Hindawi
Journal of Mathematics
Volume 2023, Article ID 4926900, 12 pages
https://doi.org/10.1155/2023/4926900

Research Article

Fuzzy Spectral Spaces and Fuzzy Congruences of a Heyting ADL

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Received 27 October 2022; Revised 18 December 2022; Accepted 19 December 2022; Published 13 January 2023

Academic Editor: Tareq Al-shami

In this paper, the space $\mathfrak{F}_p$ of fuzzy prime ideals of Heyting almost distributive lattice H is studied, and it is shown that the collection of all sets $\mathfrak{M}(\eta)$ is a topology on $\mathfrak{F}_p$, where η is a fuzzy ideal on H and $\mathfrak{M}(\eta) = \left\{\theta \in \mathfrak{F}_p | \eta \nsubseteq \theta\right\}$. The only compact subset of the space $\mathfrak{F}_p$ is given. A fuzzy congruence relation Θ on H is defined, and the homomorphism between the set of all fuzzy ideals of H and the set of all fuzzy ideals of H/Θ is established. Furthermore, we established an isomorphism between fuzzy spectrum of H and fuzzy spectrum of H/Θ.

1. Introduction

The class of distributive lattices has many interesting properties, which lattices, in general, do not have. For this reason, Swamy and Rao [1] introduced the concept of an almost distributive lattice (ADL) as a common abstraction of lattice and ring theoretic generalizations of a Boolean algebra. In [1], it is proved that the commutativity of $\vee$, the commutativity of $\wedge$, the right distributivity of $\vee$ over $\wedge$, and the absorption law $(s \wedge t) \vee s = t$ are all equivalent to each other and whenever any one of these properties holds, an ADL Abecomes a distributive lattice. The concept of an ideal in A was introduced analogous to that in a distributive lattice, and it was observed that the set $PI(R)$ of all principal ideals of A forms a distributive lattice.

These results enable us to extend many existing concepts from the class of distributive lattices to the class of ADLs. Rao et al. [2] introduced the concept of Heyting almost distributive lattices (HADLs) as a generalization of Heyting algebra in the class of ADLs, and HADL is characterized in terms of the set of all its principal ideals. Rao and Ravi Kumar [3] proved that given any multiplicatively closed subset S of A disjoint from an ideal I of A, there exists a prime ideal belonging to I which is disjoint from S.

Sambasiva Rao and Rao in [4] studied topological properties of the space of prime ideals of a HADL. These authors proved that there is an isomorphism between a Heyting algebra and the open base of its prime spectrum. This mapping is epimorphism, and it is isomorphism if and only if the HADL is a Heyting algebra.

In [5], the concept of fuzzy set theory as a generalization of classical set theory was introduced by Zadeh. Rosenfield [6] started the pioneering work in the domain of fuzzification of algebraic objects on fuzzy groups. In particular, Bo and Wangming [7] and Santhi Sundar Raj et al. [8] have laid down the foundation for fuzzy ideals of a lattice and an ADL, respectively. In [9, 10], Alemayehu et al. introduced the concepts of belligerent fuzzy GE-filter of GE-algebras and beta-fuzzy filters of stone almost distributive lattices, respectively. Santhi Sundra Raj et al. [8] introduced the concept of fuzzy prime ideals of almost distributive lattices. In 1998, Swamy and Raju [11] introduced the concept of fuzzy ideals and fuzzy congruences of distributive lattices and showed that there is a one-to-one correspondence between the lattice of fuzzy ideals and the lattice of fuzzy congruences of A. In [12], Alaba and Addis studied the fuzzy congruence relations of an ADL A, and they gave the smallest fuzzy congruence on A such that its quotient is a distributive lattice.

This paper comprises of four sections. The first two sections deal with the introductory and preliminary concepts. In Section 3 of this paper, we studied some properties of fuzzy prime ideals of a HADL H, and it is shown that the collection $\sum = \{\mathfrak{M}(\eta)$: η is a fuzzy ideal of $H\}$ is a topology on $\mathfrak{F}_p$. The fact that the set $\mathfrak{M}(x_\alpha) = \{\theta \in \mathfrak{F}_p | x_\alpha \notin \theta\}$ is the only compact subset of the space is proved. Furthermore, it is shown that an ADL A is a distributive lattice if and only if the mapping between A and $\{\mathfrak{M}(x_\alpha)\}_{x_\alpha \in Y}$ where $Y = \{x_\alpha | x \in A, \alpha \in (0,1]\}$ is injective. In Section 4, a fuzzy congruence relation Θ on a HADL H is defined, and the homomorphism between the set $FI(H)$ of all fuzzy ideals of H and the set $FI(H/\Theta)$ of all fuzzy ideals of H/Θ is established, where $H/\Theta = \{\Theta_x | x \in H\}$. Furthermore, we proved that the mapping between fuzzy spectrum of $H(F\mathrm{spec}(H))$ and fuzzy spectrum of $H/\Theta(F\mathrm{spec}(H/\Theta))$ is bijective.

2. Preliminaries

In this section, we recall basic definitions and results which will be used in this article.

Definition 1. An algebra $(A, \vee, \wedge, 0)$ of type $(2,2,0)$ is called an almost distributive lattice if it satisfies the following conditions: for all r, s and $t \in A$,

(1) $0 \wedge r = 0$

(2) $r \vee 0 = r$

(3) $r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$

(4) $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$

(5) $(r \vee s) \wedge t = (r \wedge t) \vee (s \wedge t)$

(6) $(r \vee s) \wedge s = s$

Let $r, s \in A$ and " $\leq$ " be a partial ordering on A. We read r is less than or equal to s and write $r \leq s$ if $r \wedge s = r$. Equivalently, $r \vee s = s$. If an element m is a maximal with respect to the partial ordering $\leq$ on A, then m is said to be maximal.

Theorem 1. *Let A be an ADL and $m \in A$. Then, the following are equivalent:*

(1) m is maximal with respect to $\leq$

(2) $m \vee s = m$, for all $s \in A$

(3) $m \wedge s = s$, for all $s \in A$

Definition 2. Let $(A, \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Suppose $\longrightarrow$ is a binary operation on A satisfying the following conditions:

(1) $r \longrightarrow r = m$

(2) $(r \longrightarrow s) \wedge s = s$

(3) $r \wedge (r \longrightarrow s) = r \wedge s \wedge m$

(4) $r \longrightarrow (s \wedge t) = (r \longrightarrow s) \wedge (r \longrightarrow t)$

(5) $(r \vee s) \longrightarrow t = (r \longrightarrow t) \wedge (s \longrightarrow t)$ for all $r, s, t \in A$

Then, $(A, \vee, \wedge, \longrightarrow, 0, m)$ is called a Heyting almost distributive lattice or simply a HADL.

Lemma 1. *Let $(A, \vee, \wedge, \longrightarrow, 0, m)$ be a HADL A. Then, for $r, s, t, c \in H$, the following hold:*

(1) $(r \longrightarrow t) \wedge c = (r \wedge c \longrightarrow t \longrightarrow c) \wedge c$

(2) $r \wedge c = s \wedge c \Rightarrow (r \longrightarrow t) \wedge c = (s \longrightarrow t) \wedge c$

(3) $r \wedge c = s \wedge c \Rightarrow (t \longrightarrow r) \wedge c = (t \longrightarrow s) \wedge c$

Definition 3. Let $(A, \vee, \wedge, 0, m)$ and $(A', \vee, \wedge, 0', m')$ be two HADLs. Then, the mapping α: $A \longrightarrow A'$ is called a homomorphism of A into A' if for any $s, t \in A$, the following conditions hold:

(1) $\alpha(s \wedge t) = \alpha(s) \wedge \alpha(t)$

(2) $\alpha(s \vee t) = \alpha(s) \vee \alpha(t)$

(3) $\alpha(s \longrightarrow t) = \alpha(s) \longrightarrow \alpha(t)$

(4) $\alpha(0) = 0'$

For any set A, a function ν: $A \longrightarrow [0,1]$ is called a fuzzy subset of A, where $[0,1]$ is a unit interval of real numbers.

Definition 4. Let η be a fuzzy subset of an ADL A. For any $\alpha \in [0,1]$, we denote the level subset η_α, i.e.,

$$\eta_\alpha = \{x \in A\colon \alpha \leq \eta(x)\}. \quad (1)$$

Definition 5. A fuzzy subset η of an ADL A is said to be a fuzzy ideal of A if and only if

(1) $\eta(0) = 1$

(2) $\eta(s \vee t) = \eta(s) \wedge \eta(t)$, for all $s, t \in A$

From [13], remember that for each $x \in L$ and $\alpha \in [0,1)$, the fuzzy subset x_α of L given by

$$x_\alpha(b) = \begin{cases} \alpha, & \text{if } x = b, \\ 0, & \text{if otherwise,} \end{cases} \quad (2)$$

for all $b \in L$ is called a fuzzy point of H. In this case, an element x of L is called the support of x_α and α is its value. For a fuzzy point x_α of H and a fuzzy subset μ of H, we write $x_\alpha \in \mu$ (or $x_\alpha \subseteq \mu$) to say that $\mu(x) \geq \alpha$.

3. Fuzzy Prime Spectrum of HADLs

This section deals with some properties of fuzzy prime ideals of HADL. The fact that the set $\mathfrak{M}(x_\alpha) = \{\theta \in \mathfrak{F}_p | x_\alpha \notin \theta\}$ (where $\mathfrak{F}_p$ is the class of all fuzzy prime ideals of a HADL H) is the only compact subset of the space $\mathfrak{F}_p$ is proved. Furthermore, it is shown that an ADL A is a distributive lattice if and only if the mapping between A and $\{\mathfrak{M}(x_\alpha)\}_{\alpha \in Y}$ where $Y = \{x_\alpha | x \in A, \alpha \in (0,1]\}$ is injective. Unless otherwise mentioned H stands for Heyting almost distributive lattices.

Definition 6. A fuzzy subset η of H is called a fuzzy ideal if for all $s,t \in H$, it satisfies the following conditions:

(1) $\eta(0) = 1$
(2) $\eta(s \vee t) \geq \eta(s) \wedge \eta(t)$
(3) $\eta(s \wedge t) \geq \eta(s) \vee \eta(t)$

From this definition, we have got that η is a fuzzy ideal of H if and only if one of the following conditions holds:

(1) $\eta(0) = 1$ and $\eta(s \vee t) = \eta(s) \wedge \eta(t)$, for all $s,t \in H$
(2) $\eta(0) = 1$ and $\eta(s \vee t) \geq \eta(s) \wedge \eta(t)$ and $\eta(s \wedge t) \geq \eta(s) \vee \eta(t)$, for all $s,t \in H$

Definition 7. A fuzzy subset η of H is called a fuzzy filter if for all $s,t \in H$ and a maximal element m, it satisfies the following conditions:

(1) $\eta(m) = 1$
(2) $\eta(s \vee t) \geq \eta(s) \vee \eta(t)$
(3) $\eta(s \wedge t) \geq \eta(s) \wedge \eta(t)$

From this definition, we have got that λ is a fuzzy filter of H if and only if one of the following conditions holds:

(1) $\lambda(m) = 1$ and $\lambda(s \wedge t) = \lambda(s) \wedge \eta(t)$, for all $s,t \in H$
(2) $\lambda(m) = 1$ and $\lambda(s \vee t) \geq \lambda(s) \vee \lambda(t)$ and $\lambda(s \wedge t) \geq \eta(s) \wedge \eta(t)$, for all $s,t \in H$

Example 1. Let $H = \{0, r, s, t\}$. *Define the binary operations* $\vee$, $\wedge$, *and* $\longrightarrow$ *on* A as follows:

$\vee$	0	r	s	t
0	0	r	s	t
r	r	r	r	r
s	s	s	s	s
t	t	r	s	t

$\wedge$	0	r	s	t
0	0	0	0	0
r	0	r	s	t
s	0	r	s	t
t	0	t	t	t

$\rightarrow$	0	r	s	t
0	r	r	r	r
r	0	r	s	t
s	0	r	r	t
t	0	r	r	r

Then, $(H, \vee, \wedge, \longrightarrow, 0)$ is a Heyting ADL with maximal elements r and s.

For fuzzy subsets μ and λ of A, define $\mu(0) = 1$, $\mu(t) = 0.8$ and $\mu(r) = \mu(s) = 0.5$, and $\lambda(r) = \lambda(s) = 1$, $\lambda(t) = 0.5$, and $\lambda(0) = 0.3$. Then, it can be easily checked that μ is a fuzzy ideal of A and λ is a fuzzy filter of A.

Theorem 2. *Let* η *be a fuzzy ideal in* H *and* $s,t \in H$. *Then,* $\eta(s \wedge t) = \eta(t \wedge s)$.

Proof. Suppose that η is a fuzzy ideal of H and $s,t \in H$. Then, $\eta(s \wedge t) \leq \eta(s \wedge t \wedge s) = \eta(t \wedge s \wedge s) = \eta(t \wedge s)$.

With similar procedure, $\eta(t \wedge s) \leq \eta(s \wedge t)$. Hence, $\eta(t \wedge s) = \eta(s \wedge t)$. □

Theorem 3. *Let* η *be a fuzzy filter in* H *and* $s,t \in H$. *Then,* $\eta(s \vee t) = \eta(t \vee s)$.

Remark 1. In Example 1, *we have* $r \wedge s = s$, $s \wedge r = r$, $r \wedge s \neq s \wedge r$, *and* $\mu(r \wedge s) = \mu(s \wedge r)$.

We denote the class of all fuzzy ideals of H by $FI(H)$.

Theorem 4. *The set* $FI(H)$ *forms a Heyting algebra.*

Proof. Clearly, $FI(H)$ is a bounded distributive lattice with greatest element χ_H and least element χ_0. In addition to this, the intersection of a class of fuzzy ideals is a fuzzy ideal. This implies that it forms a complete lattice. For any two fuzzy ideals $\eta, \theta \in FI(H)$, define an operation $\longrightarrow$ on $FI(H)$ as follows:

$$\eta \longrightarrow \theta = \sup\{\xi \in FI(H) | \eta \cap \xi \subseteq \theta\}. \quad (3)$$

Let μ, η and θ be fuzzy ideals of H such that $\eta \cap \mu \subseteq \theta$. Now we prove that $\eta \longrightarrow \theta$ is the largest fuzzy ideal of H. Since $\mu \cap \eta \subseteq \theta$ and $\theta(0) = 1$, we have got that $\eta \longrightarrow \theta(0) = \mathrm{Sup}\{\mu(0): \mu \in FI(H), \eta \cap \mu \subseteq \theta\} \geq \theta(0) = 1$. Again for any $x, y \in H$, $(\eta \longrightarrow \theta)(x) \wedge (\eta \longrightarrow \theta)(y)$

$$\begin{aligned} &= \mathrm{Sup}\{\mu(x): \mu \in FI(H), \eta \cap \mu \subseteq \theta\} \wedge \mathrm{Sup}\{\sigma(x): \sigma \in FI(H), \eta \cap \sigma \subseteq \theta\} \\ &= \mathrm{Sup}\{\mu(x) \wedge \sigma(y): \mu, \sigma \in FI(H), \eta \cap \mu \subseteq \theta, \eta \cap \sigma \subseteq \theta\} \\ &\leq \mathrm{Sup}\{(\mu \vee \sigma)(x) \wedge (\mu \vee \sigma)(y): \mu, \sigma \in FI(H), \eta \cap \mu \subseteq \theta, \eta \cap \sigma \subseteq \theta\}. \end{aligned} \quad (4)$$

For each $\mu, \sigma \in FI(H)$ such that $\eta \cap \mu \subseteq \theta$ and $\eta \cap \sigma \subseteq \theta, \mu \vee \sigma \in FI(H)$ and $(\mu \vee \sigma) \cap \eta \subseteq \theta$. Then, $(\eta \longrightarrow \theta)(x) \wedge (\eta \longrightarrow \theta)(y) \leq \text{Sup}\{\lambda(x) \wedge \lambda(y): \lambda \in FI(H), \eta \cap \lambda \subseteq \theta\} = \text{Sup}\{\lambda(x \vee y): \lambda \in FI(H), \eta \cap \lambda \subseteq \theta\} = (\eta \longrightarrow \theta)(x \vee y)$. Thus, $(\eta \longrightarrow \theta)(x \vee y) \geq (\eta \longrightarrow \theta)(x) \wedge (\eta \longrightarrow \theta)(y)$. Now we show that $(\eta \longrightarrow \theta)(x) \leq (\eta \longrightarrow \theta)(x \wedge y)$ and $(\eta \longrightarrow \theta)(y) \leq (\eta \longrightarrow \theta)(x \wedge y)$. Thus, $(\eta \longrightarrow \theta)(x) = \text{Sup}\{\mu(x): \mu \in FI(H), \eta \cap \mu \subseteq \theta\} \leq \text{Sup}\{\mu(x \wedge y): \mu \in FI(H), \eta \cap \mu \subseteq \theta\} = (\eta \longrightarrow \theta)(x \wedge y)$. Similarly, $(\eta \longrightarrow \theta)(y) \leq (\eta \longrightarrow \theta)(x \wedge y)$. So, $(\eta \longrightarrow \theta)(x \wedge y) \geq (\eta \longrightarrow \theta)(x) \vee (\eta \longrightarrow \theta)(y)$. Hence, $\eta \longrightarrow \theta$ is a fuzzy ideal of H. From the definition of $\eta \longrightarrow \theta$ and the given condition $\eta \cap \mu \subseteq \theta$, it is clear that $\eta \cap (\eta \longrightarrow \theta) \subseteq \theta$. Suppose that σ is a fuzzy ideal of H containing $\eta \longrightarrow \theta$ such that $\eta \cap \sigma \subseteq \theta$. It contradicts the definition of $\eta \longrightarrow \theta$. Hence, $\eta \longrightarrow \theta$ is the largest fuzzy ideal of H such that $\eta \cap \sigma \subseteq \theta$ for any $\sigma \in FI(H)$.

Therefore, $(FI(H), \cup, \cap, \longrightarrow, \chi_{\{0\}}, \chi_H)$ is a Heyting algebra. □

Lemma 2. *Let η be a fuzzy ideal of H, $s \in H$, and $\beta \in [0, 1)$. If $\eta(s) \leq \beta$, then there exists a prime fuzzy ideal θ of H such that $\eta \subseteq \theta$ and $\theta(s) \leq \beta$.*

Let $\mathfrak{F}_p$ denote the set of all fuzzy prime ideals of H. For any fuzzy ideal η of H, define $\mathfrak{M}(\mu) = \{\theta \in \mathfrak{F}_p | \mu \nsubseteq \theta\}$, and for any fuzzy point x_α, define $\mathfrak{M}(x_\alpha) = \{\theta \in \mathfrak{F}_p \mid x_\alpha \notin \theta\}$. Then, we have the following results.

Theorem 5. *Let η be a fuzzy ideal of H. Then, $x_\alpha \in \eta$ if and only if $\mathfrak{M}(x_\alpha) \subseteq \mathfrak{M}(\eta)$.*

Proof. Let $x_\alpha \in \eta$ and $\theta \in \mathfrak{M}(x_\alpha)$. Then, $x_\alpha \notin \theta$. Hence, we get $\eta \nsubseteq \theta$. Thus, $\theta \in \mathfrak{M}(\eta)$. So, $\mathfrak{M}(x_\alpha) \subseteq \mathfrak{M}(\eta)$. Conversely, assume that $\mathfrak{M}(x_\alpha) \subseteq \mathfrak{M}(\eta)$. Suppose that $x_\alpha \notin \eta$. This implies that $\eta(x) < \alpha$. Then, from Lemma 2, there exists $\theta \in \mathfrak{F}_p$ such that $\eta \subseteq \theta$ and $\theta(x) < \alpha$. This shows that $x_\alpha \notin \theta$ so that $\theta \in \mathfrak{M}(x_\alpha)$. It follows that $\theta \in \mathfrak{M}(\eta)$ where $\eta \nsubseteq \theta$ which is a contradiction. Therefore, $x_\alpha \in \eta$. □

Lemma 3. *For any fuzzy ideals η and θ of H, the following conditions hold:*

(1) $\eta \subseteq \theta$ if and only if $\mathfrak{M}(\eta) \subseteq \mathfrak{M}(\theta)$

(2) $\mathfrak{M}(\eta) \cap \mathfrak{M}(\theta) = \mathfrak{M}(\eta \cap \theta)$

(3) $\mathfrak{M}(\eta) \cup \mathfrak{M}(\theta) = \mathfrak{M}(\eta \vee \theta)$

(4) $\mathfrak{M}(\eta \longrightarrow \theta) = \mathfrak{M}(\theta) \cup \{\mathfrak{F}_p - \mathfrak{M}(\eta)\}$

Proof

(1) Suppose $\mathfrak{M}(\eta \longrightarrow \theta) = \mathfrak{M}(\theta) \cup \{\mathfrak{F}_p - \mathfrak{M}(\eta)\}$ and θ are fuzzy ideals of H such that $\eta \subseteq \theta$. Let $\lambda \in \mathfrak{M}(\eta)$. This implies that $\eta \nsubseteq \lambda$. This implies that $\theta \nsubseteq \lambda$, and so $\lambda \in \mathfrak{M}(\theta)$. Hence, $\mathfrak{M}(\eta) \subseteq \mathfrak{M}(\theta)$. Conversely, suppose that $\mathfrak{M}(\eta) \subseteq \mathfrak{M}(\theta)$. We prove that $\eta \subseteq \theta$. Assume $\eta \nsubseteq \theta$. This implies that $\eta(x) > \theta(x)$ for some $x \in H$. Put $\eta(x) = \alpha$. This implies that $x_\alpha \in \eta$ and $x_\alpha \notin \theta$. By Theorem 5, we have $\mathfrak{M}(x_\alpha) \subseteq \mathfrak{M}(\eta)$ and $\mathfrak{M}(x_\alpha) \nsubseteq \mathrm{M}(\theta)$. Hence, $\mathfrak{M}(\eta) \nsubseteq \mathrm{M}(\theta)$, which is contradiction. Hence, $\eta \subseteq \theta$.

(2) Clearly, $\mathfrak{M}(\eta \cap \theta) \subseteq \mathfrak{M}(\eta) \cap \mathfrak{M}(\theta)$. Conversely, let $\lambda \in \mathfrak{F}_p$. Suppose that $\lambda \in \mathfrak{M}(\eta) \cap \mathfrak{M}(\theta)$. Then, $\lambda \in \mathfrak{M}(\eta)$ and $\lambda \in \mathfrak{M}(\theta)$. This means $\eta \nsubseteq \lambda$ and $\theta \nsubseteq \lambda$ where $\eta \cap \theta \nsubseteq \lambda$. Consequently, $\lambda \in \mathfrak{M}(\eta \cap \theta)$. Therefore, $\mathfrak{M}(\eta) \cap \mathfrak{M}(\theta) \subseteq \mathfrak{M}(\eta \cap \theta)$.

(3) Clearly, $\mathfrak{M}(\eta) \cup \mathfrak{M}(\theta) \subseteq \mathfrak{M}(\eta \vee \theta)$. On the other hand, if $\lambda \in \mathfrak{M}(\eta \vee \theta)$, then $\eta \vee \theta \nsubseteq \lambda$. Suppose that $\eta \subseteq \lambda$ and $\theta \subseteq \lambda$. This shows that $\eta \vee \theta \subseteq \lambda$ which is a contradiction. Thus, $\eta \nsubseteq \lambda$ or $\theta \nsubseteq \lambda$. Following this, $\lambda \in \mathfrak{M}(\eta)$ or $\lambda \in \mathfrak{M}(\theta)$ where $\lambda \in \mathfrak{M}(\eta) \cup \mathfrak{M}(\theta)$. It follows that $\mathfrak{M}(\eta \vee \theta) \subseteq \mathfrak{M}(\eta) \cup \mathfrak{M}(\theta)$. Hence, $\mathfrak{M}(\eta \vee \theta) = \mathfrak{M}(\eta) \cup \mathfrak{M}(\theta)$.

(4) Let $\lambda \in \mathfrak{M}(\eta \longrightarrow \theta)$. Then, $\eta \longrightarrow \theta \nsubseteq \lambda$. This implies that there exists $\sigma \in FI(H)$ such that $\sigma \nsubseteq \lambda$ and $\eta \cap \sigma \subseteq \theta$. This implies that $\sigma(x) > \lambda(x)$ for some $x \in H$.

$$
\begin{aligned}
&\text{If } \eta \subseteq \lambda, \text{ then we get } \lambda \notin \mathfrak{M}(\eta) \\
&\Rightarrow \lambda \in \mathfrak{M}(\theta) \cup \{\mathfrak{F}_p - \mathfrak{M}(\eta)\} \\
&\Rightarrow \mathfrak{M}(\eta \longrightarrow \theta) \subseteq \mathfrak{M}(\theta) \cup \{\mathfrak{F}_p - \mathfrak{M}(\eta)\} \\
&\text{If } \eta \nsubseteq \lambda, \text{ then there exists, } y \in H \text{ such that } \eta(y) > \lambda(y) \\
&\Rightarrow (\sigma \cap \eta)(x \wedge y) \geq \sigma(x \wedge y) \wedge \eta(x \wedge y) > \sigma(x) \wedge \eta(y) > \lambda(x) \wedge \lambda(y) \\
&= \lambda(x \wedge y) \\
&\Rightarrow \sigma \cap \eta \nsubseteq \lambda \\
&\text{Since } \sigma \cap \eta \subseteq \theta, \theta \nsubseteq \lambda \text{ and so } \lambda \in \mathfrak{M}(\theta) \\
&\Rightarrow \lambda \in \mathfrak{M}(\theta) \cup \{\mathfrak{F}_p - \mathfrak{M}(\eta)\} \\
&\Rightarrow \mathfrak{M}(\eta \longrightarrow \theta) \subseteq \mathfrak{M}(\theta) \cup \{\mathfrak{F}_p - \mathfrak{M}(\eta)\}.
\end{aligned}
\tag{5}
$$

Conversely, $\mathfrak{M}(\sigma) \subseteq \mathfrak{M}(\theta) \cup \{\mathfrak{F}_p - \mathfrak{M}(\eta)\}$ for some fuzzy ideals of H. Let $\lambda \in \mathfrak{F}_p$ be such that $\lambda \in \mathfrak{M}(\eta \cap \sigma)$.

$$\begin{aligned} &\Rightarrow \lambda \in \mathfrak{M}(\eta) \text{ and } \lambda \in \mathfrak{M}(\sigma) \\ &\Rightarrow \lambda \notin (\mathfrak{F}_p - \mathfrak{M}(\eta)) \text{ and } \lambda \in \mathfrak{M}(\sigma) \\ &\Rightarrow \lambda \in \mathfrak{M}(\theta). \end{aligned} \tag{6}$$

Thus, $\mathfrak{M}(\eta \cap \sigma) \subseteq \mathfrak{M}(\theta)$.

$$\begin{aligned} &\Rightarrow \mathfrak{M}(\eta \cap \sigma) \subseteq \mathfrak{M}(\theta) \\ &\Rightarrow \eta \cap \sigma \subseteq \theta \\ &\Rightarrow \sigma \subseteq \eta \longrightarrow \theta \\ &\Rightarrow \mathfrak{M}(\sigma) \subseteq \mathfrak{M}(\eta \longrightarrow \theta) \\ &\Rightarrow (\mathfrak{F}_p - \mathfrak{M}(\eta)) \cup \mathfrak{M}(\theta) \subseteq \mathfrak{M}(\eta \longrightarrow \theta). \end{aligned} \tag{7}$$

Hence, $\mathfrak{M}(\theta) \cup \{\mathfrak{F}_p - \mathfrak{M}(\eta)\} = \mathfrak{M}(\eta \longrightarrow \theta)$. □

Next, we discus the space of fuzzy prime ideals of HADL analogous with the hull-kernel topology.

Theorem 6. *Let $\Sigma = \{\mathfrak{M}(\mu)$: μ is a fuzzy ideal of $(H\}$. Then, Σ is a topology on $\mathfrak{F}_p$.*

Proof. Consider the fuzzy subsets λ_1, λ_2 of H defined as $\lambda_1(x) = 0$ and $\lambda_2(x) = 1$ for all $x \in H$. Clearly, $(\lambda_1]$ and λ_2 are fuzzy ideals of H. $(\lambda_1] \subseteq \eta$ for all $\eta \in \mathfrak{F}_p$, which is impossible. Thus, $\mathfrak{M}((\lambda_1]) = \varnothing$. Since each $\eta \in \mathfrak{F}_p$ is nonconstant, $\lambda_2 \nsubseteq \eta$ for all $\eta \in \mathfrak{F}_p$. Thus, $\mathfrak{F}_p(\lambda_2) = \mathfrak{F}_p$. This implies $\varnothing, \mathfrak{F}_p \in \Sigma$.

Also, for any fuzzy ideals λ_1 and λ_2 of H, by (2) in Lemma 3, we have $\mathfrak{M}(\lambda_1) \cap \mathfrak{M}(\lambda_2) = \mathfrak{M}(\lambda_1 \cap \lambda_2)$. This shows that Σ is closed under finite intersections. Next, let $\{\lambda_i, i \in \Omega\}$ be any family of fuzzy ideals of H. Now we prove that $\cup_{i\in\Omega}\mathfrak{M}(\lambda_i) = \mathfrak{M}((\cup_{i\in\Omega}\lambda_i])$. Let $\eta \in \mathfrak{M}((\cup_{i\in\Omega}\lambda_i])$; then, $(\cup_{i\in\Omega}\lambda_i] \nsubseteq \eta$, which implies that $\lambda_i \nsubseteq \eta$ for some $i \in \Omega$. Otherwise, if $\lambda_i \subseteq \eta$ for each $i \in \Omega$, it will be true that $(\cup_{i\in\Omega}\lambda_i] \subseteq \eta$. Thus, $\eta \in \cup_{i\in\Omega}\mathfrak{M}(\lambda_i)$ where $\mathfrak{M}((\cup_{i\in\Omega}\lambda_i]) \subseteq \cup_{i\in\Omega}\mathfrak{M}(\lambda_i)$. Clearly, $\cup_{i\in\Omega}\mathfrak{M}(\lambda_i) \subseteq \mathfrak{M}((\cup_{i\in\Omega}\lambda_i])$. Hence, $\cup_{i\in\Omega}\mathfrak{M}(\lambda_i) = \mathfrak{M}((\cup_{i\in\Omega}\lambda_i])$. Therefore, Σ is closed under arbitrary unions, and hence it is topology on $\mathfrak{F}_p$. □

The topological space $(\mathfrak{F}_p; \Sigma)$ is called the fuzzy prime spectrum of H, and it is denoted by $F\mathrm{spec}(H)$.

Theorem 7. *Let η and μ be fuzzy subsets of H. Then, the following conditions hold:*

(1) $\mathfrak{F}_p(\eta) = \mathfrak{F}_p(\mu) \Rightarrow (\eta] = (\mu]$

(2) $\mathfrak{F}_p(\eta) = \mathfrak{F}_p((\eta])$

Corollary 1. *For any $x, y \in H$ and $\alpha \in (0, 1]$, the following conditions hold:*

(1) If $x \leq y$, then $\mathfrak{M}(x_\alpha) \subseteq \mathfrak{M}(y_\alpha)$

(2) $\mathfrak{M}(x_\alpha) \cap \mathfrak{M}(y_\alpha) = \mathfrak{M}((x \wedge y)_\alpha)$

(3) $\mathfrak{M}(x_\alpha) \cup \mathfrak{M}(y_\alpha) = \mathfrak{M}((x \vee y)_\alpha)$

(4) $\cup_{x\in H, \alpha\in((0,1])}\mathfrak{M}(x_\alpha) = \mathfrak{F}_p$

(5) $\mathfrak{M}(x_\alpha) \subseteq \mathfrak{M}((x \vee m)_\alpha)$

(6) $\mathfrak{M}((x_\alpha]) = \mathfrak{M}(x_\alpha)$

(7) $\mathfrak{M}(0_\alpha) = \varnothing$

(8) $\mathfrak{M}((x \longrightarrow y)_\alpha) = \mathfrak{M}(y_\alpha) \cup \{\mathfrak{F}_p - \mathfrak{M}(x_\alpha)\}$

From Corollary 1, we have the following.

Corollary 2. *Let $\mathscr{B} = \{\mathfrak{M}(x_\beta): x \in H, \beta \in (0, 1]\}$. Then, $\mathscr{B}$ forms a base for a topology on Σ.*

Corollary 3. *$\mathfrak{F}_p$ is a compact space.*

For any fuzzy subset σ of H, $\mathfrak{F}_p(\sigma) = \{\eta \in \mathfrak{F}_p: \eta \nsubseteq \sigma\}$ is open set of $\mathfrak{F}_p$ and $(\mathscr{C})(\sigma) = \mathfrak{F}_p - \mathfrak{M}(\sigma)$ is a closed set of $\mathfrak{F}_p$. Also, every closed set in is of the form $\mathscr{C}(\sigma)$ for all fuzzy subsets of H. Then, we have the following.

Theorem 8. *Let $A \subseteq \mathfrak{F}_p$. Then, the closure of A (i.e., $\overline{A}$) is given by $\overline{A} = \mathscr{C}(\cap_{\eta\in A}\eta)$.*

Theorem 9. *Let $x \in H$ and $\beta \in [0, 1]$. Then, the set $\{\mathfrak{M}(x_\beta)\}$ forms a Heyting algebra. Moreover, $Y = \{x_\beta\}$ is epimorphic to this Heyting algebra.*

Proof. From (2) and (3) of Corollary 1 it is clear that

$$\mathfrak{M}^o = \langle \{\mathfrak{M}(x_\beta)\}_{x_{\beta\in Y}}, \cup, \cap, \varnothing, \mathfrak{F}_p \rangle, \tag{8}$$

is a bounded distributive lattice. Now define the operation $\longrightarrow$ on $\mathfrak{M}^o$ by

$$\mathfrak{M}(x_\beta) \longrightarrow \mathfrak{M}(y_\beta) = \mathfrak{M}((x \longrightarrow y)_\beta). \tag{9}$$

Then, it can be routinely verified that $\mathfrak{M}^o$ is a Heyting algebra. Define a mapping $g: H \longrightarrow \{\mathfrak{M}(x_\beta)\}_{x_\beta \in Y}$ by $g(x) = \mathfrak{M}(x_\beta)$ for any $x \in H$. Since $\theta(0) = 1 \geq \beta, 0_\beta \notin \theta$. Hence, $g(0) = \mathfrak{M}(0_\beta) = \varnothing$. Using Corollary 3.9, it can be easily obtained that g is a homomorphism in the sense of Heyting algebras. From the definition of g, clearly it is surjective. Therefore, Y is epimorphic to $\mathfrak{M}^o$. □

Theorem 10. *An ADL A is a distributive lattice if and only if the above map g is an injective.*

Proof. If A is a distributive lattice, then it is a known fact that the map g is an isomorphism. Conversely, assume that g is injective. For any $r, s \in A$, we have always $g(r \wedge s) = \mathfrak{M}((r \wedge s)_\alpha) = \mathfrak{M}(r_\alpha) \cap \mathfrak{M}(s_\alpha) = \mathfrak{M}(s_\alpha) \cap \mathfrak{M}(r_\alpha) = \mathfrak{M}((s \wedge r)_\alpha) = g(s \wedge r)$. Since g is injective, we get $r \wedge s = s \wedge r$. Therefore, A is a distributive lattice. □

Let us denote the class of all HADLs by $\mathfrak{H}$ and the category of all homomorphisms from an ADL A_1 into an ADL A_2 by $\mathrm{Hom}(A_1, A_2)$. Then, we have the following result.

Lemma 4. *Let $A_1, A_2 \in \mathfrak{H}$ and $g \in Hom(A_1, A_2)$. Let $\mathfrak{F}_p^1$ and $\mathfrak{F}_p^2$ be the sets of prime fuzzy ideals of A_1 and A_2, respectively. Then, for any $\eta \in \mathfrak{F}_p^2$, $g^{-1}(\eta) \in \mathfrak{F}_p^1$.*

Proof. Let $g \in \mathrm{Hom}(A_1, A_2)$ and $\eta \in \mathfrak{F}_p^2, 0_1 \in A_1$. Then,

$$\begin{aligned}\left(g^{-1}(\eta)\right)(0_1) &= \eta(g(0_1))\\ &= \eta(0_2)\\ &= 1.\end{aligned} \tag{10}$$

For any $r, s \in A_1$,

$$\begin{aligned}\left(g^{-1}(\eta)\right)(r\vee s) &= \eta(g(r\vee s))\\ &= \eta(g(r)\vee g(s))\\ &\geq \eta\left(g(r)\wedge\eta(g(s)) \geq g^{-1}(\eta)(r)\wedge g^{-1}(\eta)(s)\right).\end{aligned} \tag{11}$$

Also, for any $r, s \in A_1$,

$$\begin{aligned}\left(g^{-1}(\eta)\right)(r\wedge s) &= \eta(g(r\wedge s))\\ &= \eta(g(r)\wedge g(s))\\ &\geq \eta(g(r))\vee\eta(g(s))\\ &\geq g^{-1}(\eta)(r)\vee g^{-1}(\eta)(s).\end{aligned} \tag{12}$$

This shows that $g^{-1}(\eta)$ is a fuzzy ideal of A_1. Next, we show that $g^{-1}(\eta)$ is prime fuzzy ideal of A_1. Let $\lambda, \nu \in FI(A_1)$, such that $\lambda \cap \nu \subseteq g^{-1}(\eta)$. This implies $g(\lambda) \cap g(\nu) = g(\lambda \cap \nu) \subseteq \eta$. Since η is a prime fuzzy ideal of A_2, $g(\lambda) \subseteq \eta$ and $g(\nu) \subseteq \eta$. This implies $\lambda \subseteq g^{-1}(\eta)$ and $\nu \subseteq g^{-1}(\eta)$. Hence, it concludes that $g^{-1}(\eta) \in \mathfrak{F}_p^1$. □

Theorem 11. *Let H_1 and H_2 be HADLs and $g \in Hom(H_1, H_2)$. Then, the mapping T_g: $\mathfrak{F}_p^2 \longrightarrow \mathfrak{F}_p^1$ defined by $T_g(\eta) = g^{-1}(\eta)$ for all $\eta \in \mathfrak{F}_p^2$ is continuous.*

Proof. From Lemma 4, T_g is well defined. For arbitrary x_α, we know that $\mathfrak{M}(x_\alpha)$ is an open set in $\mathfrak{F}_p^1$. Then, we have the following:

$$\begin{aligned}\eta \in \mathfrak{M}(g(x_\alpha)) &\Leftrightarrow g(x_\alpha) \notin \eta\\ &\Leftrightarrow x_\alpha \notin g^{-1}(\eta)\\ &\Leftrightarrow g^{-1}(\eta) \in \mathfrak{M}(x_\alpha)\\ &\Leftrightarrow T_g(\eta) \in \mathfrak{M}(x_\alpha)\\ &\Leftrightarrow \eta \in T_g^{-1}(\mathfrak{M}(x_\alpha)).\end{aligned} \tag{13}$$

Therefore, it follows that $T_g^{-1}(\mathfrak{M}(x_\alpha)) = \mathfrak{M}(g(x_\alpha))$, which is an open set in $\mathfrak{F}_p^2$. Hence, T_g is a continuous mapping from $\mathfrak{F}_p^2$ onto $\mathfrak{F}_p^1$. □

4. Isomorphism between the Fuzzy Spectral Spaces of a HADL and Its Fuzzy Congruence Lattice

In this paper, a fuzzy congruence relation Θ on H is defined and the homomorphism between the set $FI(H)$ of all fuzzy ideals of H and the set $FI(H/\Theta)$ of all fuzzy ideals of H is established, where $H/\Theta = \{\Theta_x | x \in H\}$. Furthermore, we proved that the mapping between fuzzy spectrum of $H(F\mathrm{spec}(H))$ and fuzzy spectrum of $H/\Theta(F\mathrm{spec}(H/\Theta))$ is bijective.

Definition 8. A fuzzy equivalence relation Θ on a HADL H is a called fuzzy congruence relation on H, if the following are satisfied:

(1) $\Theta(s\wedge p, r\wedge q) \geq \Theta(s, r)\wedge\Theta(p, q)$, for all $p, q, r, s \in H$
(2) $\Theta(s\vee p, r\vee q) \geq \Theta(s, r)\wedge\Theta(p, q)$, for all $p, q, r, s \in H$
(3) $\Theta(s \longrightarrow p, r \longrightarrow q) \geq \Theta(s, r)\wedge\Theta(p, q)$, for all $p, q, r, s \in H$

The following theorem is the characterization of Definition 8.

Theorem 18. *A fuzzy equivalence relation Θ on HADL H is a fuzzy congruence on H if and only if $\Theta(s, r) \leq \Theta(s\wedge p, r\wedge p)\wedge\Theta(p\wedge s, p\wedge r)\wedge\Theta(p\vee s, p\vee r)\wedge\Theta(s\vee p, r\vee p)\wedge\Theta(s \longrightarrow p, r \longrightarrow p)\wedge\Theta(p \longrightarrow s, p \longrightarrow r)$, for all $p, r, s \in H$.*

Proof. Suppose that Θ is a congruence relation on H. Then, for any $s, t, p \in H$, $\Theta(s, t) = \Theta(s, t) \wedge \Theta(p, p) \leq \Theta(s \wedge p, t \wedge p)$ and also $\Theta(s, t) = \Theta(p, p) \wedge \Theta(s, t) \leq \Theta(p \wedge s, p \wedge t)$. Hence, $\Theta(s, t) \leq \Theta(s \wedge p, t \wedge p) \wedge \Theta(p \wedge s, p \wedge t)$. With similar procedure, $\Theta(s, r) \leq \Theta(p \vee s, p \vee r) \wedge \Theta(s \vee p, r \vee p)$ and $\Theta(s, r) \leq \Theta(s \longrightarrow p, r \longrightarrow p) \wedge \Theta(p \longrightarrow s, p \longrightarrow r)$. Finally, we get $\Theta(s, r) \leq \Theta(s \wedge p, r \wedge p) \wedge \Theta(p \wedge s, p \wedge r) \wedge \Theta(p \vee s, p \vee r) \wedge \Theta(s \vee p, r \vee p) \wedge \Theta(s \longrightarrow p, r \longrightarrow p) \wedge\Theta(p \longrightarrow s, p \longrightarrow r)$ for all $x, y, z \in H$.

Conversely, suppose that $\Theta(s, r) \leq \Theta(s\wedge p, r\wedge p)\wedge\Theta(p \wedge s, p \wedge r)\wedge\Theta(p\vee s, p\vee r)\wedge\Theta(s\vee p, r\vee p)\wedge\Theta(s \longrightarrow p, r \longrightarrow p) \wedge\Theta(p \longrightarrow s, p \longrightarrow r)$ is true for all $r, s, p \in H$.

This implies $\Theta(s, r) \leq \Theta(s \wedge p, r \wedge p)$, $\Theta(s, r) \leq \Theta(p \wedge s, p \wedge r)$, $\Theta(s, r) \leq \Theta(p \vee s, p \vee r)$, $\Theta(s, r) \leq \Theta(s \vee p, r \vee p)$, $\Theta(s, r) \leq \Theta(s \longrightarrow p, r \longrightarrow p)$, $\Theta(s, r) \leq \Theta(p \longrightarrow s, p \longrightarrow r)$ for all $s, r, p \in H$. Now $\Theta(s, r) \wedge \Theta(p, q) = \Theta(s, r) \wedge \Theta(p, p)\wedge\Theta(r, r) \wedge\Theta(p, q) \leq \Theta(s \wedge p, r \wedge p) \wedge\Theta(r \wedge p, r \wedge q) \leq \Theta(s \wedge p, r \wedge q)$ by transitivity. Similarly, $\Theta(s\vee p, r\vee q) \geq \Theta(s, r)\wedge\Theta(p, q)$ and $\Theta(s \longrightarrow p, r \longrightarrow q) \geq \Theta(s, r)\wedge\Theta(p, q)$. Hence, Θ is a congruence relation. □

Example 2. *Let $H = \{0, r, s, t\}$. Define the binary operations $\vee$, $\wedge$, and $\longrightarrow$ on A* as follows:

$\vee$	0	r	s	t
0	0	r	s	t
r	r	r	r	r
s	s	s	s	s
t	t	r	s	t

$\wedge$	0	r	s	t
0	0	0	0	0
r	0	r	s	t
s	0	r	s	t
t	0	t	t	t

$\rightarrow$	0	r	s	t
0	s	r	s	t
r	0	s	s	t
s	0	r	s	t
t	0	s	s	s

Then, $(H, \vee, \wedge, \longrightarrow, 0)$ is a Heyting ADL with maximal elements r and s.

Define a fuzzy relation Θ on H as $\Theta(0,0) = \Theta(r,r) = \Theta(s,s) = \Theta(t,t) = \Theta(r,s) = \Theta(s,r) = 1$, $\Theta(r,t) = \Theta(t,r) = \Theta(s,t) = \Theta(t,s) = 0.7$, and $\Theta(0,r) = \Theta(r,0) = \Theta(s,0) = \Theta(0,s) = \Theta(t,0) = \Theta(0,t) = 0.5$. Then, Θ is a fuzzy congruence relation on H.

Lemma 5. *Let Θ be an equivalence relation on H. Then, Θ is a congruence relation on H if and only if its characteristic function χ_Θ is a fuzzy congruence on H.*

Lemma 6. *A fuzzy relation Θ on H is a fuzzy congruence on H if and only if every level subset Θ_λ of Θ at $\lambda \in [0,1]$ is a congruence relation on H.*

Theorem 19. *Let $(H, \vee, \wedge, \longrightarrow, 0, m)$ be a HADL H and μ be a fuzzy filter of H. Define $\Theta_\mu(s,t) = \sup\{\mu(x)\colon s\wedge x = t\wedge x \text{ for all } x \in H\}$. Then, Θ_μ is a fuzzy congruence relation on H.*

Proof. Clearly, Θ_μ is reflexive and symmetric. Next, we show that Θ_μ is transitive.

For any HADL H, suppose that $s\wedge x = t\wedge x$ and $t\wedge y = r\wedge y$ for any $s, t, x, y \in H$. Then, $s\wedge x\wedge y = t\wedge x\wedge y = x\wedge t\wedge y = x\wedge r\wedge y = r\wedge x\wedge y$.

$$\begin{aligned}\Theta_\mu(s,t)\wedge\Theta_\mu(t,r) &= \sup\{\mu(x)\colon s\wedge x = t\wedge x \text{ for all } x \in H\}\wedge \\ &\quad \sup\{\mu(y)\colon t\wedge y = r\wedge y \text{ for all } y \in H\} \\ &= \sup\{\mu(x)\wedge\mu(y)\colon s\wedge x = t\wedge x, t\wedge y = r\wedge y \text{ for all } x, y \in H\} \\ &\le \sup\{\mu(x\wedge y)\colon s\wedge x\wedge y = r\wedge x\wedge y, \forall x, y \in H\} \\ &= \Theta_\mu(s,r).\end{aligned} \tag{14}$$

Thus, Θ_μ is transitive. Hence, Θ_μ is a fuzzy equivalence relation on H. Next, we prove the computability of $\wedge, \vee, \longrightarrow$ \!.

Suppose that $s\wedge x = r\wedge x$ for some $s, r, x \in H$; then, we get $s\wedge q\wedge x = q\wedge s\wedge x = q\wedge r\wedge x = r\wedge q\wedge x$ and $(s\vee q)\wedge x = (s\wedge x)\vee(q\wedge x) = (r\wedge x)\vee(q\wedge x) = (r\vee q)\wedge x$ for any $q \in H$. By Lemma 1, we have $(s \longrightarrow q)\wedge x = [s\wedge x \longrightarrow q\wedge x]\wedge x = [r\wedge x \longrightarrow q\wedge x]\wedge x = (r \longrightarrow q)\wedge x$ for any $q \in H$. Now,

$$\begin{aligned}\Theta_\mu(s,t) &= \sup\{\mu(x)\colon s\wedge x = r\wedge x, x \in H\} \\ &\le \sup\{\mu(x)\colon (s\vee q)\wedge x = (r\vee q)\wedge x, x \in H\} \\ &= \Theta_\mu(s\vee q, t\vee q), \\ \Theta_\mu(s,t) &= \sup\{\mu(x)\colon s\wedge x = r\wedge x, x \in H\} \\ &\le \sup\{\mu(x)\colon s\wedge q\wedge x = r\wedge q\wedge x, x \in H\} \\ &= \Theta_\mu(s\wedge q, t\wedge q), \\ \Theta_\mu(s,t) &= \sup\{\mu(x)\colon s\wedge x = r\wedge x, x \in H\} \\ &\le \sup\{\mu(x)\colon (s \longrightarrow q)\wedge x = (r \longrightarrow q)\wedge x, t\vee q\} \\ &= \Theta_\mu(s \longrightarrow q, t \longrightarrow q).\end{aligned} \tag{15}$$

Similarly, we have $\Theta_\mu(s,t) \le \Theta_\mu(q\wedge s, q\wedge t)$, $\Theta_\mu(s,t) \le \Theta_\mu(q\vee s, q\wedge t)$, and $\Theta_\mu(s,t) \le \Theta_\mu(q \longrightarrow s, q \longrightarrow t)$. Thus, by Theorem 18, Θ_μ is fuzzy congruence relation on H. □

Theorem 20. *Let Θ be a fuzzy congruence relation on H. A fuzzy subset μ_Θ defined by $\mu_\Theta(r) = \Theta(r,m)$ for all $r \in H$ is a fuzzy filter of H.*

Proof. $\mu_\Theta(m) = \Theta(m,m) = 1$. For all $r,s \in H$,

$$\begin{aligned}\mu_\Theta(r\vee s) &= \Theta(r\vee s, m)\\ &= \Theta(r\vee s, m\vee s)\\ &\ge \Theta(r,m)\wedge\Theta(s,s)\\ &= \mu_\Theta(r).\end{aligned} \tag{16}$$

Similarly, $\mu_\Theta(r\vee s) \ge \mu_\Theta(r)$, and also we have $\mu_\Theta(r\vee s) \ge \mu_\Theta(r)\vee\mu_\Theta(s)$.

$$\begin{aligned}\mu_\Theta(r\wedge s) &= \Theta(r\wedge s, m)\\ &= \Theta(r\wedge s, m\wedge m)\\ &\ge \Theta(r,m)\wedge\Theta(s,m)\\ &= \mu_\Theta(r)\wedge\mu_\Theta(s).\end{aligned} \tag{17}$$

Hence, μ_Θ is a fuzzy filter of H. □

The set $H/\Theta = \{\Theta_t | t \in H\}$ of all congruence classes is an ADL H induced by a fuzzy congruence Θ with respect to the following operations:

$$\begin{aligned}\Theta_r\wedge\Theta_s &= \Theta_{r\wedge s},\\ \Theta_r\vee\Theta_s &= \Theta_{r\vee s},\\ \Theta_r \longrightarrow \Theta_s &= \Theta_{r\longrightarrow s},\end{aligned} \tag{18}$$

where $\Theta_r = \{s \in H\colon \Theta(r,s) = 1\}$ is a congruence class of r. Clearly, we get that $\Theta_{r\wedge s} = \Theta_{s\wedge r}$ and $\Theta_{r\vee s} = \Theta_{s\vee r}$. Here for any $r,s \in H$, it can be observed that $\Theta_r = \Theta_s \Longleftrightarrow \Theta(s,t) = 1$, $\Theta_s \cap \Theta_r = \varnothing$ or $\Theta_s = \Theta_r$, and $(H/\Theta, \vee, \wedge, \longrightarrow, \Theta_0, \Theta_m)$ is a HADL.

Definition 9. For any fuzzy ideal ν of a HADL H and a fuzzy ideal η of H/Θ, we define the following:

(1) $\xi(\nu)(\Theta_s) = \sup\{\nu(t) | \Theta(s,t) = 1 \text{ for some } t \in H\}$
(2) $\omega(\eta)(s) = \sup\{\eta(\Theta_t) | \Theta(s,t) = 1 \text{ for some } t \in H\}$

Lemma 7. *For any fuzzy ideal ν of H and a fuzzy ideal η of H/Θ, we have the following:*

(1) $\xi(\nu)(\Theta_s) \ge \nu(s)$
(2) $\omega(\eta)$ is a fuzzy ideal in H
(3) $\xi(\nu)$ is a fuzzy ideal in H/Θ
(4) ω and ξ are isotones

Proof. Let $\nu, \lambda \in FI(H)$ and η be a fuzzy ideal of H/Θ; then,

(1) Since $\Theta(s,s) = 1$, $\xi(\nu)(\Theta_s) = \sup\{\nu(t) | \Theta(s,t) = 1, \text{ for some } t \in H\} \ge \nu(s)$.

(2)
$$\begin{aligned}\omega(\eta)(0) &= \sup\{\eta(\Theta_t) | \Theta(0,t) = 1 \text{ for some } t \in H\}\\ &\ge \eta(\Theta_0)\\ &= 1.\end{aligned} \tag{19}$$

Hence, $\omega(\eta)(0) = 1$. Now take $s,t \in H$. Then, we get

$$\begin{aligned}\omega(\eta)(s)\wedge\omega(\eta)(t) &= \sup\left\{\eta\left(\Theta_{z_1}\right) | \Theta(s,z_1) = 1 \text{ for some } z_1 \in H\right\}\wedge\\ &\quad \sup\left\{\eta\left(\Theta_{z_2}\right) | \Theta(t,z_2) = 1 \text{ for some } z_2 \in H\right\}\\ &= \sup\left\{\eta\left(\Theta_{z_1}\right)\wedge\eta\left(\Theta_{z_1}\right) | \Theta(s,z_1) = 1, \Theta(t,z_2) = 1, \text{for some } z_1, z_2 \in H\right\}\\ &\le \sup\left\{\eta\left(\Theta_{z_1}\vee\Theta_{z_2}\right) | \Theta(s\vee t, z_1\vee z_2) = 1, \text{for some } z_1, z_2 \in H\right\}\\ &= \sup\left\{\eta\left(\Theta_{(z_1\vee z_2)}\right) | \Theta(s\vee t, z_1\vee z_2) = 1, \text{ for some } z_1, z_2 \in H\right\}\\ &= \omega(\eta)(s\vee t).\end{aligned} \tag{20}$$

Hence, $\omega(\eta)(s\vee t) \ge (\omega(\eta))(s)\wedge(\omega(\eta))(t)$. Next,

$$\begin{aligned}\omega(\eta)(s)\vee\omega(\eta)(t) &= \sup\{\eta(\Theta_{z_1})|\,\Theta(s,z_1)=1 \text{ for some } z_1\in H\}\\ &\quad\vee\sup\{\eta(\Theta_{z_2})|\,\Theta(t,z_2)=1, \text{ for some } z_2\in H\}\\ &= \sup\{\eta(\Theta_{z_1})\vee\eta(\Theta_{z_1})|\,\Theta(s,z_1)=1,\Theta(t,z_2)=1, \text{for some } z_1,z_2\in H\}\\ &\le \sup\{\eta(\Theta_{z_1}\wedge\Theta_{z_2})|\,\Theta(s\wedge t,z_1\wedge z_2)=1, \text{for some } z_1,z_2\in H\}\\ &= \sup\{\eta(\Theta_{(z_1\wedge z_2)})|\,\Theta(s\wedge t,z_1\wedge z_2)=1, \text{for some } z_1,z_2\in H\}\\ &= \omega(\eta)(s\wedge t).\end{aligned} \tag{21}$$

Hence, $(\omega(\eta))(s\wedge t)\ge(\omega(\eta))(s)\vee(\omega(\eta))(t)$. Therefore, $\omega(\eta)$ is a fuzzy ideal in H.

(3) $(\xi(\nu))(\Theta_0)=\sup\{\nu(t)|\Theta(0,t)=1 \text{ for some } t\in H\}\ge\nu(0)=1$. This shows that $(\xi(\nu))(\Theta_0)=1$. For some $s,t\in H$, we get

$$\begin{aligned}\omega(\eta)(s)\wedge\omega(\eta)(t) &= \sup\{\{\eta(\Theta_{z_1})|\Theta(s,z_1)=1 \text{ for some } z_1\in H\}\wedge\sup\{\eta(\Theta_{z_2})|\Theta(t,z_2)=1 \text{ for some } z_2\in H\}\\ &= \sup\{\eta(\Theta_{z_1})\wedge\eta(\Theta_{z_1})|\Theta(s,z_1)=1,\Theta(t,z_2)=1, \text{for some } z_1,z_2\in H\}\\ &\le \sup\{\eta(\Theta_{z_1}\vee\Theta_{z_2})|\Theta(s\vee t,z_1\vee z_2)=1, \text{for some } z_1,z_2\in H\}\\ &= \sup\{\eta(\Theta_{(z_1\vee z_2)})|\Theta(s\vee t,z_1\vee z_2)=1, \text{for some } z_1,z_2\in H\}\\ &= \omega(\eta)(s\vee t).\end{aligned} \tag{22}$$

Hence, $(\xi(\nu))(\Theta_s\vee\Theta_t)\ge(\xi(\nu))(\Theta_s)\wedge(\xi(\nu))(\Theta_t)$. Similarly,

$$\begin{aligned}\xi(\nu)(\Theta_s)\vee\xi(\nu)(\Theta_t) &= \sup\{\nu(z_1)|\Theta(s,z_1)=1, \text{ for some } z_1\in H\}\\ &\quad\vee\sup\{\nu(z_2)|\Theta(t,z_2)=1 \text{ for some } z_2\in H\}\\ &= \sup\{\nu(z_1)\vee\nu(z_2)|\Theta(s,z_1)=1,\Theta(t,z_2)=1, \text{for some } z_1,z_2\in H\}\\ &\le \sup\{\nu(z_1\wedge z_2)|\Theta(s\wedge t,z_1\wedge z_2)=1, \text{ for some } z_1,z_2\in H\}\\ &= \xi(\nu)(\Theta_{(s\wedge t)})\\ &= \xi(\nu)(\Theta_s\wedge\Theta_t).\end{aligned} \tag{23}$$

Hence, $\xi(\nu)(\Theta_s\wedge\Theta_t)\ge\xi(\nu)(\Theta_s)\vee\xi(\nu)(\Theta_t)$.

Therefore, $\xi(\nu)$ is a fuzzy ideal in H/Θ.

(4) Let $\nu\subseteq\lambda$. Then,

$$\begin{aligned}\xi(\nu)(\Theta_s) &= \sup\{\nu(t)|\Theta(s,t)=1 \text{ for some } t\in H\}\\ &\le \sup\{\lambda(t)|\Theta(s,t)=1 \text{ for some } t\in H\}\\ &= \xi(\lambda)(\Theta_s).\end{aligned} \tag{24}$$

This shows that $\xi(\nu)\subseteq\xi(\lambda)$. Hence, ξ is isotone. Similarly, ω is isotone. □

Example 3. From Example 2, congruence class of H induced by a fuzzy relation Θ is $\Theta_0=\{0\}$, $\Theta_r=\{r,s\}=\Theta_s$, and $\Theta_t=\{t\}$. Thus, $H/\Theta=\{\Theta_0,\Theta_r,\Theta_t\}$. Also, for any fuzzy subset η on H, in Example 2 defined as $\eta(r)=\eta(s)=0.2$, $\eta(t)=0.4$ and $\eta(0)=1$ is a fuzzy ideal of H. We can easily show that $\xi(\eta)$ is a fuzzy ideal of H/Θ and $\omega(\xi(\eta))$ is a fuzzy ideal of H .

Theorem 21. *The mapping $\xi\colon FI(H)\longrightarrow FI(H/\Theta)$ is a homomorphism.*

Proof. Let $r,s,t\in H$. Consider $\nu,\lambda\in FI(H)$ and $\xi\colon FI(H)\longrightarrow FI(H/\Theta)$. Clearly, $\nu\cap\lambda\subseteq\nu,\lambda$. Since ξ is isotone, we get $\xi(\nu\cap\lambda)\subseteq\xi(\nu)\cap\xi(\lambda)$. Furthermore, $(\xi(\nu)\cap\xi(\lambda))(\Theta_r)$

$$
\begin{aligned}
&= \sup\{\nu(s)|\Theta(r,s)=1\}\wedge\sup\{\lambda(t)|\Theta(r,t)=1\}\\
&= \sup\{\nu(s)|\Theta(r,r\wedge s)=1\}\wedge\sup\{\lambda(t)|\Theta(r\wedge s,s\wedge t)=1\}\\
&\leq \sup\{\nu(s\wedge t)|\Theta(r,r\wedge s)=1\}\wedge\sup\{\lambda(s\wedge t)|\Theta(r\wedge s,s\wedge t)=1\}\\
&= \sup\{\nu(s\wedge t)\wedge\lambda(s\wedge t)|\Theta(r,r\wedge s)=1,\Theta(r\wedge s,s\wedge t)=1,\}\\
&= \sup\{(\nu\cap\lambda)(s\wedge t)|\Theta(r,s\wedge t)=1\}\\
&= \xi(\nu\cap\lambda)(\Theta_r).
\end{aligned}
\tag{25}
$$

Hence, $\xi(\nu)\cap\xi(\lambda)\subseteq\xi(\nu\cap\lambda)$. Therefore, $\xi(\nu\cap\lambda)=\xi(\nu)\cap\xi(\lambda)$. Since $\nu,\lambda\subseteq\nu\vee\lambda$, we get $\xi(\nu)\vee\xi(\lambda)\subseteq\xi(\nu\vee\lambda)$. Now, $(\xi(\nu)\vee\xi(\lambda))(\Theta_r)$

$$
\begin{aligned}
&= \sup\{\nu(s)|\Theta(r,s)=1\}\vee\sup\{\lambda(t)|\Theta(r,t)=1\}\\
&= \sup\{\nu(s)|\Theta(r,r\wedge s)=1\}\vee\sup\{\lambda(t)|\Theta(r\wedge s,s\wedge t)=1\}\\
&\leq \sup\{\nu(s\wedge t)|\Theta(r,r\wedge s)=1\}\vee\sup\{\lambda(s\wedge t)|\Theta(r\wedge s,s\wedge t)=1\}\\
&= \sup\{\nu(s\wedge t)\vee\lambda(s\wedge t)|\Theta(r,r\wedge s)=1,\Theta(r\wedge s,s\wedge t)=1,\}\\
&= \sup\{(\nu\vee\lambda)(s\wedge t)|\Theta(r,s\wedge t)=1\}\\
&= \xi(\nu\vee\lambda)(\Theta_r).
\end{aligned}
\tag{26}
$$

This shows that $\xi(\nu)\subseteq\xi(\lambda)$. Hence, $\xi(\nu\vee\lambda)=\xi(\nu)\vee\xi(\lambda)$. Therefore, ξ is a homomorphism. □

Theorem 22. *Let ν and λ be fuzzy ideals of H. The map $\nu\longrightarrow\omega o\xi(\nu)$ is closure operator on $FI(H)$. That is,*

(1) $\nu\subseteq\omega o\xi(\nu)$

(2) $\nu\subseteq\lambda$ implies $\omega o\xi(\nu)\subseteq\omega o\xi(\lambda)$

(3) $\omega o\xi[\omega o\xi(\nu)]=\omega o\xi(\nu)$

Proof

(1) For any $r,s,t\in H$, we get

$$
\begin{aligned}
((\omega o\xi)(\nu))(r) &= \xi(\nu)(\Theta_r)\\
&= \sup\{\nu(s)|\Theta(r,s)=1\})\\
&\geq \nu(r).
\end{aligned}
\tag{27}
$$

Hence, $\nu\subseteq\omega o\xi(\nu)$.

(2) Let $\nu\subseteq\lambda$. Since ω and ξ are isotone, we have $(\omega o\xi)(\nu)\subseteq(\omega o\xi)(\lambda)$.

$$
\begin{aligned}
(3)\ \omega o\xi[\omega o\xi(\nu)](r) &= \xi[\omega o\xi(\nu)](\Theta_r)\\
&= \sup\{((\omega o\xi)(\nu))(s)|\Theta(r,s)=1 \text{ for some } s\in H\}\\
&= \sup\{\sup\{(\xi(\nu))(\Theta_s)|\Theta(r,s)=1 \text{ for some } s\in H\}\}\\
&= \sup\{(\xi(\nu))(\Theta_s)||\Theta(r,s)=1 \text{ for some } s\in H\}\\
&= (\xi(\nu))(\Theta_s)\\
&= \omega o\xi(\nu)(r).
\end{aligned}
\tag{28}
$$

□

Theorem 23. *For any fuzzy ideal ν of H/Θ, $\xi o\omega(\nu)=\nu$.*

Proof. Clearly, $\nu\subseteq\xi o\omega(\nu)$. Conversely, for any $r,t\in H$, $\Theta(r,t)=1$ if and only if $\Theta_r=\Theta_t$. Then, we get

$$
\begin{aligned}
(\xi o\omega(\nu))(\Theta_r) &= \sup\{(\omega(\nu))(s)|\Theta(r,s)=1, \text{ for some } s\in H\}\\
&= \sup\{\sup\{\nu(\Theta_t)|\Theta(s,t)=1\}|\Theta(r,s)=1\}\\
&= \sup\{\nu(\Theta_t)|\Theta(r,s)=1,\Theta(s,t)=1\}\\
&\leq \sup\{\nu(\Theta_t)|\Theta(r,t)=1\}\\
&= \nu(\Theta_r).
\end{aligned}
\tag{29}
$$

This implies $\xi o\omega(\nu)\subseteq\nu$. Hence, $\xi o\omega(\nu)=\nu$. □

Lemma 8. *For any prime fuzzy ideal ν of H, $\omega o\xi(\nu)=\nu$.*

Proof. For any $r, s, t \in H$, we get the following:

$$\begin{aligned}\omega o\xi(\nu)(r) &= \sup\{\xi(\nu)(\Theta_s)|\Theta(r,s)=1,\}\\ &= \sup\{\sup\{\nu(t)|\Theta(s,t)=1\}|\Theta(r,s)=1\}\\ &= \sup\{\nu(t)|\Theta(r,s)=1, \Theta(s,t)=1 \leq \sup\{\nu(t)|\Theta(r,t)=1\} = \nu(r).\end{aligned} \tag{30}$$

Hence, $\omega o\xi(\nu)\subseteq\nu$. On the other hand, from (1) in Theorem 22, we get $\nu\subseteq\omega o\xi(\nu)$. □

Lemma 9. *Let ν, η be any fuzzy ideal of H/Θ. Then, $\omega(\nu\cap\eta) = \omega(\nu)\cap\omega(\eta)$.*

Theorem 24. *The following conditions hold on H:*

(1) For any prime fuzzy ideal μ of H, $\xi(\mu)$ is a prime fuzzy ideal in H/Θ

(2) For any prime fuzzy ideal μ of H/Θ, $\omega(\mu)$ is a prime fuzzy ideal in H

Proof

(1) Suppose that μ is a prime fuzzy ideal of H and for any fuzzy ideals ν, λ of H/Θ such that $\nu\cap\lambda\subseteq\xi(\mu)$. By Lemma 1 and Theorem 4.15, $\omega(\nu)\cap\ \omega(\lambda)\subseteq \omega o\xi(\mu) = \mu$. Since μ is prime fuzzy ideal, $\omega(\nu)\subseteq\mu$ or $\omega(\lambda)\subseteq\mu$. Again by Theorem 4.9 and Theorem 4.14, $\nu = \xi o\omega(\nu)\subseteq\xi(\mu)$ or $\lambda = \xi o\omega(\lambda)\subseteq\xi(\mu)$. Hence, $\xi(\mu)$ is a prime fuzzy ideal of H.

(2) Suppose that μ is a prime fuzzy ideal of H/Θ and for any fuzzy ideals η, π of H such that $\eta\cap\pi\subseteq\omega(\mu)$. Since ξ is an isotone and μ is a prime fuzzy ideal of H/Θ, $\xi(\eta)\cap\xi(\pi) = \xi(\eta\cap\pi)\subseteq\xi o\omega(\mu) = \mu$. This implies $\xi(\eta)\subseteq\mu$ or $\xi(\pi)\subseteq\mu$. Thus, we have $\eta\subseteq\omega o\xi(\eta)\subseteq\omega(\mu)$ or $\pi\subseteq\omega o\xi(\pi)\subseteq\omega(\mu)$. Hence, $\omega(\mu)$ is a prime fuzzy ideal of H. □

$F\mathrm{Spec}(H/\Theta)$ denotes the set of all prime fuzzy ideals of H/Θ. Then, we have the following result.

Theorem 25. *The mapping α: $FSpec(H) \longrightarrow F\ Spec\ (H/\Theta)$ defined for all $\mu\in FSpec(H)$ by $\alpha(\mu) = \xi(\mu)$ is bijective.*

Proof. Let $\mu, \nu \in H$ such that $\mu = \nu$. Since ξ is isotone, $\alpha(\mu) = \xi(\mu) = \xi(\nu) = \alpha(\nu)$. Thus, α is well defined. Next, we prove that α is one-to-one. Let $\mu, \nu \in \mathrm{Spec}(H)$ such that $\alpha(\mu) = \alpha(\nu)$. By Lemma 8, we have $\mu = \omega o\xi(\mu) = \omega o\xi(\nu) = \nu$. Hence, α is one-to-one. Let $\eta\in F\mathrm{Spec}(H/\Theta)$. Then, we get $\eta = \xi o\omega(\eta)$, where $\omega(\eta)$ is a prime fuzzy ideal in H. Therefore, α is onto and so f is bijective. □

5. Conclusion

In this paper, we discussed the concept of the space $\mathfrak{F}_p$ of fuzzy prime ideals of Heyting almost distributive lattice H, and it is shown that the collection $\Sigma = \{\mathfrak{M}(\eta)\colon \eta \text{ is a fuzzy ideal of } H\}$ is a topology on $\mathfrak{F}_p$, where $\mathfrak{M}(\eta) = \left\{\theta\in\mathfrak{F}_p|\eta\nsubseteq\theta\right\}$. The fact that the set $\mathfrak{M}(x_\alpha) = \left\{\theta\in\mathfrak{F}_p|x_\alpha\notin\theta\right\}$ is the only compact subset of the space $\mathfrak{F}_p$ is proved. A fuzzy congruence relation Θ on H is defined, and the homomorphism between the set of all fuzzy ideals of H and the set of all fuzzy ideals of the class of all congruence classes of H (i.e., H/Θ) is established. Furthermore, we proved that the mapping between fuzzy spectrum of H and fuzzy spectrum of H/Θ is a bijective. We will further extend these concepts to fuzzy prime spectrum of semi-Heyting algebras and fuzzy prime spectrum of skew lattices.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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