a+10b+16a^3\lambda^2+60af_0\right)\right)=0,\\
&\quad-18a^2f_1^3\lambda\mu+6\mu^2\left(a^2+b^2+d^2+bd+a(b-c+d)+af_0(a+2b+6af_0)\right)f_2-af_1f_2\lambda\mu\left(13a+26b+88a^3\lambda^2+156af_0\right)\\
&\quad+4a\lambda^2f_2^2\left(a+2b8a^3\lambda^2+12af_0\right)+2af_1^2\left(\mu^2\left(a+2b+13a^3\lambda^2+12af_0\right)+18af_2\lambda^2\right)=0,\\
&\quad-2a\left(6af_1^2\mu^2\left(2a^2\lambda\mu-f_1\right)-4\mu f_1\left(\mu\left(a+2b+22a^3\lambda^2+12af_0\right)-12af_1\lambda\right)f_2\right.\\
&\quad\left.+\lambda\left(5\mu\left(a+2b+16a^3\lambda^2+12af_0\right)-27af_1\lambda\right)f_2^2\right)=0,\\
&2a\left(4a^3f_1^2\mu^4+2af_1f_2\mu^2\left(-38a^2\lambda\mu+15f_1\right)+\mu\left(\mu\left(3a+6b+148a^3\lambda^2+36af_0\right)-69af_1\lambda\right)f_2^2+12a\lambda^2f_2^3\right)=0\\
&12a^2f_2\mu\left(-5\lambda f_2\left(4a^2\mu^2+f_2\right)+f_1\left(4a^2\mu^3+7\mu f_2\right)\right)=0\\
&36a^2\mu^2f_2^2\left(2a^2\mu^2+f_2\right)=0.
\end{aligned} \tag{49}$$

We obtain the follwoing two sets of solutions when solving the former system of equations:

Set 1

$$\begin{aligned}
&f_1=2a^2\lambda\mu, f_2=-2a^2\mu^2, b=\frac{1}{2}\left(-a-2a^3\lambda^2+12af_0\right),\\
&d=\frac{1}{4}\left(-a+2a^3\lambda^2-12af_0-\left(-11a^2+16ac-4a^4\lambda^2-12a^6\lambda^4-24a^2f_0\right.\right.\\
&\quad\left.\left.-112a^4\lambda^2f_0-336a^2f_0^2\right)\right.
\end{aligned} \tag{50}$$

Set 2

$$\begin{aligned}
&f_1=2a^2\lambda\mu, f_2=-2a^2\mu^2, b=\frac{1}{2}\left(-a-2a^3\lambda^2+12af_0\right),\\
&d=\frac{1}{4}\left(-a+2a^3\lambda^2-12af_0+\left(-11a^2+16ac-4a^4\lambda^2-12a^6\lambda^4-24a^2f_0\right.\right.\\
&\quad\left.\left.-112a^4\lambda^2f_0-336a^2f_0^2\right)^{1/2}\right).
\end{aligned} \tag{51}$$

The traveling wave solution presented in what follows.

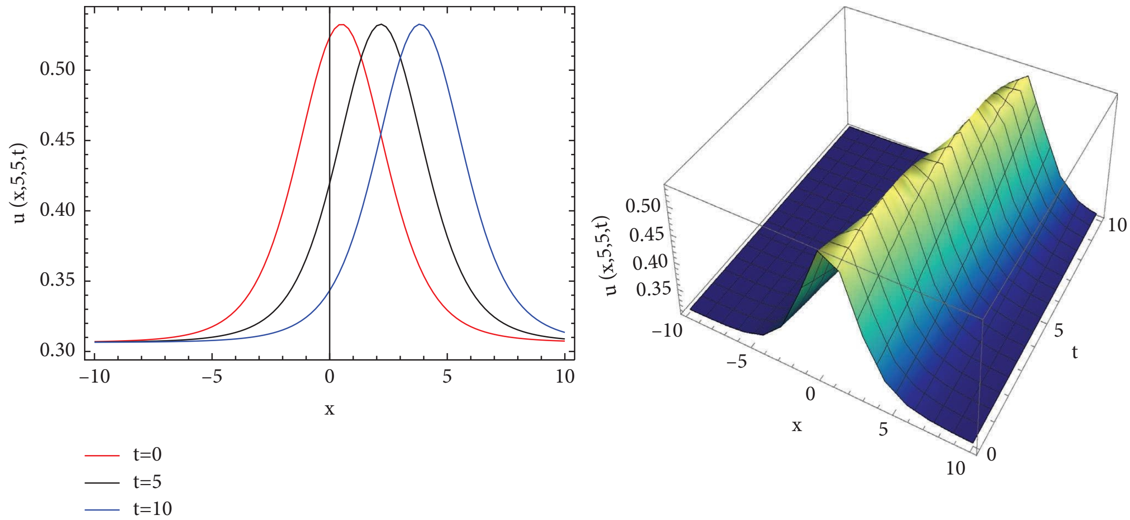

FIGURE 1: 2D and 3D graphs, respectively, corresponding to (33) with (31) employing the $\exp(-\psi(\eta))$ expansion method at $a = 1.8, c = 0.6, A_0 = 0.3, \lambda = 0.3, \mu = 0.001$.

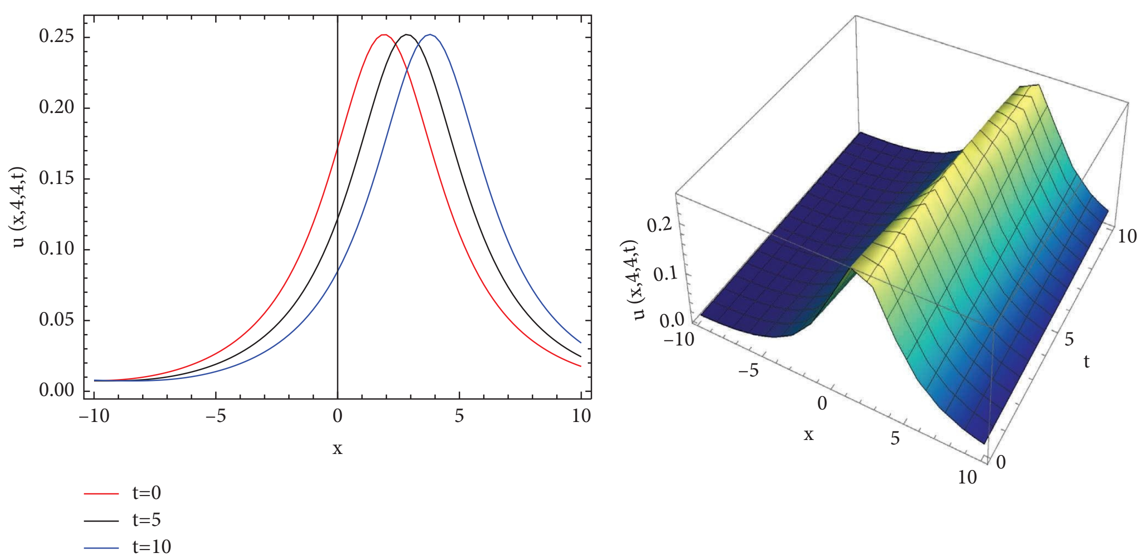

FIGURE 2: 2D and 3D graphs, respectively, corresponding to (35) with (31) employing the $\exp(-\psi(\eta))$ expansion method at $a = 1.6, c = 0.3, A_0 = 0.01, \lambda = 0.003, \mu = 0.002$.

$$u(x,y,z,t) = f_0 + f_1\left(\frac{1}{(\mu/\lambda) + \alpha e^{\lambda(ax+by+dz-ct)}}\right) + f_2\left(\frac{1}{(\mu/\lambda) + \alpha e^{\lambda(ax+by+dz-ct)}}\right)^2. \tag{52}$$

4. Graphical Illustrations

Graphs depict the behavior of the solutions that we were able to obtain. This section includes the presentation of some figures that highlight various potential solutions.

In Figure 1, we exhibit the graph of (33) with (31) applying the $\exp(-\psi(\eta))$ expansion method at $a = 1.8, c = 0.6, A_0 = 0.3, \lambda = 0.3, \mu = 0.001$. In Figure 2, (35) with (31) is plotted employing the $\exp(-\psi(\eta))$ expansion method at $a = 1.6, c = 0.3, A_0 = 0.01, \lambda = 0.003, \mu = 0.002$. The graph of (37) with (31) is presented in Figure 3 applying the $\exp(-\psi(\eta))$ expansion method at $a = 1.8$, $c = 0.6, A_0 = 0.02, \lambda = 0.01, \mu = 0$. Figure 4 present the graph of (38)with (31) using the $\exp(-\psi(\eta))$ expansion method at $a = 1.8, c = 0.5, A_0 = 0.01, \mu = 0.1$. In Figure 5, we offer the graph of (39) with (31) using the $\exp(-\psi(\eta))$ expansion method at $a = 1.8, c = 0.5, A_0 = 0.01$. Figure 6 present the graph of (44) with (42) employing the $(G'/kG' + G + r)$ expansion method at $m_0 = 0.05, a = 1, b = 0.9, c = 0.3, k = 1, \lambda = \sqrt{5}, \mu = 1.1, p_2 = -1.87671 p_1$. In Figure 7 we plot (46) with (42) employing the $(G'/kG' + G + r)$ expansion method at $m_0 = 0.01, a = 0.6, b = 0.7, c = 0.5, k = 1, \quad \lambda = \sqrt{2}, \mu =$

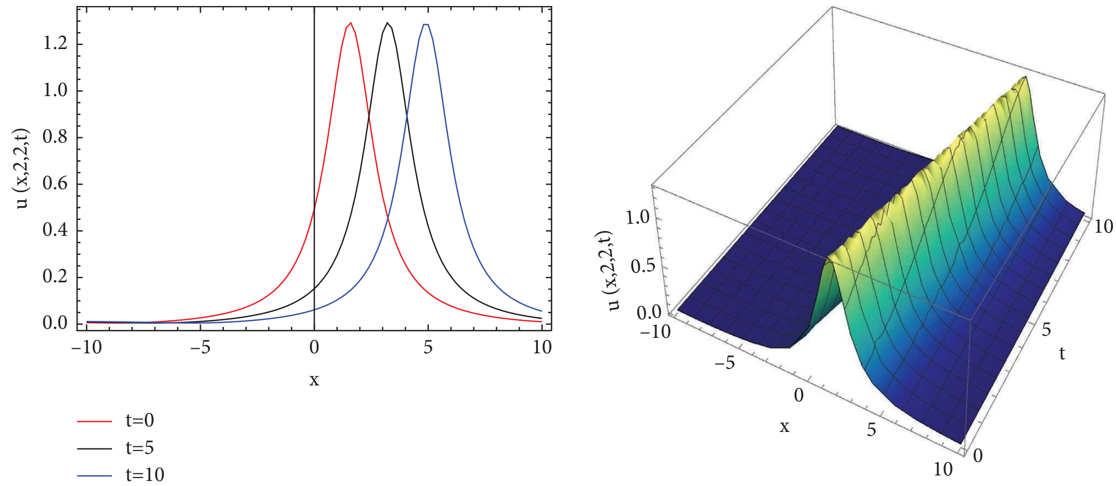

Figure 3: 2D and 3D graphs, respectively, corresponding to (37) with (31) employing the $\exp(-\psi(\eta))$ expansion method at $a = 1.8, c = 0.6, A_0 = 0.02, \lambda = 0.01, \mu = 0$.

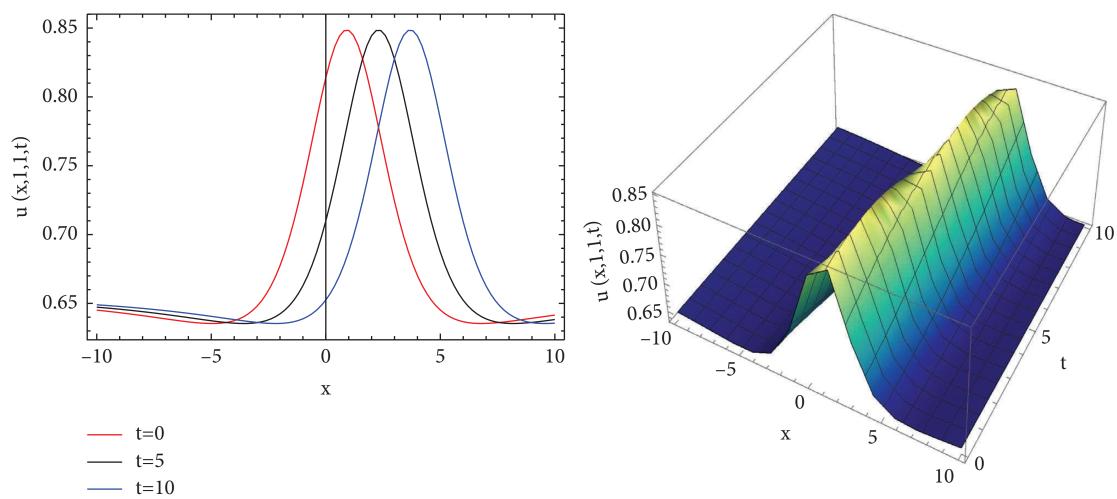

Figure 4: 2D and 3D graphs, respectively, corresponding to (38) with (31) employing the $\exp(-\psi(\eta))$ expansion method at $a = 1.8, c = 0.5, A_0 = 0.01, \mu = 0.1$.

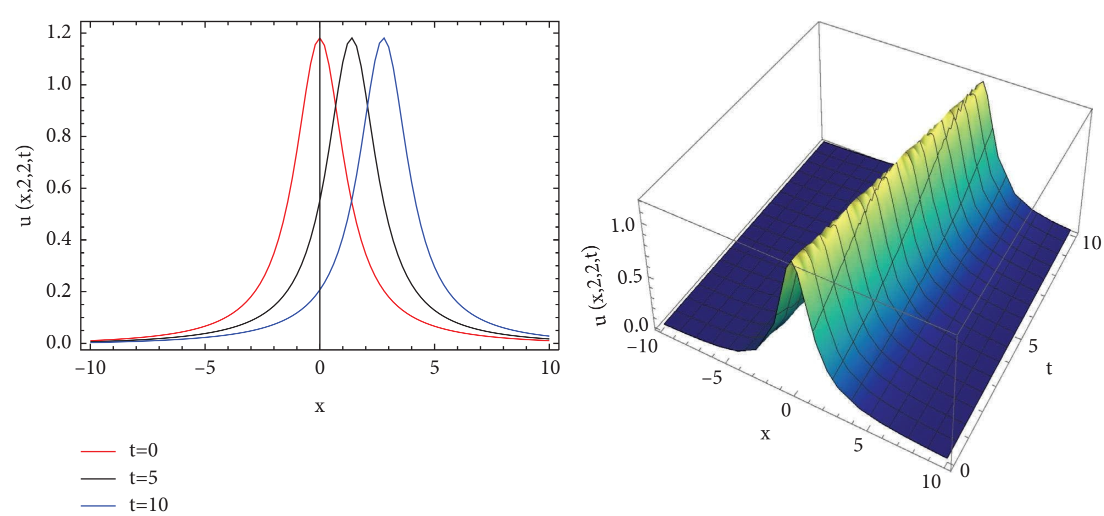

Figure 5: 2D and 3D graphs, respectively, corresponding to (39) with (31) employing the $\exp(-\psi(\eta))$ expansion method at $a = 1.8, c = 0.5, A_0 = 0.01$.

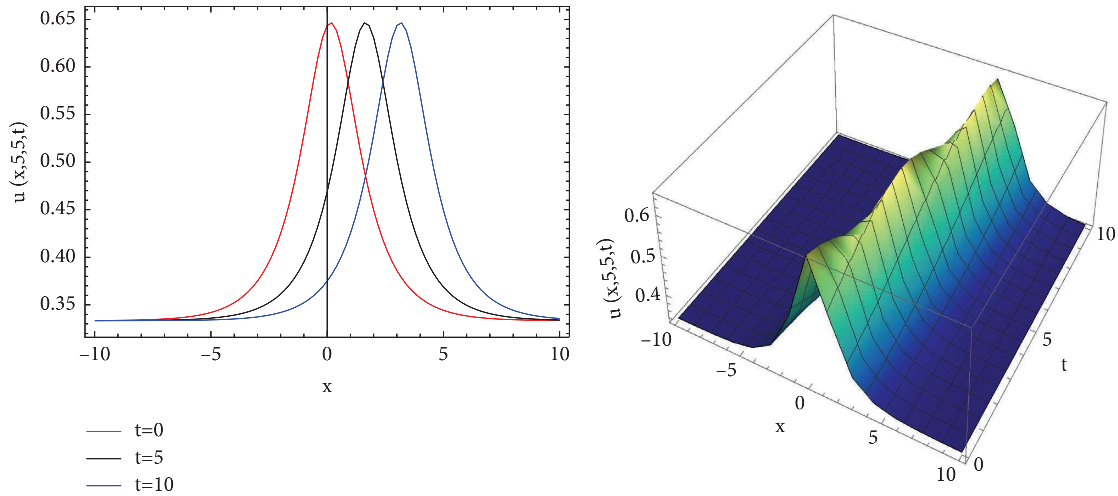

FIGURE 6: 2D and 3D graphs, respectively, corresponding to (44) with (42) employing the $(G'/kG'+G+r)$ expansion method at $m_0 = 0.05, a = 1, b = 0.9, c = 0.3, k = 1, \lambda = \sqrt{5}, \mu = 1.1, p_2 = -1.87671 p_1$.

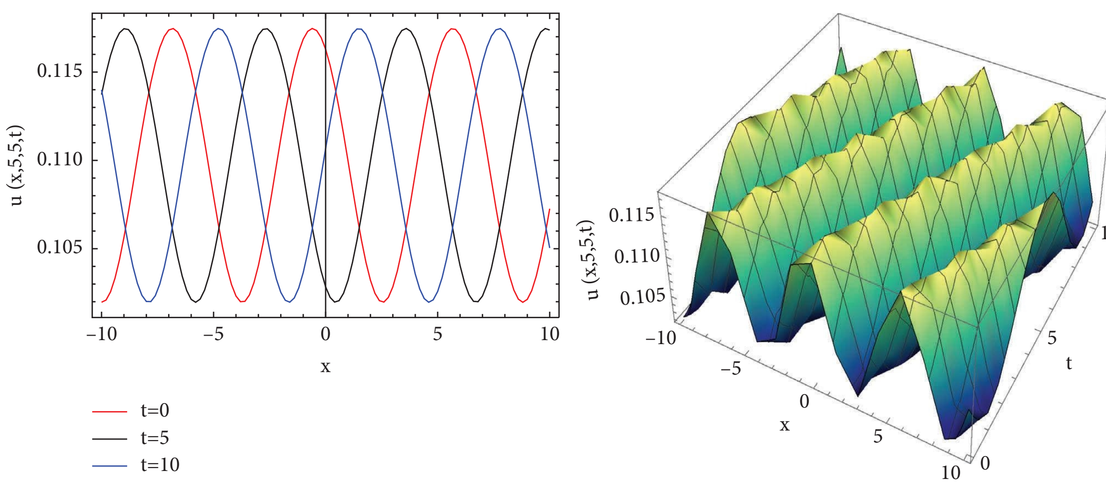

FIGURE 7: 2D and 3D graphs, respectively, corresponding to (46) with (42) employing the $(G'/kG'+G+r)$ expansion method at $m_0 = 0.01, a = 0.6, b = 0.7, c = 0.5, k = 1, \lambda = \sqrt{2}, \mu = 1.2, p_2 = 2.85654 p_1$.

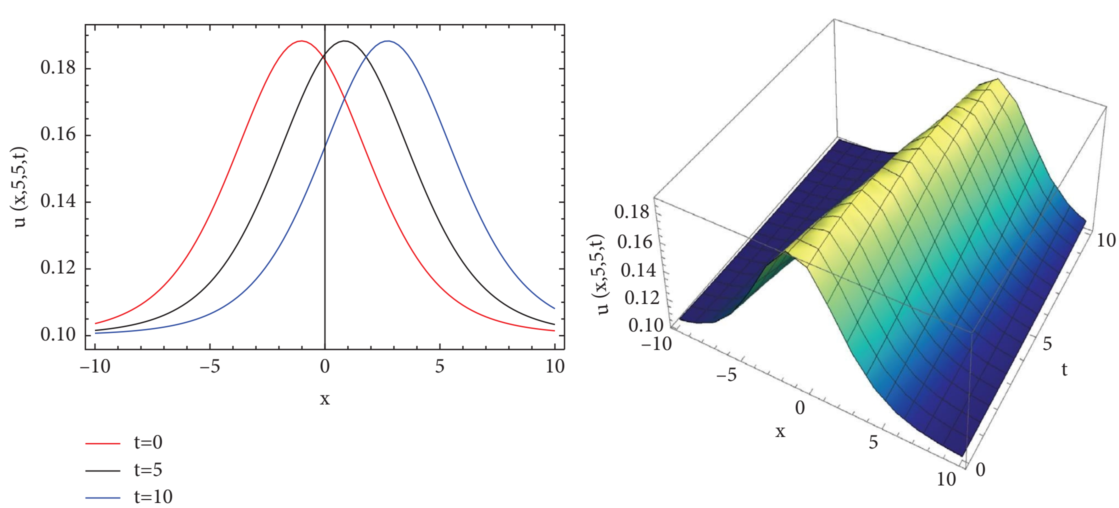

FIGURE 8: 2D and 3D graphs, respectively, corresponding to (52) with (50) employing the Bernoulli sub-ODE method at $f_0 = 0.1, \lambda = 0.2, c = 0.6, \mu = 0.1, \alpha = 4, a = 1.6$.

1.2, $p_2 = 2.85654 p_1$. At the end, Figure 8 present the graph of (52) with (50) employing Bernoulli sub-ODE method at $f_0 = 0.1, \lambda = 0.2, c = 0.6, \mu = 0.1, \alpha = 4, a = 1.6$.

5. Discussion

Using graphs as a means to display and explain potential solutions are one of the most effective approaches. The profile of every two-dimensional graph given the values of the parameters that are supplied, as depicted with each graph, is in the shape of a bell. The behavior of a soliton is seen in Figures 1–6, 8, and as time passes $t = 0, 5, 10$, the wave moves to the right. On the other hand, as time passes from $t = 0$ to $t = 5$, the wave moves to the left, as shown in Figure 7. In conclusion, we are able to conclude that we have demonstrated three useful approaches to solving the $(3 + 1)$-dimensional Sakovich equation. Also, the solutions that we presented are fully consistent with the properties of the soliton waves, as they keep their shape over time, and even when two or more waves overlap, separation occurs and each wave still maintains its shape and properties. This is evidenced by the fact that the soliton waves are able to maintain their shape and properties even when they are superimposed on one another. The graphs that were presented make the physical explanations very simple to see, which is another thing that is obvious.

6. Conclusion

In this study, we investigate the new three-and-a-half-plus-dimensional Sakovich equation, which has the potential to be applied to the description of additional dispersion and nonlinear effects in order to accommodate a wide range of applications. We do three different analytical procedures, including the $\exp(-\psi(\eta))$ expansion methods, the novel method $(G'/kG' + G + r)$ expansion method, and the Bernoulli sub-ODE method. We are able to obtain soliton solutions in a variety of forms, such as rational, exponential, hyperbolic, and trigonometric, which introduce singular, dark, bright, periodic, and other types of optical solitons. The findings demonstrate the efficacy as well as the potential of the strategies that were employed. In conclusion, the solutions that we have shown in this work are fully compatible with the analytical solutions that were acquired in the previous work [18]. We have high hopes that this work will be extended to include fractional derivatives in subsequent work, and that we will also be able to solve this equation using a variety of numerical approaches.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Acknowledgments

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