Research Article

Novel MCDM Methods and Similarity Measures for Extended Fuzzy Parameterized Possibility Fuzzy Soft Information with Their Applications

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Complicated uncertainties arising in the multicriteria decision-making (MCDM) problems that show distinct possible satisfaction of the subjects to favorable and equally unfavorable parameters with varying preferences require reliable decision-making under comprehensive mathematical tools. For such complications, this work aims to develop a novel fuzzy parameterized possibility fuzzy bipolar soft set model as a fuzzy parameterized bipolar soft extension of possibility fuzzy sets. The proposed model efficiently depicts the possibility of fuzzy belongingness of alternatives under fuzzy parameterized bipolar parameters (or attributes). The respective operations and properties such as subset, complement, union, and intersection are presented along with their numerical illustrations. Two logical operations namely “AND” and “OR” operations followed by two corresponding MCDM algorithms have been developed and implemented. Furthermore, similarity measures between fuzzy parameterized possibility fuzzy bipolar soft sets are proposed and applied to an agricultural land selection scenario. Finally, a comparative analysis of current work with existing ones is discussed in detail to show the eminent quality of the proposed work over them.

1. Introduction

In these days, uncertainty theories and their soft computing hybrid fusions are emerging and playing a significant role in almost every domain, including medicine, agriculture, economics, real estate, business, and engineering. Rapid growth was seen in the literature of uncertainty theories after the production of fuzzy set theory in 1965, which was the pioneer mathematical tool overall traditional tools of that time such as probability theory. These fuzzy sets for the first time provided an insight into the partial truth or partial belongingness of an element, leading to the solutions of many uncertain problems wandering between the bounds of absolute truth and absolute false. Besides, bipolarity seems to pervade human understanding of preference and information, and bipolar representations look very useful in the development of intelligent technologies. Bipolarity refers to the explicit handling of positive and negative sides of information. Therefore, for dealing with the bipolarity of fuzzy datasets, the idea of bipolar fuzzy sets was presented by Zhang in 1994. Later, in 1998, Zhang introduced the Yin-Yang bipolar fuzzy sets by further refining his theory for bipolarity in fuzziness. To date, a lot of research has been completed using the idea of bipolar fuzzy sets.

To deal with uncertain MCDM circumstances, Molodtsov launched another uncertainty theory of soft sets as a new mathematical mechanism for dealing with different ambiguities. The soft set model provided a parametric approach to decision-making by considering parameterized families of sets. It made the model free of the limitations that influenced existing approaches. Due to the presence of soft information in the bipolar form, the theory of bipolar soft (BS) sets was initiated by Shabir and Naz in 2013 as a natural generalization of soft sets. In recent years, several researchers have been attracted to fuzzy MCDM models by establishing many new hybrid approaches. The fuzzy MCDM techniques explain how parametric information is to be analyzed to determine the ranking of alternatives or an appropriate
alternative since fuzzy MCDM models have been widely used in the literature of different fields ranging from medical to social sciences. In all the existing hybrid-soft-set models, alternatives are categorized regarding the parameters. However, it can be easily analyzed that in most cases, some parameters have more preferences over others, and thus, higher degrees of less preferable parameter families may affect the decisions. To sort out this issue, several fuzzy parameterized soft set models have been seen in the literature. In 2010, Çağman, Çıtak, and Enginoğlu came up with their fuzzy parameterized fuzzy set theory allowing preferential parameterization. To further consider the possibility of a certain degree of belongingness of alternatives or elements in the soft environment, Alkhazaleh, Salleh, and Hassan presented their possibility fuzzy soft set theory in 2011 that assigned a fuzzy possibility degree besides each alternative's fuzzy belongingness degree.

All the existing models lack precision when subjected to BS knowledge under fuzzy parameterized possibility fuzzy environment. From the above discussion, one can readily observe that a hybrid model having the efficiency to deal with BS data with fuzzy parameterized possibility fuzzy information is still not proposed. Taking into account the deficiencies of the existing hybrid models, we propose a novel direction for research towards emerging era of MCDM approaches.

1.1. Literature Review. Zadeh [1] was the first who introduced the theory of fuzzy sets (FSs). These FSs provided means to depict and handle uncertainties as a decision-making revolution. Many recent works appear as extensions and applications of these FSs in different domains and directions. For instance, Atif et al. [2] discussed some covering-based fuzzy-rough sets besides their decision-making applications. Juan and Qiang [3] proposed interval-valued hesitant fuzzy linguistic MCDM method using Heronian mean operators. Azam et al. [4] used a rather extended version, i.e., complex intuitionistic FS theory to evaluate information security management.

Three decades after the introduction of FSs, Zhang [5] generalized the theory of fuzzy sets to bipolar fuzzy theory for dealing with the bipolarity of fuzzy information (see also [6]). Further refining it, Zhang [7] developed the notion of Yin-Yang bipolar fuzzy sets in 1998. To deal with MCDM information, Molodtsov [8] launched the soft set theory as a new mathematical mechanism for dealing with different types of uncertain situations. After the production of the soft set model, Molodtsov [8] claimed that his uncertainty theory can be easily fused with other uncertain mathematical tools to form more general hybridized models, which may provide more accurate and surprising results than parent models. After that, several researchers have paid a lot of attention to the soft set theory and its hybrid-soft-set models. For example, Maji et al. [9] was the first who examined the applicability of the soft set model for decision-making situations. Additionally, they [10] proposed a hybrid model called fuzzy soft set (FSS) by the fusion of fuzzy and soft sets. Later, Roy and Maji [11] presented an FSS theoretic approach to solving decision-making problems. Due to the occurrence of soft information in a bipolar format, Shahib and Naz [12] introduced the theory of bipolar soft (or BS) sets as a natural generalization of soft sets. Later on, Malik and Shabir [13] discussed the roughness of the fuzzy BS set theory and presented different new results. Al-Shami [14] discussed relationship between bipolar soft sets and ordinary points along with the decision-making applications. Akram et al. [15] proposed a MCDM model called rough $m$-polar fuzzy BS sets. Ali and Ansari [16] combined BS sets with Fermatean fuzzy sets and developed a new hybrid model called Fermatean fuzzy BS sets for MCDM. Al-Shami and Mhamedi [17] defined some types of belonging and non-belonging relationship between ordinary points and double framed soft sets. Recently, Al-Shami et al. [18] provided $(a, b)$-FSSs as a novel generalization of FSSs.

On the other hand, for dealing with fuzzy MCDM problems, a novel hybrid model called fuzzy parameterized (FP) soft sets was introduced by Çağman and Enginoğlu [19], where fuzzy membership values are associated with parameters to better describe their relative preferences or weights. Moreover, they [20] generalized their work to FP-FSSs. The existing fuzzy MCDM models fail to deal with FP intuitionistic fuzzy information. To overcome this issue, more generalizations of this useful idea have been proposed, including intuitionistic FP soft sets [21], intuitionistic FP-FSSs [22], intuitionistic FP intuitionistic FSSs [23], FP $q$-rung orthopair fuzzy soft expert sets [24], and FP fuzzy soft expert sets [25]. All the aforementioned FP soft set approaches are inefficient to deal with interval-valued representations of datasets. To solve this problem, Aydin and Enginoğlu [26] introduced a more extended model, namely, interval-valued intuitionistic FP interval-valued intuitionistic FSSs, and explored an MCDM application. From another perspective to solve the fuzzy MCDM problems, Alkhazaleh et al. [27] proposed the concepts of the possibility fuzzy soft sets. Later, Bashir et al. [28] further generalized this concept by constructing a hybrid model called the possibility intuitionistic fuzzy soft sets. Garg and Arora [29] used some new information measures for possibility intuitionistic fuzzy soft set decision-making to develop efficient MCDM algorithms. Several hybrid possibility fuzzy extensions of different models have been completed, for example, possibility fuzzy soft expert sets [30], possibility belief interval-valued soft sets [31], possibility fuzzy soft ordered semigroups for ideals [32], possibility Pythagorean FSSs [33], possibility $m$-polar FSSs [34], possibility Pythagorean bipolar FSSs [35], possibility multi-FSSs [36], possibility neutrosophic soft sets [37], inverse possibility FSSs [38].

The similarity measure and distance measure are significant topics for uncertainty theories to deal with different kinds of datasets. In data science, the similarity measure is a way of measuring how data samples are related or closed to each other. Currently, similarity measures among the existing fuzzy extensions of hybrid-soft-set models have been widely studied due to their daily-life applications in different fields, including clustering, image processing, and pattern recognition. For instance, Majumdar and Samanta
studied the concepts of similarity measures among soft sets [39] and FSSs [40]. Kharal [41] presented certain operations for soft sets under similarity and distance measures and applied them to solve a decision-making problem, that is, the financial diagnosis of firms. Jiang et al. [42] introduced some distance measures among intuitionistic FSSs and developed certain entropies on intuitionistic FSSs and interval-valued FSSs. Wang and Qu [43] introduced similarity measures and fuzzy numbers similarity measures based risk analysis. Later, Gohain et al. [44] investigated similarity measures for Pythagorean fuzzy sets and explored their applications to MCDM. Gogoi and Chutia [45] presented crop selection applications of intuitionistic fuzzy numbers and their applications in agriculture. Gohain et al. [47] initiated some new similarity measures for intuitionistic FSSs along with their applications.

1.2. Motivations and Contributions. The motivations of this study are given as follows:

1. Fuzzy parameterized extensions of hybrid-soft-set models are arising as powerful tools but a hybrid model which can deal with FP-possibility fuzzy BS information is still missing in the literature.

2. The similarity measure phenomenon is used to measure how much different datasets are related. That’s why similarity measure of the proposed model is also necessary.

The following are major contributions of this work:

1. A novel hybrid MCDM model called fuzzy parameterized possibility fuzzy bipolar soft sets ($\mathcal{FPFBS}$) is developed, which is a FP-BS extension of possibility FSS model

2. Certain fundamental properties of the launched hybrid model, including subset, complement, union, and intersection, are investigated and illustrated with numerical examples

3. Two basic operations, including the “AND” operation and the “OR” operation, are also studied and explained via a brief numerical example, which are supported by their respective algorithms

4. Further, a new concept of similarity measures between the $\mathcal{FPFBS}$ is discussed

5. To validate the potentiality and consistency of the initiated model, we explore a daily-life example of an agricultural land selection problem

6. A comparative analysis of current work with existing ones is discussed in detail to show the eminent quality of the proposed work over them

1.3. Organization. The remaining structure of the paper is formulated as follows: in section 2, we first recall some basic notions and then present a new hybrid model, namely, fuzzy parameterized possibility fuzzy bipolar soft sets (or $\mathcal{FPFBS}$). We also investigate some basic properties of the proposed model in this section. In Section 3, we discuss two basic operations, including the “AND” operation and the “OR” operation and demonstrate them with the help of a brief numerical example. In Section 4, we study a new concept of similarity measures between the $\mathcal{FPFBS}$ and explore a daily-life application, that is, agricultural land selection for cropping sugarcane. In Section 5, we compare our initiated model with some existing models in both qualitative and quantitative formats. In Section 6, we give the concluding remarks and certain future orientations of our work.

2. Fuzzy Parameterized Possibility Fuzzy Bipolar Soft Sets

This section first reviews the basic notions to support the further study of this work. Then, a major hybrid model called fuzzy parameterized possibility fuzzy bipolar soft sets (or $\mathcal{FPFBS}$) is presented.

Definition 1. [12] Let $\mathcal{U}$ and $\mathcal{E}$ be two sets of objects and parameters (or attributes), respectively. For any $\mathcal{X} \subseteq \mathcal{E}$, a 4-tuple $(\mathcal{P}, \mathcal{Q}, \mathcal{X}, \mathcal{E})$ is referred to as bipolar soft set (or BS set) on $\mathcal{U}$, where $\mathcal{P}$, $\mathcal{Q}$ are functions provided as $\mathcal{P}: \mathcal{X} \rightarrow \mathcal{U}$ and $\mathcal{Q}: \mathcal{X} \rightarrow \mathcal{U}$ such that $\mathcal{P}(e) \cap \mathcal{Q}(e) = \emptyset$ for all $e \in \mathcal{X}$ and $e \in \mathcal{E}$. Here, $\mathcal{U}$ denotes the power set of $\mathcal{U}$, and $\mathcal{X}$ represents the “Not set” of parameters, which contains parameters opposite to those contained in $\mathcal{X}$.

Definition 2. [27] Let $\mathcal{U}$ and $\mathcal{E}$ be two sets of objects and attributes, respectively. Let $\lambda: \mathcal{E} \rightarrow [0, 1]$ be a fuzzy set (possibility function) on $\mathcal{E}$ and $\mathcal{P}: \mathcal{E} \rightarrow \mathcal{U}^\mathcal{U}$ be another mapping, where $\mathcal{U}^\mathcal{U}$ represents the family of all fuzzy subsets of $\mathcal{U}$. For any $\mathcal{X} \subseteq \mathcal{E}$, a pair $(\mathcal{P}, \mathcal{X})$ is called the possibility fuzzy soft set (or PFSS) over soft universe $(\mathcal{U}, \mathcal{E})$ where the mapping $\mathcal{P}: \mathcal{X} \rightarrow \mathcal{U}^\mathcal{U} \times [0, 1]$ is defined as follows:

$$\mathcal{P}_i(e_j) = \left(\frac{\mathcal{P}_i(e_j)}{\mathcal{P}_i(e_j) + \lambda(e_j)}, \lambda(e_j)\right), \text{ for all } \mathcal{P}_i \in \mathcal{U}, e_j \in \mathcal{X},$$

(1)

where $\mathcal{P}_i(e_j)$ denotes the membership degrees of the objects ($\mathcal{P}_i, i = 1, 2, \ldots, n$) in $\mathcal{P}_i(e_j)$ and $\lambda(e_j)$ denotes the strength of possibility of those memberships of objects of $\mathcal{P}_i$ in $\mathcal{P}_i(e_j)$ with $j = 1, 2, \ldots, m$. In set form, the PFSS $(\mathcal{P}_i, \mathcal{X})$ is given by
\[
(\tilde{\mathcal{P}}_\omega, \tilde{\mathcal{X}}) = \left\{ \left( \tilde{e}_1, \left\langle \left( \frac{\tilde{u}_1}{\mathcal{P}(\tilde{e}_1)(\tilde{u}_1)}, \lambda(\tilde{e}_1)_{\tilde{u}_1} \right), \left( \frac{\tilde{u}_2}{\mathcal{P}(\tilde{e}_1)(\tilde{u}_2)}, \lambda(\tilde{e}_1)_{\tilde{u}_2} \right), \ldots, \left( \frac{\tilde{u}_n}{\mathcal{P}(\tilde{e}_1)(\tilde{u}_n)}, \lambda(\tilde{e}_1)_{\tilde{u}_n} \right) \right\rangle \right) \right\},
\]

\[
(\tilde{e}_2, \left\langle \left( \frac{\tilde{u}_1}{\mathcal{P}(\tilde{e}_2)(\tilde{u}_1)}, \lambda(\tilde{e}_2)_{\tilde{u}_1} \right), \left( \frac{\tilde{u}_2}{\mathcal{P}(\tilde{e}_2)(\tilde{u}_2)}, \lambda(\tilde{e}_2)_{\tilde{u}_2} \right), \ldots, \left( \frac{\tilde{u}_n}{\mathcal{P}(\tilde{e}_2)(\tilde{u}_n)}, \lambda(\tilde{e}_2)_{\tilde{u}_n} \right) \right\rangle \right) \right\},
\]

\vdots

\[
(\tilde{e}_m, \left\langle \left( \frac{\tilde{u}_1}{\mathcal{P}(\tilde{e}_m)(\tilde{u}_1)}, \lambda(\tilde{e}_m)_{\tilde{u}_1} \right), \left( \frac{\tilde{u}_2}{\mathcal{P}(\tilde{e}_m)(\tilde{u}_2)}, \lambda(\tilde{e}_m)_{\tilde{u}_2} \right), \ldots, \left( \frac{\tilde{u}_n}{\mathcal{P}(\tilde{e}_m)(\tilde{u}_n)}, \lambda(\tilde{e}_m)_{\tilde{u}_n} \right) \right\rangle \right) \right\}.
\]

We are now ready to construct the notion of fuzzy parameterized possibility fuzzy bipolar soft sets (or \(\mathcal{FPPFS} \mathcal{BS} \mathcal{S} \)) which is given as follows.

**Definition 3.** Let \(\mathcal{U}\) and \(\mathcal{E}\) be the sets of objects and attributes, respectively. For any \(\mathcal{X} \subseteq \mathcal{E}\), a 4-tuple \((\tilde{\mathcal{P}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega)\) is called fuzzy parameterized possibility fuzzy bipolar soft set (or \(\mathcal{FPPFS} \mathcal{BS} \mathcal{S}\)) where \(\tilde{\mathcal{P}}_\omega : \tilde{\mathcal{X}}_\omega \rightarrow \mathcal{S} \mathcal{F} \mathcal{U} \times [0, 1]\) and \(\tilde{\mathcal{X}}_\omega : \tilde{\mathcal{X}}_\omega \rightarrow \mathcal{S} \mathcal{F} \mathcal{U} \times [0, 1]\) are two functions, which are given by

\[
\tilde{\mathcal{P}}_\omega \left( \tilde{e}^{\omega}(\tilde{e}) \right) = \left\{ \left( \frac{\tilde{u}}{\mathcal{P}(\tilde{e})(\tilde{u})}, \lambda^\omega(\tilde{e})_{\tilde{u}} \right) \mid \tilde{u} \in \tilde{\mathcal{U}}, \mathcal{P}(\tilde{e})(\tilde{u}) \in [0, 1] \right\},
\]

\[
\tilde{\mathcal{X}}_\omega \left( \tilde{e}^{\omega}(\tilde{e}) \right) = \left\{ \left( \frac{\tilde{u}}{\mathcal{X}(\tilde{e})(\tilde{u})}, \theta^\omega(\tilde{e})_{\tilde{u}} \right) \mid \tilde{u} \in \tilde{\mathcal{U}}, \mathcal{X}(\tilde{e})(\tilde{u}) \in [0, 1] \right\},
\]

for all \(\omega(\tilde{e}), \omega(\tilde{e}) \in [0, 1]\). In set form, the \(\mathcal{FPPFS} \mathcal{BS} \mathcal{S}\) \((\tilde{\mathcal{P}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega)\) is represented as the union of two FP-possibility FSSs, that is, \((\tilde{\mathcal{P}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega) = (\tilde{\mathcal{P}}_\omega, \tilde{\mathcal{X}}_\omega) \cup (\tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega)\).

This novel concept is demonstrated via the following example:

**Example 1.** Let \(\mathcal{U} = \{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_5\}\) be the set of five washing machines and let \(\mathcal{X} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\} \subseteq \mathcal{E}\) be the set of favorable parameters where \(\tilde{e}_1 = \text{Efficient}, \tilde{e}_2 = \text{Manual}, \tilde{e}_3 = \text{Bipolar} \).

Automatic, \(\tilde{e}_3 = \text{Digital}\) and \(\tilde{X} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}\) be the “not set” of parameters where \(\tilde{e}_1 = \text{Inefficient}, \tilde{e}_2 = \text{Manual}, \tilde{e}_3 = \text{Analog}\). Consider the weights provided by an expert (decision-maker) to these sets of parameters as \(\tilde{X}_1 = \{0.45, 0.38, 0.61\}\) and \(\tilde{X}_2 = \{0.71, 0.50, 0.15\}\). Then, the judgments of the decision-maker about the machines with respect to available parameters are provided in the form of a \(\mathcal{FPPFS} \mathcal{BS} \mathcal{S}\) \((\tilde{\mathcal{P}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega, \tilde{\mathcal{X}}_\omega)\) where

\[
\tilde{\mathcal{P}}_1^{\tilde{e}_1} = \begin{bmatrix}
(\tilde{u}_1, 0.67), 0.24 & (\tilde{u}_2, 0.57), 0.34 & (\tilde{u}_3, 0.32), 0.78 & (\tilde{u}_4, 0.17), 0.36 & (\tilde{u}_5, 0.21), 0.45 \\
\end{bmatrix},
\]

\[
\tilde{\mathcal{P}}_1^{\tilde{e}_2} = \begin{bmatrix}
(\tilde{u}_1, 0.42), 0.55 & (\tilde{u}_2, 0.36), 0.20 & (\tilde{u}_3, 0.61), 0.34 & (\tilde{u}_4, 0.55), 0.28 & (\tilde{u}_5, 0.48), 0.30 \\
\end{bmatrix},
\]

\[
\tilde{\mathcal{P}}_1^{\tilde{e}_3} = \begin{bmatrix}
(\tilde{u}_1, 0.56), 0.43 & (\tilde{u}_2, 0.69), 0.55 & (\tilde{u}_3, 0.50), 0.42 & (\tilde{u}_4, 0.67), 0.15 & (\tilde{u}_5, 0.64), 0.39 \\
\end{bmatrix}.
\]
\[
\mathcal{F}_{\theta}^{-0.71} = \begin{pmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\bar{u}_3 \\
\bar{u}_4 \\
\bar{u}_5
\end{pmatrix} = \begin{pmatrix}
0.42, 0.55 \\
0.36, 0.20 \\
0.61, 0.34 \\
0.55, 0.28 \\
0.48, 0.30
\end{pmatrix},
\]
\[
\mathcal{F}_{\theta}^{-0.50} = \begin{pmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\bar{u}_3 \\
\bar{u}_4 \\
\bar{u}_5
\end{pmatrix} = \begin{pmatrix}
0.67, 0.24 \\
0.57, 0.34 \\
0.32, 0.36 \\
0.17, 0.36 \\
0.21, 0.45
\end{pmatrix},
\]
\[
\tilde{\mathcal{F}}_{\theta}^{-0.15} = \begin{pmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\bar{u}_3 \\
\bar{u}_4 \\
\bar{u}_5
\end{pmatrix} = \begin{pmatrix}
0.16, 0.80 \\
0.29, 0.17 \\
0.30, 0.64 \\
0.46, 0.58 \\
0.25, 0.66
\end{pmatrix}.
\]

In matrix form, the \( \mathcal{F}_{\theta}^{-0.71} \) is provided as follows:

\[
\begin{align*}
\mathcal{F}_{\theta}^{-0.71} &= \begin{pmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\bar{u}_3 \\
\bar{u}_4 \\
\bar{u}_5
\end{pmatrix} = \begin{pmatrix}
0.67, 0.24 \\
0.42, 0.55 \\
0.56, 0.43 \\
0.32, 0.78 \\
0.30, 0.42
\end{pmatrix},
\end{align*}
\]

Now, we investigate some fundamental operations on \( \mathcal{F}_{\theta}^{-0.71} \) such as subset-relation, complement, union, and intersection with corresponding numerical examples.

**Definition 4.** Let \( \mathcal{F}_{\theta}^{-0.71} \) and \( \mathcal{F}_{\theta}^{-0.50} \) be two \( \mathcal{F}_{\theta}^{-0.71} \) over the soft universe \( (\mathcal{U}, \mathcal{E}) \).

Then, the \( \mathcal{F}_{\theta}^{-0.71} \) subset of \( \mathcal{F}_{\theta}^{-0.50} \) is said to be \( \mathcal{F}_{\theta}^{-0.71} \) subset of \( \mathcal{F}_{\theta}^{-0.50} \), written as \( \mathcal{F}_{\theta}^{-0.71} \subseteq \mathcal{F}_{\theta}^{-0.50} \), if all \( \bar{e} \in \mathcal{X} \) and \( \bar{e} \in \mathcal{X} \) hold:

1. \( \mathcal{X}_1 \subseteq \mathcal{Y}_1 \) and \( \mathcal{X}_2 \subseteq \mathcal{Y}_2 \)
2. \( \lambda(\bar{e}) \) is a fuzzy subset of \( \lambda(\bar{e}) \), and \( \theta(\bar{e}) \) is a fuzzy subset of \( \theta(\bar{e}) \)
3. \( \mathcal{P}_1(\bar{e}^{\alpha_1}) \) is a possibility fuzzy subset of \( \mathcal{P}_1(\bar{e}^{\alpha_1}) \), and \( \mathcal{Q}_1(\bar{e}^{\alpha_2}) \) is a possibility fuzzy subset of \( \mathcal{Q}_1(\bar{e}^{\alpha_2}) \)

Note that \( \mathcal{F}_{\theta}^{-0.71}, \mathcal{F}_{\theta}^{-0.50} \) is a super set of \( \mathcal{F}_{\theta}^{-0.71} \), written as \( \mathcal{F}_{\theta}^{-0.71}, \mathcal{F}_{\theta}^{-0.50} \) of \( \mathcal{F}_{\theta}^{-0.71} \).

The following example explains the idea of sub sethood between \( \mathcal{F}_{\theta}^{-0.71} \).

**Example 2.** Let \( \mathcal{U} = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_5\} \) be a set of five bikes, and \( \mathcal{E} = \{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\} \) be a set of parameters where \( \bar{e}_1 = \) cheap, \( \bar{e}_2 = \) disc brakes, \( \bar{e}_3 = \) red, \( \bar{e}_4 = \) classic technology. Let \( \mathcal{X}_1 = \{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\} \) and \( \mathcal{X}_2 = \{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\} \) be the set of favorable parameters and its corresponding “not set” of parameters along with weights, respectively, where \( \bar{e}_1 \) = expensive, \( \bar{e}_2 \) = rim brakes, \( \bar{e}_3 \) = black, \( \bar{e}_4 \) = modern technology. Assume that estimations of the available alternatives regarding parameters are provided by an expert in the form of \( \mathcal{F}_{\theta}^{-0.71} \) over the soft universe \( (\mathcal{U}, \mathcal{E}) \), which is given by

\[
\begin{align*}
\mathcal{F}_{\theta}^{-0.71} &= \begin{pmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\bar{u}_3 \\
\bar{u}_4 \\
\bar{u}_5
\end{pmatrix} = \begin{pmatrix}
0.36, 0.40 \\
0.66, 0.50 \\
0.47, 0.20 \\
0.59, 0.46 \\
0.50, 0.32
\end{pmatrix},
\end{align*}
\]
Now suppose another $\mathcal{FPPP}_S$ of $(\mathcal{R}_1, \mathcal{S}_1, \mathcal{Y}_{\sigma_1}, \mathcal{Y}_{\sigma_2})$ where $\mathcal{Y}_{\sigma_1} = \{e_1, e_2, e_3, e_4\}$ and $\mathcal{Y}_{\sigma_2} = \{e_1, e_2, e_3, e_4\}$, which is defined as follows

\[
\begin{align*}
\mathcal{R}_1(e_1) &= \left\{ \begin{pmatrix} \tilde{u}_1 & 0.50 \\ \tilde{u}_2 & 0.58 \\ \tilde{u}_3 & 0.35 \\ \tilde{u}_4 & 0.52 \\ \tilde{u}_5 & 0.46 \end{pmatrix}, \begin{pmatrix} \tilde{u}_1 & 0.43 \\ \tilde{u}_2 & 0.67 \\ \tilde{u}_3 & 0.55 \\ \tilde{u}_4 & 0.62 \\ \tilde{u}_5 & 0.55 \end{pmatrix}, \begin{pmatrix} \tilde{u}_1 & 0.43 \\ \tilde{u}_2 & 0.67 \\ \tilde{u}_3 & 0.55 \\ \tilde{u}_4 & 0.62 \\ \tilde{u}_5 & 0.55 \end{pmatrix}, \begin{pmatrix} \tilde{u}_1 & 0.43 \\ \tilde{u}_2 & 0.67 \\ \tilde{u}_3 & 0.55 \\ \tilde{u}_4 & 0.62 \\ \tilde{u}_5 & 0.55 \end{pmatrix}, \begin{pmatrix} \tilde{u}_1 & 0.43 \\ \tilde{u}_2 & 0.67 \\ \tilde{u}_3 & 0.55 \\ \tilde{u}_4 & 0.62 \\ \tilde{u}_5 & 0.55 \end{pmatrix} \right\}, \\
&\vdots \\
\mathcal{R}_1(e_4) &= \left\{ \begin{pmatrix} \tilde{u}_1 & 0.30 \\ \tilde{u}_2 & 0.48 \\ \tilde{u}_3 & 0.24 \\ \tilde{u}_4 & 0.70 \\ \tilde{u}_5 & 0.65 \end{pmatrix}, \begin{pmatrix} \tilde{u}_1 & 0.30 \\ \tilde{u}_2 & 0.48 \\ \tilde{u}_3 & 0.24 \\ \tilde{u}_4 & 0.70 \\ \tilde{u}_5 & 0.65 \end{pmatrix}, \begin{pmatrix} \tilde{u}_1 & 0.30 \\ \tilde{u}_2 & 0.48 \\ \tilde{u}_3 & 0.24 \\ \tilde{u}_4 & 0.70 \\ \tilde{u}_5 & 0.65 \end{pmatrix}, \begin{pmatrix} \tilde{u}_1 & 0.30 \\ \tilde{u}_2 & 0.48 \\ \tilde{u}_3 & 0.24 \\ \tilde{u}_4 & 0.70 \\ \tilde{u}_5 & 0.65 \end{pmatrix}, \begin{pmatrix} \tilde{u}_1 & 0.30 \\ \tilde{u}_2 & 0.48 \\ \tilde{u}_3 & 0.24 \\ \tilde{u}_4 & 0.70 \\ \tilde{u}_5 & 0.65 \end{pmatrix} \right\}. \\
\end{align*}
\]

It shows that $(\mathcal{P}_1, \mathcal{S}_0, \mathcal{X}_{w_1}, \mathcal{X}_{w_2})$ is a $\mathcal{FPPP}_S$ subset of $(\mathcal{R}_1, \mathcal{S}_1, \mathcal{Y}_{\sigma_1}, \mathcal{Y}_{\sigma_2})$.

In the following definition, we discuss the condition of equality between $\mathcal{FPPP}_S$.

**Definition 5.** Let $(\mathcal{P}_1, \mathcal{S}_0, \mathcal{X}_{w_1}, \mathcal{X}_{w_2})$ and $(\mathcal{R}_1, \mathcal{S}_1, \mathcal{Y}_{\sigma_1}, \mathcal{Y}_{\sigma_2})$ be two $\mathcal{FPPP}_S$ over the soft universe $(\mathcal{U}, \mathcal{S})$. Then, the $\mathcal{FPPP}_S$ $(\mathcal{P}_1, \mathcal{S}_0, \mathcal{X}_{w_1}, \mathcal{X}_{w_2})$ and $(\mathcal{R}_1, \mathcal{S}_1, \mathcal{Y}_{\sigma_1}, \mathcal{Y}_{\sigma_2})$ are called equal $\mathcal{FPPP}_S$, written as $(\mathcal{P}_1, \mathcal{S}_0, \mathcal{X}_{w_1}, \mathcal{X}_{w_2}) = (\mathcal{R}_1, \mathcal{S}_1, \mathcal{Y}_{\sigma_1}, \mathcal{Y}_{\sigma_2})$, if and only if

1. $\mathcal{X} = \mathcal{Y}$ and $\mathcal{X} = \mathcal{X}$
2. $\lambda(e)$ is equal to $\lambda_1(e)$ and $\theta(e)$ is equal to $\theta_1(e)$ for all $e \in \mathcal{X}$ and $\mathcal{X} \in \mathcal{X}$
3. $\mathcal{P}_1(e^{\omega_1}(e)) = \mathcal{R}_1(e^{\omega_1}(e))$ and $\mathcal{S}_0(e^{\omega_1}(e))$ is equal to $\mathcal{S}_0(e^{\omega_1}(e))$, for all $e^{\omega_1}(e) \in \mathcal{X}_{w_1}$ and $\mathcal{S}_0(e^{\omega_1}(e)) \in \mathcal{X}_{w_2}$.
Two extreme results of $\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$ are studied in the following definition:

**Definition 6.** A $\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$ is called null-$\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$, represented as $(\Phi, \mathcal{U}, \mathcal{X}, \mathcal{X}_2)$ where \( \Phi: \mathcal{X}_1 \rightarrow \mathcal{F}^{\mathcal{B}} \times [0, 1] \) and \( \mathcal{U}: \mathcal{X}_2 \rightarrow \mathcal{F}^{\mathcal{B}} \times [0, 1] \) are mappings, which satisfy the following assertions:

\[
\Phi_0(e^{-w_1(\bar{e})}) = \left\{ \left( \frac{\bar{u}}{0}, 0 \right) \mid \bar{u} \in \mathcal{U} \right\}, \text{ for all } e^{-w_1(\bar{e})} \in \mathcal{X}_1,
\]

\[
\tilde{\mathcal{U}}_1(e^{-w_2(\bar{e})}) = \left\{ \left( \frac{\bar{u}}{1}, 0 \right) \mid \bar{u} \in \mathcal{U} \right\}, \text{ for all } e^{-w_2(\bar{e})} \in \mathcal{X}_2.
\]

Similarly, a $\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$ is called absolute $\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$, represented as $(\mathcal{U}, \Phi, \mathcal{X}, \mathcal{X}_2)$ where

\[
\tilde{\mathcal{U}}_1(e^{-w_1(\bar{e})}) = \left\{ \left( \frac{\bar{u}}{1}, 0 \right) \mid \bar{u} \in \mathcal{U} \right\}, \text{ for all } e^{-w_1(\bar{e})} \in \mathcal{X}_1,
\]

\[
\Phi_0(e^{-w_2(\bar{e})}) = \left\{ \left( \frac{\bar{u}}{0}, 0 \right) \mid \bar{u} \in \mathcal{U} \right\}, \text{ for all } e^{-w_2(\bar{e})} \in \mathcal{X}_2.
\]

Now, an important property (complement) of $\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$ is provided in the following definition:

**Definition 7.** The complement of a $\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$ $(\tilde{\mathcal{P}}^t_1, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})$ over the soft universe $(\mathcal{U}, \mathcal{B})$, denoted by $(\tilde{\mathcal{P}}^t_1, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})^c$, is defined as $(\tilde{\mathcal{P}}^t_1, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})^c = (\tilde{\mathcal{P}}^t_1, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})$ where the functions $\tilde{\mathcal{P}}^t_1$ and $\tilde{\mathcal{X}}_{w_1}$ are, respectively, given by $\tilde{\mathcal{P}}^t_1: \tilde{\mathcal{X}}_{w_1} \rightarrow \mathcal{F}^{\mathcal{B}} \times [0, 1]$ and $\tilde{\mathcal{X}}_{w_1}: \tilde{\mathcal{X}}_{w_2} \rightarrow \mathcal{F}^{\mathcal{B}} \times [0, 1]$ and are defined as follows:

\[
\tilde{\mathcal{P}}^t_1(e^{-w_1(\bar{e})}) \left( \frac{\bar{u}}{1} \right) = \left( \frac{\bar{u}}{1 - \tilde{\mathcal{P}}(\bar{e})} \right), \quad \text{and} \quad \tilde{\mathcal{X}}_{w_1}(e^{-w_1(\bar{e})}) \left( \frac{\bar{u}}{1 - \tilde{\mathcal{X}}(\bar{e})} \right) = \left( \frac{\bar{u}}{1 - \tilde{\mathcal{X}}(\bar{e})} \right).
\]

for all $\bar{u} \in \mathcal{U}$, $\bar{e} \in \mathcal{X}$ and $\bar{e} \in \mathcal{X}$.

The following example demonstrates the concept of complement for $\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$.

**Example 3.** Reconsider the $\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$ $(\mathcal{P}_1, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})$ as provided in Example 1. Then, the complement of $\mathcal{FPPF}_{\mathcal{B}}\mathcal{S}\mathcal{S}$ $(\mathcal{P}_1, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})$ is computed as follows:

\[
\tilde{\mathcal{P}}^t_1(e^{-0.55}) = \left\{ \left( \frac{\bar{u}}{0.33}, 0.76 \right), \left( \frac{\bar{u}}{0.43}, 0.66 \right), \left( \frac{\bar{u}}{0.68}, 0.22 \right), \left( \frac{\bar{u}}{0.83}, 0.64 \right), \left( \frac{\bar{u}}{0.79}, 0.55 \right) \right\},
\]

\[
\tilde{\mathcal{P}}^t_1(e^{-0.62}) = \left\{ \left( \frac{\bar{u}}{0.58}, 0.45 \right), \left( \frac{\bar{u}}{0.64}, 0.80 \right), \left( \frac{\bar{u}}{0.39}, 0.66 \right), \left( \frac{\bar{u}}{0.45}, 0.72 \right), \left( \frac{\bar{u}}{0.52}, 0.70 \right) \right\},
\]

\[
\tilde{\mathcal{P}}^t_1(e^{-0.39}) = \left\{ \left( \frac{\bar{u}}{0.44}, 0.57 \right), \left( \frac{\bar{u}}{0.31}, 0.45 \right), \left( \frac{\bar{u}}{0.50}, 0.58 \right), \left( \frac{\bar{u}}{0.33}, 0.85 \right), \left( \frac{\bar{u}}{0.36}, 0.61 \right) \right\}.
\]
\[
\mathcal{\tilde{X}}_{\tilde{\mathbf{e}}_1}(-0.29) = \left( \begin{array}{c}
\tilde{u}_1 \\
\tilde{u}_2 \\
\tilde{u}_3 \\
\tilde{u}_4 \\
\tilde{u}_5
\end{array} \right) = \left( \begin{array}{c}
0.58 \\
0.64 \\
0.39 \\
0.45 \\
0.52
\end{array} \right),
\]
\[
\mathcal{\tilde{X}}_{\tilde{\mathbf{e}}_2}(-0.50) = \left( \begin{array}{c}
\tilde{u}_1 \\
\tilde{u}_2 \\
\tilde{u}_3 \\
\tilde{u}_4 \\
\tilde{u}_5
\end{array} \right) = \left( \begin{array}{c}
0.33 \\
0.43 \\
0.68 \\
0.83 \\
0.79
\end{array} \right),
\]
\[
\mathcal{\tilde{X}}_{\tilde{\mathbf{e}}_3}(-0.85) = \left( \begin{array}{c}
\tilde{u}_1 \\
\tilde{u}_2 \\
\tilde{u}_3 \\
\tilde{u}_4 \\
\tilde{u}_5
\end{array} \right) = \left( \begin{array}{c}
0.84 \\
0.71 \\
0.70 \\
0.54 \\
0.75
\end{array} \right).
\]

In matrix form, the complement of \( \mathcal{\tilde{F}}_{\tilde{\mathbf{e}}_1}, \tilde{\mathcal{\tilde{X}}}, \tilde{\mathcal{\tilde{X}}}_3 \) is provided as follows:

\[
\mathcal{\tilde{\tilde{X}}} = \begin{pmatrix}
\tilde{v}_1 \\ 
\tilde{v}_2 \\ 
\tilde{v}_3 \\
\end{pmatrix} = \begin{pmatrix}
\tilde{v}_1^{0.55} 
n\tilde{v}_2^{62} 
\tilde{v}_3^{39}
\end{pmatrix} \begin{pmatrix}
u_1 
(0.33, 0.76) 
(0.58, 0.45) 
(0.44, 0.57)
\end{pmatrix} \begin{pmatrix}
u_2 
(0.43, 0.66) 
(0.64, 0.80) 
(0.31, 0.45)
\end{pmatrix} \begin{pmatrix}
u_3 
(0.68, 0.22) 
(0.39, 0.66) 
(0.50, 0.58)
\end{pmatrix} \begin{pmatrix}
u_4 
(0.83, 0.64) 
(0.45, 0.72) 
(0.33, 0.85)
\end{pmatrix} \begin{pmatrix}
u_5 
(0.79, 0.55) 
(0.52, 0.70) 
(0.36, 0.61)
\end{pmatrix}
\]

\[
\mathcal{\tilde{Y}} = \begin{pmatrix}
\tilde{v}_1 \\ 
\tilde{v}_2 \\ 
\tilde{v}_3 \\
\end{pmatrix} = \begin{pmatrix}
\tilde{v}_1^{0.29} 
n\tilde{v}_2^{50} 
\tilde{v}_3^{85}
\end{pmatrix} \begin{pmatrix}
u_1 
(0.58, 0.45) 
(0.33, 0.76) 
(0.84, 0.20)
\end{pmatrix} \begin{pmatrix}
u_2 
(0.64, 0.80) 
(0.43, 0.66) 
(0.71, 0.83)
\end{pmatrix} \begin{pmatrix}
u_3 
(0.68, 0.22) 
(0.39, 0.66) 
(0.70, 0.36)
\end{pmatrix} \begin{pmatrix}
u_4 
(0.45, 0.72) 
(0.83, 0.64) 
(0.54, 0.42)
\end{pmatrix} \begin{pmatrix}
u_5 
(0.52, 0.70) 
(0.79, 0.55) 
(0.75, 0.34)
\end{pmatrix}
\]

We now investigate two fundamental properties of the \( \mathcal{\tilde{F}}_{\tilde{\mathbf{e}}_1}, \tilde{\mathcal{\tilde{X}}}, \tilde{\mathcal{\tilde{X}}}_3 \), namely, union and intersection, and explain them via illustrative examples.

**Definition 8.** The union of two \( \mathcal{\tilde{F}}_{\tilde{\mathbf{e}}_1}, \tilde{\mathcal{\tilde{X}}}, \tilde{\mathcal{\tilde{X}}}_3 \) (\( \tilde{\mathcal{\tilde{X}}}, \tilde{\mathcal{\tilde{X}}}_3 \)) over the universe \( (\tilde{\mathcal{\tilde{X}}}, \tilde{\mathcal{\tilde{X}}}_3) \), written as \( (\tilde{\mathcal{\tilde{X}}}, \tilde{\mathcal{\tilde{X}}}_3) \), is a \( \mathcal{\tilde{F}}_{\tilde{\mathbf{e}}_1}, \tilde{\mathcal{\tilde{X}}}, \tilde{\mathcal{\tilde{X}}}_3 \) (\( \tilde{\mathcal{\tilde{X}}}, \tilde{\mathcal{\tilde{X}}}_3 \)) where \( \tilde{\mathcal{\tilde{X}}}_1 = \tilde{\mathcal{\tilde{X}}}_1 \cup \tilde{\mathcal{\tilde{X}}}_3 \cup \tilde{\mathcal{\tilde{X}}}_4 \) and \( \tilde{\mathcal{\tilde{X}}}_4 : \tilde{\mathcal{\tilde{X}}}_1 \rightarrow \tilde{\mathcal{\tilde{X}}}_4 \) and \( \tilde{\mathcal{\tilde{X}}}_4 \rightarrow \tilde{\mathcal{\tilde{X}}}_4 \times [0, 1] \) are mappings, which are defined by

\[
\tilde{\mathcal{\tilde{X}}}_1 = \tilde{\mathcal{\tilde{X}}}_1 \cup \tilde{\mathcal{\tilde{X}}}_3 \cup \tilde{\mathcal{\tilde{X}}}_4,
\]

\[
\tilde{\mathcal{\tilde{X}}}_4 = \tilde{\mathcal{\tilde{X}}}_4 \times [0, 1].
\]

for all \( \tilde{\mathbf{e}} \in \tilde{\mathcal{\tilde{X}}} \) and \( \tilde{\mathbf{e}} \in \tilde{\mathcal{\tilde{X}}} \).

**Example 4.** Consider \( \tilde{\mathcal{\tilde{X}}} = \{ \tilde{u}_1, \tilde{u}_3, \ldots, \tilde{u}_5 \} \) as the universal set of five laptops and let \( \tilde{\mathcal{\tilde{X}}} = \{ \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \} \) be the set of parameters related to the alternatives in \( \tilde{\mathcal{\tilde{X}}} \) where \( \tilde{e}_1 = \) plastic – built, \( \tilde{e}_2 = \) thick, \( \tilde{e}_3 = \) modern technology, \( \tilde{e}_4 = \) budget range and \( \tilde{\mathcal{\tilde{X}}} = \{ \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \} \) where \( \tilde{e}_1 = \) matel– built, \( \tilde{e}_2 = \) thin, \( \tilde{e}_3 = \) classic technology, \( \tilde{e}_4 = \) expensive. Let \( \tilde{\mathcal{\tilde{X}}} = \{ \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \} \subseteq \tilde{\mathcal{\tilde{X}}} \) and \( \tilde{\mathcal{\tilde{X}}} = \{ \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \} \subseteq \tilde{\mathcal{\tilde{X}}} \), and their corresponding weighted versions are given as \( \tilde{\mathcal{\tilde{X}}}_{\tilde{e}_1} = \).
\[ \mathcal{X}_{w_1} = \left\{ \epsilon_{1}, \epsilon_{2}, \epsilon_{3} \right\} \text{ and } \mathcal{X}_{w_2} = \left\{ \epsilon_{1}, \epsilon_{2}, \epsilon_{3} \right\}. \]

Then, a HCPPDSS (\( \mathcal{P}_{1}, \mathcal{Q}_{0}, \mathcal{R}_{0}, \mathcal{S}_{0} \)) is provided as follows:

\[
\mathcal{P}_{1}(\epsilon_{1}) = \left\{ \frac{\tilde{u}_1}{0.41}, 0.65, \frac{\tilde{u}_2}{0.29}, 0.54, \frac{\tilde{u}_3}{0.65}, 0.37, \frac{\tilde{u}_4}{0.72}, 0.45, \frac{\tilde{u}_5}{0.68}, 0.56 \right\},
\]

\[
\mathcal{P}_{1}(\epsilon_{2}) = \left\{ \frac{\tilde{u}_1}{0.63}, 0.51, \frac{\tilde{u}_2}{0.59}, 0.78, \frac{\tilde{u}_3}{0.47}, 0.70, \frac{\tilde{u}_4}{0.65}, 0.82, \frac{\tilde{u}_5}{0.72}, 0.80 \right\},
\]

\[
\mathcal{P}_{1}(\epsilon_{3}) = \left\{ \frac{\tilde{u}_1}{0.44}, 0.57, \frac{\tilde{u}_2}{0.31}, 0.45, \frac{\tilde{u}_3}{0.50}, 0.58, \frac{\tilde{u}_4}{0.33}, 0.85, \frac{\tilde{u}_5}{0.36}, 0.61 \right\},
\]

\[
\mathcal{Q}_{0}(\epsilon_{1}) = \left\{ \frac{\tilde{u}_1}{0.58}, 0.45, \frac{\tilde{u}_2}{0.64}, 0.80, \frac{\tilde{u}_3}{0.39}, 0.66, \frac{\tilde{u}_4}{0.45}, 0.72, \frac{\tilde{u}_5}{0.52}, 0.70 \right\},
\]

\[
\mathcal{Q}_{0}(\epsilon_{2}) = \left\{ \frac{\tilde{u}_1}{0.33}, 0.76, \frac{\tilde{u}_2}{0.43}, 0.66, \frac{\tilde{u}_3}{0.68}, 0.22, \frac{\tilde{u}_4}{0.83}, 0.64, \frac{\tilde{u}_5}{0.79}, 0.55 \right\},
\]

\[
\mathcal{Q}_{0}(\epsilon_{3}) = \left\{ \frac{\tilde{u}_1}{0.84}, 0.20, \frac{\tilde{u}_2}{0.71}, 0.83, \frac{\tilde{u}_3}{0.70}, 0.36, \frac{\tilde{u}_4}{0.54}, 0.42, \frac{\tilde{u}_5}{0.75}, 0.34 \right\}.
\]

For \( \mathcal{Y} = \left\{ \epsilon_{1}, \epsilon_{2}, \epsilon_{3}, \epsilon_{4} \right\} \subseteq \tilde{\mathcal{Y}} \) and \( \mathcal{Y} = \left\{ \epsilon_{1}, \epsilon_{2}, \epsilon_{3} \right\} \subseteq \tilde{\mathcal{Y}} \), their respective weighted versions are provided by \( \mathcal{Y}_{\sigma_1} = \left\{ \epsilon_{1}^{0.62}, \epsilon_{2}^{0.22}, \epsilon_{3}^{0.54} \right\} \) and \( \mathcal{Y}_{\sigma_2} = \left\{ \epsilon_{1}^{0.25}, \epsilon_{2}^{0.49}, \epsilon_{3}^{0.45} \right\} \). Let

\[
\mathcal{R}_{1}(\epsilon_{1}) = \left\{ \frac{\tilde{u}_1}{0.37}, 0.70, \frac{\tilde{u}_2}{0.46}, 0.25, \frac{\tilde{u}_3}{0.39}, 0.56, \frac{\tilde{u}_4}{0.58}, 0.68, \frac{\tilde{u}_5}{0.58}, 0.42 \right\},
\]

\[
\mathcal{R}_{1}(\epsilon_{3}) = \left\{ \frac{\tilde{u}_1}{0.44}, 0.73, \frac{\tilde{u}_2}{0.25}, 0.55, \frac{\tilde{u}_3}{0.56}, 0.45, \frac{\tilde{u}_4}{0.44}, 0.67, \frac{\tilde{u}_5}{0.78}, 0.20 \right\},
\]

\[
\mathcal{R}_{1}(\epsilon_{4}) = \left\{ \frac{\tilde{u}_1}{0.39}, 0.58, \frac{\tilde{u}_2}{0.47}, 0.56, \frac{\tilde{u}_3}{0.34}, 0.22, \frac{\tilde{u}_4}{0.53}, 0.66, \frac{\tilde{u}_5}{0.46}, 0.52 \right\},
\]

\[
\mathcal{S}_{0}(\epsilon_{1}) = \left\{ \frac{\tilde{u}_1}{0.69}, 0.20, \frac{\tilde{u}_2}{0.72}, 0.65, \frac{\tilde{u}_3}{0.55}, 0.74, \frac{\tilde{u}_4}{0.27}, 0.43, \frac{\tilde{u}_5}{0.40}, 0.62 \right\},
\]

\[
\mathcal{S}_{0}(\epsilon_{3}) = \left\{ \frac{\tilde{u}_1}{0.22}, 0.50, \frac{\tilde{u}_2}{0.52}, 0.69, \frac{\tilde{u}_3}{0.77}, 0.30, \frac{\tilde{u}_4}{0.78}, 0.42, \frac{\tilde{u}_5}{0.40}, 0.36 \right\},
\]

\[
\mathcal{S}_{0}(\epsilon_{4}) = \left\{ \frac{\tilde{u}_1}{0.71}, 0.33, \frac{\tilde{u}_2}{0.60}, 0.24, \frac{\tilde{u}_3}{0.65}, 0.45, \frac{\tilde{u}_4}{0.77}, 0.82, \frac{\tilde{u}_5}{0.62}, 0.44 \right\}.
\]
By Definition 8, the union \((\mathcal{M}_t, \mathcal{N}_t, \mathcal{G}_t, \mathcal{C}_t) = (\mathcal{P}_1, \mathcal{Q}_t, \mathcal{X}_t, \mathcal{Y}_t) \cup (\mathcal{R}_1, \mathcal{S}_t, \mathcal{Y}_t, \mathcal{Y}_t)\) of the available \(\mathcal{X} \mathcal{Y} \mathcal{X} \mathcal{Y} \mathcal{X} \mathcal{Y} \mathcal{S} \mathcal{S} \mathcal{S}\) is computed as follows:

\[
\mathcal{M}_t(\varepsilon_{0.45,0.62}) = \left\{ \begin{array}{c}
\frac{\tilde{u}_1}{\max(0.41,0.37)}, \frac{\tilde{u}_2}{\max(0.29,0.46)}, \\
\frac{\tilde{u}_3}{\max(0.37,0.56)}, \frac{\tilde{u}_4}{\max(0.72,0.58)}, \\
\frac{\tilde{u}_5}{\max(0.68,0.58)}
\end{array} \right\}.
\]

Similarly,

\[
\mathcal{M}_t(\varepsilon_{0.62}) = \left\{ \begin{array}{c}
\frac{\tilde{u}_1}{0.41}, \frac{\tilde{u}_2}{0.46}, \frac{\tilde{u}_3}{0.65}, \frac{\tilde{u}_4}{0.72}, \\
\frac{\tilde{u}_5}{0.68}
\end{array} \right\}.
\]

In matrix form, we can write the union \((\mathcal{M}_t, \mathcal{N}_t, \mathcal{G}_t, \mathcal{C}_t) = (\mathcal{P}_1, \mathcal{Q}_t, \mathcal{X}_t, \mathcal{Y}_t) \cup (\mathcal{R}_1, \mathcal{S}_t, \mathcal{Y}_t, \mathcal{Y}_t)\) as follows:
\[
\mathcal{M}_{\delta} = \begin{pmatrix}
\tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 & \tilde{u}_4 & \tilde{u}_5 \\
0.62 & 0.41, 0.70 & 0.46, 0.54 & 0.65, 0.56 & 0.72, 0.68 & 0.68, 0.56 \\
0.33 & 0.63, 0.51 & 0.59, 0.78 & 0.47, 0.70 & 0.65, 0.82 & 0.72, 0.80 \\
0.28 & 0.44, 0.73 & 0.31, 0.55 & 0.56, 0.58 & 0.44, 0.85 & 0.78, 0.61 \\
0.54 & 0.39, 0.58 & 0.47, 0.56 & 0.34, 0.22 & 0.53, 0.66 & 0.46, 0.52
\end{pmatrix}
\]
\[
\mathcal{N}_{\gamma} = \begin{pmatrix}
\tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 & \tilde{u}_4 & \tilde{u}_5 \\
0.35 & 0.69, 0.45 & 0.72, 0.80 & 0.55, 0.74 & 0.45, 0.72 & 0.52, 0.70 \\
0.60 & 0.33, 0.76 & 0.43, 0.66 & 0.68, 0.22 & 0.83, 0.64 & 0.79, 0.55 \\
0.57 & 0.84, 0.50 & 0.71, 0.83 & 0.77, 0.36 & 0.78, 0.42 & 0.75, 0.44 \\
0.45 & 0.71, 0.33 & 0.60, 0.24 & 0.65, 0.45 & 0.77, 0.82 & 0.62, 0.44
\end{pmatrix}
\]

\textbf{Definition 9.} The intersection of two \(\mathcal{FPPFBSS}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{S}}_1, \tilde{\mathcal{X}}_1, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Z}}_1)\) and \((\mathfrak{A}_1, \mathfrak{S}_1, \mathfrak{Y}_1, \mathfrak{Z}_1)\) over the soft universe \((\mathcal{U}, \mathcal{F})\), written as \((\tilde{\mathcal{P}}_1, \tilde{\mathcal{S}}_1, \tilde{\mathcal{X}}_1, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Z}}_1) \cap \mathcal{FPPFBSS}(\mathfrak{A}_1, \mathfrak{S}_1, \mathfrak{Y}_1, \mathfrak{Z}_1)\), is a \(\mathcal{FPPFBSS}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{S}}_1, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Z}}_1)\) where

\[
\mathfrak{D}_{\delta_1} = \tilde{\mathcal{Y}}_1 \cap \tilde{\mathcal{Z}}_1, \quad \mathfrak{D}_{\delta_2} = \tilde{\mathcal{X}}_1 \cap \tilde{\mathcal{Y}}_1, \quad \mathfrak{D}_{\delta_3} = \tilde{\mathcal{Z}}_1 \cap \tilde{\mathcal{Y}}_1, \quad \mathfrak{D}_{\delta_4} = \mathfrak{FPU} \times [0, 1] \quad \text{and} \quad \mathfrak{D}_{\delta_5} : \mathfrak{D}_{\delta_4} \rightarrow \mathfrak{FPU} \times [0, 1]
\]

are functions, which are defined as follows:

\[
\mathfrak{D}_{\delta_1} = \left( \begin{array}{c}
\mathfrak{D}_{\delta_1}(\tilde{e}) \\
\mathfrak{D}_{\delta_2}(\tilde{e}) \\
\mathfrak{D}_{\delta_3}(\tilde{e}) \\
\mathfrak{D}_{\delta_4}(\tilde{e}) \\
\mathfrak{D}_{\delta_5}(\tilde{e})
\end{array} \right)
\]

for all \(\tilde{e} \in \mathfrak{D}_{\delta_1}\) and \(\tilde{e} \in \mathfrak{D}_{\delta_5}\).

\textbf{Example 5.} Reconsider the \(\mathcal{FPPFBSS}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{S}}_1, \tilde{\mathcal{X}}_1, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Z}}_1)\) and \((\mathfrak{A}_1, \mathfrak{S}_1, \mathfrak{Y}_1, \mathfrak{Z}_1)\) as provided in Example 4. Then, by Definition 9, their intersection \((\tilde{\mathcal{P}}_1, \tilde{\mathcal{S}}_1, \tilde{\mathcal{X}}_1, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Z}}_1) \cap \mathcal{FPPFBSS}(\mathfrak{A}_1, \mathfrak{S}_1, \mathfrak{Y}_1, \mathfrak{Z}_1)\) is a \(\mathcal{FPPFBSS}(\tilde{\mathcal{P}}_1, \tilde{\mathcal{S}}_1, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Z}}_1)\), which is computed as follows:

\[
\mathfrak{D}_{\delta} = \begin{pmatrix}
\tilde{u}_1 \\
\tilde{u}_2 \\
\tilde{u}_3 \\
\tilde{u}_4 \\
\tilde{u}_5
\end{pmatrix}
\]

where

\[
\tilde{u}_1 = \begin{pmatrix}
\min(0.41, 0.37) & \min(0.65, 0.70)
\end{pmatrix}
\]

\[
\tilde{u}_2 = \begin{pmatrix}
\min(0.29, 0.46) & \min(0.54, 0.25)
\end{pmatrix}
\]

\[
\tilde{u}_3 = \begin{pmatrix}
\min(0.65, 0.39) & \min(0.37, 0.56)
\end{pmatrix}
\]

\[
\tilde{u}_4 = \begin{pmatrix}
\min(0.72, 0.58) & \min(0.45, 0.68)
\end{pmatrix}
\]

\[
\tilde{u}_5 = \begin{pmatrix}
\min(0.68, 0.58) & \min(0.56, 0.42)
\end{pmatrix}
\]

Similarly,
\[\mathcal{D}_\delta(\varepsilon_3) = \left\{ \begin{pmatrix} \bar{u}_1 \\ 0.44 \\ 0.57 \\ \bar{u}_2 \\ 0.25 \\ 0.45 \\ \bar{u}_3 \\ 0.50 \\ 0.45 \\ \bar{u}_4 \\ 0.33 \\ 0.67 \\ \bar{u}_5 \\ 0.36 \\ 0.20 \end{pmatrix} \right\}, \]
\[\mathcal{D}_\gamma(\varepsilon_1) = \left\{ \begin{pmatrix} \bar{u}_1 \\ 0.58 \\ 0.20 \\ \bar{u}_2 \\ 0.64 \\ 0.65 \\ \bar{u}_3 \\ 0.39 \\ 0.66 \\ \bar{u}_4 \\ 0.27 \\ 0.43 \\ \bar{u}_5 \\ 0.40 \\ 0.62 \end{pmatrix} \right\}, \]
\[\mathcal{D}_\gamma(\varepsilon_3) = \left\{ \begin{pmatrix} \bar{u}_1 \\ 0.22 \\ 0.20 \\ \bar{u}_2 \\ 0.52 \\ 0.69 \\ \bar{u}_3 \\ 0.70 \\ 0.30 \\ \bar{u}_4 \\ 0.54 \\ 0.42 \\ \bar{u}_5 \\ 0.40 \\ 0.34 \end{pmatrix} \right\}. \]

In matrix form, we can write their intersection \((\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2}) \cap (\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)\) as follows:

\[
\tilde{\mathcal{P}}_\delta = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 \\
0.37 & 0.65 & 0.29 & 0.25 & 0.39 & 0.37 & 0.58 & 0.45 & 0.58 & 0.42 \\
0.44 & 0.57 & 0.25 & 0.45 & 0.50 & 0.45 & 0.33 & 0.67 & 0.36 & 0.20 \\
0.58 & 0.20 & 0.64 & 0.65 & 0.39 & 0.66 & 0.27 & 0.43 & 0.40 & 0.62 \\
0.22 & 0.20 & 0.52 & 0.69 & 0.70 & 0.30 & 0.54 & 0.42 & 0.40 & 0.35 \end{pmatrix}
\]

In the following, we investigate some basic results such as commutativity, associativity, and distributivity among the \(\mathbb{PPP}_D^S\).

**Proposition 1.** Let \((\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})\), \((\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)\), and \((\mathcal{L}_1, \tilde{\mathcal{H}}, \tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2)\) be any three \(\mathbb{PPP}_D^S\) over the soft universe \((\tilde{\mathcal{U}}, \tilde{\mathcal{E}})\). Then,

1. \((\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2}) \cup (\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2) = (\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2) \cup (\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})\)
2. \((\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2}) \cap (\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2) = (\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2) \cap (\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})\)

**Proof.** (1) Using Definition 8, we have

\[
\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2} \cup \mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2 \Rightarrow \mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2 \cup \mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2}
\]

(3) \((\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2}) \cap (\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2) = (\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2) \cap (\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2})\)

(4) \((\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2}) \cap (\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2) \cap (\mathcal{L}_1, \tilde{\mathcal{H}}, \tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2) = (\mathcal{L}_1, \tilde{\mathcal{H}}, \tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2) \cap (\mathcal{P}_1, \tilde{\mathcal{U}}, \tilde{\mathcal{X}}_{w_1}, \tilde{\mathcal{X}}_{w_2}) \cap (\mathcal{R}_1, \tilde{\mathcal{E}}, \tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)\)

where \(\mathcal{P}_1 = \tilde{\mathcal{X}}_{w_1} \cup \tilde{\mathcal{Y}}_1, \mathcal{R}_1 = \tilde{\mathcal{X}}_{w_2} \cup \tilde{\mathcal{Y}}_2, \mathcal{L}_1 \to \mathbb{PPP} \times [0,1]\) and \(\mathcal{M}_1: \tilde{\mathcal{E}}, \mathcal{E}_2 \to \mathbb{PPP} \times [0,1]\) are mappings. Then, we get
\[ \mathcal{M}_\delta \left( e^{\max \left( \omega_1 (\tilde{e}), \sigma_1 (\tilde{e}) \right)} \right) = \left\{ \left( \frac{\tilde{u}}{\max \left( \overline{P}(\tilde{e})(\tilde{u}), \overline{R}(\tilde{e})(\tilde{u}) \right)}, \max \left( \lambda (\tilde{e}), \lambda_1 (\tilde{e}) \right) \right) \left| \tilde{u} \in \overline{U} \right. \right\} \\
= \left\{ \left( \frac{\tilde{u}}{\max \left( \overline{R}(\tilde{e})(\tilde{u}), \overline{P}(\tilde{e})(\tilde{u}) \right)}, \max \left( \lambda_1 (\tilde{e}), \lambda (\tilde{e}) \right) \right) \left| \tilde{u} \in \overline{U} \right. \right\} \\
= \mathcal{M}_\delta \left( e^{\max \left( a_1 (\tilde{e}), a_0 (\tilde{e}) \right)} \right), \\
\mathcal{N}_\gamma \left( e^{\max \left( \omega_2 (\tilde{e}), \sigma_2 (\tilde{e}) \right)} \right) = \left\{ \left( \frac{\tilde{u}}{\max \left( \overline{Q}(\tilde{e})(\tilde{u}), \overline{S}(\tilde{e})(\tilde{u}) \right)}, \max \left( \theta (\tilde{e}), \theta_1 (\tilde{e}) \right) \right) \left| \tilde{u} \in \overline{U} \right. \right\} \\
= \left\{ \left( \frac{\tilde{u}}{\max \left( \overline{S}(\tilde{e})(\tilde{u}), \overline{Q}(\tilde{e})(\tilde{u}) \right)}, \max \left( \theta_1 (\tilde{e}), \theta (\tilde{e}) \right) \right) \left| \tilde{u} \in \overline{U} \right. \right\} \\
= \mathcal{N}_\gamma \left( e^{\max \left( \sigma_2 (\tilde{e}), a_2 (\tilde{e}) \right)} \right), \tag{24} \]

for all \( \tilde{e} \in \mathcal{E} \) and \( \tilde{e} \in \mathcal{G} \). Thus, it is clear from the above arguments that

\[ \left( \overline{I}_2, \overline{Q}_0, \overline{X}_w, \overline{X}_w \right) \cup \left( \overline{R}_1, \overline{S}_0, \overline{Y}_1, \overline{Y}_1 \right) = \left( \overline{I}_1, \overline{Q}_1, \overline{X}_w, \overline{X}_w \right). \tag{25} \]

(2) Similar to part 1. 
(3) Using Definition 8, we get

\[ \mathcal{M}_\delta: \mathcal{E}_0 \longrightarrow \mathcal{F} \mathcal{U} \times [0, 1] \quad \text{and} \quad \mathcal{N}_\gamma: \mathcal{E}_0 \longrightarrow \mathcal{F} \mathcal{U} \times [0, 1] \quad \text{are mappings. Then, we have} \]

\[ \mathcal{M}_\delta \left( e^{\max \left( \omega_1 (\tilde{e}), \max \left( a_1 (\tilde{e}), a_0 (\tilde{e}) \right) \right)} \right) = \left\{ \left( \frac{\tilde{u}}{\max \left( \overline{P}(\tilde{e})(\tilde{u}), \max \left( \overline{R}(\tilde{e})(\tilde{u}), \overline{T}(\tilde{e})(\tilde{u}) \right) \right)}, \max \left( \lambda (\tilde{e}), \max \left( \lambda_1 (\tilde{e}), \lambda_2 (\tilde{e}) \right) \right) \right\} \\
= \left\{ \left( \frac{\tilde{u}}{\max \left( \overline{R}(\tilde{e})(\tilde{u}), \max \left( \overline{P}(\tilde{e})(\tilde{u}), \overline{T}(\tilde{e})(\tilde{u}) \right) \right)}, \max \left( \lambda_1 (\tilde{e}), \max \left( \lambda_1 (\tilde{e}), \lambda_2 (\tilde{e}) \right) \right) \right\} \\
= \mathcal{M}_\delta \left( e^{\max \left( \omega_1 (\tilde{e}), \sigma_1 (\tilde{e}), a_1 (\tilde{e}), a_0 (\tilde{e}) \right)} \right), \]
Proposition 3. Let \((\mathfrak{B}_1, \tilde{\mathfrak{B}}_0, \tilde{\mathfrak{B}}_2, \tilde{\mathfrak{B}}_3), (\mathfrak{R}_1, \tilde{\mathfrak{R}}_0, \tilde{\mathfrak{R}}_2, \tilde{\mathfrak{R}}_3), (\mathfrak{Y}_1, \tilde{\mathfrak{Y}}_0, \tilde{\mathfrak{Y}}_2, \tilde{\mathfrak{Y}}_3), \) and \((\mathfrak{T}_1, \tilde{\mathfrak{T}}_0, \tilde{\mathfrak{T}}_2, \tilde{\mathfrak{T}}_3)\) be any three \(\mathcal{FPF}\mathcal{F}\mathcal{S}\mathcal{S}\) over \((\tilde{\mathcal{U}}, \tilde{\mathcal{E}})\). Then, the results given in the following hold:

(1) \((\mathfrak{B}_1, \tilde{\mathfrak{B}}_0, \tilde{\mathfrak{B}}_2, \tilde{\mathfrak{B}}_3) \cup (\mathfrak{R}_1, \tilde{\mathfrak{R}}_0, \tilde{\mathfrak{R}}_2, \tilde{\mathfrak{R}}_3) \cap (\mathfrak{Y}_1, \tilde{\mathfrak{Y}}_0, \tilde{\mathfrak{Y}}_2, \tilde{\mathfrak{Y}}_3) = (\mathfrak{T}_1, \tilde{\mathfrak{T}}_0, \tilde{\mathfrak{T}}_2, \tilde{\mathfrak{T}}_3)\)

(2) \((\mathfrak{B}_1, \tilde{\mathfrak{B}}_0, \tilde{\mathfrak{B}}_2, \tilde{\mathfrak{B}}_3) \cap (\mathfrak{R}_1, \tilde{\mathfrak{R}}_0, \tilde{\mathfrak{R}}_2, \tilde{\mathfrak{R}}_3) \cup (\mathfrak{Y}_1, \tilde{\mathfrak{Y}}_0, \tilde{\mathfrak{Y}}_2, \tilde{\mathfrak{Y}}_3) = (\mathfrak{T}_1, \tilde{\mathfrak{T}}_0, \tilde{\mathfrak{T}}_2, \tilde{\mathfrak{T}}_3)\)

Proof. It can be immediately followed using similar arguments as in Proposition 1. \(\square\)

3. OR and AND Operations between \(\mathcal{FPF}\mathcal{F}\mathcal{S}\mathcal{S}\) with Applications

In this section, we first give the concepts of OR and AND operations between \(\mathcal{FPF}\mathcal{F}\mathcal{S}\mathcal{S}\) and then explain them with examples which are supported by corresponding algorithms.

Definition 10. Let \((\mathfrak{B}_1, \tilde{\mathfrak{B}}_0, \tilde{\mathfrak{B}}_2, \tilde{\mathfrak{B}}_3)\) and \((\mathfrak{R}_1, \tilde{\mathfrak{R}}_0, \tilde{\mathfrak{R}}_2, \tilde{\mathfrak{R}}_3)\) be two \(\mathcal{FPF}\mathcal{F}\mathcal{S}\mathcal{S}\) over the soft universe \((\tilde{\mathcal{U}}, \tilde{\mathcal{E}})\), then the operation ”AND” between them, denoted by \((\mathfrak{B}_1, \tilde{\mathfrak{B}}_0, \tilde{\mathfrak{B}}_2, \tilde{\mathfrak{B}}_3) \cap (\mathfrak{R}_1, \tilde{\mathfrak{R}}_0, \tilde{\mathfrak{R}}_2, \tilde{\mathfrak{R}}_3), \) is defined as follows:

\[
(\mathfrak{B}_1, \tilde{\mathfrak{B}}_0, \tilde{\mathfrak{B}}_2, \tilde{\mathfrak{B}}_3) \cap (\mathfrak{R}_1, \tilde{\mathfrak{R}}_0, \tilde{\mathfrak{R}}_2, \tilde{\mathfrak{R}}_3) = (\mathfrak{B}_1, \tilde{\mathfrak{B}}_0, \tilde{\mathfrak{B}}_2, \tilde{\mathfrak{B}}_3, \mathfrak{R}_1, \tilde{\mathfrak{R}}_0, \tilde{\mathfrak{R}}_2, \tilde{\mathfrak{R}}_3),
\]
where for all \((\bar{\alpha}^{w_1(a)}, \bar{\beta}^{\sigma_1(b)}) \in \bar{X}_{w_1} \times \bar{Y}_{\sigma_1}, (\bar{\alpha}^{w_2(a)}, \bar{\beta}^{\sigma_2(b)}) \in \bar{X}_{w_2} \times \bar{Y}_{\sigma_2}\), we have

\[
\Gamma_1 (\bar{\alpha}, \bar{\beta}) \min (w_1(a), \sigma_1(b)) = \tilde{\mathcal{P}}_1 (\bar{\alpha}^{w_1(a)}) \cap \tilde{\mathcal{R}}_1 (\bar{\beta}^{\sigma_1(b)}),
\]

\[
Y_\pi (\bar{\alpha}, \bar{\beta}) \min (w_2(a), \sigma_2(b)) = \tilde{\mathcal{S}}_1 (\bar{\alpha}^{w_2(a)}) \cap \tilde{\mathcal{S}}_1 (\bar{\beta}^{\sigma_2(b)}).
\]

(29)

In the following, we now provide algorithm 1 using the concept of AND operation and explore an application.

**Example 6.** Consider five different solar panels evaluated by a team of experts for star rating purpose which are manufactured by different companies. Let \(\bar{\mathcal{U}} = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_5\}\) be the universal set, \(\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5\}\) be the set of five main specifications which describe their features, and \(\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5\}\) be the “not set” of parameters. Let \(\bar{\mathcal{X}} = \{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\} \subseteq \bar{\mathcal{S}}\) be the favorable set of parameters and the weights provided by expert team are 0.33, 0.40, and 0.62, respectively. The weights provided by experts to the “not set” of favorable parameters \(\mathcal{X} = \{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\} \subseteq \mathcal{S}\) are 0.50, 0.25, and 0.30. The evaluation report of the experts team is given in the form of a \(\mathcal{HPPP_{5\theta}}\) of \(\delta (\mathcal{P}_1, \mathcal{Q}_0, \bar{\mathcal{X}}_{w_1}, \bar{\mathcal{X}}_{w_2})\), which is given as follows:

\[
\tilde{\mathcal{P}}_1 (\mathcal{e}_1^{0.33}) = \left\{ \left( \frac{\bar{\mathcal{U}}_1}{0.32}, \frac{\bar{\mathcal{U}}_2}{0.56}, \frac{\bar{\mathcal{U}}_3}{0.20}, \frac{\bar{\mathcal{U}}_4}{0.45} \right), \left( \frac{\bar{\mathcal{U}}_1}{0.28}, \frac{\bar{\mathcal{U}}_3}{0.36}, \frac{\bar{\mathcal{U}}_4}{0.63}, \frac{\bar{\mathcal{U}}_5}{0.47} \right) \right\},
\]

\[
\tilde{\mathcal{P}}_1 (\mathcal{e}_2^{0.40}) = \left\{ \left( \frac{\bar{\mathcal{U}}_1}{0.54}, \frac{\bar{\mathcal{U}}_2}{0.42}, \frac{\bar{\mathcal{U}}_3}{0.69}, \frac{\bar{\mathcal{U}}_4}{0.38}, \frac{\bar{\mathcal{U}}_5}{0.56}, \frac{\bar{\mathcal{U}}_6}{0.73} \right), \left( \frac{\bar{\mathcal{U}}_1}{0.56}, \frac{\bar{\mathcal{U}}_3}{0.63}, \frac{\bar{\mathcal{U}}_4}{0.71} \right) \right\},
\]

\[
\tilde{\mathcal{P}}_1 (\mathcal{e}_3^{0.62}) = \left\{ \left( \frac{\bar{\mathcal{U}}_1}{0.35}, \frac{\bar{\mathcal{U}}_2}{0.22}, \frac{\bar{\mathcal{U}}_3}{0.36}, \frac{\bar{\mathcal{U}}_4}{0.49}, \frac{\bar{\mathcal{U}}_5}{0.24}, \frac{\bar{\mathcal{U}}_6}{0.52} \right), \right\},
\]

\[
\tilde{\mathcal{Q}}_\bar{\mathcal{E}} (\mathcal{e}_1^{0.50}) = \left\{ \left( \frac{\bar{\mathcal{U}}_1}{0.49}, \frac{\bar{\mathcal{U}}_2}{0.36}, \frac{\bar{\mathcal{U}}_3}{0.57}, \frac{\bar{\mathcal{U}}_4}{0.30}, \frac{\bar{\mathcal{U}}_5}{0.36}, \frac{\bar{\mathcal{U}}_6}{0.63} \right), \left( \frac{\bar{\mathcal{U}}_1}{0.43}, \frac{\bar{\mathcal{U}}_3}{0.61} \right) \right\},
\]

\[
\tilde{\mathcal{Q}}_\bar{\mathcal{E}} (\mathcal{e}_2^{0.25}) = \left\{ \left( \frac{\bar{\mathcal{U}}_1}{0.24}, \frac{\bar{\mathcal{U}}_2}{0.36}, \frac{\bar{\mathcal{U}}_3}{0.57}, \frac{\bar{\mathcal{U}}_4}{0.59}, \frac{\bar{\mathcal{U}}_6}{0.76}, \frac{\bar{\mathcal{U}}_7}{0.55} \right), \left( \frac{\bar{\mathcal{U}}_1}{0.70}, \frac{\bar{\mathcal{U}}_3}{0.46} \right) \right\},
\]

\[
\tilde{\mathcal{Q}}_\bar{\mathcal{E}} (\mathcal{e}_3^{0.30}) = \left\{ \left( \frac{\bar{\mathcal{U}}_1}{0.75}, \frac{\bar{\mathcal{U}}_2}{0.11}, \frac{\bar{\mathcal{U}}_3}{0.52}, \frac{\bar{\mathcal{U}}_4}{0.27}, \frac{\bar{\mathcal{U}}_5}{0.45}, \frac{\bar{\mathcal{U}}_6}{0.33} \right), \left( \frac{\bar{\mathcal{U}}_1}{0.66}, \frac{\bar{\mathcal{U}}_3}{0.25} \right) \right\}.
\]

(30)

Now let for better results the evaluation report of another experts team about these solar panels is given in the following in the form of another \(\mathcal{HPPP_{5\theta}}\) of \(\delta (\mathcal{P}_1, \mathcal{Q}_0, \bar{\mathcal{Y}}_{\sigma_1}, \bar{\mathcal{Y}}_{\sigma_2})\) with favorable set of parameters \(\bar{\mathcal{Y}} = \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\} \subseteq \bar{\mathcal{S}}\), according to the 2nd team with weights 0.62, 0.22, and 0.54, respectively. Let \(\mathcal{Y} = \{\mathcal{Y}_1, \mathcal{Y}_2\} \subseteq \mathcal{S}\) be its "not set" with weights 0.25, 0.49, and 0.45, respectively.
(1) **Input:** $\mathcal{U}$, the universal set, $\mathcal{E}$, the set of parameters, $(\mathcal{P}_1, \mathcal{Q}_1, \mathcal{X}_1, \mathcal{Y}_1)$ and $(\mathcal{R}_1, \mathcal{S}_1, \mathcal{X}_2, \mathcal{Y}_2)$, the FPF $\mathbb{F}_\mathbb{F}_\mathbb{F}_\mathbb{M}_\mathbb{M}_\mathbb{M}_\mathbb{M}$ over the soft universe $(\mathcal{U}, \mathcal{E})$, where $\alpha, \beta, \gamma, \delta$ are the possibility functions, and $\omega_1, \omega_2, \alpha_1, \alpha_2$ represent the weights.

(2) Find the “AND” operation $(\Gamma_1, \mathcal{Y}_1, \mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_1, \mathcal{Y}_1)$ between the available FPF $\mathbb{F}_\mathbb{F}_\mathbb{F}_\mathbb{M}_\mathbb{M}_\mathbb{M}_\mathbb{M}$.

(3) Select the highest membership grade in each set $\Gamma_1(\alpha, \beta)^{\min(\omega_1, \omega_2)}$ and $\mathcal{Y}_1(\alpha, \beta)^{\min(\omega_1, \omega_2)}$. We do not consider the numerical grades of the alternatives against the pairs where both the parameters are same.

(4) Select the possibility functional values $\xi_i$ and $\pi_i$ from the highest membership grade tables for the set of parameters and its “not set,” respectively.

(5) Select the weight values $e_i$ and $\epsilon_i$ from the sets having highest membership grade values.

(6) Calculate the positive ($\mathcal{S}_1^+$) and negative ($\mathcal{S}_1^-$) score values of each alternative by using the formulas: $\mathcal{S}_1^+(\xi_i) = \sum_{i=1}^{n} \Gamma_1^{\h}(\alpha, \beta)^{\epsilon_i} \times \xi_i \times e_i$ and $\mathcal{S}_1^-(\xi_i) = \sum_{i=1}^{n} \Gamma_1^{\h}(\alpha, \beta)^{\epsilon_i} \times \xi_i \times \epsilon_i$ for all $\mathcal{U}_i \in \mathcal{U}$. Here, $\Gamma_1^{\h}(\alpha, \beta)^{\epsilon_i}$ and $\mathcal{Y}_1^{\h}(\alpha, \beta)^{\epsilon_i}$ are the highest membership grade values.

(7) Compute the ultimate score values using the formula: $\Delta_i = \mathcal{S}_1^+ + \mathcal{S}_1^-$. (8) Find $n$, for which $\Delta_n = \max \Delta_i$.

(9) **Output:** The object $\mathcal{U}_n$ will be the best decision object. If there exist more than one values of $n$, then anyone of the $\mathcal{U}_n$’s can be selected.

**Algorithm 1:** Selection of the most suitable choice using “AND” operation

\[
\mathcal{R}_1^+(\epsilon_1) = \left\{ \begin{array}{l}
\mathcal{U}_1 \cdot 0.61, \\
\mathcal{U}_3 \cdot 0.16, \\
\mathcal{U}_3 \cdot 0.47, \\
\mathcal{U}_4 \cdot 0.59, \\
\mathcal{U}_5 \cdot 0.33,
\end{array} \right. \\
\mathcal{R}_1^-(\epsilon_2) = \left\{ \begin{array}{l}
\mathcal{U}_1 \cdot 0.54, \\
\mathcal{U}_3 \cdot 0.16, \\
\mathcal{U}_3 \cdot 0.36, \\
\mathcal{U}_4 \cdot 0.58, \\
\mathcal{U}_5 \cdot 0.69,
\end{array} \right. \\
\mathcal{R}_1^+(\epsilon_4) = \left\{ \begin{array}{l}
\mathcal{U}_1 \cdot 0.30, \\
\mathcal{U}_3 \cdot 0.47, \\
\mathcal{U}_4 \cdot 0.57, \\
\mathcal{U}_5 \cdot 0.73,
\end{array} \right. \\
\mathcal{S}_1^+(\epsilon_1) = \left\{ \begin{array}{l}
\mathcal{U}_1 \cdot 0.60, \\
\mathcal{U}_3 \cdot 0.65, \\
\mathcal{U}_4 \cdot 0.34, \\
\mathcal{U}_5 \cdot 0.53,
\end{array} \right. \\
\mathcal{S}_1^+(\epsilon_2) = \left\{ \begin{array}{l}
\mathcal{U}_1 \cdot 0.41, \\
\mathcal{U}_3 \cdot 0.60, \\
\mathcal{U}_4 \cdot 0.21, \\
\mathcal{U}_5 \cdot 0.27,
\end{array} \right. \\
\mathcal{S}_1^+(\epsilon_4) = \left\{ \begin{array}{l}
\mathcal{U}_1 \cdot 0.51, \\
\mathcal{U}_3 \cdot 0.36, \\
\mathcal{U}_4 \cdot 0.73, \\
\mathcal{U}_5 \cdot 0.35,
\end{array} \right.
\]
Similarly,

\[
\Gamma_\varepsilon(\bar{e}_1, \bar{e}_2)^{0.22} = \left\{ \left( \frac{\tilde{u}_1}{0.32}, 0.54 \right), \left( \frac{\tilde{u}_2}{0.16}, 0.45 \right), \left( \frac{\tilde{u}_3}{0.47}, 0.28 \right), \left( \frac{\tilde{u}_4}{0.35}, 0.36 \right), \left( \frac{\tilde{u}_5}{0.59}, 0.11 \right) \right\},
\]

\[
\Gamma_\varepsilon(\bar{e}_1, \bar{e}_4)^{0.33} = \left\{ \left( \frac{\tilde{u}_1}{0.30}, 0.49 \right), \left( \frac{\tilde{u}_2}{0.20}, 0.45 \right), \left( \frac{\tilde{u}_3}{0.25}, 0.13 \right), \left( \frac{\tilde{u}_4}{0.44}, 0.36 \right), \left( \frac{\tilde{u}_5}{0.37}, 0.43 \right) \right\},
\]

\[
\Gamma_\varepsilon(\bar{e}_2, \bar{e}_1)^{0.40} = \left\{ \left( \frac{\tilde{u}_1}{0.28}, 0.42 \right), \left( \frac{\tilde{u}_2}{0.37}, 0.16 \right), \left( \frac{\tilde{u}_3}{0.30}, 0.47 \right), \left( \frac{\tilde{u}_4}{0.49}, 0.59 \right), \left( \frac{\tilde{u}_5}{0.49}, 0.33 \right) \right\},
\]

\[
\Gamma_\varepsilon(\bar{e}_2, \bar{e}_2)^{0.22} = \left\{ \left( \frac{\tilde{u}_1}{0.35}, 0.42 \right), \left( \frac{\tilde{u}_2}{0.16}, 0.46 \right), \left( \frac{\tilde{u}_3}{0.38}, 0.36 \right), \left( \frac{\tilde{u}_4}{0.35}, 0.58 \right), \left( \frac{\tilde{u}_5}{0.63}, 0.11 \right) \right\},
\]

\[
\Gamma_\varepsilon(\bar{e}_2, \bar{e}_4)^{0.40} = \left\{ \left( \frac{\tilde{u}_1}{0.30}, 0.42 \right), \left( \frac{\tilde{u}_2}{0.38}, 0.47 \right), \left( \frac{\tilde{u}_3}{0.25}, 0.13 \right), \left( \frac{\tilde{u}_4}{0.44}, 0.57 \right), \left( \frac{\tilde{u}_5}{0.37}, 0.43 \right) \right\},
\]

\[
\Gamma_\varepsilon(\bar{e}_3, \bar{e}_1)^{0.40} = \left\{ \left( \frac{\tilde{u}_1}{0.28}, 0.49 \right), \left( \frac{\tilde{u}_2}{0.22}, 0.16 \right), \left( \frac{\tilde{u}_3}{0.30}, 0.47 \right), \left( \frac{\tilde{u}_4}{0.24}, 0.59 \right), \left( \frac{\tilde{u}_5}{0.27}, 0.33 \right) \right\},
\]

\[
\Gamma_\varepsilon(\bar{e}_3, \bar{e}_2)^{0.22} = \left\{ \left( \frac{\tilde{u}_1}{0.35}, 0.49 \right), \left( \frac{\tilde{u}_2}{0.16}, 0.36 \right), \left( \frac{\tilde{u}_3}{0.41}, 0.36 \right), \left( \frac{\tilde{u}_4}{0.24}, 0.58 \right), \left( \frac{\tilde{u}_5}{0.27}, 0.11 \right) \right\},
\]

\[
\Gamma_\varepsilon(\bar{e}_3, \bar{e}_4)^{0.54} = \left\{ \left( \frac{\tilde{u}_1}{0.30}, 0.49 \right), \left( \frac{\tilde{u}_2}{0.22}, 0.36 \right), \left( \frac{\tilde{u}_3}{0.25}, 0.13 \right), \left( \frac{\tilde{u}_4}{0.24}, 0.57 \right), \left( \frac{\tilde{u}_5}{0.27}, 0.43 \right) \right\}.
\]

Now for all \((\bar{e}_1, \bar{e}_1) \in \bar{F}_{e_1} \times \bar{F}_{e_2},\) we have

\[
Y_\pi(\bar{e}_1, \bar{e}_1)^{\text{min}(0.50,0.25)} = \left\{ \left( \frac{\tilde{u}_1}{\text{min}(0.49,0.60)}, \text{min}(0.36,0.11) \right), \left( \frac{\tilde{u}_3}{\text{min}(0.55,0.63)}, \text{min}(0.71,0.56) \right), \left( \frac{\tilde{u}_4}{\text{min}(0.30,0.46)}, \text{min}(0.57,0.65) \right), \left( \frac{\tilde{u}_5}{\text{min}(0.36,0.18)}, \text{min}(0.63,0.34) \right), \left( \frac{\tilde{u}_5}{\text{min}(0.43,0.31)}, \text{min}(0.61,0.53) \right) \right\},
\]

\[
Y_{\varepsilon}(\bar{e}_1, \bar{e}_1)^{0.25} = \left\{ \left( \frac{\tilde{u}_1}{0.49}, 0.11 \right), \left( \frac{\tilde{u}_2}{0.55}, 0.56 \right), \left( \frac{\tilde{u}_3}{0.30}, 0.57 \right), \left( \frac{\tilde{u}_4}{0.18}, 0.34 \right), \left( \frac{\tilde{u}_5}{0.31}, 0.53 \right) \right\}.
\]
Similarly,

\[
Y_d(\tilde{e}_1, \tilde{e}_2)^{0.49} = \left\{ \left( \frac{\tilde{u}_1}{0.13}, 0.36 \right), \left( \frac{\tilde{u}_2}{0.43}, 0.60 \right), \left( \frac{\tilde{u}_3}{0.30}, 0.21 \right), \left( \frac{\tilde{u}_4}{0.36}, 0.33 \right), \left( \frac{\tilde{u}_5}{0.31}, 0.27 \right) \right\},
\]

\[
Y_d(\tilde{e}_1, \tilde{e}_4)^{0.45} = \left\{ \left( \frac{\tilde{u}_1}{0.49}, 0.24 \right), \left( \frac{\tilde{u}_2}{0.51}, 0.15 \right), \left( \frac{\tilde{u}_3}{0.30}, 0.36 \right), \left( \frac{\tilde{u}_4}{0.36}, 0.63 \right), \left( \frac{\tilde{u}_5}{0.43}, 0.35 \right) \right\},
\]

\[
Y_d(\tilde{e}_2, \tilde{e}_1)^{0.25} = \left\{ \left( \frac{\tilde{u}_1}{0.24}, 0.11 \right), \left( \frac{\tilde{u}_2}{0.36}, 0.56 \right), \left( \frac{\tilde{u}_3}{0.46}, 0.13 \right), \left( \frac{\tilde{u}_4}{0.18}, 0.34 \right), \left( \frac{\tilde{u}_5}{0.31}, 0.46 \right) \right\},
\]

\[
Y_d(\tilde{e}_2, \tilde{e}_3)^{0.25} = \left\{ \left( \frac{\tilde{u}_1}{0.13}, 0.41 \right), \left( \frac{\tilde{u}_2}{0.36}, 0.57 \right), \left( \frac{\tilde{u}_3}{0.59}, 0.13 \right), \left( \frac{\tilde{u}_4}{0.70}, 0.33 \right), \left( \frac{\tilde{u}_5}{0.31}, 0.27 \right) \right\},
\]

\[
Y_d(\tilde{e}_2, \tilde{e}_4)^{0.25} = \left\{ \left( \frac{\tilde{u}_1}{0.24}, 0.24 \right), \left( \frac{\tilde{u}_2}{0.36}, 0.15 \right), \left( \frac{\tilde{u}_3}{0.56}, 0.13 \right), \left( \frac{\tilde{u}_4}{0.68}, 0.55 \right), \left( \frac{\tilde{u}_5}{0.53}, 0.35 \right) \right\},
\]

\[
Y_d(\tilde{e}_3, \tilde{e}_1)^{0.25} = \left\{ \left( \frac{\tilde{u}_1}{0.60}, 0.11 \right), \left( \frac{\tilde{u}_2}{0.52}, 0.54 \right), \left( \frac{\tilde{u}_3}{0.46}, 0.27 \right), \left( \frac{\tilde{u}_4}{0.18}, 0.33 \right), \left( \frac{\tilde{u}_5}{0.31}, 0.25 \right) \right\},
\]

\[
Y_d(\tilde{e}_3, \tilde{e}_2)^{0.30} = \left\{ \left( \frac{\tilde{u}_1}{0.13}, 0.11 \right), \left( \frac{\tilde{u}_2}{0.43}, 0.54 \right), \left( \frac{\tilde{u}_3}{0.61}, 0.21 \right), \left( \frac{\tilde{u}_4}{0.45}, 0.33 \right), \left( \frac{\tilde{u}_5}{0.31}, 0.25 \right) \right\},
\]

\[
Y_d(\tilde{e}_3, \tilde{e}_4)^{0.30} = \left\{ \left( \frac{\tilde{u}_1}{0.62}, 0.11 \right), \left( \frac{\tilde{u}_2}{0.51}, 0.15 \right), \left( \frac{\tilde{u}_3}{0.56}, 0.27 \right), \left( \frac{\tilde{u}_4}{0.45}, 0.33 \right), \left( \frac{\tilde{u}_5}{0.53}, 0.25 \right) \right\}.
\]

(35)

In matrix notation,
Now to find the best solar panel, we first identify the highest-membership values (that is, as mentioned by overline marks in matrix formats) and highest values concerning possibility function in all rows of matrices $\Gamma_c$ and $\Upsilon_\pi$. Then, we represent all the identified information in a tabular format and calculate the positive scores (i.e., for set of parameters) $S^+_i$ for all objects by taking sum of all the products of these highest-membership degrees with relevant possibility values and weights (see Table 1).

In a similar manner, the negative scores $S^-_i$ are computed in the following using the data displayed in Table 2.

Now, we are ready to compute the final scores (see Table 3) by using the formula $\Delta_i = S^+_i + S^-_i$.

![Table 1](https://example.com/table1.png)

![Table 2](https://example.com/table2.png)

![Table 3](https://example.com/table3.png)

The solar panel with the highest final score will be selected. Thus, the company will select the solar panel $u_4$. 

$$\begin{align*}
\Gamma_c &= \begin{pmatrix}
(\check{c}_1, \check{c}_1)^{0.33} & (0.28, 0.56) & (0.20, 0.16) & (0.30, 0.28) & (0.49, 0.36) & (0.49, 0.33) \\
(\check{c}_1, \check{c}_2)^{0.22} & (0.32, 0.54) & (0.16, 0.45) & (0.47, 0.28) & (0.35, 0.36) & (0.59, 0.11) \\
(\check{c}_1, \check{c}_4)^{0.33} & (0.30, 0.49) & (0.20, 0.45) & (0.25, 0.13) & (0.44, 0.36) & (0.37, 0.43) \\
(\check{c}_2, \check{c}_1)^{0.40} & (0.28, 0.42) & (0.37, 0.16) & (0.30, 0.47) & (0.49, 0.59) & (0.49, 0.33) \\
(\check{c}_2, \check{c}_2)^{0.22} & (0.35, 0.42) & (0.16, 0.46) & (0.38, 0.36) & (0.35, 0.58) & (0.63, 0.11) \\
(\check{c}_2, \check{c}_4)^{0.40} & (0.30, 0.42) & (0.38, 0.47) & (0.25, 0.13) & (0.44, 0.57) & (0.37, 0.43) \\
(\check{c}_3, \check{c}_1)^{0.62} & (0.28, 0.49) & (0.22, 0.16) & (0.30, 0.47) & (0.24, 0.59) & (0.27, 0.33) \\
(\check{c}_3, \check{c}_2)^{0.22} & (0.35, 0.49) & (0.16, 0.36) & (0.41, 0.36) & (0.24, 0.58) & (0.27, 0.11) \\
(\check{c}_3, \check{c}_4)^{0.54} & (0.30, 0.49) & (0.22, 0.36) & (0.25, 0.13) & (0.24, 0.57) & (0.27, 0.43)
\end{pmatrix}
\end{align*}$$

$$\begin{align*}
\Upsilon_\pi &= \begin{pmatrix}
(\check{c}_1, \check{c}_1)^{0.25} & (0.49, 0.11) & (0.55, 0.56) & (0.30, 0.57) & (0.18, 0.34) & (0.31, 0.53) \\
(\check{c}_1, \check{c}_2)^{0.49} & (0.13, 0.36) & (0.43, 0.60) & (0.30, 0.21) & (0.36, 0.33) & (0.31, 0.27) \\
(\check{c}_1, \check{c}_4)^{0.45} & (0.49, 0.24) & (0.51, 0.15) & (0.30, 0.36) & (0.36, 0.63) & (0.43, 0.35) \\
(\check{c}_2, \check{c}_1)^{0.25} & (0.24, 0.11) & (0.36, 0.56) & (0.46, 0.13) & (0.18, 0.34) & (0.31, 0.46) \\
(\check{c}_2, \check{c}_2)^{0.25} & (0.13, 0.41) & (0.36, 0.57) & (0.59, 0.13) & (0.70, 0.33) & (0.31, 0.27) \\
(\check{c}_2, \check{c}_4)^{0.25} & (0.24, 0.24) & (0.36, 0.15) & (0.56, 0.13) & (0.68, 0.55) & (0.53, 0.35) \\
(\check{c}_3, \check{c}_1)^{0.25} & (0.60, 0.11) & (0.52, 0.54) & (0.46, 0.27) & (0.18, 0.33) & (0.31, 0.25) \\
(\check{c}_3, \check{c}_2)^{0.30} & (0.13, 0.11) & (0.43, 0.54) & (0.61, 0.21) & (0.45, 0.33) & (0.31, 0.25) \\
(\check{c}_3, \check{c}_4)^{0.30} & (0.62, 0.11) & (0.51, 0.15) & (0.56, 0.27) & (0.45, 0.33) & (0.53, 0.25)
\end{pmatrix}
\end{align*}$$

$$\begin{align*}
S^+_1 &= \check{S} \left( \check{u}_1 \right) \\
&= (0.25 \times 0.60 \times 0.11) + (0.30 \times 0.62 \times 0.11) \\
&= 0.0370, \\
S^-_2 &= (0.49 \times 0.43 \times 0.60) + (0.45 \times 0.51 \times 0.15) \\
&= 0.1608, \\
S^+_3 &= (0.25 \times 0.46 \times 0.13) + (0.30 \times 0.61 \times 0.21) \\
&= 0.0605, \\
S^-_4 &= 0.25 \times 0.68 \times 0.55 \\
&= 0.0935, \\
S^-_5 &= 0.
\end{align*}$$
Table 1: Highest-grade table for the set of parameters.

<table>
<thead>
<tr>
<th>${\tilde{e}_1, \tilde{e}_2}$ weights</th>
<th>Objects</th>
<th>Highest-grade values</th>
<th>Possibility value</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\tilde{e}_1, \tilde{e}_2}^{0.33}$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>${\tilde{e}_1, \tilde{e}_2}^{0.22}$</td>
<td>$u_3$</td>
<td>0.59</td>
<td>0.11</td>
</tr>
<tr>
<td>${\tilde{e}_1, \tilde{e}_2}^{0.33}$</td>
<td>$u_4$</td>
<td>0.44</td>
<td>0.36</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.40}$</td>
<td>$u_5$</td>
<td>0.49</td>
<td>0.59</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.22}$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.62}$</td>
<td>$u_6$</td>
<td>0.44</td>
<td>0.57</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.22}$</td>
<td>$u_7$</td>
<td>0.30</td>
<td>0.47</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.54}$</td>
<td>$u_8$</td>
<td>0.41</td>
<td>0.36</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.20}$</td>
<td>$u_9$</td>
<td>0.30</td>
<td>0.49</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.20}$</td>
<td>$u_1$</td>
<td>0.30</td>
<td>0.49</td>
</tr>
</tbody>
</table>

Table 2: Highest-grade table for the “not set” of parameters.

<table>
<thead>
<tr>
<th>${\tilde{e}_1, \tilde{e}_2}$ weights</th>
<th>Objects</th>
<th>Highest-grade values</th>
<th>Possibility value</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\tilde{e}_1, \tilde{e}_2}^{0.25}$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>${\tilde{e}_1, \tilde{e}_2}^{0.49}$</td>
<td>$u_2$</td>
<td>0.43</td>
<td>0.60</td>
</tr>
<tr>
<td>${\tilde{e}_1, \tilde{e}_2}^{0.45}$</td>
<td>$u_3$</td>
<td>0.51</td>
<td>0.15</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.25}$</td>
<td>$u_5$</td>
<td>0.46</td>
<td>0.13</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.25}$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.25}$</td>
<td>$u_6$</td>
<td>0.68</td>
<td>0.55</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.25}$</td>
<td>$u_7$</td>
<td>0.60</td>
<td>0.11</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.30}$</td>
<td>$u_8$</td>
<td>0.61</td>
<td>0.21</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.30}$</td>
<td>$u_9$</td>
<td>0.62</td>
<td>0.11</td>
</tr>
<tr>
<td>${\tilde{e}_2, \tilde{e}_4}^{0.30}$</td>
<td>$u_1$</td>
<td>0.62</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Table 3: Final score table.

<table>
<thead>
<tr>
<th>Objects</th>
<th>Positive scores</th>
<th>Negative scores</th>
<th>Final scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>0.0794</td>
<td>0.0370</td>
<td>0.1164</td>
</tr>
<tr>
<td>$u_2$</td>
<td>0</td>
<td>0.1608</td>
<td>0.1608</td>
</tr>
<tr>
<td>$u_3$</td>
<td>0.1199</td>
<td>0.0605</td>
<td>0.1804</td>
</tr>
<tr>
<td>$u_4$</td>
<td>0.2682</td>
<td>0.0935</td>
<td>0.3617</td>
</tr>
<tr>
<td>$u_5$</td>
<td>0.0790</td>
<td>0</td>
<td>0.0790</td>
</tr>
</tbody>
</table>

\[
\mathbb{S}^+_1 = \mathbb{S}^+(u_1) \\
= 0.54 \times 0.30 \times 0.49 \\
= 0.0794, \\
\mathbb{S}^+_2 = 0, \\
\mathbb{S}^+_3 = (0.62 \times 0.30 \times 0.47) + (0.22 \times 0.41 \times 0.36) \\
= 0.1199, \\
\mathbb{S}^+_4 = (0.40 \times 0.44 \times 0.57) + (0.40 \times 0.49 \times 0.59) + (0.33 \times 0.44 \times 0.36) \\
= 0.2682, \\
\mathbb{S}^+_5 = (0.40 \times 0.49 \times 0.33) + (0.22 \times 0.59 \times 0.11) \\
= 0.0790. \\
\]

(37)

Definition 11. Let $(\Psi_1, \tilde{\Xi}_1, \tilde{\mathcal{X}}_1, \tilde{\mathcal{Y}}_1)$ and $(\Psi_2, \tilde{\Xi}_2, \tilde{\mathcal{X}}_2, \tilde{\mathcal{Y}}_2)$ be two posets over the soft universe $(\mathcal{U}, \mathcal{S})$, then the operation "OR" between them, denoted by $(\Psi_1, \tilde{\Xi}_1, \tilde{\mathcal{X}}_1, \tilde{\mathcal{Y}}_1) \lor (\Psi_2, \tilde{\Xi}_2, \tilde{\mathcal{X}}_2, \tilde{\mathcal{Y}}_2)$, is defined as follows:

\[
(\Psi_1, \tilde{\Xi}_1, \tilde{\mathcal{X}}_1, \tilde{\mathcal{Y}}_1) \lor (\Psi_2, \tilde{\Xi}_2, \tilde{\mathcal{X}}_2, \tilde{\mathcal{Y}}_2) = (\Psi_1 \lor \Psi_2, \tilde{\Xi}_1 \lor \tilde{\Xi}_2, \tilde{\mathcal{X}}_1 \lor \tilde{\mathcal{X}}_2, \tilde{\mathcal{Y}}_1 \lor \tilde{\mathcal{Y}}_2) \\
\]

(39)
where for all \((\tilde{a}^{\omega_1(a)}, \tilde{b}^{\omega_1(b)}), (\tilde{a}^{\omega_2(a)}, \tilde{b}^{\omega_2(b)}) \in \tilde{X}_{\omega_1} \times \tilde{Y}_{\sigma_1}, (\tilde{a}^{\omega_3(a)}, \tilde{b}^{\omega_3(b)}) \in \tilde{X}_{\omega_2} \times \tilde{Y}_{\sigma_2}\), we have

\[
\Psi_{\xi}(\tilde{a}, \tilde{b})_{\max(\omega_1(a), \omega_2(a), \omega_3(a))} = \left\{ \begin{array}{c}
\frac{\tilde{u}_1}{\max(0.32, 0.28)}, \\
\frac{\tilde{u}_2}{\max(0.56, 0.61)}, \\
\frac{\tilde{u}_3}{\max(0.28, 0.47)}, \\
\frac{\tilde{u}_4}{\max(0.45, 0.16)}
\end{array} \right\},
\]

(40)

\[
\Omega_{\xi}(\tilde{a}, \tilde{b})_{\max(\omega_1(a), \omega_2(a), \omega_3(a))} = \left\{ \begin{array}{c}
\frac{\tilde{u}_1}{\max(0.32, 0.28)}, \\
\frac{\tilde{u}_2}{\max(0.56, 0.61)}, \\
\frac{\tilde{u}_3}{\max(0.28, 0.47)}, \\
\frac{\tilde{u}_4}{\max(0.45, 0.16)}
\end{array} \right\},
\]

Now, we discuss an application of “OR” operation by using algorithm given as follows:

**Example 7.** Reconsider the \(\mathcal{F}_\mathcal{F}_{\mathcal{P}_\mathcal{F}_{\mathcal{S}_\mathcal{S}_\mathcal{G}}}(\tilde{a}, \tilde{b}, \tilde{X}, \tilde{Y})\) and \((\tilde{R}_{\lambda_1}, \tilde{E}_{\rho_1}, \tilde{X}_{\sigma_1}, \tilde{Y}_{\omega_1})\) and \((\tilde{R}_{\lambda_2}, \tilde{E}_{\rho_2}, \tilde{X}_{\sigma_2}, \tilde{Y}_{\omega_2})\) over the soft universe \((\mathcal{U}, \mathcal{S})\) as given in Example 6. Then, the “OR” operation between them is given by \((\tilde{\Psi}_{\xi}, \tilde{\Omega}_{\xi}, \tilde{X}_{\omega_1} \times \tilde{Y}_{\sigma_1}, \tilde{X}_{\omega_2} \times \tilde{Y}_{\omega_2})\) where for all \((\tilde{e}_1, \tilde{e}_2) \in \tilde{X}_{\omega_1} \times \tilde{Y}_{\sigma_1}\), we get

\[
\Psi_{\xi}(\tilde{e}_1, \tilde{e}_2)_{\max(0.33, 0.62)} = \left\{ \begin{array}{c}
\frac{\tilde{u}_1}{\max(0.32, 0.28)}, \\
\frac{\tilde{u}_2}{\max(0.56, 0.61)}, \\
\frac{\tilde{u}_3}{\max(0.28, 0.47)}, \\
\frac{\tilde{u}_4}{\max(0.45, 0.16)}
\end{array} \right\}.
\]

(41)

Similarly, all the remaining values are computed and displayed in the following matrix format:

\[
\Psi_{\xi} = \begin{bmatrix}
(0.32, 0.61) & (0.37, 0.45) & (0.56, 0.47) & (0.63, 0.59) & (0.59, 0.47) \\
(0.35, 0.56) & (0.20, 0.46) & (0.56, 0.36) & (0.63, 0.58) & (0.69, 0.47) \\
(0.32, 0.56) & (0.38, 0.47) & (0.56, 0.28) & (0.63, 0.57) & (0.59, 0.47) \\
(0.54, 0.61) & (0.50, 0.69) & (0.38, 0.61) & (0.56, 0.73) & (0.63, 0.71) \\
(0.54, 0.54) & (0.50, 0.69) & (0.47, 0.61) & (0.56, 0.73) & (0.69, 0.71) \\
(0.54, 0.49) & (0.50, 0.69) & (0.38, 0.61) & (0.56, 0.73) & (0.63, 0.71) \\
(0.35, 0.61) & (0.37, 0.36) & (0.41, 0.49) & (0.49, 0.74) & (0.49, 0.52) \\
(0.35, 0.54) & (0.22, 0.46) & (0.47, 0.49) & (0.35, 0.74) & (0.69, 0.52) \\
(0.35, 0.49) & (0.38, 0.47) & (0.41, 0.49) & (0.44, 0.74) & (0.37, 0.52)
\end{bmatrix}
\]

(42)

Now for all \((\tilde{e}_1, \tilde{e}_2) \in \tilde{X}_{\omega_2} \times \tilde{Y}_{\omega_2}\), we have
(1) Input: \( \mathcal{U} \), the universal set; \( \mathcal{E} \), the set of parameters; \( (\Psi_1, \Xi_1, \Upsilon_1, \Psi_2, \Xi_2, \Upsilon_2) \) and \( (\Rho_1, \Xi_3, \Upsilon_3) \), the fuzzy set of fuzzy sets over the soft universe \( (\mathcal{W}, \mathcal{R}) \), where \( \lambda, \Lambda, \delta, \Sigma \) are the possibility functions and \( \omega_1, \omega_2, \sigma_1, \sigma_2 \) represent the weights.

(2) Find the “OR” operation \( (\Psi, \Omega, \Xi, \Upsilon)^\top \times \Upsilon \times \Xi \times \Psi \) between the given fuzzy sets over the soft universe \( (\mathcal{W}, \mathcal{R}) \).

(3) Select the highest membership grade in each set \( \Psi_i(u_1, u_2, u_3) \) and \( \Omega_i(u_i, u_i, u_i) \) and \( \Xi_i(u_i, u_i, u_i) \), \( \Upsilon_i(u_i, u_i, u_i) \), respectively.

(4) Select the highest membership grade in each set, “not,” respectively.

(5) Select the weight values \( \kappa_i \) and \( \lambda_i \) from the sets having highest membership grade values.

(6) Calculate the positive \( (\Xi^+) \) and negative \( (\Xi^-) \) score values of each alternative by using the formulas: \( \Xi^+(u_1, u_2, u_3) = \sum \max_0^1 \Gamma_i^+ (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \times \tilde{u}_1 \times \tilde{u}_2 \times \tilde{u}_3 \) and \( \Xi^- (u_1, u_2, u_3) = \sum \max_0^1 \Gamma_i^- (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \times \tilde{u}_1 \times \tilde{u}_2 \times \tilde{u}_3 \) for all \( u_i \in \mathcal{U} \). Here, \( \Gamma_i^+ (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \) and \( \Gamma_i^- (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \) are the highest membership grade values.

(7) Compute the ultimate score values using the formula: \( \Delta_i = \Xi_i^+ + \Xi_i^- \).

(8) Find \( n \), for which \( \Delta_n = \max \Delta_i \).

(9) Output: The object \( u_n \) will be the best decision object. If there exist more than one values of \( n \), then anyone of the \( u_n \) ’s can be chosen.

**Algorithm 2:** Selection of the most suitable choice using “OR” operation

\[
\Omega_{\text{OR}}(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = \begin{cases} 
\left( \frac{\tilde{u}_1}{\max(0.49, 0.60)}, \max(0.36, 0.11), \frac{\tilde{u}_2}{\max(0.55, 0.65)}, \max(0.71, 0.56) \right), \\
\left( \frac{\tilde{u}_1}{\max(0.30, 0.46)}, \max(0.57, 0.65), \frac{\tilde{u}_2}{\max(0.36, 0.18)}, \max(0.63, 0.34) \right), \\
\left( \frac{\tilde{u}_3}{\max(0.43, 0.31)}, \max(0.61, 0.53) \right) \end{cases}
\]

\[
\Omega_{\text{OR}}(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = \begin{cases} 
\left( \frac{\tilde{u}_1}{0.60}, 0.36 \right), \left( \frac{\tilde{u}_2}{0.63}, 0.71 \right), \left( \frac{\tilde{u}_3}{0.46}, 0.65 \right), \left( \frac{\tilde{u}_4}{0.36}, 0.63 \right), \left( \frac{\tilde{u}_5}{0.43}, 0.61 \right) \end{cases}
\]

Similarly, all the remaining values are computed and displayed in the following matrix format:

\[
\Omega_{\text{OR}} = \begin{pmatrix} 
(0.60, 0.36) & (0.63, 0.71) & (0.46, 0.65) & (0.36, 0.63) & (0.43, 0.61) \\
(0.49, 0.41) & (0.55, 0.71) & (0.68, 0.57) & (0.70, 0.63) & (0.43, 0.61) \\
(0.62, 0.36) & (0.55, 0.71) & (0.56, 0.57) & (0.68, 0.73) & (0.53, 0.61) \\
(0.60, 0.67) & (0.63, 0.57) & (0.59, 0.65) & (0.76, 0.55) & (0.70, 0.53) \\
(0.24, 0.67) & (0.43, 0.60) & (0.68, 0.21) & (0.76, 0.55) & (0.70, 0.46) \\
(0.62, 0.67) & (0.51, 0.57) & (0.59, 0.36) & (0.76, 0.73) & (0.70, 0.46) \\
(0.75, 0.11) & (0.63, 0.56) & (0.61, 0.65) & (0.45, 0.34) & (0.66, 0.53) \\
(0.75, 0.41) & (0.52, 0.60) & (0.68, 0.27) & (0.70, 0.33) & (0.66, 0.27) \\
(0.75, 0.24) & (0.52, 0.54) & (0.61, 0.36) & (0.68, 0.73) & (0.66, 0.35) 
\end{pmatrix}
\]

Now to find the best solar panel, we first identify the highest-membership values (that is, as mentioned by over-line marks in matrix formats) and highest values concerning possibility function in all rows of matrices \( \Psi_i \) and \( \Omega_i \). Then, we represent all the identified information in a tabular format and calculate the positive scores (i.e., for set of parameters) \( \Xi_i^+ \) for all objects by taking sum of all the products of these highest-membership degrees with relevant possibility values and weights (see Table 4).

In a similar manner, the negative scores \( \Xi_i^- \) are computed with the help of Table 5 given as follows:

Thus, the final scores are computed using the formula \( \Delta_i = \Xi_i^+ + \Xi_i^- \) and provided by the Table 6.
The solar panel with the highest final score will be selected. Thus, the company will select the solar panel $u_4$.
Remark 1. Note that by applying both the Algorithms 1 and 2 on the Example 6, we get the same optimal result, that is, \( \bar{u}_4 \).

4. Similarity Measure between \( \text{FPPFSs} \)

In this section, we first introduce certain notions concerning the similarity measure between \( \text{FPPFSs} \) and then explore its application in daily life.

\[
\delta\left(\tilde{\mathbf{P}}_1, \tilde{\mathbf{R}}_1\right) = M\left(\tilde{\mathbf{P}}\left(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})\right)(u), \tilde{\mathbf{R}}\left(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}})\right)(u)\right) \times M\left(\lambda\left(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})\right), \lambda_1\left(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}})\right)\right) \times M\left(\omega_1(\bar{\mathfrak{e}}), \sigma_1(\bar{\mathfrak{e}})\right),
\]

Such that

\[
M\left(\tilde{\mathbf{P}}\left(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})\right)(u), \tilde{\mathbf{R}}\left(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}})\right)(u)\right) = \max_{i=1}^{n} M_i\left(\tilde{\mathbf{P}}\left(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})\right)(u), \tilde{\mathbf{R}}\left(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}})\right)(u)\right),
\]

\[
M\left(\lambda\left(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})\right), \lambda_1\left(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}})\right)\right) = \max_{i=1}^{n} M_i\left(\lambda\left(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})\right), \lambda_1\left(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}})\right)\right),
\]

\[
M\left(\omega_1(\bar{\mathfrak{e}}), \sigma_1(\bar{\mathfrak{e}})\right) = \max_{i=1}^{n} M_i\left(\omega_1(\bar{\mathfrak{e}}), \sigma_1(\bar{\mathfrak{e}})\right),
\]

where

\[
M_i\left(\tilde{\mathbf{P}}\left(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})\right)(u), \tilde{\mathbf{R}}\left(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}})\right)(u)\right) = 1 - \frac{\sum_{j=1}^{n} \|\tilde{\mathbf{P}}_{ij}(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})) - \tilde{\mathbf{R}}_{ij}(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}}))\|}{\sum_{j=1}^{n} \|\tilde{\mathbf{P}}_{ij}(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})) + \tilde{\mathbf{R}}_{ij}(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}}))\|},
\]

\[
M_i\left(\lambda\left(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})\right), \lambda_1\left(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}})\right)\right) = 1 - \frac{\sum_{j=1}^{n} \|\lambda_{ij}(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})) - \lambda_{1ij}(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}}))\|}{\sum_{j=1}^{n} \|\lambda_{ij}(\bar{\mathfrak{e}}_1^{\omega_i}(\bar{\mathfrak{e}})) + \lambda_{1ij}(\bar{\mathfrak{e}}_1^{\sigma_i}(\bar{\mathfrak{e}}))\|},
\]

\[
M_i\left(\omega_1(\bar{\mathfrak{e}}), \sigma_1(\bar{\mathfrak{e}})\right) = 1 - \frac{\left|\omega_{1i}(\bar{\mathfrak{e}}) - \sigma_{1i}(\bar{\mathfrak{e}})\right|}{\left|\omega_{1i}(\bar{\mathfrak{e}}) + \sigma_{1i}(\bar{\mathfrak{e}})\right|},
\]

and

\[
\mathbb{E} \bar{\omega} = (0.50 \times 0.70 \times 0.63) + (0.50 \times 0.68 \times 0.73) + (0.25 \times 0.76 \times 0.55) + (0.45 \times 0.76 \times 0.73)
\]

\[
= 0.8229,
\]

\[
\mathbb{E} \bar{\omega} = 0.
\]

**Definition 12.** For any two \( \text{FPPFSs} \) \( \tilde{\mathbf{P}}_1, \tilde{\mathbf{R}}_1 \) and \( \tilde{\mathbf{P}}_2, \tilde{\mathbf{R}}_2 \) over the soft universe \( \left\{\mathcal{U}, \mathcal{S}\right\} \), the similarity measure between them, represented by \( \delta\left(\tilde{\mathbf{P}}_1, \tilde{\mathbf{R}}_1\right), \delta\left(\tilde{\mathbf{P}}_2, \tilde{\mathbf{R}}_2\right) \), is defined as follows:
\[
\delta(\tilde{\mathbf{L}}_{\tilde{\varphi}}, \tilde{\mathbf{G}}_{\tilde{\beta}}) = M^*(\tilde{\mathbf{L}}(e^{\omega_1(\tilde{\epsilon})}) (u), \tilde{\mathbf{G}}(e^{\sigma_1(\tilde{\epsilon})}) (u)) \times M^*(\delta(e^{\omega_1(\tilde{\epsilon})}, \delta_1(e^{\sigma_1(\tilde{\epsilon})})) \times M^*(\omega_2(\tilde{\epsilon}), \sigma_2(\tilde{\epsilon})),
\]

(54)

Such that

\[
M^*(\tilde{\mathbf{L}}(e^{\omega_1(\tilde{\epsilon})}) (u), \tilde{\mathbf{G}}(e^{\sigma_1(\tilde{\epsilon})}) (u)) = \max M_i^*(\tilde{\mathbf{L}}(e^{\omega_1(\tilde{\epsilon})}) (u), \tilde{\mathbf{G}}(e^{\sigma_1(\tilde{\epsilon})}) (u)),
\]

\[
M^*(\delta(e^{\omega_1(\tilde{\epsilon})}, \delta_1(e^{\sigma_1(\tilde{\epsilon})})) = \max M_i^*(\delta(e^{\omega_1(\tilde{\epsilon})}, \delta_1(e^{\sigma_1(\tilde{\epsilon})})),
\]

\[
M^*(\omega_2(\tilde{\epsilon}), \sigma_2(\tilde{\epsilon})) = \max M_i^*(\omega_2(\tilde{\epsilon}), \sigma_2(\tilde{\epsilon})),
\]

where

\[
M_i^*(\tilde{\mathbf{L}}(e^{\omega_1(\tilde{\epsilon})}) (u), \tilde{\mathbf{G}}(e^{\sigma_1(\tilde{\epsilon})}) (u)) = 1 - \frac{\sum_{i=1}^{n} \left| \tilde{\mathbf{L}}_{ij}(e^{\omega_1(\tilde{\epsilon})}) - \tilde{\mathbf{G}}_{ij}(e^{\sigma_1(\tilde{\epsilon})}) \right|}{\sum_{i=1}^{n} \tilde{\mathbf{L}}_{ij}(e^{\omega_1(\tilde{\epsilon})}) + \tilde{\mathbf{G}}_{ij}(e^{\sigma_1(\tilde{\epsilon})})},
\]

(56)

\[
M_i^*(\delta(e^{\omega_1(\tilde{\epsilon})}, \delta_1(e^{\sigma_1(\tilde{\epsilon})}))) = 1 - \frac{\sum_{i=1}^{n} \left| \delta_{ij}(e^{\omega_1(\tilde{\epsilon})}) - \delta_{ij}(e^{\sigma_1(\tilde{\epsilon})}) \right|}{\sum_{i=1}^{n} \delta_{ij}(e^{\omega_1(\tilde{\epsilon})}) + \delta_{ij}(e^{\sigma_1(\tilde{\epsilon})})},
\]

\[
M_i^*(\omega_2(\tilde{\epsilon}), \sigma_2(\tilde{\epsilon})) = 1 - \frac{\left| \omega_{2i}(\tilde{\epsilon}) - \sigma_{2i}(\tilde{\epsilon}) \right|}{\omega_{2i}(\tilde{\epsilon}) + \sigma_{2i}(\tilde{\epsilon})},
\]

Definition 13. Let \((\tilde{\mathbf{P}}_1, \tilde{\mathbf{L}}_{\tilde{\varphi}}, \tilde{\mathbf{G}}_{\tilde{\beta}})\) and \((\tilde{\mathbf{R}}_{\tilde{\alpha}}, \tilde{\mathbf{H}}_{\tilde{\gamma}})\) be two \(\mathcal{F} \mathcal{P} \mathcal{P} \mathcal{F}_{\tilde{\beta}}\) over the soft universe \((\mathcal{U}, \mathcal{E})\).

Then, we say that they are significantly similar if

\[
\delta(\tilde{\mathbf{P}}_1, \tilde{\mathbf{R}}_{\tilde{\alpha}}) \geq \frac{1}{2} \text{ and } \delta(\tilde{\mathbf{L}}_{\tilde{\varphi}}, \tilde{\mathbf{H}}_{\tilde{\gamma}}) \geq \frac{1}{2}.
\]

or

\[
\delta(\tilde{\mathbf{P}}_1, \tilde{\mathbf{R}}_{\tilde{\alpha}}) + \delta(\tilde{\mathbf{L}}_{\tilde{\varphi}}, \tilde{\mathbf{H}}_{\tilde{\gamma}}) \geq \frac{1}{2}.
\]

(57)
Example 8. Consider again the data of solar panels as given in Example 6, then assume that two novel FPPFBSSs $(\tilde{P} \overset{\lambda}{\sim} \tilde{\lambda}, \tilde{Q} \overset{\vartheta}{\sim} \tilde{\vartheta}, \tilde{X} \overset{\omega}{\sim} \tilde{\omega}_1, \tilde{\omega}_2)$ and $(\tilde{R} \overset{\lambda_1}{\sim} \tilde{\lambda}_1, \tilde{S} \overset{\vartheta_1}{\sim} \tilde{\vartheta}_1, \tilde{Y} \overset{\sigma}{\sim} \tilde{\sigma}_1, \tilde{\sigma}_2)$ are provided as follows:

\[\begin{align*}
\tilde{P} & = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0.69, 0.45 \\ 0.72, 0.80 \end{pmatrix}, \\
\tilde{Q} & = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0.55, 0.74 \\ 0.45, 0.72 \end{pmatrix}, \\
\tilde{R} & = \begin{pmatrix} u_5 \end{pmatrix} = \begin{pmatrix} 0.52, 0.70 \end{pmatrix},
\end{align*}\]

\[\begin{align*}
\tilde{R} & = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0.60 \end{pmatrix}, \\
\tilde{Q} & = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0.68, 0.22 \\ 0.83, 0.64 \end{pmatrix}, \\
\tilde{R} & = \begin{pmatrix} u_5 \end{pmatrix} = \begin{pmatrix} 0.79, 0.55 \end{pmatrix}, \quad (58)
\end{align*}\]

In the following, we now compute the similarity measurement between the FPPFBSs $(\tilde{P} \overset{\lambda}{\sim} \tilde{\lambda}, \tilde{Q} \overset{\vartheta}{\sim} \tilde{\vartheta}, \tilde{X} \overset{\omega}{\sim} \tilde{\omega}_1, \tilde{\omega}_2)$ and $(\tilde{R} \overset{\lambda_1}{\sim} \tilde{\lambda}_1, \tilde{S} \overset{\vartheta_1}{\sim} \tilde{\vartheta}_1, \tilde{Y} \overset{\sigma}{\sim} \tilde{\sigma}_1, \tilde{\sigma}_2)$, which is given by $(\tilde{\delta} (\tilde{P}, \tilde{R}), \tilde{\delta} (\tilde{Q}, \tilde{S}), \tilde{\delta} (\tilde{X}, \tilde{Y}))$ where

\[\begin{align*}
\tilde{\delta}(\tilde{P}, \tilde{R}) &= \mathcal{M} \begin{pmatrix} \tilde{P} \overset{\omega}{\sim} \tilde{\omega}_1 \end{pmatrix}(u), \begin{pmatrix} \tilde{R} \overset{\sigma_1}{\sim} \tilde{\omega}_1 \end{pmatrix}(u) \end{pmatrix} \times \mathcal{M} \begin{pmatrix} \tilde{R} \overset{\omega_1}{\sim} \tilde{\omega}_1 \end{pmatrix}(u), \begin{pmatrix} \tilde{R} \overset{\omega}{\sim} \tilde{\omega}_1 \end{pmatrix}(u) \end{pmatrix} \times \mathcal{M} \begin{pmatrix} \tilde{R} \overset{\omega}{\sim} \tilde{\omega}_1 \end{pmatrix}(u), \begin{pmatrix} \tilde{R} \overset{\sigma_1}{\sim} \tilde{\omega}_1 \end{pmatrix}(u) \end{pmatrix}.
\end{align*}\]

Now by equation (53), we have

\[\mathcal{M}_1(\omega_1(e), \sigma_1(e)) = 1 - \frac{|\omega_{11}(e) - \sigma_{11}(e)|}{|\omega_{11}(e) + \sigma_{11}(e)|},\]

\[= 1 - \frac{|0.35 - 0.85|}{|0.35 + 0.85|} = 0.5833.\]
Now using equation (52), for \( i = 1, 2, 3 \), we calculate the following:

\[
\mathcal{M}_1 \left( \lambda \left( e^{\omega_i(e)} \right), \lambda_1 \left( e^{\sigma_i(e)} \right) \right) = 1 - \frac{\sum_{i=1}^{5} | \lambda_{1i} \left( e^{\omega_i(e)} \right) - \lambda_{1i} \left( e^{\sigma_i(e)} \right) |}{\sum_{i=1}^{5} | \lambda_{1i} \left( e^{\omega_i(e)} \right) + \lambda_{1i} \left( e^{\sigma_i(e)} \right) |} = 1 - \frac{0.45 - 0.40 + 0.74 - 0.45 + 0.70 - 0.81}{0.45 + 0.40 + 0.80 + 0.75 + 0.70 + 0.81} = 0.9508.
\]

Similarly, \( \mathcal{M}_1 \left( \lambda \left( e^{\omega_i(e)} \right), \lambda_1 \left( e^{\sigma_i(e)} \right) \right) = 0.9653 \), and \( \mathcal{M}_2 \left( \lambda \left( e^{\omega_i(e)} \right), \lambda_1 \left( e^{\sigma_i(e)} \right) \right) = 0.8035 \). To accumulate the above three equations, we compute the following using equation (49):

\[
\mathcal{M}(\lambda \left( e^{\omega_i(e)} \right), \lambda_1 \left( e^{\sigma_i(e)} \right)) = \max \left( \mathcal{M}_1 \left( \lambda \left( e^{\omega_i(e)} \right), \lambda_1 \left( e^{\sigma_i(e)} \right) \right), \mathcal{M}_2 \left( \lambda \left( e^{\omega_i(e)} \right), \lambda_1 \left( e^{\sigma_i(e)} \right) \right), \mathcal{M}_3 \left( \lambda \left( e^{\omega_i(e)} \right), \lambda_1 \left( e^{\sigma_i(e)} \right) \right) \right) = 0.9653.
\]

Now,

\[
\mathcal{M}_1 \left( \bar{\Psi} \left( e^{\omega_i(e)} \right), \bar{\mathcal{R}} \left( e^{\sigma_i(e)} \right) \right) = 1 - \frac{\sum_{i=1}^{5} | \bar{\Psi}_{1i} \left( e^{\omega_i(e)} \right) - \bar{\mathcal{R}}_{1i} \left( e^{\sigma_i(e)} \right) |}{\sum_{i=1}^{5} | \bar{\Psi}_{1i} \left( e^{\omega_i(e)} \right) + \bar{\mathcal{R}}_{1i} \left( e^{\sigma_i(e)} \right) |} = 1 - \frac{0.69 - 0.58 + 0.72 - 0.64 + 0.69 + 0.58}{0.69 + 0.58 + 0.72 + 0.64 + 0.69 + 0.58} = 0.9022.
\]

Similarly, \( \mathcal{M}_2 \left( \bar{\Psi} \left( e^{\omega_i(e)} \right), \bar{\mathcal{R}} \left( e^{\sigma_i(e)} \right) \right) = 0.9569 \), and \( \mathcal{M}_3 \left( \bar{\Psi} \left( e^{\omega_i(e)} \right), \bar{\mathcal{R}} \left( e^{\sigma_i(e)} \right) \right) = 0.9394 \). By equation (48), we get

\[
\mathcal{M} \left( \bar{\Psi} \left( e^{\omega_i(e)} \right), \bar{\mathcal{R}} \left( e^{\sigma_i(e)} \right) \right) = \max \left( \mathcal{M}_1 \left( \bar{\Psi} \left( e^{\omega_i(e)} \right), \bar{\mathcal{R}} \left( e^{\sigma_i(e)} \right) \right), \mathcal{M}_2 \left( \bar{\Psi} \left( e^{\omega_i(e)} \right), \bar{\mathcal{R}} \left( e^{\sigma_i(e)} \right) \right), \mathcal{M}_3 \left( \bar{\Psi} \left( e^{\omega_i(e)} \right), \bar{\mathcal{R}} \left( e^{\sigma_i(e)} \right) \right) \right) = 0.9569.
\]

Thus,

\[
\delta \left( \bar{\Psi}_I, \bar{\Lambda}_I \right) = \mathcal{M} \left( \bar{\Psi} \left( e^{\omega_i(e)} \right), \bar{\mathcal{R}} \left( e^{\sigma_i(e)} \right) \right) \times \mathcal{M} \left( \lambda \left( e^{\omega_i(e)} \right), \lambda_1 \left( e^{\sigma_i(e)} \right) \right) \times \mathcal{M} \left( \omega_1 \left( e \right), \sigma_1 \left( e \right) \right) = 0.9569 \times 0.9653 \times 0.8713 = 0.8048 > \frac{1}{2}.
\]
In the similar manner, the similarity measure for the “not set” of parameters of both $\mathcal{FPPFS}$ is $\tilde{\mathcal{S}}(\tilde{\Omega}_\varphi, \tilde{\Omega}_\psi) = 0.6166 > (1/2)$. Clearly, the similarity measure between two types of datasets evaluated by two experts in the form of different $\mathcal{FPPFS}$ is greater than (1/2). Hence, the given $\mathcal{FPPFS}$ are significantly similar.

**Proposition 4.** Let $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{x}_1, \tilde{x}_0), (\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{y}_1, \tilde{y}_0)$, and $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{\nu}_1, \tilde{\nu}_0)$ be any three $\mathcal{FPPFS}$ over $(\mathcal{U}, \mathcal{S})$. Then, the following assertions satisfied:

1. $\delta\left(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{x}_1, \tilde{x}_0\right)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{y}_1, \tilde{y}_0)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{\nu}_1, \tilde{\nu}_0)$\right) = \delta\left(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{x}_1, \tilde{x}_0\right) = \delta\left(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{y}_1, \tilde{y}_0\right) = \delta\left(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{\nu}_1, \tilde{\nu}_0\right) = 1$

2. $0 \leq \delta\left(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{x}_1, \tilde{x}_0\right)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{y}_1, \tilde{y}_0)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{\nu}_1, \tilde{\nu}_0)$\right) \leq 1$

3. $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{x}_1, \tilde{x}_0)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{y}_1, \tilde{y}_0)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{\nu}_1, \tilde{\nu}_0)$\right) = 1$

4. $\delta\left(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{x}_1, \tilde{x}_0\right)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{\nu}_1, \tilde{\nu}_0)$\right) = \delta\left(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{x}_1, \tilde{x}_0\right)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{y}_1, \tilde{y}_0)$\right) = \delta\left(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{x}_1, \tilde{x}_0\right)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{\nu}_1, \tilde{\nu}_0)$\right)$ = 1$

5. $\delta\left(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{x}_1, \tilde{x}_0\right)$, $(\tilde{\Omega}_1, \tilde{\Omega}_0, \tilde{y}_1, \tilde{y}_0)$\right) = 0$

**Proof.** Its can be easily proved by Definitions 12 and 13.

4.1. Application of Similarity Measure to Agricultural Land Selection. Sugarcane is considered as one of the most significant crops in Pakistan. The importance of sugarcane is more than a subsistence crop. To increase the production of sugarcane, the cultivation is fully based on the suitability of land and some related parameters, including water availability, nutrient availability index, soil texture and coarse surface materials, rooting condition, and topography. Suppose that an agricultural organization plans to find a suitable land for the production of sugarcane from four given alternatives $A, B, C, D$. To approach the best option, the organization appoints a team of experts to complete this complex task. Let $\mathcal{U}$ be a universal set of four alternatives (lands) and $\mathcal{S} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_6\}$ be the set of six parameters where

- $\tilde{x}_1 = $ Good composition of soil
- $\tilde{x}_2 = $ Nearest to local market
- $\tilde{x}_3 = $ Availability of water
- $\tilde{x}_4 = $ Good rooting condition
- $\tilde{x}_5 = $ Favorable environmental factors
- $\tilde{x}_6 = $ Suitable topographical condition

Now, let $\mathcal{S} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_6\}$ be the “not set” of parameters where

- $\tilde{x}_1 = $ Bad composition of soil
- $\tilde{x}_2 = $ Far from local market
- $\tilde{x}_3 = $ Unavailability of water
- $\tilde{x}_4 = $ Bad rooting condition
- $\tilde{x}_5 = $ Unfavorable environmental factors
- $\tilde{x}_6 = $ Unsuitable topographical condition

The characteristics of an ideal agricultural land are provided by the organization in the form of a $\mathcal{FPPFS}$ as displayed in Table 7. The team of experts evaluates each option ($A, B, C, D$) with respect to favorable parameters and provides its estimations as $\mathcal{FPPFS}$ as $\mathcal{A}_A$, $\mathcal{A}_B$, $\mathcal{A}_C$, and $\mathcal{A}_D$, which are, respectively, given in Tables 8–11.

To find the most suitable alternative which is closest to the ideal agricultural land, we calculate the similarity measure for the first alternative, that is, $\tilde{\mathcal{S}}(\mathcal{A}_A, \mathcal{A}_B)$, $\tilde{\mathcal{S}}(\mathcal{A}_B, \mathcal{A}_C)$, $\tilde{\mathcal{S}}(\mathcal{A}_C, \mathcal{A}_D)$, and $\tilde{\mathcal{S}}(\mathcal{A}_D, \mathcal{A}_A)$, which is given as follows:

<table>
<thead>
<tr>
<th>$\lambda(e_{(2)}^{(0)})$</th>
<th>$\lambda(\mathcal{A}_B)$</th>
<th>$\lambda(\mathcal{A}_C)$</th>
<th>$\lambda(\mathcal{A}_D)$</th>
<th>$\lambda(\mathcal{A}_A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{x}_1$</td>
<td>1.00</td>
<td>0.85</td>
<td>0.75</td>
<td>0.60</td>
</tr>
<tr>
<td>$\tilde{x}_2$</td>
<td>0.90</td>
<td>0.96</td>
<td>0.64</td>
<td>0.55</td>
</tr>
<tr>
<td>$\tilde{x}_3$</td>
<td>0.85</td>
<td>0.93</td>
<td>0.49</td>
<td>0.40</td>
</tr>
<tr>
<td>$\tilde{x}_4$</td>
<td>0.90</td>
<td>0.89</td>
<td>0.38</td>
<td>0.20</td>
</tr>
<tr>
<td>$\tilde{x}_5$</td>
<td>1.00</td>
<td>0.90</td>
<td>0.20</td>
<td>1.00</td>
</tr>
<tr>
<td>$\tilde{x}_6$</td>
<td>0.99</td>
<td>1.00</td>
<td>0.20</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Table 8: Estimations on the alternative "A" in the form of a $\mathcal{F} \mathcal{P} \mathcal{P} \mathcal{F} \mathcal{B} \mathcal{S} \mathcal{S}$.

<table>
<thead>
<tr>
<th>$\lambda_A$</th>
<th>$\mathcal{A} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\lambda_A$</th>
<th>$\mathcal{A} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\lambda_A$</th>
<th>$\mathcal{A} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\mathcal{A} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_{-0.60}$</td>
<td>0.45</td>
<td>0.50</td>
<td>$\lambda_{-0.22}$</td>
<td>0.33</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.55}$</td>
<td>0.60</td>
<td>0.75</td>
<td>$\lambda_{-0.34}$</td>
<td>0.24</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.42}$</td>
<td>0.28</td>
<td>0.60</td>
<td>$\lambda_{-0.41}$</td>
<td>0.58</td>
<td>0.44</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.73}$</td>
<td>0.64</td>
<td>0.20</td>
<td>$\lambda_{-0.10}$</td>
<td>0.69</td>
<td>0.62</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.62}$</td>
<td>0.12</td>
<td>0.72</td>
<td>$\lambda_{-0.27}$</td>
<td>0.23</td>
<td>0.80</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.80}$</td>
<td>0.56</td>
<td>0.90</td>
<td>$\lambda_{-0.31}$</td>
<td>0.10</td>
<td>0.50</td>
<td></td>
</tr>
</tbody>
</table>

Table 9: Estimations on the alternative "B" in the form of a $\mathcal{F} \mathcal{P} \mathcal{P} \mathcal{F} \mathcal{B} \mathcal{S} \mathcal{S}$.

<table>
<thead>
<tr>
<th>$\lambda_B$</th>
<th>$\mathcal{B} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\lambda_B$</th>
<th>$\mathcal{B} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\lambda_B$</th>
<th>$\mathcal{B} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\mathcal{B} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_{-0.60}$</td>
<td>0.60</td>
<td>0.60</td>
<td>$\lambda_{-0.22}$</td>
<td>0.15</td>
<td>0.54</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.55}$</td>
<td>0.50</td>
<td>0.70</td>
<td>$\lambda_{-0.34}$</td>
<td>0.69</td>
<td>0.26</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.42}$</td>
<td>0.48</td>
<td>0.40</td>
<td>$\lambda_{-0.41}$</td>
<td>0.63</td>
<td>0.70</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.73}$</td>
<td>0.70</td>
<td>0.59</td>
<td>$\lambda_{-0.10}$</td>
<td>0.57</td>
<td>0.49</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.62}$</td>
<td>0.89</td>
<td>0.74</td>
<td>$\lambda_{-0.27}$</td>
<td>0.39</td>
<td>0.58</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.80}$</td>
<td>0.64</td>
<td>0.88</td>
<td>$\lambda_{-0.31}$</td>
<td>0.14</td>
<td>0.80</td>
<td></td>
</tr>
</tbody>
</table>

Table 10: Estimations on the alternative "C" in the form of a $\mathcal{F} \mathcal{P} \mathcal{P} \mathcal{F} \mathcal{B} \mathcal{S} \mathcal{S}$.

<table>
<thead>
<tr>
<th>$\lambda_C$</th>
<th>$\mathcal{C} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\lambda_C$</th>
<th>$\mathcal{C} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\lambda_C$</th>
<th>$\mathcal{C} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\mathcal{C} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_{-0.60}$</td>
<td>0.33</td>
<td>0.40</td>
<td>$\lambda_{-0.22}$</td>
<td>0.55</td>
<td>0.55</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.55}$</td>
<td>0.42</td>
<td>0.66</td>
<td>$\lambda_{-0.34}$</td>
<td>0.42</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.42}$</td>
<td>0.51</td>
<td>0.73</td>
<td>$\lambda_{-0.41}$</td>
<td>0.38</td>
<td>0.54</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.73}$</td>
<td>0.68</td>
<td>0.49</td>
<td>$\lambda_{-0.10}$</td>
<td>0.32</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.62}$</td>
<td>0.89</td>
<td>0.50</td>
<td>$\lambda_{-0.27}$</td>
<td>0.13</td>
<td>0.42</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.80}$</td>
<td>0.91</td>
<td>0.43</td>
<td>$\lambda_{-0.31}$</td>
<td>0.12</td>
<td>0.30</td>
<td></td>
</tr>
</tbody>
</table>

Table 11: Estimations on the alternative "D" in the form of a $\mathcal{F} \mathcal{P} \mathcal{P} \mathcal{F} \mathcal{B} \mathcal{S} \mathcal{S}$.

<table>
<thead>
<tr>
<th>$\lambda_D$</th>
<th>$\mathcal{D} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\lambda_D$</th>
<th>$\mathcal{D} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\lambda_D$</th>
<th>$\mathcal{D} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
<th>$\mathcal{D} \left( e_{\sigma_i}^{(e)} \right)$ (u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_{-0.60}$</td>
<td>0.77</td>
<td>0.60</td>
<td>$\lambda_{-0.22}$</td>
<td>0.25</td>
<td>0.52</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.55}$</td>
<td>0.80</td>
<td>0.46</td>
<td>$\lambda_{-0.34}$</td>
<td>0.70</td>
<td>0.23</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.42}$</td>
<td>0.40</td>
<td>0.73</td>
<td>$\lambda_{-0.41}$</td>
<td>0.54</td>
<td>0.76</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.73}$</td>
<td>0.50</td>
<td>0.49</td>
<td>$\lambda_{-0.10}$</td>
<td>0.53</td>
<td>0.48</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.62}$</td>
<td>0.80</td>
<td>0.50</td>
<td>$\lambda_{-0.27}$</td>
<td>0.50</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>$\xi_{-0.80}$</td>
<td>0.70</td>
<td>0.74</td>
<td>$\lambda_{-0.31}$</td>
<td>0.40</td>
<td>0.58</td>
<td></td>
</tr>
</tbody>
</table>
Table 12: Qualitative comparison with existing models.

<table>
<thead>
<tr>
<th>Hybrid models</th>
<th>Weights</th>
<th>Bipolarity of parameters</th>
<th>Possibility fuzzy information</th>
<th>Membership</th>
<th>MCDM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soft sets [8]</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Bipolar soft sets [12]</td>
<td>×</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Fuzzy parameterized soft sets [19]</td>
<td>✓</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Fuzzy parameterized fuzzy soft sets [20]</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Fuzzy parameterized interval-valued fuzzy soft sets [48]</td>
<td>✓</td>
<td>×</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Possibility fuzzy soft sets [27]</td>
<td>×</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Possibility Pythagorean fuzzy soft sets [33]</td>
<td>×</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Possibility Pythagorean bipolar fuzzy soft sets [35]</td>
<td>×</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Possibility multifuzzy soft sets [36]</td>
<td>×</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Possibility m-polar fuzzy soft sets [34]</td>
<td>×</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Proposed fuzzy parameterized possibility fuzzy bipolar soft sets</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 13: Scores for Application 4.1 obtained by existing models.

<table>
<thead>
<tr>
<th>Hybrid models</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Rankings</th>
</tr>
</thead>
<tbody>
<tr>
<td>PFSSs [27]</td>
<td>0.5100</td>
<td>0.6679</td>
<td>0.5858</td>
<td>0.6427</td>
<td>B &gt; D &gt; C &gt; A</td>
</tr>
<tr>
<td>FPSSs [20]</td>
<td>0.4188</td>
<td>0.5485</td>
<td>0.4811</td>
<td>0.5278</td>
<td>B &gt; D &gt; C &gt; A</td>
</tr>
<tr>
<td>Proposed (FPFFPBSS)</td>
<td>0.4113</td>
<td>0.5460</td>
<td>0.4164</td>
<td>0.5283</td>
<td>B &gt; D &gt; C &gt; A</td>
</tr>
</tbody>
</table>

Figure 1: Scores comparison of PFSSs [27] and FPSSs [20] with proposed \(FPFFPBSS\).
\[
\delta \left( \tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2 \right) = \mathcal{M} \left( \tilde{\mathbf{P}}^{(e^{-\omega_1(\xi)})} (u), \tilde{\mathbf{P}}^{(e^{-\sigma_1(\xi)})} (u) \right) \\
\quad \times \mathcal{M} \left( \lambda^{(e^{-\omega_1(\xi)})}, \lambda^{(e^{-\sigma_1(\xi)})} \right) \\
\quad \times \mathcal{M} \left( \omega_1 (\epsilon), \sigma_1 (\epsilon) \right), \\
\delta \left( \tilde{\mathbf{Q}}_1, \tilde{\mathbf{Q}}_2 \right) = \mathcal{M}^* \left( \tilde{\mathbf{Q}}^{(e^{-\omega_1(\xi)})} (u), \tilde{\mathbf{Q}}^{(e^{-\sigma_1(\xi)})} (u) \right) \\
\quad \times \mathcal{M}^* \left( \delta^{(e^{-\omega_1(\xi)})}, \delta^{(e^{-\sigma_1(\xi)})} \right) \\
\quad \times \mathcal{M}^* \left( \omega_2 (\epsilon), \sigma_2 (\epsilon) \right), \\
(67)
\]

where
\[
\mathcal{M} \left( \tilde{\mathbf{P}}^{(e^{-\omega_1(\xi)})} (u), \tilde{\mathbf{P}}^{(e^{-\sigma_1(\xi)})} (u) \right) = 0.6393, \\
\mathcal{M} \left( \lambda^{(e^{-\omega_1(\xi)})}, \lambda^{(e^{-\sigma_1(\xi)})} \right) = 0.7978, \\
\mathcal{M} \left( \omega_1 (\epsilon), \sigma_1 (\epsilon) \right) = 0.8212, \\
\mathcal{M}^* \left( \tilde{\mathbf{Q}}^{(e^{-\omega_1(\xi)})} (u), \tilde{\mathbf{Q}}^{(e^{-\sigma_1(\xi)})} (u) \right) = 0.7714, \\
\mathcal{M}^* \left( \delta^{(e^{-\omega_1(\xi)})}, \delta^{(e^{-\sigma_1(\xi)})} \right) = 0.8029, \\
\mathcal{M}^* \left( \omega_2 (\epsilon), \sigma_2 (\epsilon) \right) = 0.6519. \\
(68)
\]

So, \( \delta \left( \tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2 \right) = 0.4188 \) and \( \delta \left( \tilde{\mathbf{Q}}_1, \tilde{\mathbf{Q}}_2 \right) = 0.4038. \)

Similarly,
\[
\delta \left( \tilde{\mathbf{Q}}_1, \tilde{\mathbf{Q}}_2 \right) = 0.5485, \\
\delta \left( \tilde{\mathbf{Q}}_1, \tilde{\mathbf{Q}}_2 \right) = 0.5436, \\
\delta \left( \tilde{\mathbf{Q}}_1, \tilde{\mathbf{Q}}_2 \right) = 0.4810, \\
\delta \left( \tilde{\mathbf{Q}}_1, \tilde{\mathbf{Q}}_2 \right) = 0.3518, \\
\delta \left( \tilde{\mathbf{Q}}_1, \tilde{\mathbf{Q}}_2 \right) = 0.5278, \\
\delta \left( \tilde{\mathbf{Q}}_1, \tilde{\mathbf{Q}}_2 \right) = 0.5287. \\
(69)
\]

From the above results, it is clear that two alternatives B and C are significantly similar to the ideal agricultural land; however, their ranking can be determined by taking average of both similarity measures for set of parameters and its "not set." So, the overall ranking is computed as \( B > D > C > A. \) Thus, the option "B" is most suitable for the cultivation of sugarcane.

5. Discussion

Fuzzy parameterized, possibility fuzzy, and bipolar soft environments are three different formats to deal with different types of datasets, but the developed model has ability to tackle all these three kinds of datasets accumulatively. We now compare our initiated model with existing models in both quantitative and qualitative formats. First, a qualitative comparative analysis is provided in Table 12.

Now to make a quantitative comparative analysis of the developed model with existing models, we have applied the methodologies of pre-existing FPSSs [20] and PFSSs [27] approaches to the dataset of explored application in Section 4.1. The computed results are displayed in Table 13 and Figure 1. One can easily observe from Figure 1 that by the implementation of proposed model and existing hybrid models, including PFSSs [27] and FPFSSs [20] on the Application 4.1, the object B is the best option, and the object A is the worst option. Considering more generalized dataset by the proposed method and obtained similar rankings by applying developed approach, and other exiting methods prove its validity. All in all, the proposed approach generalizes the previous approaches by considering bipolarity, parameter preference, and possibility. It makes this work more complete, reliable, and efficient.

6. Conclusions and Future Plans

By the critical analysis of soft computing hybrid models which have been proposed in the last two decades, one can easily observe a big increase in the number of authentic problem-solving techniques for different variants of datasets. For instance, these days, a natural extension of soft sets, namely, the BS set model, is emerging as a more efficient mathematical tool when combined with other mathematical tools such as fuzzy sets, rough sets, Pythagorean fuzzy sets, Fermatean fuzzy sets, q-rung orthopair fuzzy sets, m-polar fuzzy sets. All the fusions of BS sets with the above-discussed models cannot be used to deal with several practical situations involving FP-possibility fuzzy information. That is why the existing concepts and their limitations are the main motivations of this work. To overcome these, we have proposed the notion of a new hybrid model for soft computing called fuzzy parameterized possibility fuzzy bipolar soft sets (or \( \mathbb{FPFSS}_B \)) by integrating the BS sets and PFSSs. We have also investigated some fundamental theoretical properties and results of the developed hybrid model, which are complement, subset relation, union, and intersection. Moreover, we have studied two fundamental operations called the "AND" operation and the "OR" operation and described them with illustrative numerical examples. Further, we have developed two new algorithms based on the "AND" operation and "OR" operation between \( \mathbb{FPFSS}_B \). After that, we have employed the launched algorithms to solve a decision-making problem which explain a clear demonstration to prove the effectiveness and feasibility of the developed approaches and related results. Thereafter, we have presented the concepts of checking the similarity measures between two or more \( \mathbb{FPFSS}_B \), which are
supported by a daily-life application that is actually studied the performance of an agricultural land regarding different essential parameters for a certain crop. Finally, we have compared our developed model with some existing models to show its authenticity over them. For future works, our research can be extended to

(i) Fuzzy parameterized possibility fuzzy bipolar soft topology
(ii) Aggregation operators for fuzzy parameterized possibility fuzzy bipolar soft information
(iii) Fuzzy parameterized possibility fuzzy bipolar soft expert sets
(iv) Fuzzy parameterized possibility Pythagorean fuzzy bipolar soft sets
(v) Intuitionistic fuzzy parameterized possibility intuitionistic fuzzy bipolar soft sets

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that he has no conflicts of interest.

References