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Research Article

Block-Graceful Designs

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In this article, we adapt the edge-graceful graph labeling definition into block designs and define a block design (V, B) with $|V| = v$ and $|B| = b$ as block-graceful if there exists a bijection $f\colon B \longrightarrow \{1, 2, \ldots, b\}$ such that the induced mapping $f^{+}\colon V \longrightarrow \mathbb{Z}_v$ given by $f^{+}(x) = \sum_{\substack{x\in A\\ A\in B}} f(A) (\mathrm{mod}\, v)$ is a bijection. A quick observation shows that every (v, b, r, k, λ)–BIBD that is generated from a cyclic difference family is block-graceful when $(v, r) = 1$. As immediate consequences of this observation, we can obtain block-graceful Steiner triple system of order v for all $v \equiv 1 (\mathrm{mod}\, 6)$ and block-graceful projective geometries, i.e., $((q^{d+1} - 1)/(q - 1), (q^{d} - 1)/(q - 1), (q^{d-1} - 1)/(q - 1))$–BIBDs. In the article, we give a necessary condition and prove some basic results on the existence of block-graceful (v, k, λ)–BIBDs. We consider the case $v \equiv 3 (\mathrm{mod}\, 6)$ for Steiner triple systems and give a recursive construction for obtaining block-graceful triple systems from those of smaller order which allows us to get infinite families of block-graceful Steiner triple systems of order v for $v \equiv 3 (\mathrm{mod}\, 6)$. We also consider affine geometries and prove that for every integer $d \geq 2$ and $q \geq 3$, where q is an odd prime power or $q = 4$, there exists a block-graceful $(q^{d}, q, 1)$–BIBD. We make a list of small parameters such that the existence problem of block-graceful labelings is completely solved for all pairwise non-isomorphic BIBDs with these parameters. We complete the article with some open problems and conjectures.

1. Introduction

A *graph* is a pair (V, E), where V is a set of elements called vertices and E is a set of paired vertices called edges. A graph labeling is an assignment of integers to the vertices, edges, or both of a graph subject to certain conditions. Since mid-1960s, over 200 graph labeling techniques have been studied in over 3000 papers. For a dynamic survey on graph labeling, see [1]. On the other hand, design labeling has not been studied much except for the studies of Akbari et al. in [2, 3] where they adapt the zero-sum flow graph labeling definition into block designs and the study of Colbourn in [4] where he studies labeling of blocks of a design such that the point weights are as equal as possible. In this article, we adapt the edge-graceful graph labeling definition into block designs.

A *design* (or *block design*) is a pair (V, B), where V is a finite set of points or symbols, and B is a collection (i.e., multiset) of nonempty subsets of V called blocks.

Let v, k, and λ be the positive integers such that $v > k \geq 2$. A (v, k, λ)*–balanced incomplete block design* ((v, k, λ)–BIBD) is a design (V, B) such that $|V| = v$, each block contains exactly k points, and every pair of distinct points is contained in exactly λ blocks. In a (v, k, λ)–BIBD, every point occurs in exactly $r = \lambda(v - 1)/(k - 1)$ blocks and there are exactly $b = vr/k = \lambda(v^2 - v)/(k^2 - k)$ blocks. The conditions that $\lambda(v - 1) \equiv 0 (\mathrm{mod}\, k - 1)$ and $\lambda v(v - 1) \equiv 0 (\mathrm{mod}\, k(k - 1))$ are the necessary conditions for the existence of a (v, k, λ)–BIBD. In this article, we will sometimes include the parameters r and b and denote a (v, k, λ)–BIBD as a (v, b, r, k, λ)–BIBD. The *complement* $(V, \overline{B})$ of a (v, b, r, k, λ)–BIBD (V, B), where $\overline{B} = \{V \setminus A\colon\ A \in B\}$, is a $(v, b, b - r, v - k, b - 2r + \lambda)$–BIBD [5, 6].

A graph (V, E) with $|V| = p$ and $|E| = q$ is called *edge-graceful* if there exists a bijection $f\colon E \longrightarrow \{1, 2, \ldots, q\}$ such that the induced mapping $f^+\colon V \longrightarrow \mathbb{Z}_p$ given by $f^+(x) = \sum_{\substack{y \in V \\ \{x, y\} \in E}} f(\{x, y\}) (\mathrm{mod}\, p)$ is a bijection [1].

We define a block design (V, B) with $|V| = v$ and $|B| = b$ as *block-graceful* if there exists a bijection $f\colon B \longrightarrow \{1, 2, \ldots, b\}$ such that the induced mapping $f^+\colon V \longrightarrow \mathbb{Z}_v$ given by $f^+(x) = \sum_{\substack{x \in A \\ A \in B}} f(A) (\mathrm{mod}\, v)$ is a bijection. The function f is called a *block-graceful labeling*.

In the following example and throughout the article, blocks will be denoted without using braces or commas.

Example 1. A block-graceful labeling of the $(9, 3, 1)$–BIBD with $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{012, 345, 678, 036, 147, 258, 048, 156, 237, 057, 138, 246\}$ is shown below:

$$
\begin{array}{lll}
 & & f^+(0) = 3 + 11 + 8 + 7 = 29 \equiv 2 (\mathrm{mod}\, 9) \\
f(012) = 3 & f(048) = 8 & f^+(1) = 3 + 12 + 5 + 4 = 24 \equiv 6 (\mathrm{mod}\, 9) \\
f(345) = 6 & f(156) = 5 & f^+(2) = 3 + 10 + 2 + 1 = 16 \equiv 7 (\mathrm{mod}\, 9) \\
f(678) = 9 & f(237) = 2 & f^+(3) = 6 + 11 + 2 + 4 = 23 \equiv 5 (\mathrm{mod}\, 9) \\
 & & f^+(4) = 6 + 12 + 8 + 1 = 27 \equiv 0 (\mathrm{mod}\, 9) \\
f(036) = 11 & f(057) = 7 & f^+(5) = 6 + 10 + 5 + 7 = 28 \equiv 1 (\mathrm{mod}\, 9) \\
f(147) = 12 & f(138) = 4 & f^+(6) = 9 + 11 + 5 + 1 = 26 \equiv 8 (\mathrm{mod}\, 9) \\
f(258) = 10 & f(246) = 1 & f^+(7) = 9 + 12 + 2 + 7 = 30 \equiv 3 (\mathrm{mod}\, 9) \\
 & & f^+(8) = 9 + 10 + 8 + 4 = 31 \equiv 4 (\mathrm{mod}\, 9).
\end{array}
\tag{1}
$$

In this article, we study the existence problem of block-graceful (v, k, λ)–BIBDs. In Section 2, we give a necessary condition for the existence of a block-graceful (v, k, λ)–BIBD and prove some basic results. We study the existence problem of block-graceful designs for Steiner triple systems in Section 3, for affine and projective geometries in Section 4, and for designs with small parameters in Section 5. We complete the article in Section 6 with proposing some open problems and conjectures.

2. Preliminaries and Basic Results

In this section, we give a necessary condition for the existence of a block-graceful (v, k, λ)–BIBD and prove some basic results.

Theorem 1. *If a (v, b, r, k, λ)–BIBD is block-graceful, then $r(b + 1) - v$ is odd.*

Proof. Let (V, B) be a block-graceful (v, b, r, k, λ)–BIBD. Then, the sum of the point weights must be $0 + 1 + \cdots + v - 1 \equiv v(v - 1)/2 (\mathrm{mod}\, v)$. On the other hand, since the labels of blocks are $1, 2, \ldots, b$ and each block contains exactly k points, the sum of the point weights is $k(1 + 2 + \cdots + b) \equiv kb(b + 1)/2 \equiv vr(b + 1)/2 (\mathrm{mod}\, v)$ since $bk = vr$. Hence, $(r(b + 1) - (v - 1))/2$ must be an integer, and the result follows.

For example, as a consequence of Theorem 1, we can see that no $(v, 4, 2)$–BIBD for $v \equiv 4 (\mathrm{mod}\, 6)$ can be block-graceful.

Let v, k, and λ be the positive integers such that $v > k \geq 2$, and G be a finite additive group of order v. A (v, k, λ)–*difference family* is a collection $[B_1, B_2, \ldots, B_n]$ of k element subsets of G such that the multiset union $\cup_{i=1}^{n} [x - y\colon x, y \in B_i, x \neq y]$ contains every nonzero element of G exactly λ times. The difference family is called *cyclic* if G is cyclic. The development $\mathrm{Dev}(B_1, B_2, \ldots, B_n) = [B_i + g\colon 1 \leq i \leq n \text{ and } g \in G]$, where $B_i + g = \{x + g\colon x \in B_i\}$, generates the block set of a (v, k, λ)–BIBD on the point set G [7, 8]. □

Theorem 2. *Every (v, b, r, k, λ)–BIBD that is generated from a cyclic difference family is block-graceful when $(v, r) = 1$.*

Proof. Let $[B_1, B_2, \ldots, B_n]$ be a cyclic (v, k, λ)– difference family on $\mathbb{Z}_v$ and let $B = \mathrm{Dev}(B_1, B_2, \ldots, B_n)$. Define a labeling $f\colon B \longrightarrow \{1, 2, \ldots, b\}$ such that $f(B_i + g) = (i - 1)v + g + 1$ for any $g \in \mathbb{Z}_v$ and $1 \leq i \leq n$. (Here, g is taken as an integer rather than an element of $\mathbb{Z}_v$ while computing $(i - 1)v + g + 1$). For any $a \in \mathbb{Z}_v$ we get $f^+(a) \equiv f^+(0) + ar (\mathrm{mod}\, v)$. Since $(v, r) = 1$, the numbers $r, 2r, \ldots, (v - 1)r$ are all different mod v, and the result follows. □

Theorem 3. *If (V, B) is a block-graceful (v, b, r, k, λ)–BIBD, then its complement $(V, \overline{B})$ is also block-graceful.*

Proof. Suppose that (V, B) has a block-graceful labeling $f\colon B \longrightarrow \{1, 2, \ldots, b\}$ with the induced mapping $f^+\colon V \longrightarrow \mathbb{Z}_v$ on points. Define $g\colon \overline{B} \longrightarrow \{1, 2, \ldots, b\}$ such that for every block $A \in B$, $g(V \setminus A) = f(A)$. Then, the weight of any point $x \in V$ in the complement design will be $g^+(x) = (1+2+\cdots+b) - f^+(x) \equiv b(b+1)/2 - f^+(x) \pmod v$. Since $f^+(x)$ s are different mod v, $g^+(x)$ s must also be different mod v, and the result follows. □

Theorem 4. *If there exists a block-graceful (v,b,r,k,λ)–BIBD and m is a positive integer such that the parameters $(v,mb,mr,k,m\lambda)$ satisfy the necessary condition in Theorem 1, then there exists a block-graceful $(v,mb,mr,k,m\lambda)$–BIBD.*

Proof. Suppose that there exists a (v,b,r,k,λ)–BIBD (V,B) with a block-graceful labeling $f\colon B \longrightarrow \{1,2,\ldots,b\}$, and the parameters $(v,mb,mr,k,m\lambda)$ satisfy the necessary condition in Theorem 1. A $(v,mb,mr,k,m\lambda)$–BIBD can be obtained by taking m copies of each block in (V,B).

If m is odd, we can label the s^{th} copy of a block $A \in B$ as $f(A) + (s-1)b$ for $1 \le s \le (m+1)/2$ and $b+1-f(A) + (s-1)b$ for $(m+3)/2 \le s \le m$. Clearly, this will label all blocks by the integers from 1 to mb. The weight of any point $x \in V$ in the $(v,mb,mr,k,m\lambda)$–BIBD will be

$$\sum_{s=1}^{(m+1)/2}\left(f^+(x) + r(s-1)b\right) + \sum_{s=(m+3)/2}^{m}\left(r(b+1) - f^+(x) + r(s-1)b\right) = f^+(x) + \frac{r(m-1)(mb+b+1)}{2}, \tag{2}$$

and hence these values will all be different mod v, since $f^+(x)$ s are all different mod v.

On the other hand, if m is even, then v must be odd in order for the parameters $(v,mb,mr,k,m\lambda)$ to satisfy the necessary condition in Theorem 1. Then, we can label the s^{th} copy of a block $A \in B$ as $f(A) + (s-1)b$ for $1 \le s \le (m+2)/2$ and $(b+1-f(A)) + (s-1)b$ for $(m+4)/2 \le s \le m$. Then, this will label all blocks by the integers from 1 to mb, and the weight of any point $x \in V$ in the $(v,mb,mr,k,m\lambda)$–BIBD will be

$$\sum_{s=1}^{(m+2)/2}\left(f^+(x) + r(s-1)b\right) + \sum_{s=(m+4)/2}^{m}\left(r(b+1) - f^+(x) + r(s-1)b\right) = 2f^+(x) + \frac{r\left(m^2b - 2b + m - 2\right)}{2}, \tag{3}$$

and hence, these values will all be different mod v, since $f^+(x)$ s are all different mod v and v is odd.

As a consequence of Theorems 3 and 4, to determine the set of parameters for which a block-graceful BIBD exists, it is sufficient to consider only the (v,k,λ)–BIBDs, where $k \le v/2$ and λ is the minimum value satisfying the necessary conditions for the existence of a (v,k,λ)–BIBD and the condition in Theorem 1. □

3. Steiner Triple Systems

A $(v,3,\lambda)$–BIBD is called a *triple system* and is denoted by TS (v,λ). The size of V is called the *order* of the triple system. A TS $(v,1)$ is called a *Steiner triple system* and is denoted by STS (v). An STS (v) exists if and only if $v \equiv 1,3 \pmod 6$ [6, 8]. In this section, we consider the existence problem of block-graceful Steiner triple systems.

It is well known that a cyclic $(v,3,1)$– difference family exists for all $v \equiv 1 \pmod 6$ [7, 8]. As a consequence of this result and Theorem 2, we immediately get the following theorem.

Theorem 5. *There exists a block-graceful STS (v) for all $v \equiv 1 \pmod 6$.*

Proof. Since the conditions of Theorem 2 are satisfied, the result follows.

The case $v \equiv 3 \pmod 6$ seems to be more complicated. We tried to construct block-graceful STS (v) s for $v \equiv 3 \pmod 6$ by making use of the well known Bose construction [8], 3–GDDs of type 3^u for odd u, or 3–GDDs of type 6^u with adding 3 points [9], but none of these approaches worked out nicely. There are also well-known recursive constructions of Steiner triple systems. One of them is the construction of an STS $(2v+1)$ from an STS (v), but this construction is also not useful in constructing block-graceful Steiner triple systems because being different modulo v provides us nothing about being different modulo $2v+1$. Another known recursive construction is the construction of an STS (mv) by using an STS (m) and an STS (v). This construction seems to be useful in terms of block-graceful labelings and we give the following construction of block-graceful designs for triple systems in general which can be used to obtain infinite families of block-graceful STS (v) s for $v \equiv 3 \pmod 6$. □

Theorem 6. *Suppose that there exists a TS (m,λ) and a block-graceful TS (v,λ). Then, there exists a block-graceful TS (mv,λ) if the following conditions hold:*

(i) $m|v$,

(ii) Either v is odd or v/m is even,

(iii) $(\lambda,m) = 1$,

(iv) There exists an integer j such that $(j,v) = 1$ and $(j + v/m, m) = 1$.

Proof. Let (V,B) be a TS (v,λ) (i.e., a $(v,b,r,3,\lambda)$–BIBD), where $V = \{0,1,\ldots,v-1\}$, and $f\colon B \longrightarrow \{1,2,\ldots,b\}$ is a block-graceful labeling with the induced mapping

$f^+: V \longrightarrow \mathbb{Z}_v$ on points. Also, let (U, A) be a TS (m, λ) (an $(m, b', r', 3, \lambda)$–BIBD) where $U = \{0, 1, \ldots, m-1\}$ and $A = \{\{x_s, y_s, z_s\}: 1 \le s \le b'\}$, where $x_s < y_s < z_s$ for $1 \le s \le b'$. Define $X = V \times U$, $C_1 = \{\{(c,i), (d,i), (e,i)\}: \{c,d,e\} \in B, i \in U\}$, $C_2 = \{\{(c, x_s), (d, y_s), (c + d \,(\mathrm{mod}\, v), z_s)\}: c, d \in V, 1 \le s \le b'\}$, and $C = C_1 \cup C_2$. Then, it is well known and can be easily seen that (X, C) is a $TS(mv, \lambda)$ with $b'' = \lambda mv(mv-1)/6$ blocks.

We define $g: C \longrightarrow \{1, 2, \ldots, b''\}$ as follows. Take $g(\{(c,i), (d,i), (e,i)\}) = mf(\{c,d,e\}) - i$ for all $\{c,d,e\} \in B$ and $i \in U$ and $g(\{(c, x_s), (d, y_s), (c+d \,(\mathrm{mod}\, v), z_s)\}) = \lambda mv (v-1)/6 + (s-1)v^2 + 1 + vc + d$ for $c, d \in V$ and $1 \le s \le b'$. We will show that g is a block-graceful labeling of (X, C).

It can be easily seen that g is a bijection. We must show that the point weights are all different mod mv. For any $a \in V$ and $i \in U$, labels of the blocks in C_1 contribute $mf^+(a) - ri$ to the point weight of (a, i). Note that $r = \lambda(v-1)/2$ is relatively prime with m since $m|v$ and $(\lambda, m) = 1$. Since $f^+(a)$s are all different mod v, we see that the values $mf^+(a) - ri$ are all different mod mv. We will now show that the contribution of labels of the blocks in C_2 to the point weights will just permute the values $mf^+(a) - ri$, and hence, the point weights will all be different mod mv. For $1 \le s \le b'$, let $C_{2,s} = \{\{(c, x_s), (d, y_s), (c + d \,(\mathrm{mod}\, v), z_s)\}: c, d \in V\}$ so that $C_2 = \cup_{s=1}^{b'} C_{2,s}$. It is sufficient to show that for all s with $1 \le s \le b'$, the contribution of labels of the blocks in $C_{2,s}$ to the point weights just permute the values $mf^+(a) - ri$.

For any $a \in V$, contribution of labels of the blocks in $C_{2,s}$ to the point weight of (a, x_s) is

$$\begin{aligned}\sum_{d=0}^{v-1} \frac{\lambda mv(v-1)}{6} + (s-1)v^2 + 1 + va + d &= \frac{\lambda mv^2(v-1)}{6} + (s-1)v^3 + v + v^2 a + \frac{(v-1)v}{2} \\ &= mvb + v^2((s-1)v + a) + \frac{v(v+1)}{2} \\ &\equiv \frac{v(v+1)}{2} \,(\mathrm{mod}\ mv),\end{aligned} \tag{4}$$

since $m|v$, and hence, $v^2 \equiv 0 \,(\mathrm{mod}\ mv)$. Recall that the weights of the points (a, x_s) from the labels of the blocks in C_1 were $mf^+(a) - rx_s$. Adding the constant value $v(v+1)/2$ will just permute these point weights among themselves, since it is a multiple of m by conditions (i) and (ii) in the statement of the theorem.

For any $a \in V$, contribution of labels of the blocks in $C_{2,s}$ to the point weight of (a, y_s) is

$$\begin{aligned}\sum_{c=0}^{v-1} \left(\frac{\lambda mv(v-1)}{6} + (s-1)v^2 + 1 + vc + a \right) &= \frac{\lambda mv^2(v-1)}{6} + (s-1)v^3 + v + \frac{(v-1)v^2}{2} + va \\ &= mvb + (s-1)v^3 + v + \frac{(v-1)v^2}{2} + va \\ &\equiv va + \frac{v\left(v^2 - v + 2\right)}{2} \,(\mathrm{mod}\ mv).\end{aligned} \tag{5}$$

The weight of the point (a, y_s) from the labels of the blocks in C_1 was $mf^+(a) - ry_s$. So, for $a \in V$, we get the point weights $\{-ry_s, m - ry_s, 2m - ry_s, \ldots, (v-1)m - ry_s\}$ in some order. By condition (iv), there exists an integer j such that $(j, v) = 1$ and $(j + v/m, m) = 1$. Since $(j, v) = 1$, we can rename the points in V if necessary so that the weight of the point (a, y_s) from the labels of the blocks in C_1 and the blocks in C_2 that are considered so far is $jam - ry_s$ modulo mv for all $a \in V$. Adding va to the point weight of (a, y_s) will make its point weight $a(jm + v) - ry_s$. Since $(j, v) = 1$, we have $(j, v/m) = 1$, and hence, $(j + v/m, v/m) = 1$. Since we also have $(j + v/m, m) = 1$, we get $(j + v/m, v) = 1$. Therefore, $(jm + v, mv) = m$. This shows that the point weights $a(jm + v) - ry_s$ are all distinct modulo mv, and they form the elements in the set $\{-ry_s, m - ry_s, 2m - ry_s, \ldots, (v-1)m - ry_s\}$, i.e., the point weights obtained from the labels of the blocks in C_1 are just permuted among themselves. Finally adding the constant $v(v^2 - v + 2)/2$ which is a multiple of m to these point weights permute them among themselves again.

For any $a \in V$, contribution of labels of the blocks in $C_{2,s}$ to the point weight of (a, z_s) is

$$\sum_{c=0}^{v-1} \sum_{\substack{d=0 \\ c+d \equiv a \pmod v}}^{v-1} \left(mb + (s-1)v^2 + 1 + vc + d\right) = mvb + (s-1)v^3 + v + \frac{(v-1)v^2}{2} + \frac{(v-1)v}{2} \equiv \frac{v\left(v^2+1\right)}{2} \pmod{mv}. \tag{6}$$

The weights of the points (a, z_s) from the labels of the blocks in C_1 were $mf^+(a) - rz_s$. Adding the constant value $v(v^2+1)/2$, which is a multiple of m by conditions (i) and (ii), will just permute these point weights among themselves.

As a conclusion, g is a block-graceful labeling of (X, C), and the proof is complete.

If we take $\lambda = 1$ in Theorem 6, condition (iii) is automatically satisfied, and since the existence of an STS (v) requires v being odd, condition (ii) is also satisfied. Hence, we get the following recursive construction for block-graceful Steiner triple systems. □

Corollary 1. *If* $m \equiv 1, 3 \pmod 6$, *there exists a block-graceful STS* (v) *where* $m|v$, *and there exists an integer* j *such that* $(j, v) = 1$ *and* $(j + v/m, m) = 1$; *then, there exists a block-graceful STS* (mv).

If we take m as prime in Corollary 1, the given conditions can be satisfied by choosing $j = 1$ if $v/m \not\equiv -1 \pmod m$, and $j = -1$ otherwise. Moreover, if m is not prime but all prime factors of m are $1, 3 \pmod 6$, then to obtain a block-graceful STS (mv) from a block-graceful STS (v), we can apply Corollary 1 for each prime factor of m with their multiplicities. Hence, we get the following result.

Corollary 2. *If* m *is a positive integer all of whose prime factors are* $1, 3 \pmod 6$ *and there exists a block-graceful STS* (v) *where* $m|v$, *then there exists a block-graceful STS* (mv).

If m has a prime factor which is $5 \pmod 6$, then it is not easy in general to show that there exists an integer j satisfying the conditions in Corollary 1. As a special case, if we take $m = 15$, then we can choose $j = 1$ if $v/m \equiv 0, 1, 3, 6, 7, 10, 12, 13 \pmod{15}$, $j = -1$ if $v/m \equiv 2, 5, 8, 9, 14 \pmod{15}$, $j = -2$ if $v/m \equiv 4 \pmod{15}$, and $j = 2$ if $v/m \equiv 11 \pmod{15}$. We get the following result.

Corollary 3. *If there exists a block-graceful STS* (v) *where* $15|v$, *then there exists a block-graceful STS* $(15v)$.

In Example A.11 in Appendix A, we show that all $(15, 3, 1)$–BIBDs are block-graceful. Also, we construct a block-graceful $(21, 3, 1)$–BIBD in Example A.15. Using Corollaries 2 and 3 together with Examples 1, A.11, and A.15, we can get infinite families of block-graceful Steiner triple systems of order v where $v \equiv 3 \pmod 6$.

Theorem 7. *There exist block-graceful Steiner triple systems of order*

(i) 3^t *for every integer* $t \geq 2$,

(ii) $3^t 5^u$ *for all positive integers* t *and* u *with* $t \geq u$, *and*

(iii) $3^t 7^u$ *for all positive integers* t *and* u.

Proof

(i) Take a block-graceful STS(9) (Example 1) and apply Corollary 2 $(t-2)$ times with $m = 3$.

(ii) Take a block-graceful STS(15) (Example A.11) and apply Corollary 3 $(u-1)$ times, and Corollary 2 $(t-u)$ times with $m = 3$.

(iii) Take a block-graceful STS(21) (Example A.15) and apply Corollary 2 $(t-1)$ times with $m = 3$ and $(u-1)$ times with $m = 7$. □

4. Affine and Projective Geometries

An $(n^2 + n + 1, n + 1, 1)$–BIBD with $n \geq 2$ is called a *projective plane of order n*, and an $(n^2, n, 1)$–BIBD with $n \geq 2$ is called an *affine plane of order n*. It is known that for every prime power $q \geq 2$, there exists a projective plane of order q and an affine plane of order q, which can be constructed using the 2-dimensional projective geometry $PG_2(q)$ and affine geometry $AG_2(q)$ [6].

A generalization to higher dimensions shows that for any $d \geq 2$, the points and hyperplanes of the d-dimensional projective geometry $PG_d(q)$ form a $((q^{d+1}-1)/(q-1), (q^{d+1}-1)/(q-1), (q^d-1)/(q-1), (q^d-1)/(q-1), (q^{d-1}-1)/(q-1))$–BIBD, and it is known that these BIBDs can be generated from cyclic difference families, see [6, 10]. Since $((q^{d+1}-1)/(q-1), (q^d-1)/(q-1)) = 1$, we immediately get the following theorem as a corollary of Theorem 2.

Theorem 8. *There exists a block-graceful* $((q^{d+1}-1)/(q-1), (q^d-1)/(q-1), (q^{d-1}-1)/(q-1))$*–BIBD for every prime power* $q \geq 2$ *and integer* $d \geq 2$.

On the other hand, construction of block-graceful affine geometries seems to be more complicated and we will consider this problem in this section. Besides using the points and lines of the 2-dimensional affine geometries, affine planes can also be constructed using mutually orthogonal Latin squares. A *Latin square of order n* with entries from an n-set X is an $n \times n$ array L in which every cell contains an element of X such that every row of L is a permutation of X and every column of L is a permutation of X. Suppose that L_1 is a Latin square of order n with entries from X and L_2 is a Latin square of order n with entries from Y. We say that L_1 and L_2 are orthogonal Latin squares provided that for every $x \in X$ and for every $y \in Y$, there is a unique cell (i, j) such that $L_1(i, j) = x$ and $L_2(i, j) = y$. A set of Latin squares of order n consisting of $L_1, \ldots, L_s$ is said to be mutually orthogonal Latin squares if L_i and L_j are orthogonal for all $1 \le i < j \le s$. A set of s mutually orthogonal Latin squares of order n is denoted by s MOLS (n) [6, 11].

An affine plane of order n can be constructed from $n-1$ MOLS (n) as follows. Let $\{L_b\}_{b=1}^{n-1}$ be a collection of $n-1$ MOLS (n) with entries from $X = \{0, 1, 2, \ldots, n-1\}$ such that the rows and columns of the Latin squares are indexed from 0 to $n-1$. Let $P = X \times X$. For $0 \le a \le n-1$ and $1 \le b \le n-1$ define $B_{a,b} = \{(r,s)\colon L_b(r,s) = a\}$, for $0 \le r \le n-1$ define $B_{r,n} = \{(r,s)\colon 0 \le s \le n-1\}$, and for $0 \le s \le n-1$ define $B_{s,n+1} = \{(r,s)\colon 0 \le r \le n-1\}$. Finally, let $B = \{B_{a,b}\colon 0 \le a \le n-1 \text{ and } 1 \le b \le n+1\}$. Then, (P, B) is an affine plane of order n [6, 8].

It is known that for every prime power $q \ge 2$, there exist $q-1$ MOLS (q) [11]. Then, the construction described above gives affine planes of order q for every prime power $q \ge 2$. We will make a block-graceful labeling of the blocks in this construction when q is an odd prime power and prove the following theorem.

Theorem 9. *There exists a block-graceful affine plane of order q for every odd prime power q.*

Proof. Let (P, B) be an affine plane of order q constructed using $q-1$ MOLS (q) as described above where we assume without any loss that the symbols in each Latin square have been permuted if necessary such that $L_b(0,s) = s$ for $1 \le b \le q-1$ and $0 \le s \le q-1$. Note that this does not effect orthogonality. We define a labeling $f\colon B \longrightarrow \{1, 2, \ldots, q^2 + q\}$ as follows:

(i) For $0 \le a \le q-1$ and $1 \le b \le q-1$, define $f(B_{a,b}) = qa + b$,

(ii) For $0 \le r \le q-1$, define $f(B_{r,q}) = q(r+1)$, and

(iii) For $0 \le s \le q-1$, define $f(B_{s,q+1}) = q^2 + s + 1$.

It can be easily seen that f is a bijection. We will show that the point weights are all different mod q^2.

Let $(r,s) \in X \times X$. If $r = 0$, then by our assumption $L_b(0,s) = s$ for $1 \le b \le q-1$ and $0 \le s \le q-1$. Hence, the blocks containing the point $(0,s)$ are $B_{s,b}$ for $1 \le b \le q-1$, $B_{0,q}$, and $B_{s,q+1}$. Therefore, weight of the point $(0,s)$ is

$$\begin{aligned} f^{+}(0,s) &= \left(\sum_{b=1}^{q-1} (qs + b)\right) + q + q^2 + s + 1 \\ &= (q-1)qs + \frac{(q-1)q}{2} + q + q^2 + s + 1. \end{aligned} \tag{7}$$

On the other hand, if $r \ne 0$, then $L_b(r,s) \ne s$ for $1 \le b \le q-1$ since $L_b(0,s) = s$ and L_b is a Latin square. Assume that $L_b(r,s) = L_c(r,s) = x$ for some $x \in X$ and $1 \le b < c \le q-1$. Since we also have $L_b(0,x) = L_c(0,x) = x$, this contradicts the orthogonality of L_b and L_c. Hence, we must have $L_b(r,s) \ne L_c(r,s)$ for $1 \le b < c \le q-1$. This means that for $1 \le b \le q-1$, the symbol $L_b(r,s)$ takes all values except for s exactly once. Therefore, weight of the point (r,s) is

$$\begin{aligned} f^{+}(r,s) &= \left(\sum_{b=1}^{q-1} \sum_{\substack{a=0 \\ L_b(r,s)=a}}^{q-1} (qa + b)\right) + q(r+1) + q^2 + s + 1 \\ &= q\left(\frac{(q-1)q}{2} - s\right) + \frac{(q-1)q}{2} + q(r+1) + q^2 + s + 1. \end{aligned} \tag{8}$$

Since q is odd, we can combine (7) and (8), and say that for all $(r,s) \in X \times X$,

$$f^{+}(r,s) \equiv -qs + \frac{(q-1)q}{2} + q(r+1) + s + 1 \equiv \frac{(q-1)q}{2} + q + 1 + q(r-s) + s \pmod{q^2}. \tag{9}$$

Assume that $f^{+}(r_1,s_1) \equiv f^{+}(r_2,s_2) \pmod{q^2}$ for $(r_1,s_1), (r_2,s_2) \in X \times X$. Then, $q(r_1 - s_1) + s_1 \equiv q(r_2 - s_2) + s_2 \pmod{q^2}$ which implies that $s_1 \equiv s_2 \pmod{q}$. Since $0 \le s_1, s_2 \le q-1$, we must have $s_1 = s_2$. Then, we get $r_1 \equiv r_2 \pmod{q}$, and for the same reason, we must have $r_1 = r_2$. This proves that the point weights must all be different mod q^2, which completes the proof. □

Theorem 10. *If there exists a block-graceful $(q^d, q, 1)$–BIBD, where q is a prime power and $d \ge 2$, then there exists a block-graceful $(q^{d+1}, q, 1)$–BIBD.*

Proof. Let (V, B) be a $(q^d, q, 1)$–BIBD where $r = (q^d - 1)/(q-1)$, $V = \mathbb{F}_{q^d}$, the finite field with q^d elements, and $f\colon B \longrightarrow \{1, 2, \ldots, (q^{d-1}(q^d - 1))/(q-1)\}$ is a block-graceful labeling with the induced mapping $f^{+}\colon V \longrightarrow \mathbb{Z}_{q^d}$ on points. Let W be the subfield of order q in V and let $X = V \times W$.

For all $A \in B$ and $y \in W$, let $C_{A,y} = A \times \{y\}$ and define $D_1 = \{C_{A,y}\colon A \in B \text{ and } y \in W\}$. For all $a, b \in V$, let $C_{a,b} = \{(ya + b, y)\colon y \in W\}$, and define $D_2 = \{C_{a,b}\colon a, b \in V\}$ and $D = D_1 \cup D_2$.

We will first show that (X, D) is a $(q^{d+1}, q, 1)$–BIBD. Clearly, $|X| = q^{d+1}$ and every block contains q points. Let (x_1, y_1) and (x_2, y_2) be two distinct points in X. If $y_1 = y_2$, there is a unique block $A \in B$ containing x_1 and x_2, and hence, the block C_{A,y_1} is the only block containing the points (x_1, y_1) and (x_2, y_2). On the other hand, if $y_1 \ne y_2$, then the only block containing the points (x_1, y_1) and (x_2, y_2) is $C_{a,b}$, where $a = (x_1 - x_2)(y_1 - y_2)^{-1}$ and $b = x_1 - y_1 a$. This shows that (X, D) is a $(q^{d+1}, q, 1)$–BIBD.

Take any bijections $h_1\colon V \longrightarrow \{0, 1, 2, \ldots, q^d - 1\}$ and $h_2\colon W \longrightarrow \{0, 1, 2, \ldots, q-1\}$ and define $g\colon D \longrightarrow \{1, 2, \ldots, (q^d(q^{d+1} - 1))/(q-1)\}$ as follows. For any $A \in B$ and $y \in W$, define $g(C_{A,y}) = qf(A) - h_2(y)$, and for any $a, b \in V$, define $g(C_{a,b}) = (q^d(q^d - 1))/(q-1) + 1 + q^d h_1$

Table 1: Small parameters.

v	b	r	k	λ	Nd	Nbg	Reference
6	10	5	3	2	1	1	Example A.1
6	20	10	3	4	4	0	Theorem 1
6	30	15	3	6	6	6	Example A.2
7	7	3	3	1	1	1	Theorem 8
7	14	6	3	2	4	4	Example A.3
7	21	9	3	3	10	10	Example A.4
7	28	12	3	4	35	35	Example A.5
8	14	7	4	3	4	4	Example A.6
9	12	4	3	1	1	1	Theorem 9
9	24	8	3	2	36	36	Example A.7
9	18	8	4	3	11	11	Example A.8
10	7	3	4	2	3	0	Theorem 1
10	18	9	5	4	21	21	Example A.9
11	11	5	5	2	1	1	Theorem 2
13	26	6	3	1	2	2	Example A.10
13	13	4	4	1	1	1	Theorem 8
15	35	7	3	1	80	80	Example A.11
15	15	7	7	3	5	5	Example A.12
16	20	5	4	1	1	1	Example A.13
16	16	6	6	2	3	0	Theorem 1
19	19	9	9	4	6	6	Example A.14
21	21	5	5	1	1	1	Theorem 8

$(b) + h_1(a)$. It can be easily seen that g is a bijection. We will prove that g is a block-graceful labeling of (X, D) by showing that the point weights are all different mod q^{d+1}.

Let $(x, y) \in X$. The blocks in D_1 containing the point (x, y) are $C_{A,y}$ such that $x \in A$. Therefore, the contribution of the labels of the blocks in D_1 to the point weight of (x, y) is

$$\sum_{x\in A} g\left(C_{A,y}\right) = \sum_{x\in A} \left(qf(A) - h_2(y)\right) = qf^{+}(x) - rh_2(y). \tag{10}$$

Since $f^{+}(x)$ s are all different mod q^d and $(q, r) = 1$, the values $qf^{+}(x) - rh_2(y)$ must be all different mod q^{d+1}. We will now show that the contribution of labels the blocks in D_2 to the point weights will just permute these values, and hence, the point weights will all be different mod q^{d+1}. We consider the cases $y = 0$ and $y \neq 0$ separately.

If $y = 0$, then the blocks in D_2 containing the point $(x, 0)$ are $C_{a,x}$ for all $a \in V$. Therefore, the contribution of the labels of the blocks in D_2 to the point weight of $(x, 0)$ is

$$\begin{aligned}\sum_{a\in V} g\left(C_{a,x}\right) &= \sum_{a\in V}\left(\frac{q^d\left(q^d-1\right)}{q-1} + 1 + q^d h_1(x) + h_1(a)\right)\\ &= \frac{q^{2d}\left(q^d-1\right)}{q-1} + q^d + q^{2d}h_1(x) + \frac{\left(q^d-1\right)q^d}{2}\\ &\equiv \frac{\left(q^d+1\right)q^d}{2}\left(\operatorname{mod} q^{d+1}\right).\end{aligned} \tag{11}$$

If $y \neq 0$, then the blocks in D_2 containing the point (x, y) are $C_{a,b}$ for $a, b \in V$ that satisfy $x = ya + b$. Since $y \neq 0$, as a runs through all elements of V, $b = x - ya$ also runs through all elements of V. Therefore, the contribution of the labels of the blocks in D_2 to the point weight of (x, y) is

$$\begin{aligned}\sum_{\substack{a,b\in V\\ x=ya+b}} g\left(C_{a,b}\right) &= \sum_{\substack{a,b\in V\\ x=ya+b}}\left(\frac{q^d\left(q^d-1\right)}{q-1} + 1 + q^d h_1(b) + h_1(a)\right)\\ &= \frac{q^{2d}\left(q^d-1\right)}{q-1} + q^d + \frac{\left(q^d-1\right)q^{2d}}{2} + \frac{\left(q^d-1\right)q^d}{2}\\ &\equiv \frac{\left(q^d+1\right)q^d}{2}\left(\operatorname{mod} q^{d+1}\right).\end{aligned} \tag{12}$$

Therefore, contribution of the labels of the blocks in D_2 to the point weight of (x, y) is $(q^d + 1)q^d/2$ for any $(x, y) \in X$. Adding this constant value to the weights of the points will just permute them among themselves. As a conclusion, g is a block-graceful labeling of (X, D), and the proof is complete.

As a consequence of Theorems 9-10 and Example A.13, we get the following theorem. □

Theorem 11. *If $d \geq 2$ is an integer and q is an odd prime power or $q = 4$, then there exists a block-graceful $(q^d, q, 1)$–BIBD.*

5. Designs with Small Parameters

In this section, we consider (v, b, r, k, λ)–BIBDs with small parameters where $2k \leq v \leq 21$, $r \leq 15$, and $b \leq 35$, and the number Nd of pairwise nonisomorphic (v, b, r, k, λ)–BIBDs satisfies $1 \leq Nd \leq 80$. The results for these parameters are summarized in Table 1, where Nbg denotes the number of pairwise nonisomorphic (v, b, r, k, λ)–BIBDs that are block-graceful.

Three of the parameters given in Table 1 do not satisfy the necessary condition for the existence of a block-graceful BIBD given in Theorem 1. For some of the parameters that satisfy the necessary condition, Theorems 2, 8, and 9 provide block-graceful labelings. For the remaining parameters, we obtain block-graceful labelings via computer search, and these labelings are presented in Appendix A.

6. Conclusions

In this article, we defined and studied block-graceful labelings for some classes of BIBDs. In Section 3, after observing that one can easily obtain block-graceful Steiner triple systems of order v for all $v \equiv 1 \pmod 6$, we constructed infinite families of block-graceful Steiner triple systems of order v for $v \equiv 3 \pmod 6$. The case $v \equiv 3 \pmod 6$ is far from complete and remains as an open problem.

Open Problem 1. Does there exist a block-graceful Steiner triple system of order v for all $v \equiv 3 \pmod 6$?

In Section 4, we considered affine and projective geometries and after observing that for every prime power $q \geq 2$ and integer $d \geq 2$, a block-graceful $((q^{d+1}-1)/(q-1), (q^d-1)/(q-1), (q^{d-1}-1)/(q-1))$–BIBD can be easily obtained, we proved that for every prime power q and integer $d \geq 2$, where q is odd or $q = 4$, there exists a block-graceful $(q^d, q, 1)$–BIBD. The case when $q = 2^s$ for $s \geq 3$ remains as an open problem for $(q^d, q, 1)$–BIBDs.

Open Problem 2. Does there exist a block-graceful $(q^d, q, 1)$–BIBD for all $q = 2^s$, $d, s \geq 2$? In particular, does there exist a block-graceful affine plane of order 2^s for all $s \geq 2$?

In Section 5, we made a list of small parameters such that the existence problem of block-graceful labelings is completely solved for all pairwise nonisomorpic BIBDs with these parameters. We have not found any BIBD that satisfies the necessary condition in Theorem 1 and is not block-graceful. Based on this result, we make the following conjectures.

Conjecture 1 (Weak). *There exists a block-graceful (v, k, λ)–BIBD for all v, k, and λ that satisfy the necessary conditions for the existence of a (v, k, λ)–BIBD and the condition in Theorem 1.*

Conjecture 2 (Strong). *All BIBDs satisfying the necessary condition in Theorem 1 are block-graceful.*

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Supplementary Materials

Example A.1: A block-graceful labeling of the unique (6, 3, 2)–BIBD. Example A.2: Block-graceful labelings of the 6 nonisomorphic (6, 3, 6)–BIBDs. Example A.3: Block-graceful labelings of the 4 nonisomorphic (7, 3, 2)–BIBDs. Example A.4: Block-graceful labelings of the 10 nonisomorphic (7, 3, 3)–BIBDs. Example A.5: Block-graceful labelings of the 35 nonisomorphic (7, 3, 4)–BIBDs. Example A.6: Block-graceful labelings of the 4 nonisomorphic (8, 4, 3)–BIBDs. Example A.7: Block-graceful labelings of the 36 nonisomorphic (9, 3, 2)–BIBDs. Example A.8: Block-graceful labelings of the 11 nonisomorphic (9, 4, 3)–BIBDs. Example A.9: Block-graceful labelings of the 21 nonisomorphic (10, 5, 4)–BIBDs. Example A.10: Block-graceful labelings of the 2 nonisomorphic (13, 3, 1)–BIBDs. Example A.11: Block-graceful labelings of the 80 nonisomorphic (15, 3, 1)–BIBDs. Example A.12: Block-graceful labelings of the 5 nonisomorphic (15, 7, 3)–BIBDs. Example A.13: A block-graceful labeling of the unique (16, 4, 1)–BIBD. Example A.14: Block-graceful labelings of the 6 nonisomorphic (19, 9, 4)–BIBDs. Example A.15: A block-graceful labeling of a (21, 3, 1)–BIBDs. (*Supplementary Materials*)

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