1. Introduction

The theory of bundles of infinitely near points was introduced in 1953 by Andre Weil in [1] and has become a subject of significant interest in differential geometry. A commutative, associative, unitary real algebra $A$ is called a Weil algebra if it is a finite-dimensional local algebra of the form $A = \mathbb{R} \oplus \mathcal{M}$ (i.e., $\dim(A/\mathcal{M}) = 1$) where $\mathcal{M}$ is its only maximal ideal (see [2], from page 625). As an example, one defines the algebra $D = \mathbb{R}[x]/(x^2)$ of dual numbers whose maximal ideal is $\mathcal{M} = x\mathbb{R}$.

Let $M$ be a smooth manifold and $A$ a Weil algebra. We discuss the differential forms in the Weil bundles $(M^A, \pi, M)$, and we established a link between differential forms in $M^A$ and $M$ as well as their cohomology. We also discuss the cohomology in the general theory. This approach consists of sending a geometric structure from $M$ to $M^A$ (regarded as an $A$–manifold, i.e., $\dim_r M^A = \dim_r M$) as developed in [5–10] where the authors studied the prolongations of vector fields and differential forms, linear connections, symplectic structures, and pseudo-Riemannian structures. Many directions have been developed from the last decades for these manifolds such as affine manifold structures studied in [2] and principal fiber bundles studied in [11], and nice applications to Grassmann bundles can be found in [12].

Instead of regarding $M^A$ as an $A$–manifold, we discuss in this paper differential forms and de Rham cohomology on $M^A$ without any prolongation. This approach consists of regarding $M^A$ as an $\mathbb{R}$–manifold (i.e., $\dim_r M^A = \dim(A) \cdot \dim(M)$) (see [13]). More specifically, if $\Omega^k(M^A, \mathbb{R})$ denotes the space of $k$–forms in $M^A$, we introduce the map

$$D: \Omega^k(M^A, \mathbb{R}) \longrightarrow \Omega^{k+1}(M^A, \mathbb{R}),$$

sending a $k$–form from $M^A$ to a $k$–form in $M$. Conversely, we introduce the map

$$C: \Omega^k(M) \longrightarrow \Omega^k(M^A, \mathbb{R}),$$

sending a $k$–form from $M$ to a $k$–form in $M^A$. These two maps are central and enable to extend the de Rham complex in $M^A$ by introducing the operator

$$\tilde{d} = C \circ d D: \Omega^k(M^A, \mathbb{R}) \longrightarrow \Omega^{k+1}(M^A, \mathbb{R}),$$

where $d$ is the de Rham operator in $M$.
in $M^A$ where $d: \Omega^k(M) \to \Omega^{k+1}(M)$ is the de Rham operator in $M$, and we prove that $d$ defines indeed the de Rham cohomology operator in $M^A$.

2. Basic Notions

Definition 1. (Weil functor)

Let $A$ be a Weil algebra with maximal ideal $\mathfrak{m}$, $M$ be a $C^\infty$–smooth manifold. Denote by $\text{Mfd}$ the category of smooth manifolds. By the Weil functor of $M$, we mean a functor $T^A: \text{Mfd} \to \text{Mfd}$ such that

1. for any $M \in \text{Ob}(\text{Mfd})$,
\[
T^AM = \bigcup_{x \in M} M^A_x,
\]
with projection $T^AM \to M$ and fibers $M^A_x$ for any $x \in M$.

2. for any $M, N \in \text{Ob}(\text{Mfd})$ and $f: M \to N$, we have $T^A f: T^AM \to T^AN$ such that for any $x \in M, T^A f(M^A_x) \subset N^A_{f(x)}$ and the following diagram commutes
\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\pi_\text{M} & \downarrow & \pi_\text{N} \\
M^A & \xrightarrow{T^A f} & N^A
\end{array}
\]

Remark 1

1. Denote by $T^AM = M^A$ and $T^A\mathbb{R}^n = \mathbb{A}^n$.

2. If $f: M \to \mathbb{R}$ is a function, then
\[
f^A: M^A \to A,
\]
such that for any $\varepsilon \in M^A_x$, we have
\[
f^A(\varepsilon) = \varepsilon(f) = f(\varepsilon) + t \varepsilon \text{ for } t \in \mathfrak{m}.
\]

3. Claim: if $\tilde{f}: M^A \to \mathbb{R}$ is a function and $\varepsilon_1, \varepsilon_2 \in M^A_x$, then $\tilde{f}(\varepsilon_1) = \tilde{f}(\varepsilon_2)$. This is a very important claim and will be widely used throughout this paper.

4. Let $\{x_1, \ldots, x_n\}$ be a basis for $A$ and $M$ be a manifold such that $\{x_1, \ldots, x_n\}$ is a system of local coordinate around $x \in U \subset M$, then there exists $\varepsilon \in \pi^{-1}(U) \subset M^A_x$ and functions $x_{i,j}: \pi^{-1}(U) \to \mathbb{R}$, $i = 1, \ldots, n; j = 1, \ldots, m$ such that for
\[
\varepsilon(x_i) = \sum_{j=1}^m x_{i,j}(\varepsilon) \cdot \alpha_j \text{ if } i = 1, \ldots, n.
\]
The functions $\{x_{1,1}, \ldots, x_{n,m}\}$ are a system of local coordinate around $x \in \pi^{-1}(U) \subset M^A_x$. It is clear that $\dim_\mathbb{A}(M^A) = n \cdot m$.

5. If $M$ and $N$ are smooth manifolds and $h: M \to N$ a smooth map (resp. diffeomorphism) then
\[
h^A: M^A \to N^A, \varepsilon \to h^A(\varepsilon).
\]
such that $\forall \phi \in C^\infty(N), h^A(\varepsilon)(\phi) = \varepsilon(h^A(\varepsilon))$ is a smooth map (resp. diffeomorphism).

(6) Given a Weil bundle $(M^A, \pi, M)$ with $\pi: M^A \to M$ and $\pi^{-1}(x) = M^A_x \forall x \in M$, define a special section $\alpha: M \to M^A$ of $\pi$ such that for any $x \in M, \alpha(x) = x^A$.

Lemma 1. Let $\tilde{f}: M^A \to \mathbb{R}$ be a function on $M^A$ and $\alpha: M \to M^A$ be the special section of the Weil bundle $(M^A, \pi, M)$ (i.e., $\alpha(x) = x^A$). Then, $\tilde{f} \circ \alpha = \tilde{f}$.

Proof. Let $\varepsilon \in M^A_x$, then there exists $x \in M$ such that $\varepsilon \in M^A_x$. For this, $x, \alpha(x) = x^A \in M^A_x$.
\[
\tilde{f} \circ \alpha(\varepsilon) = \tilde{f} \circ \alpha(\varepsilon) = \tilde{f}(x^A) = \tilde{f}(\varepsilon),
\]
and will be widely used throughout this paper.

3. Revisiting Tangent Spaces

Let $M$ be a smooth manifold and $\mathbb{D} = \mathbb{R}[y]/(y^2)$ be the ring of dual numbers, then $\mathbb{M}^D$ can be identified with the tangent $TM$. Let $x \in M$, then the tangent space $T_xM$ can be identified with the space $M^D_x$ of $\mathbb{D}$–points of $M$ near to $x$ by: if $\varepsilon \in \mathbb{M}^D_x, \nu \in T_xM$ and $f \in C^\infty(M)$, then
\[
\epsilon(f) = f(x) + (v(f)) \cdot y.
\]

Let $A$ be a Weil algebra, then the tangent bundle on $M^A$ can be identified as $(M^A)^D \cong (M^D)^A$. If
\[
\mu: \mathbb{R} \times \mathbb{M}^D \to \mathbb{M}^D, (b, \varepsilon) \mapsto xe,
\]
is the external multiplication of $\mathbb{M}^D$, then one can see in [3], Definition 1 that the map
\[
\mu^A: A \times (M^A)^D \to (M^A)^D, (a, \epsilon_i) \mapsto a \epsilon_i,
\]
gives to $(M^A)^D$ the structure of $A$–module. Since $\mathbb{R} \subset A$, then one can define naturally the multiplication
\[
\tilde{\mu}: \mathbb{R} \times (M^A)^D \to (M^A)^D, (t, \epsilon_i) \mapsto t \epsilon_i,
\]
which gives to $(M^A)^D$ the structure of $\mathbb{R}$–vector space.

Definition 2. By a tangent vector on $\varepsilon$, we mean a linear map
\[
v: C^\infty(M^A, \mathbb{R}) \to \mathbb{R},
\]
satisfying the Leibniz rule, i.e., $\forall f, g \in C^\infty(M^A)$
\[
v(f \cdot g) = (f(\varepsilon))v(g) + v(f)g(\varepsilon).
\]
Such a map is called a derivation. We denote
\[
T_xM^A = \{v: C^\infty(M^A, \mathbb{R}) \to \mathbb{R} \mid v \text{ is a derivation}\}.
\]

Remark 2. Let $\{x_1, \ldots, x_n\}$ be a system of local coordinates around a neighborhood of $x \in M$ and $\{x_{i,j} \mid i = 1, \ldots, n; j = 1, \ldots, \dim(A)\}$ be a system of local coordinate around $\varepsilon \in M^A_x$. Denote by $\{\partial/\partial x_i \mid x, i = 1, \ldots, n\}$ a basis of $T_xM$ where
Remark 4. Let \( v \in T_xM \), i.e.,
\[
v: C^\infty(M) \to \mathbb{R},
\]
is a derivation. Define
\[
\nu^A: C^\infty(M^A, A) \to A,
\]
such that for any \( f \in C^\infty(M) \), we have
\[
\nu^A(f^A) = [\nu(f)]^A.
\]
Since \( \nu(f) \in \mathbb{R} \) and \( [\nu(f)]^A \in A \), one can write \([\nu(f)]^A\) as an \( \mathbb{R} \)-linear combination of basis elements of \( A \).

Definition 3. Define
\[
\nu^A(f^A) = [\nu(f)]^A = \nu(f) \cdot \alpha_i + r,
\]
where \( r \in \mathcal{M} \).

Remark 4. Denote by \( C^\infty(M^A, \mathbb{R}) \) the set of functions from \( M^A \) to \( \mathbb{R} \), by \( C^\infty(M^A, A) \) those of functions from \( M^A \) to \( A \). Define

\[

\begin{align*}
T_1: C^\infty(M^A, \mathbb{R}) & \to C^\infty(M^A, A), \\
T_2: C^\infty(M^A, A) & \to C^\infty(M^A, \mathbb{R}),
\end{align*}
\]
such that
\[

T_1(f) = (\tilde{f}^\alpha)^A, \quad \text{and}
\]
\[
T_2(g) = \phi^*g,
\]
where \( \phi: A \to \mathbb{R} \) is the linear form \( \phi = \alpha_i^* \) such that \( \alpha_i^*, \ldots, \alpha_n^* \) is the dual basis of a basis \( \alpha_1, \ldots, \alpha_m \) of \( A \). Also,
\[

\begin{align*}
R_1: C^\infty(M) & \to C^\infty(M^A, \mathbb{R}), \\
R_2: C^\infty(M^A, \mathbb{R}) & \to C^\infty(M),
\end{align*}
\]
such that
\[

\begin{align*}
R_1(f) & = f^\pi, \\
R_2(g) & = \tilde{g} \alpha.
\end{align*}
\]
The above maps play a very important role in our approach and satisfy the following results, proven in [13].

(i) If \( \tilde{X} \in \mathcal{X}(M^A, \mathbb{R}) \) is a vector field on \( M^A \), regarded as a derivation from \( C^\infty(M^A, \mathbb{R}) \) to \( C^\infty(M^A, \mathbb{R}) \), then so it is for \( R_2^{-1} \tilde{X} R_1 \) regarded as a derivation from \( C^\infty(M) \) to \( C^\infty(M) \).

(ii) If \( X \in \mathcal{X}(M) \) is a vector field on \( M \), then so it is for \( T_2^{-1} X^A T_1 \in \mathcal{X}(M^A, \mathbb{R}) \) in \( M^A \) regarded as a derivation from \( C^\infty(M^A, \mathbb{R}) \) to \( C^\infty(M^A, \mathbb{R}) \).

**Proposition 1.** The map
\[
L: T_xM^A \to T_xM,
\]
such that
\[
L(\nu) = \tilde{\nu} R_1,
\]
is surjective.

**Proof.** The linearity of \( L \) is straightforward. Let us prove that \( L \) is a tangent vector at \( x \in M \). Let \( f, g \in C^\infty(M) \), then
\[

\begin{align*}
\tilde{\nu} R_1(f \cdot g) & = \tilde{\nu}(f^\pi \cdot g^\pi) = f^\pi(\nu(\tilde{g}^\pi)) + g^\pi(\nu(\tilde{f}^\pi)) = f^\pi(\nu(\tilde{g}^\pi) + g^\pi(\nu(\tilde{f}^\pi)) \\
& = f(x) \cdot \tilde{\nu} R_1(g) + g(x) \cdot \tilde{\nu} R_1(f).
\end{align*}
\]
This shows that \( L \) is well-defined. It remains to prove that \( L \) is surjective. Let \( v \in T_x M \), then \( \phi^* v^A T_1 \in T_x M^A \) and
\[
L\left( \phi^* v^A T_1 \right) = \phi^* v^A o R_1. \tag{35}
\]
We need to prove that \( \phi^* v^A T_1 o R_1 = v \). Let \( f \in \mathcal{C}^\infty (M) \), then
\[
\phi^* v^A T_1 o R_1 (f) = \phi^* \left[ \phi (f') \right]^A = v(f).
\]
Thus,
\[
L\left( \phi^* v^A T_1 \right) = v. \tag{37}
\]

**Remark 5.** For \( i = 1, \ldots, n \) define
\[
dx_i : T_x M \longrightarrow \mathbb{R}, \tag{38}
\]
and
\[
(dx)^A : (T_x M)^A \longrightarrow A,
\]
such that
\[
(dx)^A \left( \frac{\partial}{\partial x_j} \right)^A = \left[ dx_i \left( \frac{\partial}{\partial x_j} \right) \right]^A = \delta_{ij} \cdot \alpha_i + r' \forall j,
\]
where \( r' \in \mathcal{M} \), and for any \( w \in (T_x M)^A = \operatorname{Der}(\mathcal{C}^\infty (M^A), A) \), define
\[
dx^A_i (w) = \sum_{j=1}^m dx_{ij} (\phi^* w^A T_1) \cdot \alpha_j, \tag{40}\]
where
\[
dx_{ij} : \mathcal{C}^\infty (M^A, \mathbb{R}) \longrightarrow \mathbb{R}, \tag{41}\]
is a linear form. We claim that \( dx_{i1}, \ldots, dx_{im} \) is the dual basis for \( \partial/\partial x_{i_1}, \ldots, \partial/\partial x_{i_m} \) and
\[
T_x^* M^A = \langle dx_{i1}, \ldots, dx_{im} \rangle. \tag{42}\]

The map
\[
T_x^* M^A \longrightarrow T_x M,
\]
\[
dx_{ij} \mapsto dx_i,
\]
is surjective. For any \( k \in \{1, \ldots, n\} \), denote by
\[
\wedge^k T_x^* M^A = \langle dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid 1 \leq i_1 \leq \cdots \leq i_k \leq n, 1 \leq j_1 \leq \cdots \leq j_k \leq m \text{ and } (i_1, j_1) \neq (i_k, j_k) \rangle. \tag{44}\]

### 4. Differential Form and Cohomology

We denote by \( \Omega^k (M^A, \mathbb{R}) \) the space of sections of the bundle \( \wedge^k T^* M^A \).

**Definition 4.** By a \( k \)-form on \( M^A \), we mean the \( k \)-multilinear skew-symmetric map
\[
\bar{\theta} : x (M^A, \mathbb{R}) \wedge \cdots \wedge x (M^A, \mathbb{R}) \xrightarrow{k} \mathcal{C}^\infty (M^A, \mathbb{R}). \tag{45}\]

**Proposition 2.** The map
\[
C : \Omega^k (M) \longrightarrow \Omega^k (M^A, \mathbb{R}), \tag{46}\]
such that
\[
C(\theta)(\bar{X}_1, \ldots, \bar{X}_k) = \theta(R^* \bar{X}_1 \circ R_1, \ldots, R^* \bar{X}_k \circ R_1)^\pi, \tag{47}\]
is well-defined for all \( \bar{X}_1, \ldots, \bar{X}_k \in x (M^A, \mathbb{R}) \) and \( \theta \in \Omega^k (M) \). In other words, \( k \)-forms of \( M \) give rise to \( k \)-forms of \( M^A \).

**Proof.** We need to prove that \( C(\theta)(\bar{X}_1, \ldots, \bar{X}_k) \) is a \( k \)-form, i.e., a \( k \)-multilinear which is skew-symmetric. The additivity and the skew-symmetric condition are straightforward. Let \( f \in \mathcal{C}^\infty (M^A, \mathbb{R}) \), then
\[
C(\theta)(\bar{X}_1, \ldots, f \cdot \bar{X}_p, \ldots, \bar{X}_k) = \theta(R^* \bar{X}_1 \circ R_1, \ldots, R^* (f \cdot \bar{X}_p) \circ R_1, \ldots, R^* \bar{X}_k \circ R_1)^\pi. \tag{48}\]

Observe that for any \( g \in \mathcal{C}^\infty (M) \), we have
\[ R_2 \left( \bar{f} \cdot X_1 \right)^* R_1 (g) = R_2 \circ (\bar{f} \cdot X)(g') \]
\[ = R_2 \left( \bar{f} \cdot X (g') \right) \]
\[ = \left( \bar{f} \cdot X (g') \right)^* \]
\[ = \bar{f} \cdot X (g')^* \]
\[ = \bar{f} \cdot X (g')^* \]
\[ = R_2 \circ (\bar{f} \cdot X)(g'), \quad (49) \]

Then

\[ R_2 \left( \bar{f} \cdot X_1 \right)^* R_1 = \bar{f} \cdot X (g')^* \cdot R_2, \quad (50) \]

Since \( \theta \) is a \( \kappa \)-form, then \( \theta (R_2 \circ (\bar{f} \cdot X)(g')) = \bar{f} \cdot X (g')^* \cdot \theta (R_2, R_1), \]

\[ C(\bar{f}) \left( \bar{f} \cdot X_1, \ldots, \bar{f} \cdot X_k \right) = \bar{f} \cdot C(\bar{f}) \left( X_1, \ldots, X_k \right). \]

(51)

\[ D(\tilde{\theta})(X_1, \ldots, f \cdot X_i, \ldots, X_k) = \tilde{\theta}(R(X_1), \ldots, R(f \cdot X_i), \ldots, R(X_k))^* \alpha. \]

(55)

Observe that for any \( \tilde{g} \in C^\infty (M^\kappa, \mathbb{R}) \), we have

\[ R(f \cdot X_i)(\tilde{g}) = T_2^* (f \cdot X_i)^* T_1(\tilde{g}) = \phi \circ [f \cdot X_i(\tilde{g})]^\kappa = \phi f^\kappa \cdot T_2^* X_i^* T_1(\tilde{g}), \]

(56)

that is,

\[ R(f \cdot X_i) = \phi f^\kappa \cdot R(X_i), \]

and

\[ D(\tilde{\theta})(X_1, \ldots, f \cdot X_i, \ldots, X_k) = \phi f^\kappa \cdot \tilde{\theta}(R(X_1), \ldots, R(X_i), \ldots, R(X_k))^* \alpha. \]

(57)

Observe that if \( x \in M \), then

\[ \phi f^\kappa \alpha(x) = \phi [f^\kappa(x)] = \phi [f(x) \mod \mathcal{M}] = f(x), \]

(58)

then

\[ D(\tilde{\theta})(X_1, \ldots, f \cdot X_i, \ldots, X_k) = f \cdot D(\tilde{\theta})(X_1, \ldots, f \cdot X_i, \ldots, X_k). \]

(59)

Definition 5. For any \( 0 \leq k \leq \dim(M) \), define the operator

\[ \tilde{d}: \Omega^k (M^\kappa, \mathbb{R}) \longrightarrow \Omega^{k+1} (M^\kappa, \mathbb{R}), \]

(60)

such that \( \tilde{d} = C \circ d \circ D \) where \( d: \Omega^k (M) \longrightarrow \Omega^{k+1} (M) \) is the cohomology operator on \( M \).
With notations as above, we have the following:

**Theorem 1.** For any \( \bar{\theta} \in \Omega^k(M^A, \mathbb{R}) \) and \( X_1, \ldots, X_{k+1} \in \mathfrak{X}(M^A, \mathbb{R}) \), we have

\[
\overline{a}(\bar{\theta})(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{j-1} \cdot X_i \left( \overline{\theta}\left( \overline{X_1, \ldots, X_{j-1}, \ldots, X_{k+1}} \right) \right) + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \overline{\theta}\left( [\overline{X_i, \overline{X_j}}], \overline{X_1, \ldots, X_{i-1}, \ldots, X_{j+1}, \ldots, X_{k+1}} \right).
\]

(61)

**Proof.** By definition

\[
\overline{a}(\bar{\theta})(X_1, \ldots, X_{k+1}) = C \cdot \overline{d} \cdot D(\bar{\theta})(X_1, \ldots, X_{k+1}) = C \cdot \overline{d}(D(\bar{\theta}))(X_1, \ldots, X_{k+1}) = \left( \overline{d}(D(\bar{\theta}))(R_2 \cdot X_1 \circ R_1, \ldots, R_2 \cdot X_{k+1} \circ R_1) \right) \circ \pi \ast (*).
\]

(62)

Set \( Y_1 = R_2 \cdot X_1 \circ R_1, \ldots, Y_{k+1} = R_2 \cdot X_{k+1} \circ R_1 \), then

\[
\overline{a}(D(\bar{\theta}))(Y_1, \ldots, Y_{k+1}) = \sum_{i=1}^{k+1} (-1)^{j-1} \cdot Y_i \left( \overline{D(\bar{\theta})}(Y_1, \ldots, Y_{j-1}, \ldots, Y_{k+1}) \right) + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} D(\bar{\theta})\left( [Y_i, Y_j], Y_1, \ldots, Y_{i-1}, \ldots, Y_{j+1}, \ldots, Y_{k+1} \right).
\]

(63)

Observe that for each \( i \),

\[
Y_i(D(\bar{\theta})(Y_1, \ldots, Y_{j-1}, \ldots, Y_{k+1})) = Y_i\left( \overline{\theta}\left( T^\ast Y_i \circ T_1, \ldots, T^\ast Y_i \circ T_1, \ldots, T^\ast Y_i \circ T_1 \right) \right).
\]

(64)

Observe that for any \( T^\ast Y_i \circ T_1 = T^\ast R_2 \cdot X_i \circ R_1 \), let \( \bar{f} \in C^\infty(M^A, \mathbb{R}) \), then

\[
T^\ast(2^\ast X_1 \circ R_1) \circ T_1 = T^\ast(2^\ast X_1 \circ R_1) \circ (\bar{f} \ast \overline{a}) = T^\ast(2^\ast X_1 \circ R_1 \circ (\bar{f} \ast \overline{a})) = T^\ast \left[ \overline{X_1, \bar{f} \circ \overline{a}} \right] \circ T_1
\]

(65)

Thus,

\[
Y_i(D(\bar{\theta})(Y_1, \ldots, Y_{j-1}, \ldots, Y_{k+1})) = Y_i\left( \overline{\theta}\left( \overline{X_1, \ldots, X_{j-1}, \ldots, X_{k+1}} \right) \right).
\]

(67)

Observe also that
\[
Y_i(\bar{\theta}((\bar{X}_1,\ldots,\bar{X}_n,\bar{X}_{k+1})^\alpha) = R_2^* \bar{X}_i R_1(\bar{\theta}((\bar{X}_1,\ldots,\bar{X}_n,\bar{X}_{k+1})^\alpha) \\
= R_2^* \bar{X}_i(\bar{\theta}(\bar{X}_1,\ldots,\bar{X}_n,\bar{X}_{k+1})^\alpha) \\
= R_2^* \bar{X}_i(\bar{\theta}(\bar{X}_1,\ldots,\bar{X}_n,\bar{X}_{k+1})^\alpha) \\
= \bar{X}_i(\bar{\theta}(\bar{X}_1,\ldots,\bar{X}_n,\bar{X}_{k+1})^\alpha).
\]

Also,

\[
D(\bar{\theta})([Y_\rho, Y_\mu], Y_1,\ldots, Y_n, Y_{k+1}) = \bar{\theta}(R([Y_\rho, Y_\mu]), R(Y_1),\ldots,R(Y_n),\ldots,R(Y_{k+1})).
\]

Observe that

\[
R([Y_\rho, Y_\mu]) = T_2^*[Y_\rho, Y_\mu]^\alpha T_1.
\]

Let \( \tilde{f} \in C^\infty(M^A, \mathbb{R}) \), then

\[
T_2^*[Y_\rho, Y_\mu]^\alpha T_1(\tilde{f}) = T_2^*[Y_\rho, Y_\mu]^\alpha(\tilde{f}^\alpha)^A \\
= T_2^*[Y_\rho, Y_\mu]^\alpha(\tilde{f}^\alpha)^A \\
= T_2^*[Y_\rho, Y_\mu]^\alpha(\tilde{f}^\alpha)^A \\
= T_2^*[Y_\rho, Y_\mu]^\alpha(\tilde{f}^\alpha)^A.
\]

It is not difficult to see that

\[
R_2^* \bar{X}_j^* R_1(\tilde{f}^\alpha) = R_2^* \bar{X}_j(\tilde{f}^\alpha) = \bar{X}_j(\tilde{f})^\alpha, \\
\text{and } R_2^* \bar{X}_j^* R_1(\tilde{f}^\alpha) = R_2^* \bar{X}_j(\tilde{f})^\alpha
\]

Then,

\[
T_2^*[Y_\rho, Y_\mu]^\alpha T_1(\tilde{f}) = T_2^*[X_i(\tilde{X}_j(\tilde{f}))^\alpha - \bar{X}_j(\tilde{X}_i(\tilde{f}))^\alpha]^A \\
= \phi' [\bar{X}_i(\tilde{X}_j(\tilde{f}))^\alpha]^A - \phi' [\bar{X}_i(\tilde{X}_j(\tilde{f}))^\alpha]^A \\
= [X_i(\tilde{X}_j(\tilde{f})) - X_j(\tilde{X}_i(\tilde{f}))] \\
= [\bar{X}_i, \bar{X}_j](\tilde{f}), \text{ and}
\]

\[
R([Y_\rho, Y_\mu]) = [\bar{X}_i, \bar{X}_j], R(Y_i) = \bar{X}_i.
\]

Thus,
\[ D(\partial)(Y_1, Y_j, \ldots, Y_{k+1}) = \partial \big( [\tilde{X}_i, \tilde{X}_j, \ldots, \tilde{X}_{k+1}] \big) = a(2). \quad (74) \]

Replacing (1) and (2) in (*), we obtain

\[ d(\partial)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \cdot \tilde{X} (\partial \big( X_i, \ldots, \tilde{X}_i, \ldots, X_{k+1} \big)) + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \partial \big( [\tilde{X}_i, \tilde{X}_j, \ldots, \tilde{X}_i, \ldots, \tilde{X}_j, \ldots, X_{k+1}] \big). \quad (75) \]

Remark 7 (Conclusion). The previous result shows that \( d \) is the exterior derivative in \( MA \) and satisfies \( d^2 = 0 \), which makes the sequence \( (\Omega^*, (M^A, R), d) \) to a complex of differential forms on \( MA \), and we write \( H_{dR}(MA) \) for the de Rham cohomology on \( MA \) and denote by \( H(\Omega^*, R) \) the cohomology associated to the complex \( (\Omega^*(M^A, R), d) \). This gives the possibility to extend this area in different directions of differential geometry with applications.

Data Availability

The data supporting the current study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References