Hindawi
Journal of Mathematics
Volume 2023, Article ID 5035740, 8 pages
https://doi.org/10.1155/2023/5035740

Research Article

Homology of Warped Product Semi-Invariant Submanifolds of a Sasakian Space Form with Semisymmetric Metric Connection

Ibrahim Al-Dayel,[1] Meraj Ali Khan ,[1] and Mohammad Shuaib[2]

[1]*Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 65892, Riyadh 11566, Saudi Arabia*
[2]*Department of Mathematics, Aligarh Muslim University (AMU), Aligarh, India*

Correspondence should be addressed to Meraj Ali Khan; mskhan@imamu.edu.sa

Received 10 July 2023; Revised 9 September 2023; Accepted 22 September 2023; Published 10 October 2023

Academic Editor: Qingkai Zhao

This paper focuses on the investigation of semi-invariant warped product submanifolds of Sasakian space forms endowed with a semisymmetric metric connection. We delve into the study of these submanifolds and derive several fundamental results. Additionally, we explore the practical implications of our findings by applying them to the homology analysis of these submanifolds. Notably, we present a proof demonstrating the absence of stable currents for these submanifolds under a specific condition.

1. Introduction

The homology groups of a manifold provide an algebraic representation that captures important topological characteristics. These groups contain extensive topological information about the various components, voids, tunnels, and overall structure of the manifold. Consequently, homology theory finds numerous applications in diverse fields such as root construction, molecular anchoring, image segmentation, and genetic expression analysis. The relationship between submanifold theory and homological theory is widely recognized for its significance. In the seminal work by Federer and Fleming [1], it was demonstrated that any nontrivial integral homological group $H_p(M, Z)$ is connected through stable currents. Building upon this, Lawson and Simons [2] extended the investigation to submanifolds of a sphere, where they established that the existence of an integral current is precluded under a pinching condition imposed on the second fundamental form. Notably, Leung [3] and Xin [4] expanded the scope of these results from spheres to Euclidean space. Additionally, Zhang [5] conducted a study on the homology of tori, further contributing to this line of research. In a recent development, Liu and Zhang [6] presented a proof demonstrating that stable integral currents cannot exist for specific types of hypersurfaces in Euclidean spaces.

The examination of warped products in submanifold theory was initiated by Chen [7]. Chen introduced the concept of CR-warped product submanifolds within the framework of almost Hermitian manifolds. He also provided an approximation for the norm of the second fundamental form by incorporating a warping function into the expressions. This pioneering work by Chen served as an inspiration for further research. Expanding on Chen's ideas, Hasegawa and Mihai [8] explored the contact form associated with these submanifolds. They derived a similar approximation for the second fundamental form in the context of contact CR-warped product submanifolds immersed in Sasakian space forms. In a related study [9], it was concluded that the homology groups of a contact CR-warped product submanifold, immersed in an odd-dimensional sphere, are trivial. This result was attributed to the nonexistence of stable integral currents and the vanishing of homology. Taking a step forward, Sahin [10, 11] made notable progress by demonstrating that CR-warped product submanifolds in both R^n and S^6 yield identical outcomes. However, different scholars have arrived at distinct findings concerning the topological and differentiable structures of submanifolds when imposing specific constraints on the second fundamental form [4, 6, 9, 12, 13].

Nevertheless, multiple researchers have arrived at contrasting conclusions regarding the topological and differentiable characteristics of submanifolds through the utilization of advanced theories such as submanifold theory and soliton theory, among others [6, 12, 14–18]. Further inspiration for our work can be gleaned from the papers cited as references [5, 9, 19–26].

The concept of a semisymmetric linear connection on a Riemannian manifold was initially proposed by Friedmann and Schouten [27]. Subsequently, Hayden [28] defined a semisymmetric connection as a linear connection ∇ existing on an n-dimensional Riemannian manifold (M, g), where the torsion tensor T satisfies $T(\omega_1, \omega_2) = \pi(\omega_2)\omega_1 - \pi(\omega_1)\omega_2$. Here, π represents a 1-form, and $\omega_1, \omega_2 \in \mathrm{TM}$. Further exploration of semisymmetric connections was undertaken by Yano [29], who investigated semisymmetric metric connections and analyzed their properties. It was demonstrated that a conformally flat Riemannian manifold possessing a semisymmetric connection exhibits a vanishing curvature tensor. Additionally, Sular and Ozgur [30] focused on investigating warped product manifolds equipped with a semisymmetric metric connection. They specifically considered Einstein warped product manifolds featuring a semisymmetric metric connection. Their work in [24] also provided additional insights into warped product manifolds with a semisymmetric metric connection. Motivated by these previous studies, our current interest lies in examining the influence of a semisymmetric metric connection on semi-invariant warped product submanifolds and their homology within a Sasakian space form.

2. Preliminaries

Suppose $(\overline{M}, g)$ is an odd-dimensional Riemannian manifold. In that case, $\overline{M}$ is considered an almost contact metric manifold if there exists a tensor field ϕ of type $(1, 1)$ and a global vector field ξ defined on $\overline{M}$ such that the following conditions are satisfied:

$$\begin{gathered}\phi^2\omega_1 = -\omega_1 + \eta(\omega_1)\xi, g(\omega_1, \xi) = \eta(\omega_1),\\ g(\phi\omega_1, \phi\omega_2) = g(\omega_1, \omega_2) - \eta(\omega_1)\eta(\omega_2),\end{gathered} \tag{1}$$

where η represents the dual 1-form of ξ. It is widely recognized that an almost contact metric manifold is classified as a Sasakian manifold if and only if the following conditions hold:

$$\left(\overline{\overline{\nabla}}_{\omega_1}\phi\right)\omega_2 = g(\omega_1, \omega_2)\xi - \eta(\omega_2)\omega_1. \tag{2}$$

It is straightforward to observe on a Sasakian manifold $\overline{M}$ that

$$\overline{\overline{\nabla}}_{\omega_1}\xi = -\phi\omega_1, \tag{3}$$

where $\omega_1, \omega_2 \in \mathrm{T}\overline{\mathrm{M}}$ and $\overline{\overline{\nabla}}$ is the Riemannian connection with respect to g.

Now, defining a connection $\overline{\nabla}$ as

$$\overline{\nabla}_{\omega_1}\omega_2 = \overline{\overline{\nabla}}_{\omega_1}\omega_2 + \eta(\omega_2)\omega_1 - g(\omega_1, \omega_2)\xi, \tag{4}$$

such that $\overline{\nabla}g = 0$ for any $\omega_1, \omega_2 \in \mathrm{TM}$, where $\overline{\overline{\nabla}}$ is the Riemannian connection with respect to g. The connection $\overline{\nabla}$ is semisymmetric because $T(\omega_1, \omega_2) = \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2$. Using (4) in (2), we have

$$\left(\overline{\nabla}_{\omega_1}\phi\right)\omega_2 = g(\omega_1, \omega_2)\xi - g(\omega_1, \phi\omega_2)\xi - \eta(\omega_2)\omega_1 - \eta(\omega_2)\phi\omega_1, \tag{5}$$

and

$$\overline{\nabla}_{\omega_1}\xi = \omega_1 - \eta(\omega_1)\xi - \phi\omega_1. \tag{6}$$

A Sasakian manifold $\overline{M}$ is referred to as a Sasakian space form when it possesses a constant ϕ-holomorphic sectional curvature denoted by c, and it is denoted as $\overline{M}(c)$. The curvature tensor $\overline{R}$ of a Sasakian space form $\overline{M}(c)$ with a semi-symmetric metric connection can be expressed as follows [32]:

$$\begin{aligned}\overline{R}(\omega_1, \omega_2, \omega_3, \omega_4) = {} & \frac{c+3}{4}\{g(\omega_2, \omega_3)g(\omega_1, \omega_4) - g(\omega_1, \omega_3)g(\omega_2, \omega_4)\}\\ & + \frac{c-1}{4}\{\eta(\omega_1)\eta(\omega_3)g(\omega_2, \omega_4) - \eta(\omega_2)\eta(\omega_3)g(\omega_1, \omega_4)\\ & + g(\omega_1, \omega_3)\eta(\omega_2)\eta(\omega_4) - g(\omega_2, \omega_3)\eta(\omega_1)\eta(\omega_4)\\ & + g(\phi\omega_2, \omega_3)g(\phi\omega_1, \omega_4) + g(\phi\omega_3, \omega_1)g(\phi\omega_2, \omega_4)\\ & - 2g(\phi\omega_1, \omega_2)g(\phi\omega_3, \omega_4)\} + \beta(\omega_1, \omega_3)g(\omega_2, \omega_4)\\ & - \beta(\omega_2, \omega_3)g(\omega_1, \omega_4) + \beta(\omega_2, \omega_4)g(\omega_1, \omega_3)\\ & - \beta(\omega_1, \omega_4)g(\omega_2, \omega_3),\end{aligned} \tag{7}$$

for all $\omega_1, \omega_2, \omega_3, \omega_4 \in T\overline{M}$, where $\beta(\omega_1, \omega_2) = (\overline{\nabla}_{\omega_1}\omega_2 - \eta(\omega_1)\eta(\omega_2) + (1/2)g(\omega_1, \omega_2))$.

By performing a routine calculation, we can derive the Gauss and Weingarten formulas for a submanifold M that is isometrically immersed in a differentiable manifold $\overline{M}$ equipped with a semisymmetric metric connection. These formulas are as follows: $\overline{\nabla}_{\omega_1}\omega_2 = \nabla_{\omega_1}\omega_2 + h(\omega_1, \omega_2)$ and $\overline{\nabla}_{\omega_1}N = -A_N\omega_1 + \nabla^{\perp}_{\omega_1}N + \eta(N)\omega_1$, where ∇ is the induced semisymmetric metric connection on M, $N \in T^{\perp}M$ h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$, and A_N is the shape operator. The second fundamental form h and the shape operator are associated by the following formula:

$$g(h(\omega_1, \omega_2), N) = g(A_N\omega_1, \omega_2). \tag{8}$$

For the vector fields $\omega_1 \in TM$ and $\omega_3 \in T^{\perp}M$, we have the following decomposition:

$$\phi\omega_1 = P\omega_1 + F\omega_1, \tag{9}$$

and

$$\phi\omega_3 = t\omega_3 + f\omega_3, \tag{10}$$

where $P\omega_1 (t\omega_3)$ and $F\omega_1 (f\omega_3)$ are the tangential and normal parts of $\phi\omega_1 (\phi\omega_3)$, respectively.

Let R denote the Riemannian curvature tensor of the submanifold M. In the case of a semisymmetric connection, the equation of Gauss can be expressed as follows:

$$\begin{aligned}\overline{R}(\omega_1, \omega_2, \omega_3, \omega_4) = R(\omega_1, \omega_2, \omega_3, \omega_4) \\ - g(h(\omega_1, \omega_4), h(\omega_2, \omega_3)) \\ + g(h(\omega_2, \omega_4), h(\omega_1, \omega_3)),\end{aligned} \tag{11}$$

for $\omega_1, \omega_2, \omega_3, \omega_4 \in TM$.

Sular and Oẑgur investigated the warped product structures of the form $M_1 \times_f M_2$ in their work [30]. They examined these structures under the assumption of a semisymmetric metric connection and an associated vector field P on the product manifold $M_1 \times_f M_2$. Here, M_1 and M_2 denote Riemannian manifolds, and f represents a positive differentiable function defined on M_1 known as the warping function. In this context, we present several key findings from [30] in the form of the following lemma. These results hold significant relevance for the subsequent investigation.

Lemma 1. *Given a warped product manifold $M_1 \times_f M_2$ with a semisymmetric metric connection $\overline{\nabla}$, the following results hold:*

(1) If the associated vector field $P \in TM_1$, then

$$\begin{aligned}\overline{\nabla}_{\omega_1}\omega_3 &= \frac{\omega_1 f}{f}\omega_3, \\ \overline{\nabla}_{\omega_3}\omega_1 &= \frac{\omega_1 f}{f}\omega_3 + \pi(\omega_1)\omega_3.\end{aligned} \tag{12}$$

(2) If $P \in TN_2$, then

$$\begin{aligned}\overline{\nabla}_{\omega_1}\omega_3 &= \frac{\omega_1 f}{f}\omega_3, \\ \overline{\nabla}_{\omega_3}\omega_1 &= \frac{\omega_1 f}{f}\omega_3,\end{aligned} \tag{13}$$

where $\omega_1 \in TM_1, \omega_3 \in TM_2$, and π is the 1-form associated with the vector field P.

Suppose R is the curvature tensor of the semisymmetric metric connection $\overline{\nabla}$, then from equation (5) and part (ii) of Lemma 3.2 of [30], we derive

$$\begin{aligned}R(\omega_1, \omega_3)\omega_2 = \frac{H^f(\omega_1, \omega_2)}{f} + \frac{Pf}{f}g(\omega_1, \omega_2)\omega_3 \\ + \pi(P)g(\omega_1, \omega_2)\omega_3 + g\left(\omega_2, \overline{\nabla}_{\omega_1}P\right)\omega_3 \\ - \pi(\omega_1)\pi(\omega_2)\omega_3,\end{aligned} \tag{14}$$

where $\omega_1, \omega_2 \in TM_1, \omega_3 \in TM_2, P \in TM_1$, and H^f is the Hessian of the warping function f.

In (4), we defined the semisymmetric connection by setting $P = \xi$. Thus, for a warped product submanifold $M = M_1 \times_f M_2$ of a Riemannian manifold $\overline{M}$, we can derive the following relation using part (i) of Lemma 1:

$$\nabla_{\omega_1}\omega_3 = \omega_1 \ln f\omega_3, \tag{15}$$

and

$$\nabla_{\omega_3}\omega_1 = \omega_1 \ln f\omega_3 + \eta(\omega_1)\omega_3. \tag{16}$$

In addition, the (14) with (6) yields

$$\begin{aligned}R(\omega_1, \omega_3)\omega_2 = \frac{H^f(\omega_1, \omega_2)}{f}\omega_3 + \frac{\xi f}{f}g(\omega_1, \omega_2)\omega_3 + 2g(\omega_1, \omega_2)\omega_3 - 2\eta(\omega_1)\eta(\omega_2)\omega_3 \\ - g(\omega_2, \phi\omega_1)\omega_3,\end{aligned} \tag{17}$$

for $\xi, \omega_1, \omega_2 \in TM_1$, and $\omega_3 \in TM_2$.

3. Semi-Invariant Warped Product Submanifolds and Their Homology

In 1981, Bejancu [32] gave the idea of semi-invariant submanifolds in an almost contact metric manifold. An $m-$dimensional Riemannian submanifold M of a Sasakian manifold $\overline{M}$ is called a semi-invariant submanifold if ξ is tangent to M, and there exists on M a differentiable distribution D: $x \longrightarrow D_x \subset T_x M$ such that D_x is invariant under ϕ. The orthogonal complementary distribution $D_x^{\perp}$ of D_x on M is anti-invariant, i.e., $\phi D^{\perp} \subseteq T_x^{\perp} M$, where $T_x M$ and $T_x^{\perp} M$ are the tangent space and normal space at $x \in M$. In [8], Hesigawa and Mihai considered the warped product submanifold of the type $M_T \times_f M_{\perp}$ of a Sasakian manifold $\overline{M}$, where M_T is an invariant submanifold and $N_{\perp}$ is an anti-invariant submanifold, and $\xi \in \mathrm{TN}_T$. They called these types of submanifolds to contact CR-submanifold and provided some basic results. Throughout this study, we consider the warped products of the type $N_T \times_f N_{\perp}$ of a Sasakian manifold $\overline{M}$ with semisymmetric metric connection and $\xi \in \mathrm{TN}_T$. We refer to these submanifolds as semi-invariant warped product submanifolds.

Now, we start with the following initial results:

Lemma 2. *Let $M = M_T \times_f M_{\perp}$ be a semi-invariant warped product submanifold of a Sasakian manifold $\overline{M}$ endowed with a semisymmetric metric connection, then*

(i) $g(h(\phi\omega_1, \omega_3), \phi\omega_4) = \omega_1 lnf\, g(\omega_3, \omega_4) + \eta(\omega_1) g(\omega_3, \omega_4)$,

(ii) $g(h(\omega_1, \omega_2), \phi\omega_3) = 0$,

(iii) $\xi\, lnf = 0$,

$$\begin{aligned} &\forall \omega_1, \omega_2 \in \mathrm{TN}_T, \\ &\omega_3, \omega_4 \in \mathrm{TN}_{\perp}, \\ &\xi \in \mathrm{TN}_T. \end{aligned} \tag{18}$$

Proof. Using the Gauss formula and (5), we get

$$g(h(\phi\omega_1, \omega_3), \phi\omega_4) = g(\overline{\nabla}_{\omega_3}\phi\omega_1, \phi\omega_4), = g(\overline{\nabla}_{\omega_3}\omega_1, \omega_4). \tag{19}$$

Now, by formula (16), we have

$$\begin{aligned} g(h(\phi\omega_1, \omega_3), \phi\omega_4) &= g(\nabla_{\omega_3}\omega_1, \omega_4) \\ &= \omega_1 \mathrm{lnf}\, g(\omega_3, \omega_4) + \eta(\omega_1) g(\omega_3, \omega_4), \end{aligned} \tag{20}$$

this is part (i). Again using (5) and (15), and the Gauss formula, part (ii) is proved immediately. Now, by equation $\nabla_{\omega_3}\xi = \omega_3 - \eta(\omega_3) - P\omega_3$, and applying (16), we have $\xi\,\mathrm{lnf} + \eta(\xi)\omega_3 = \omega_3$ or $\xi\,\mathrm{lnf} = 0$, which is the part (iii) of the Lemma.

Next, we investigate the existence of stable currents on semi-invariant warped product submanifolds. Specifically, we establish a proof demonstrating that under certain conditions, stable currents do not exist. In this context, we present the well-known results established by Simons, Xin, and Lang. □

Lemma 3 (see [2, 6]). *Consider a compact submanifold M^n of dimension n immersed in a space form $\overline{M}(c)$ with positive curvature c. If the second fundamental form satisfies the following inequality,*

$$\sum_{i=1}^{n_1} \sum_{s=n_1+1}^{n} \left(2\left\|h(u_i, u_j)\right\|^2 - g(h(u_i, u_i), h(u_i, u_s))\right) < n_1 n_2 c, \tag{21}$$

where n_1 and n_2 are positive integers satisfying $n_1 + n_2 = n$, $\{u_1, \ldots, u_n\}$ is an orthonormal basis in $T_x M$ for any $x \in M$, h denotes the second fundamental form of M, g represents the metric tensor, and $\|h(u_i, u_j)\|$ is the norm of the second fundamental form evaluated at u_i and u_j. Under these conditions, it can be proven that there are no stable currents in M^n. Moreover, it is observed that $\widetilde{H}_{n_1}(M^n, \mathbb{Z}) = 0$ and $\widetilde{H}_{n_2}(M^n, \mathbb{Z}) = 0$, where $\widetilde{H}_j(M^n, \mathbb{Z})$ denotes the j-th homology group of M^n with integer coefficients.

Now, we have the following theorem.

Theorem 4. *Let $M^{n_1+n_2+1} = M_T^{n_1+1} \times_f M_{\perp}^{n_2}$ be a compact semi-invariant warped product submanifold of a Sasakian space form $M^{2((n_1/2)+n_2)+1}(c)$, with semisymmetric metric connection. If the following inequality holds,*

$$\begin{aligned} \Delta f - \|\nabla f\|^2_{D-\langle\xi\rangle} + \frac{n_2}{f}\|\nabla f\|^2 + \eta(\nabla f) n_2 &> \frac{2}{f}(1 + n_1) + \frac{c}{4}(2 - 3n_1) f + \frac{11}{4} n_1 f + \frac{3}{2} f \\ &+ \frac{f(n_1 + 1)}{n_2} \sum_{k=1}^{n_2} \beta(n_k, n_k) + f \sum_{i=1}^{n_1} \beta(z_i, z_i), \end{aligned} \tag{22}$$

then the $(n_1+1)-$ *stable currents are absent in* $M^{n_1+1+n_2}$. *In addition,* $H_{n_1+1}(M^n, Z) = 0, H_{n_2}(M^n, Z) = 0,$ *where* $H_j(M, Z)$ *is the* $j-$*th homology group of* M *and* n_1+1, n_2 *are the dimensions of the invariant submanifold* $M_T^{n_1+1}$ *and the anti-invariant submanifold* $M_\perp^{n_2}$, *respectively.*

Proof. Let $\{\xi, z_1, \ldots, z_{n_1}, s_1, \ldots, s_{n_2}\}$ be an orthonormal basis of TM, such that $\{\xi, z_1, \ldots, z_{n_1}\}$ is an orthonormal basis of TM_T and $\{s_1, \ldots, s_{n_2}\}$ be the basis of $\mathrm{TM}_\perp^{n_2}$. Then, by equation (7) for unit odd-dimensional sphere and (11), we have

$$\begin{aligned}\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2} g(R(z_i, s_k)z_i, s_k) &= -(n_1+1)n_2\frac{c+3}{4} - \frac{c-1}{4}n_2 - (n_1+1)\sum_{k=1}^{n_2}\beta_1(s_k, s_k)\\ &\quad - n_2\sum_{i=1}^{n_1}\beta(z_i, z_i) + \sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2} -g(h(s_k, s_k), h(z_i, z_i)) + \|h(z_i, s_k)\|^2.\end{aligned} \tag{23}$$

Therefore, we get

$$\begin{aligned}\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2} g(R(z_i, n_k)z_i, s_k) + \|h(z_i, s_k)\|^2 &= -(n_1+1)n_2\frac{c+3}{4} - \frac{c-1}{4}n_2 - (n_1+1)\sum_{k=1}^{n_2}\beta_1(s_k, s_k)\\ &\quad - n_2\sum_{i=1}^{n_1}\beta(z_i, z_i) + \sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2} -g(h(s_k, s_k), h(z_i, z_i))\\ &\quad + 2\|h(z_i, s_k)\|^2.\end{aligned} \tag{24}$$

On the other hand, from (17) and part (iii) of Lemma 2, we have

$$R(z_i, s_k)z_i = \frac{H^f(z_i, z_i)}{f}s_k + 2g(z_i, z_i)s_k - 2\eta(z_i)\eta(z_i)s_k, \tag{25}$$

where H^f is the Hessian form of f. Thus, we derive

$$\begin{aligned}\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2} g(R(z_i, s_k)z_i, s_k) &= \frac{n_2}{f}\sum_{i=1}^{n_1+1} g\left(\nabla_{z_i}\nabla f, z_i\right) + 2(n_1+1)n_2 - 2n_2\\ &= \frac{n_2}{f}\sum_{i=1}^{n_1+1} g(\nabla z_i \nabla f, z_i) + 2n_1 n_2.\end{aligned} \tag{26}$$

Putting (26) in (24), we have

$$\begin{aligned}
&\frac{n_2}{f}\sum_{i=1}^{n_1+1} g\left(\nabla_{z_i}\nabla f, z_i\right) + 2n_1 n_2 + \sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left\|h(z_i, s_k)\right\|^2 \\
&= -(n_1+1)n_2\frac{c+3}{4} - n_2\frac{c-1}{4} \\
&\quad - (n_1+1)\sum_{k=1}^{n_2}\beta(s_k, n_k) - n_2\sum_{i=1}^{n_1}\beta(z_i, z_i) \\
&\quad + \sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left(\left(-g\left(h(s_k, s_k), h(z_i, z_i)\right) + 2\left\|h(z_i, s_k)\right\|^2\right)\right).
\end{aligned} \tag{27}$$

Since

$$\Delta f = -\mathrm{div}\nabla f = -\sum_{i=1}^{n_1+1} g\left(\nabla_{z_i}\nabla f, z_i\right) - \sum_{k=1}^{n_2} g\left(\nabla_{s_k}\nabla f, s_k\right), \tag{28}$$

using (16), we derive

$$\Delta f = -\sum_{i=1}^{n_1+1} g\left(\nabla_{z_i}\nabla f, z_i\right) - \frac{n_2}{f}\|\nabla f\|^2 - \eta(\nabla f)n_2. \tag{29}$$

Using (29) in (27), we arrive at

$$\begin{aligned}
\Delta f - \frac{f}{n_2}\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left\|h(z_i, s_k)\right\|^2 &= (n_1+2)f\frac{c}{4} + \frac{11}{4}\mathrm{fn}_1 + \frac{1}{2}f \\
&\quad + \frac{f(n_1+1)}{n_2}\sum_{k=1}^{n_2}\beta(s_k, s_k) + f\sum_{i=1}^{n_1}\beta(z_i, z_i) \\
&\quad + \frac{f}{n_2}\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left(g\left(h(s_k, s_k), h(z_i, z_i)\right)\right. \\
&\quad - 2\left\|h(z_i, s_k)\right\|^2 - \frac{n_2}{f}\|\nabla f\|^2 - \eta(\nabla f)n_2,
\end{aligned} \tag{30}$$

or

$$\begin{aligned}
&\Delta f - \frac{f}{n_2}\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left\|h(z_i, s_k)\right\|^2 = \frac{c}{4}(n_1+2)f + \frac{11}{4}n_1 f + \frac{1}{2}f \\
&\quad + \frac{f(n_1+1)}{n_2}\sum_{k=1}^{n_2}\beta(s_k, s_k) + f\sum_{i=1}^{n_1}\beta(z_i, z_i) \\
&\quad - \frac{f}{n_2}\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left(2\left\|h(z_i, s_k)\right\|^2\right. \\
&\quad - g\left(h(s_k, s_k), h(z_i, z_i)\right) - \frac{n_2}{f}\|\nabla f\|^2 - \eta(\nabla f)n_2.
\end{aligned} \tag{31}$$

Moreover, if $\{s_1, \ldots, s_k\}$ is an orthonormal frame of $N_\perp$, then $\{\phi s_1, \ldots, \phi s_k\}$ is the orthonormal frame of $T^\perp M$, then

$$\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left\|h(z_i, s_k)\right\|^2 = \sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2} g\left(h(z_i, s_k), \phi s_k\right)^2. \tag{32}$$

By using part (i) of Lemma 2, we conclude

$$\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left\|h\left(z_i,s_k\right)\right\|^2=\|\nabla \ln f\|^2_{D-\langle\xi\rangle}n_2+n_2, \tag{33}$$

or

$$\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left\|h\left(z_i,s_k\right)\right\|^2=\frac{n_2}{f}\|\nabla f\|^2_{D-\langle\xi\rangle}+n_2. \tag{34}$$

Putting the above value in (31), we find

$$\begin{aligned}\frac{n_2}{f}\Delta f-\frac{n_2}{f}\|\nabla f\|^2_{D-\langle\xi\rangle}&=\frac{c}{4}n_2\left(n_1+2\right)+\frac{11}{4}n_1n_2+\frac{3}{2}n_2+\left(n_1+1\right)\sum_{k=1}^{n_2}\beta\left(s_k,s_k\right)\\&\quad+n_2\sum_{i=1}^{n_1}\beta\left(z_i,z_i\right)-\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left\{2\left\|h\left(z_i,s_k\right)\right\|^2-g\left(h\left(s_k,s_k\right),h\left(z_i,z_i\right)\right)\right\}\\&\quad-\frac{{n_2}^2}{f^2}\|\nabla f\|^2-\eta\left(\nabla f\right)\frac{q^2}{f},\end{aligned} \tag{35}$$

or

$$\begin{aligned}&\sum_{i=1}^{n_1+1}\sum_{k=1}^{n_2}\left\{2\left\|h\left(z_i,s_k\right)\right\|^2-g\left(h\left(s_k,s_k\right),h\left(z_i,z_i\right)\right)\right\}-n_1n_2c\\&=-\frac{n_2}{f}\Delta f+\frac{n_2}{f}\|\nabla f\|^2_{D-\langle\xi\rangle}-\frac{n_2^2}{f^2}\|\nabla f\|^2-\eta\left(\nabla f\right)\frac{q^2}{f}\\&\quad+\frac{c}{4}n_2\left(2-3n_1\right)+\frac{11}{4}n_1n_2+\frac{3}{2}n_2\\&\quad+\left(n_1+1\right)\sum_{k=1}^{n_2}\beta\left(s_k,s_k\right)+n_2\sum_{i=1}^{n_1}\beta\left(z_i,z_i\right).\end{aligned} \tag{36}$$

The proof is derived from (36) and Lemma 3. □

4. Conclusion

This paper has provided an in-depth investigation of semi-invariant warped product submanifolds of Sasakian space forms equipped with a semisymmetric metric connection. Through our study, we have derived several fundamental results that contribute to the understanding of these submanifolds. Furthermore, we have explored the practical implications of our findings by applying them to the homology analysis of these semi-invariant warped product submanifolds. Our analysis has revealed important insights into the homology properties of these submanifolds within the context of Sasakian space forms. One notable result we established is the proof of the absence of stable currents for these submanifolds under a specific condition. This finding has significant implications for the understanding of the geometric and topological behavior of semi-invariant warped product submanifolds in Sasakian space forms. Overall, this research contributes to the broader understanding of semi-invariant warped product submanifolds and their interaction with Sasakian space forms endowed with a semi-symmetric metric connection. The outcomes of this study pave the way for further research in this area, as well as potential applications in related fields such as differential geometry and topology [33–36].

Data Availability

No data are used in this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RP23003).

References

[1] H. Federer and W. Fleming, "Normal and integral currents," *Annals of Mathematics*, vol. 72, pp. 458–520, 1960.

[2] H. B. Lawson and J. Simons, "On stable currents and their application to global problems in real and complex geometry," *Annals of Mathematics*, vol. 98, no. 3, pp. 427–450, 1973.

[3] P. F. Leung, "On a relation between the topology and the intrinsic and extrinsic geometries of a compact submanifold," *Proceedings of the Edinburgh Mathematical Society*, vol. 28, no. 3, pp. 305–311, 1985.

[4] Y. L. Xin, "An application of integral currents to the vanishing theorems," *Scientia Sinica*, vol. 27, pp. 233–241, 1984.

[5] X. S. Zhang, "Nonexistence of stable currents in submanifolds of a product of two spheres," *Bulletin of the Australian Mathematical Society*, vol. 44, no. 2, pp. 325–336, 1991.

[6] J. Liu and Q. Y. Zhang, "Non-existence of stable currents in submanifolds of the Euclidean spaces," *Journal of Geometry*, vol. 96, no. 1-2, pp. 125–133, 2009.

[7] B. Y. Chen, "Geometry of warped product CR-submanifolds in Kaehler manifolds," *Monatshefte for Mathematik*, vol. 133, no. 3, pp. 177–195, 2001.

[8] I. Hasegawa and I. Mihai, "Contact CR-warped product submanifolds in Sasakian manifolds," *Geometriae Dedicata*, vol. 102, no. 1, pp. 143–150, 2003.

[9] B. Sahin and F. Sahin, "Homology of contact CR-warped product submanifolds of an odd-dimensional unit sphere," *Bulletin of the Korean Mathematical Society*, vol. 52, no. 1, pp. 215–222, 2015.

[10] F. Sahin, "On the topology of CR-warped product submanifolds," *International Journal of Geometric Methods in Modern Physics*, vol. 15, no. 2, Article ID 1850032, 2018.

[11] F. Sahin, "Homology of submanifolds of six dimensional sphere," *Journal of Geometry and Physics*, vol. 145, Article ID 103471, 2019.

[12] A. Ali, F. Mofarreh, C. Ozel, and W. A. M. Othman, "Homology of warped product submanifolds in the unit sphere and its applications," *International Journal of Geometric Methods in Modern Physics*, vol. 17, no. 8, Article ID 2050121, 2020.

[13] H. W. Xu and F. Ye, "Differentiable sphere theorems for submanifolds of positive K-th ricci curvature," *Manuscripta Mathematica*, vol. 138, no. 3-4, pp. 529–543, 2012.

[14] Y. Li, M. K. Gupta, S. Sharma, and S. K. Chaubey, "On ricci curvature of a homogeneous generalized matsumoto finsler space," *Mathematics*, vol. 11, p. 3365, 2023.

[15] Y. Li, S. Bhattacharyya, A. Azami, A. Saha, and S. K. Hui, "Harnack estimation for nonlinear, weighted, heat-type equation along geometric flow and applications," *Mathematics*, vol. 11, p. 2516, 2023.

[16] Y. Li and A. Caliskan, "Quaternionic shape operator and rotation matrix on ruled surfaces," *Aximos*, vol. 12, p. 486, 2023.

[17] Y. Li and E. Guler, "Hypersurfaces of revolution family supplying $\Delta\mathfrak{r} = \mathscr{A}\mathfrak{r}$ in pseudo-euclidean space $\mathbb{E}_7^3$," *AIMS Math*, vol. 8, no. 10, pp. 24957–24970, 2023.

[18] Y. Li, H. A. Kumara, M. S. Siddesha, and D. M. Naik, "Characterization of ricci almost soliton on lorentzian manifolds," *Symmetry*, vol. 15, p. 1175, 2023.

[19] Y. Li and E. Guler, "A hypersurfaces of revolution family in the five-dimensional pseudo-euclidean space $\mathbb{E}_2^5$," *Mathematics*, vol. 11, p. 3427, 2023.

[20] Y. Li, K. Eren, and S. Ersoy, "On simultaneous characterizations of partner-ruled surfaces in Minkowski 3-space," *AIMS Math*, pp. 22256–22273, 2023.

[21] Y. Li and M. Mak, "Framed natural mates of framed curves in euclidean 3-space," *Mathematics*, vol. 20233571 pages, 2013.

[22] S. Smale, "Generalized Poincares conjecture in dimensions greater than four," *Annals of Mathematics*, vol. 74, no. 2, pp. 391–406, 1961.

[23] B. Sahin, "Warped product pointwise semi-slant submanifolds of Kähler manifolds," *Portugaliae Mathematica*, vol. 70, no. 3, pp. 251–268, 2013.

[24] C. Özgür and Ozgur, "Warped warped products with a semi-symmetric non-metric Connectionroducts with a semi-symmetric non-metric connection," *Arabian Journal for Science and Engineering*, vol. 36, no. 3, pp. 461–473, 2011.

[25] H. W. Xu and E. T. Zhao, "Topological and differentiable sphere theorems for complete submanifolds," *Communications in Analysis and Geometry*, vol. 17, no. 3, pp. 565–585, 2009.

[26] T. Imai, "Notes on semi-symmetric metric connections," *Commemoration volumes for Prof. Dr. Akitsugu Kawaguchi's seventieth birthday, Vol. I. Tensor (N.S.)*, vol. 24, pp. 293–296, 1972.

[27] A. Friedmann and J. A. Schouten, "Über die Geometrie der halbsymmetrischen Übertragungen," *Mathematische Zeitschrift*, vol. 21, no. 1, pp. 211–223, 1924.

[28] H. A. Hayden, "Sub-spaces of a space with torsion," *Proceedings of the London Mathematical Society*, vol. 2–34, no. 1, pp. 27–50, 1932.

[29] K. Yano, "On semi-symmetric metric connections," *Revue Roumaine de Mathématique Pures et Appliquées*, vol. 15, pp. 1579–1586, 1970.

[30] S. Sular and C. Ozgur, "Warped products with a semi-symmetric metric connection," *Taiwanese Journal of Mathematics*, vol. 15, no. 4, pp. 1701–1719, 2011.

[31] Y. Wang, "Chen inequalities for submanifolds of complex space forms and Sasakian space forms with quarter symmetric connections," *International Journal of Geometric Methods in Modern Physics*, vol. 16, no. 8, Article ID 1950118, 2019.

[32] A. Bejancu, "CR submanifolds of a Kaehler manifold I," *Proceedings of the American Mathematical Society*, vol. 69, no. 1, pp. 135–142, 1978.

[33] B. Y. Chen and O. J. Garay, "Pointwise slant submanifolds in almost Hermitian manifolds," *Turkish Journal of Mathematics*, vol. 36, pp. 630–640, 2012.

[34] H. P. Fu and H. W. Xu, "Vanishing and topological sphere theorems for submanifolds in a hyperbolic space," *International Journal of Mathematics*, vol. 19, no. 7, pp. 811–822, 2008.

[35] T. Imai, "Hypersurfaces of a Riemannian manifold with semi-symmetric metric connection," *Tensor*, vol. 23, pp. 300–306, 1972.

[36] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore, 1984.