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Research Article

Differential Forms and Cohomology on Weil Bundles

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Let M be a smooth manifold and A a Weil algebra. We discuss the differential forms in the Weil bundles (M^A, π, M), and we established a link between differential forms in M^A and M as well as their cohomology. We also discuss the cohomology in.

1. Introduction

The theory of bundles of infinitely near points was introduced in 1953 by Andre Weil in [1] and has become a subject of significant interest in differential geometry. A commutative, associative, unitary real algebra A is called Weil algebra if it is a finite-dimensional local algebra of the form $A = \mathbb{R} \oplus \mathscr{M}$ (i.e., $\dim(A/\mathscr{M}) = 1$) where $\mathscr{M}$ is its only maximal ideal (see [2], from page 625). As an example, one defines the algebra $D = \mathbb{R}[x]/\langle x^2 \rangle$ of dual numbers whose the maximal ideal is $\mathscr{M} = x\mathbb{R}$.

Let M be a smooth manifold and $x \in M$. Given a Weil algebra A with maximal ideal $\mathscr{M}$ and basis $\alpha_1, \ldots, \alpha_m$, one defines a morphism of algebras

$$\varepsilon\colon C^\infty(M) \longrightarrow A, \tag{1}$$

such that

$$\varepsilon(f) = f(x) \cdot \alpha_1 + t = f(x) + t, \tag{2}$$

where $1 = \alpha_1 \in \mathbb{R}$ and $t \in \mathscr{M}$. Such a morphism is called A-point of M near to x, and one denotes by M_x^A the set of all A-points of M near to x. There is a functor T^A from the category of smooth manifolds to itself sending a smooth manifold M to the bundle $(T^A M, \pi, M)$ which is known as the bundle of A-points near to points in M; in this case, $T^A M = M^A = \cup_{x \in M} M_x^A$ can be regarded as a manifold with $\dim_{\mathbb{R}} M^A = \dim(A) \cdot \dim_{\mathbb{R}} M$ (see [3]). One of the questions that draw researcher's attention is the prolongation of geometric structures from M to M^A (see [4], chap. 4 for the general theory). This approach consists of sending a geometric structure from M to M^A (regarded as an $A-$manifold, i.e., $\dim_A M^A = \dim_{\mathbb{R}} M$) as developed in [5–10] where the authors studied the prolongations of vector fields and differential forms, linear connections, symplectic structures, and pseudo-Riemannian structures. Many directions have been developed from the last decades for these manifolds such as affine manifold structures studied in [2] and principal fiber bundles studied in [11], and nice applications to Grassmann bundles can be found in [12].

Instead of regarding M^A as an $A-$manifold, we discuss in this paper differential forms and de Rham cohomology on M^A without any prolongation. This approach consists of regarding M^A as an $\mathbb{R}-$manifold (i.e., $\dim_{\mathbb{R}}(M^A) = \dim(A) \cdot \dim(M)$) (see [13]). More specifically, if $\Omega^k(M^A, \mathbb{R})$ denotes the space of $k-$forms in M^A, we introduce the map

$$D\colon \Omega^k\left(M^A, \mathbb{R}\right) \longrightarrow \Omega^k(M), \tag{3}$$

sending a $k-$form from M^A to a $k-$form in M. Conversely, we introduce the map

$$C\colon \Omega^k(M) \longrightarrow \Omega^k\left(M^A, \mathbb{R}\right), \tag{4}$$

sending a $k-$form from M to a $k-$form in M^A. These two maps are central and enable to extend the de Rham complex in M^A by introducing the operator

$$\widetilde{d} = C^\circ d^\circ D\colon \Omega^k\left(M^A, \mathbb{R}\right) \longrightarrow \Omega^{k+1}\left(M^A, \mathbb{R}\right), \tag{5}$$

in M^A where $d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ is the de Rham operator in M, and we prove that $\widetilde{d}$ defines indeed the de Rham cohomology operator in M^A.

2. Basic Notions

Definition 1. (Weil functor)

Let A be a Weil algebra with maximal ideal $\mathscr{M}$, M be a C^∞−smooth manifold. Denote by Mfd the category of smooth manifolds. By the Weil functor of M, we mean a functor $T^A\colon \mathrm{Mfd} \longrightarrow \mathrm{Mfd}$ such that

(1) *for any* $M \in Ob(\mathrm{Mfd})$,

$$T^A M = \bigcup_{x\in M} M_x^A, \tag{6}$$

with projection $T^A M \longrightarrow M$ and fibers M_x^A for any $x \in M$.

(2) *for any* $M, N \in Ob(\mathrm{Mfd})$ *and* $f\colon M \longrightarrow N$, *we have* $T^A f\colon T^A M \longrightarrow T^A N$ *such that for any* $x \in M, T^A f(M_x^A) \subset N_{f(x)}^A$ *and the following diagram commutes*

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ M^A \arrow[u, "\pi_M"] \arrow[r, "T^A f"] & N^A \arrow[u, "\pi_N"'] \end{tikzcd}$$

Remark 1

(1) *Denote by* $T^A M = M^A$ *and* $T^A \mathbb{R}^n = A^n$

(2) *If* $f\colon M \longrightarrow \mathbb{R}$ *is a function, then*

$$f^A\colon M^A \longrightarrow A, \tag{7}$$

such that for any $\varepsilon \in M_x^A$, we have

$$f^A(\varepsilon) = \varepsilon(f) = f(x) + t \textit{ for } \mathrm{t} \in \mathscr{M}. \tag{8}$$

(3) Claim: if $\widetilde{f}\colon M^A \longrightarrow \mathbb{R}$ is a function and $\varepsilon_1, \varepsilon_2 \in M_x^A$, then $\widetilde{f}(\varepsilon_1) = \widetilde{f}(\varepsilon_2)$. This is a very important claim and will be widely used thoughout this paper.

(4) *Let* $\{\alpha_1, \ldots, \alpha_m\}$ *be a basis for A and M be a manifold such that* $\{x_1, \ldots, x_n\}$ *is a system of local coordinate around* $x \in U \subset M$, *then there exists* $\varepsilon \in \pi^{-1}(U) \subset M_x^A$ *and functions* $x_{i,j}\colon \pi^{-1}(U) \longrightarrow \mathbb{R}$, $i = 1, \ldots, n$; $j = 1, \ldots, m$ *such that for*

$$\varepsilon(x_i) = \sum_{j=1}^{m} x_{i,j}(\varepsilon) \cdot \alpha_j \forall = i = 1, \ldots, n. \tag{9}$$

The functions $\{x_{1,1}, \ldots, x_{n,m}\}$ *are a system of local coordinate around* $\varepsilon \in \pi^{-1}(U) \subset M_x^A$. *It is clear that* $\dim_{\mathbb{R}}(M^A) = n \cdot m$.

(5) *If M and N are smooth manifolds and* $h\colon M \longrightarrow N$ *a smooth map (resp. diffeomorphism) then*

$$h^A\colon M^A \longrightarrow N^A, \varepsilon \longrightarrow h^A(\varepsilon). \tag{10}$$

such that $\forall \phi \in C^\infty(N), h^A(\varepsilon)(\phi) = \varepsilon(\phi^\circ h)$ is a smooth map (resp. diffeomorphism).

(6)*Given a Weil bundle* (M^A, π, M) *with* $\pi\colon M^A \longrightarrow M$ *and* $\pi^{-1}(x) = M_x^A \forall x \in M$, *define a special section* $\alpha\colon M \longrightarrow M^A$ *of* π *such that for any* $x \in M, \alpha(x) = x^A$.

Lemma 1. *Let* $\widetilde{f}\colon M^A \longrightarrow \mathbb{R}$ *be a function on* M^A *and* $\alpha\colon M \longrightarrow M^A$ *be the special section of the Weil bundle* (M^A, π, M) *(i.e.,* $\alpha(x) = x^A$*). Then,* $\widetilde{f}^\circ\alpha^\circ\pi = \widetilde{f}$.

Proof. Let $\varepsilon \in M^A$, then there exists $x \in M$ such that $\varepsilon \in M_x^A$. For this, $x, \alpha(x) = x^A \in M_x^A$.

$$\widetilde{f}^\circ\alpha^\circ\pi(\varepsilon) = \widetilde{f}\circ\alpha[\pi(\varepsilon)] = \widetilde{\mathrm{f}}\circ\alpha(\mathrm{x}) = \widetilde{\mathrm{f}}\left(\mathrm{x}^A\right) = \widetilde{\mathrm{f}}(\varepsilon), \tag{11}$$

since x^A, ε are both A−points near to x (see the claim on Remark 2). □

3. Revisiting Tangent Spaces

Let M be a smooth manifold and $\mathbb{D} = \mathbb{R}[y]/\langle y^2\rangle$ be the ring of dual numbers, then $M^{\mathbb{D}}$ can be identified with the tangent TM. Let $x \in M$, then the tangent space $T_x M$ can be identified with the space $M_x^{\mathbb{D}}$ of $\mathbb{D}$−points of M near to x by: if $\varepsilon \in M_x^{\mathbb{D}}, v \in T_x M$ and $f \in C^\infty(M)$, then

$$\varepsilon(f) = f(x) + (v(f)) \cdot y. \tag{12}$$

Let A be a Weil algebra, then the tangent bundle on M^A can be identified as $(M^A)^{\mathbb{D}} \cong (M^{\mathbb{D}})^A$. If

$$\mu\colon \mathbb{R} \times M^{\mathbb{D}} \longrightarrow M^{\mathbb{D}}, (b, \varepsilon) \mapsto x\varepsilon, \tag{13}$$

is the external multiplication of $M_x^{\mathbb{D}}$, then one can see in [3], Definition 1 that the map

$$\mu^A\colon A \times \left(M^A\right)^{\mathbb{D}} \longrightarrow \left(M^A\right)^{\mathbb{D}}, (a, \varepsilon_1) \mapsto a\varepsilon_1, \tag{14}$$

gives to $(M^A)^{\mathbb{D}}$ the structure of A−module. Since $\mathbb{R} \subset A$, then one can define naturally the multiplication

$$\widetilde{\mu}\colon \mathbb{R} \times \left(M^A\right)^{\mathbb{D}} \longrightarrow \left(M^A\right)^{\mathbb{D}}, (t, \varepsilon_1) \mapsto t\varepsilon_1, \tag{15}$$

which gives to $(M^A)^{\mathbb{D}}$ the structure of $\mathbb{R}$−vector space.

Definition 2. By a tangent vector on ε, we mean a linear map

$$v\colon C^\infty\left(M^A, \mathbb{R}\right) \longrightarrow \mathbb{R}, \tag{16}$$

satisfying the Leibniz rule, i.e., $\forall f, g \in C^\infty(M^A)$

$$v(f \cdot g) = f(\varepsilon)v(g) + v(f)g(\varepsilon). \tag{17}$$

Such a map is called a derivation. We denote

$$T_\varepsilon M^A = \left\{v\colon C^\infty\left(M^A, \mathbb{R}\right) \longrightarrow \mathbb{R} \mid v \text{ is a derivation}\right\}. \tag{18}$$

Remark 2. Let $\{x_1, \ldots, x_n\}$ be a system of local coordinates around a neighborhood of $x \in M$ and $\left\{x_{i,j} \mid i = 1, \ldots, n; j = 1, \ldots, \dim(A)\right\}$ be a system of local coordinate around $\varepsilon \in M_x^A$. Denote by $\{\partial/\partial x_i \mid x, i = 1, \ldots, n\}$ a basis of $T_x M$ where

$$\begin{aligned}&\frac{\partial}{\partial x_i}\mid_x\colon C^{\infty}(M)\longrightarrow \mathbb{R}, \text{and}\\ &\left(\frac{\partial}{\partial x_i}\right)^A\mid_\varepsilon\colon C^{\infty}\left(M^A, A\right)\longrightarrow A,\end{aligned}\tag{19}$$

is the ε–derivation introduced in [6] (page 4).

Since $\{x_{1,1},\ldots,x_{n,m}\}$ is a system of local coordinate of M^A around ε, define the tangent vector

$$\frac{\partial}{\partial x_{ij}}\mid_\varepsilon\colon C^{\infty}\left(M^A,\mathbb{R}\right)\longrightarrow \mathbb{R},\tag{20}$$

of $T_\varepsilon M^A$ around ε such that $\forall g\in C^{\infty}(M^A, A)$

$$\left(\frac{\partial}{\partial x_i}\right)^A\mid_\varepsilon(g)=\sum_{j=1}^{m}\frac{\partial}{\partial x_{ij}}\mid_\varepsilon\left(\phi^{\circ}g\right)\cdot\alpha_j,\tag{21}$$

then we claim that

$$T_\varepsilon M^A=\langle\frac{\partial}{\partial\left(x_{1,1}\right)}\mid_\varepsilon,\ldots,\frac{\partial}{\partial\left(x_{n,m}\right)}\mid_\varepsilon\rangle.\tag{22}$$

Remark 3. Let $v\in T_xM$, i.e.,

$$v\colon C^{\infty}(M)\longrightarrow\mathbb{R},\tag{23}$$

is a derivation. Define

$$v^A\colon C^{\infty}\left(M^A, A\right)\longrightarrow A,\tag{24}$$

such that for any $f\in C^{\infty}(M)$, we have

$$v^A\left(f^A\right)=[v(f)]^A.\tag{25}$$

Since $v(f)\in\mathbb{R}$ and $[v(f)]^A\in A$, one can write $[v(f)]^A$ as an $\mathbb{R}$–linear combination of basis elements of A.

Definition 3. Define

$$\begin{aligned}v^A\left(f^A\right)&=[v(f)]^A\\&=v(f)\cdot\alpha_1+r,\end{aligned}\tag{26}$$

where $r\in\mathcal{M}$.

Remark 4. Denote by $C^{\infty}(M^A,\mathbb{R})$ the set of functions from M^A to $\mathbb{R}$, by $C^{\infty}(M^A, A)$ those of functions from M^A to A. Define

$$T_1\colon C^{\infty}\left(M^A,\mathbb{R}\right)\longrightarrow C^{\infty}\left(M^A, A\right),\tag{27}$$

such that

$$\begin{aligned}&T_1(\widetilde{f})=\left(\widetilde{f}^{\circ}\alpha\right)^A, \text{and}\\ &T_2\colon C^{\infty}\left(M^A, A\right)\longrightarrow C^{\infty}\left(M^A,\mathbb{R}\right),\end{aligned}\tag{28}$$

such that

$$T_2(g)=\phi^{\circ}g,\tag{29}$$

where $\phi\colon A\longrightarrow\mathbb{R}$ is the linear form $\phi=\alpha_1^*$ such that $\alpha_1^*,\ldots,\alpha_m^*$ is the dual basis of a basis $\alpha_1,\ldots,\alpha_m$ of A. Also,

$$R_1\colon C^{\infty}(M)\longrightarrow C^{\infty}\left(M^A,\mathbb{R}\right),\tag{30}$$

such that

$$\begin{aligned}&R_1(f)=f^{\circ}\pi,\\ &\text{and}\\ &R_2\colon C^{\infty}\left(M^A,\mathbb{R}\right)\longrightarrow C^{\infty}(M),\\ &R_2(\widetilde{g})=\widetilde{g}^{\circ}\alpha.\end{aligned}\tag{31}$$

The above maps play a very important role in our approach and satisfy the following results, proven in [13].

(i) If $\widetilde{X}\in\mathfrak{X}(M^A,\mathbb{R})$ is a vector field on M^A, regarded as a derivation from $C^{\infty}(M^A,\mathbb{R})$ to $C^{\infty}(M^A,\mathbb{R})$, then so it is for $R_2{}^{\circ}\widetilde{X}^{\circ}R_1$ regarded as a derivation from $C^{\infty}(M)$ to $C^{\infty}(M)$

(ii) If $X\in\mathfrak{X}(M)$ is a vector field on M, then so it is for $T_2{}^{\circ}X^{A\circ}T_1\in\mathfrak{X}(M^A,\mathbb{R})$ in M^A regarded as a derivation from $C^{\infty}(M^A,\mathbb{R})$ to $C^{\infty}(M^A,\mathbb{R})$

Proposition 1. *The map*

$$L\colon T_\varepsilon M^A\longrightarrow T_xM,\tag{32}$$

such that

$$L(\widetilde{v})=\widetilde{v}^{\circ}R_1,\tag{33}$$

is surjective.

Proof. The linearity of L is straightforward. Let us prove that L is a tangent vector at $x\in M$. Let $f,g\in C^{\infty}(M)$, then

$$\begin{aligned}\widetilde{v}^{\circ}R_1(f\cdot g)&=\widetilde{v}\left(f^{\circ}\pi\cdot g\circ\pi\right)=f^{\circ}\pi(\varepsilon)\cdot\widetilde{v}\left(g^{\circ}\pi\right)+g^{\circ}\pi(\varepsilon)\cdot\widetilde{v}\left(f^{\circ}\pi\right)=f^{\circ}\pi(\varepsilon)\cdot\widetilde{v}\left(g^{\circ}\pi\right)+g^{\circ}\pi(\varepsilon)\cdot\widetilde{v}\left(f^{\circ}\pi\right)\\&=f(x)\cdot\widetilde{v}^{\circ}R_1(g)+g(x)\cdot\widetilde{v}^{\circ}R_1(f).\end{aligned}\tag{34}$$

This shows that L is well-defined. It remains to prove that L is surjective. Let $v \in T_x M$, then $\phi^\circ v^{A^\circ} T_1 \in T_\varepsilon M^A$ and

$$L\left(\phi^\circ v^{A^\circ} T_1\right) = \phi^\circ v^A \circ T_1 \circ R_1. \tag{35}$$

We need to prove that $\phi^\circ v^{A^\circ} T_1 \circ \mathrm{R}_1 = \mathrm{v}$. Let $f \in C^\infty(M)$, then

$$\phi^\circ v^{A^\circ} T_1 \circ R_1(f) = \phi \circ [v(f^\circ \pi^\circ \alpha)]^A = \phi \circ [v(f)]^A = v(f). \tag{36}$$

Thus,

$$L\left(\phi^\circ v^{A^\circ} T_1\right) = v. \tag{37}$$

□

Remark 5. For $i = 1, \ldots, n$ define

$$\begin{aligned} &dx_i\colon T_x M \longrightarrow \mathbb{R}, \\ &\text{and} \\ &(\mathrm{d}x_i)^A\colon (T_x M)^A \longrightarrow A, \end{aligned} \tag{38}$$

such that

$$\begin{aligned} (\mathrm{d}x_i)^A\left(\frac{\partial}{\partial x_j}\right)^A &= \left[\mathrm{d}x_i\left(\frac{\partial}{\partial x_j}\right)\right]^A \\ &= \delta_{ij} \cdot \alpha_1 + r' \forall j, \end{aligned} \tag{39}$$

where $r' \in \mathscr{M}$, and for any $w \in (T_x M)^A = Der(\mathrm{C}^\infty(\mathrm{M}^\mathrm{A}, \mathrm{A}), \mathrm{A})$, define

$$\mathrm{d}x_i^A(w) = \sum_{j=1}^m \mathrm{d}x_{ij}(\phi^\circ w^\circ T_1) \cdot \alpha_j, \tag{40}$$

where

$$\mathrm{d}x_{ij}\colon C^\infty\left(M^A, \mathbb{R}\right) \longrightarrow \mathbb{R}, \tag{41}$$

is a linear form. We claim that $\mathrm{d}x_{11}, \ldots, \mathrm{d}x_{nm}$ is the dual basis for $\partial/\partial_{x_{11}} |_\varepsilon, \ldots, \partial/\partial_{x_{nm}} |_\varepsilon$ and

$$T_\varepsilon^* M^A = \langle \mathrm{d}x_{11}, \ldots, \mathrm{d}x_{nm} \rangle. \tag{42}$$

The map

$$\begin{aligned} T_\varepsilon^* M^A &\longrightarrow T_x M, \\ \mathrm{d}x_{ij} &\mapsto \mathrm{d}x_i, \end{aligned} \tag{43}$$

is surjective. For any $k \in \{1, \ldots, n\}$, denote by

$$\overset{k}{\wedge} T_\varepsilon^* M^A = \langle \mathrm{d}x_{i_1 j_1} \wedge \cdots \wedge \mathrm{d}x_{i_k j_k} \mid 1 \le i_1 \le \ldots \le i_k \le n, 1 \le j_1 \le \ldots \le j_k \le m \text{ and } (i_l, j_l) \ne (i_s, j_s) \rangle. \tag{44}$$

4. Differential Form and Cohomology

We denote by $\Omega^k(M^A, \mathbb{R})$ the space of sections of the bundle $\overset{k}{\wedge} T^* M^A$.

Definition 4. By a k–form on M^A, we mean the k–multilinear skew-symmetric map

$$\widetilde{\theta}\colon \underbrace{\mathfrak{X}(M^A, \mathbb{R}) \times \cdots \times \mathfrak{X}(M^A, \mathbb{R})}_{k} \longrightarrow C^\infty\left(M^A, \mathbb{R}\right). \tag{45}$$

Proposition 2. *The map*

$$C\colon \Omega^k(M) \longrightarrow \Omega^k\left(M^A, \mathbb{R}\right), \tag{46}$$

such that

$$C(\theta)(\widetilde{X}_1, \ldots, \widetilde{X}_k) = \theta\left(R_2^\circ \widetilde{X}_1^\circ R_1, \ldots, R_2 \circ \widetilde{X}_k \circ R_1\right)^\circ \pi, \tag{47}$$

is well-defined for all $\widetilde{X}_1, \ldots, \widetilde{X}_k \in \mathfrak{X}(M^A, \mathbb{R})$ and $\theta \in \Omega^k(M)$. In other words, k–forms of M give rise to k–forms of M^A.

Proof. We need to prove that $C(\theta)(\widetilde{X}_1, \ldots, \widetilde{X}_k)$ is a k–form, i.e., a k–multilinear which is skew-symmetric. The additivity and the skew-symmetric condition are straightforward. Let $\widetilde{f} \in C^\infty(M^A, \mathbb{R})$, then

$$C(\theta)\left(\widetilde{X}_1, \ldots, \widetilde{f} \cdot \widetilde{X}_i, \ldots, \widetilde{X}_k\right) = \theta\left(R_2^\circ \widetilde{X}_1^\circ R_1, \ldots, R_2 \circ \left(\widetilde{f} \cdot X_i\right) \circ R_1, \ldots, R_2 \circ \widetilde{X}_k \circ R_1\right)^\circ \pi. \tag{48}$$

Observe that for any $g \in C^\infty(M)$, we have

$$
\begin{aligned}
R_2{}^{\circ}\left(\widetilde{f}\cdot X_i\right)^{\circ}R_1(g) &= R_2\circ(\widetilde{\mathrm{f}}\cdot\widetilde{\mathrm{X}})(\mathrm{g}^{\circ}\pi)\\
&= R_2{}^{\circ}\left(\widetilde{f}\cdot\widetilde{X}(g^{\circ}\pi)\right)\\
&= R_2\left[\widetilde{f}\cdot\widetilde{X}(g^{\circ}\pi)\right]\\
&= \left(\widetilde{f}\cdot\widetilde{X}(g^{\circ}\pi)\right)^{\circ}\alpha\\
&= \widetilde{f}^{\circ}\alpha\cdot\widetilde{X}(g^{\circ}\pi)^{\circ}\alpha\\
&= \widetilde{f}^{\circ}\alpha\cdot R_2\circ\widetilde{X}\circ R_1(g),
\end{aligned}
\tag{49}
$$

then

$$
R_2{}^{\circ}\left(\widetilde{f}\cdot X_i\right)^{\circ}R_1 = \widetilde{f}\circ\alpha\cdot R_2\circ\widetilde{X}\circ R_1. \tag{50}
$$

Since θ is a k–form, then $\theta(R_2{}^{\circ}\widetilde{X}_1{}^{\circ}R_1,\ldots,R_2\circ(\widetilde{f}\cdot\widetilde{X}_i)\circ\mathrm{R}_1,\ldots,\mathrm{R}_2\circ\widetilde{\mathrm{X}}_\mathrm{k}\circ\mathrm{R}_1)^{\circ}\pi = \widetilde{\mathrm{f}}\circ\alpha\circ\pi\cdot\theta(\mathrm{R}_2{}^{\circ}\widetilde{\mathrm{X}}_1{}^{\circ}\mathrm{R}_1,\ldots,$ $R_2\circ(\widetilde{\mathrm{X}}_\mathrm{i}{}^{\circ}\mathrm{R}_1,\ldots,\mathrm{R}_2\circ\widetilde{X}_k\circ\mathrm{R}_1)\circ\pi = \widetilde{\mathrm{f}}\cdot\mathrm{C}(\theta)(\widetilde{\mathrm{X}}_1,\ldots,\widetilde{\mathrm{X}}_\mathrm{k})$. Thus,

$$
C(\theta)\left(\widetilde{X}_1,\ldots,\widetilde{f}\cdot\widetilde{X}_i,\ldots,\widetilde{X}_k\right) = \widetilde{f}\cdot C(\theta)(\widetilde{X}_1,\ldots,\widetilde{X}_K). \tag{51}
$$

□

Remark 6. If X is a vector field on M, it is proven in [13] (Proposition 3.5) that $T_2{}^{\circ}X^{A^{\circ}}T_1$ is a vector field on M^A where T_1 and T_2 are the maps introduced in the Remark 2. Define the map

$$
R\colon \mathfrak{X}(M)\longrightarrow\mathfrak{X}\left(M^A,\mathbb{R}\right), X\longmapsto T_2\circ X^A\circ T_1, \tag{52}
$$

then we have the following result.

Proposition 3. *The map*

$$
D\colon \Omega^k\left(M^A,\mathbb{R}\right)\longrightarrow\Omega^k(M), \tag{53}
$$

such that

$$
D(\widetilde{\theta})(X_1,\ldots,X_k) = \widetilde{\theta}(R(X_1),\ldots,R(X_k))^{\circ}\alpha, \tag{54}
$$

is well-defined for any $\widetilde{\theta}\in\Omega^k(M^A,\mathbb{R}), X_1,\ldots,X_k\in\mathfrak{X}(M)$.

Proof. The additivity and the skew-symmetry are straightforward. Let $f\in C^{\infty}(M)$, then

$$
D(\widetilde{\theta})(X_1,\ldots,f\cdot X_i,\ldots,X_k) = \widetilde{\theta}(R(X_1),\ldots,R(f\cdot X_i),\ldots,R(X_k))^{\circ}\alpha. \tag{55}
$$

Observe that for any $\widetilde{g}\in C^{\infty}(M^A,\mathbb{R})$, we have

$$
R(f\cdot X_i)(\widetilde{g}) = T_2{}^{\circ}(f\cdot X_i)^{A^{\circ}}T_1(\widetilde{g}) = \phi\circ\left[f\cdot X_i(\widetilde{g}^{\circ}\alpha)\right]^A = \phi\circ f^A\cdot T_2\circ X_i^A\circ T_1(\widetilde{g}), \tag{56}
$$

that is,

$$
\begin{aligned}
&R(f\cdot X_i) = \phi^{\circ}f^A\cdot R(X_i),\\
&\text{and}\\
&D(\widetilde{\theta})(X_1,\ldots,f\cdot X_i,\ldots,X_k) = \phi^{\circ}f^{A^{\circ}}\alpha\cdot\widetilde{\theta}(R(X_1),\ldots,R(X_i),\ldots,R(X_k))\circ\alpha.
\end{aligned}
\tag{57}
$$

Observe that if $x\in M$, then

$$
\phi^{\circ}f^{A^{\circ}}\alpha(x) = \phi\circ f^A\left(x^A\right) = \phi\left[x^A(f)\right] = \phi[f(x)\mod\mathscr{M}] = f(x), \tag{58}
$$

then

$$
D(\widetilde{\theta})(X_1,\ldots,f\cdot X_i,\ldots,X_k) = f\cdot D(\widetilde{\theta})(X_1,\ldots,\ldots,\ldots,X_k). \tag{59}
$$

□

Definition 5. For any $0\le k\le\dim(M)$, define the operator

$$
\widetilde{d}\colon \Omega^k\left(M^A,\mathbb{R}\right)\longrightarrow\Omega^{k+1}\left(M^A,\mathbb{R}\right), \tag{60}
$$

such that $\widetilde{d} = C\circ d\circ D$ where $d\colon \Omega^k(M)\longrightarrow\Omega^{k+1}(M)$ is the cohomology operator on M.

With notations as above, we have the following:

Theorem 1. *For any* $\tilde{\theta} \in \Omega^k(M^A, \mathbb{R})$ *and* $\tilde{X}_1, \ldots, \tilde{X}_{k+1} \in \mathfrak{X}(M^A, \mathbb{R})$, *we have*

$$\begin{aligned}\tilde{d}(\tilde{\theta})(\tilde{X}_1, \ldots, \tilde{X}_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i-1} \cdot \tilde{X}_i\Big(\tilde{\theta}\Big(\tilde{X}_1, \ldots, \hat{\tilde{X}}_i, \ldots, \tilde{X}_{k+1}\Big)\Big)\\ &+ \sum_{1\leq i<j\leq k+1} (-1)^{i+j}\tilde{\theta}\Big(\big[\tilde{X}_i, \tilde{X}_j\big], \tilde{X}_1, \ldots, \hat{\tilde{X}}_i, \ldots, \hat{\tilde{X}}_j, \ldots, \tilde{X}_{k+1}\Big).\end{aligned} \tag{61}$$

Proof. By definition

$$\begin{aligned}\tilde{d}(\tilde{\theta})(\tilde{X}_1, \ldots, \tilde{X}_{k+1}) &= C^\circ d^\circ D(\tilde{\theta})(\tilde{X}_1, \ldots, \tilde{X}_{k+1}) = C[d(D(\tilde{\theta}))](\tilde{X}_1, \ldots, \tilde{X}_{k+1})\\ &= \Big(d(D(\tilde{\theta}))\Big(R_2{}^\circ\tilde{X}_1{}^\circ R_1, \ldots, R_2\circ\tilde{X}_{k+1}\circ R_1\Big)\Big)\circ\pi\ (*).\end{aligned} \tag{62}$$

Set $Y_1 = R_2{}^\circ\tilde{X}_1{}^\circ R_1, \ldots, Y_{k+1} = R_2\circ\tilde{X}_{k+1}\circ R_1$, then

$$\begin{aligned}\tilde{d}(D(\tilde{\theta}))(Y_1, \ldots, Y_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i-1} \cdot Y_i\Big(D(\tilde{\theta})\Big(Y_1, \ldots, \hat{Y}_i, \ldots, Y_{k+1}\Big)\Big)\\ &+ \sum_{1\leq i<j\leq k+1} (-1)^{i+j} D(\tilde{\theta})\Big(\big[Y_i, Y_j\big], Y_1, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, \tilde{Y}_{k+1}\Big).\end{aligned} \tag{63}$$

Observe that for each i,

$$Y_i\Big(D(\tilde{\theta})\Big(Y_1, \ldots, \hat{Y}_i, \ldots, Y_{k+1}\Big)\Big) = Y_i\Big(\tilde{\theta}\Big(T_2{}^\circ Y_1^A\circ T_1, \ldots, T_2{}^\circ\widehat{Y_i^A\circ} T_1, \ldots, T_2\circ Y_{k+1}^A\circ T_1\Big)^\circ\alpha\Big). \tag{64}$$

Observe that for any

$$T_2{}^\circ Y_j^A\circ T_1 = T_2\circ\Big(R_2{}^\circ\tilde{X}_j{}^\circ R_1\Big)^A\circ T_1. \tag{65}$$

Let $\tilde{f} \in C^\infty(M^A, \mathbb{R})$, then

$$\begin{aligned}T_2{}^\circ\Big(R_2{}^\circ\tilde{X}_j{}^\circ R_1\Big)^{A^\circ} T_1(\tilde{f}) &= T_2\circ\Big(R_2{}^\circ\tilde{X}_j{}^\circ R_1\Big)^A\Big(\tilde{f}^\circ\alpha\Big)^A = T_2\circ\Big(R_2{}^\circ\tilde{X}_j\circ R_1\Big(\tilde{f}^\circ\alpha\Big)\Big)^A = T_2{}^\circ\Big[\tilde{X}_j\Big(\tilde{f}^\circ\alpha^\circ\pi\Big)\Big]^A\\ &= T_2\circ\big[\tilde{X}_j(\tilde{f})\big]^A = \phi\circ\big[\tilde{X}_j(\tilde{f})\big]^A = \tilde{X}_j(\tilde{f}).\end{aligned} \tag{66}$$

Thus,

$$Y_i\Big(D(\tilde{\theta})\Big(Y_1, \ldots, \hat{Y}_i, \ldots, Y_{k+1}\Big)\Big) = Y_i\Big(\tilde{\theta}\Big(\tilde{X}_1, \ldots, \hat{\tilde{X}}_i, \ldots, \tilde{X}_{k+1}\Big)^\circ\alpha\Big). \tag{67}$$

Observe also that

$$
\begin{aligned}
Y_i\left(\tilde{\theta}\left(\tilde{X}_1,\ldots,\hat{\tilde{X}}_i,\ldots,\tilde{X}_{k+1}\right)^{\circ}\alpha\right) &= R_2{}^{\circ}\tilde{X}_i{\circ}R_1\left(\tilde{\theta}\left(\tilde{X}_1,\ldots,\hat{\tilde{X}}_i,\ldots,\tilde{X}_{k+1}\right)^{\circ}\alpha\right)\\
&= R_2{}^{\circ}\left[\tilde{X}_i\left(\tilde{\theta}\left(\tilde{X}_1,\ldots,\hat{\tilde{X}}_i,\ldots,\tilde{X}_{k+1}\right)^{\circ}\alpha{\circ}\pi\right)\right]\\
&= R_2{}^{\circ}\left[\tilde{X}_i(\tilde{\theta})\left(\tilde{X}_1,\ldots,\hat{\tilde{X}}_i,\ldots,\tilde{X}_{k+1}\right)\right]\\
&= \tilde{X}_i(\tilde{\theta})\left(\tilde{X}_1,\ldots,\hat{\tilde{X}}_i,\ldots,\tilde{X}_{k+1}\right)^{\circ}\alpha.
\end{aligned}
\tag{68}
$$

Also,

$$
D(\tilde{\theta})\left(\left[Y_i,Y_j\right],Y_1,\ldots,\hat{Y}_i,\ldots,\hat{Y}_j,\ldots,Y_{k+1}\right) = \tilde{\theta}\left(R\left(\left[Y_i,Y_j\right]\right),R(Y_1),\ldots,\widehat{R(Y_i)},\ldots,\widehat{R(Y_j)},\ldots,R(Y_{k+1})\right).
\tag{69}
$$

Observe that

$$
R\left(\left[Y_i,Y_j\right]\right) = T_2\left[Y_i,Y_j\right]^A T_1.
\tag{70}
$$

Let $\tilde{f}\in C^{\infty}(M^A,\mathbb{R})$, then

$$
\begin{aligned}
T_2{}^{\circ}\left[Y_i,Y_j\right]^{A^{\circ}}T_1(\tilde{f}) &= T_2{\circ}\left[Y_i,Y_j\right]^A\left(\tilde{f}^{\circ}\alpha\right)^A\\
&= T_2{}^{\circ}\left(\left[Y_i,Y_j\right]\left(\tilde{f}^{\circ}\alpha\right)\right)^A\\
&= T_2\left[Y_i\left(Y_j\left(\tilde{f}^{\circ}\alpha\right)\right)-Y_j(Y_i)\left(\tilde{f}^{\circ}\alpha\right)\right]^A\\
&= T_2\left[R_2{}^{\circ}\tilde{X}_i{}^{\circ}R_1\left(R_2{}^{\circ}\tilde{X}_j{\circ}R_1\left(\tilde{f}^{\circ}\alpha\right)\right)-R_2{\circ}\tilde{X}_j{\circ}R_1\left(R_2{}^{\circ}\tilde{X}_i{\circ}R_1\right)\left(\tilde{f}^{\circ}\alpha\right)\right]^A.
\end{aligned}
\tag{71}
$$

It is not difficult to see that

$$
\begin{aligned}
R_2{}^{\circ}\tilde{X}_j{}^{\circ}R_1\left(\tilde{f}^{\circ}\alpha\right) &= R_2{\circ}\tilde{X}_j\left(\tilde{f}^{\circ}\alpha{\circ}\pi\right) = \tilde{X}_j(\tilde{f}){\circ}\pi,\\
\text{and}R_2{}^{\circ}\tilde{X}_i{}^{\circ}R_1\left(R_2{}^{\circ}\tilde{X}_j{\circ}\mathrm{R}_1\left(\tilde{\mathrm{f}}^{\circ}\alpha\right)\right) &= R_2{\circ}\tilde{X}_i{\circ}R_1\left(\tilde{X}_j(\tilde{f})^{\circ}\pi\right)\\
&= R_2{}^{\circ}\tilde{X}_i\left(\tilde{X}_j(\tilde{f})^{\circ}\pi{\circ}\alpha\right) = \tilde{X}_i\left(\tilde{X}_j(\tilde{f})\right){\circ}\alpha.
\end{aligned}
\tag{72}
$$

Then,

$$
\begin{aligned}
T_2{}^{\circ}\left[Y_i,Y_j\right]^{A^{\circ}}T_1(\tilde{f}) &= T_2\left[\tilde{X}_i\left(\tilde{X}_j(\tilde{f})\right)^{\circ}\alpha-\tilde{X}_j\left(\tilde{X}_i(\tilde{f})\right){\circ}\alpha\right]^A\\
&= \phi^{\circ}\left[\tilde{X}_i\left(\tilde{X}_j(\tilde{f})\right)^{\circ}\alpha\right]^A-\phi{\circ}\left[\tilde{X}_j\left(\tilde{X}_i(\tilde{f})\right)^{\circ}\alpha\right]^A\\
&= \tilde{X}_i\left(\tilde{X}_j(\tilde{f})\right)-\tilde{X}_j\left(\tilde{X}_i(\tilde{f})\right)\\
&= \left[\tilde{X}_i,\tilde{X}_j\right](\tilde{f}),\text{ and}\\
R\left(\left[Y_i,Y_j\right]\right) &= \left[\tilde{X}_i,\tilde{X}_j\right],R(Y_i)=\tilde{X}_i.
\end{aligned}
\tag{73}
$$

Thus,

$$D(\widetilde{\theta})\Big(\big[Y_i, Y_j\big], Y_1, \ldots, \widehat{Y}_i, \ldots, \widehat{Y}_j, \ldots, Y_{k+1}\Big) = \widetilde{\theta}\Big(\big[\widetilde{X}_i, \widetilde{X}_j\big], \widetilde{X}_1, \ldots, \widehat{\widetilde{X}}_i, \ldots, \widehat{\widetilde{X}_j}, \ldots, \widetilde{X}_{k+1}\Big) \circ \alpha \ (2). \tag{74}$$

Replacing (1) and (2) in $(*)$, we obtain

$$\begin{aligned}\widetilde{d}(\widetilde{\theta})(\widetilde{X}_1, \ldots, \widetilde{X}_{k+1}) = {} & \sum_{i=1}^{k+1} (-1)^{i-1} \cdot \widetilde{X}_i\Big(\widetilde{\theta}\Big(\widetilde{X}_1, \ldots, \widehat{\widetilde{X}}_i, \ldots, \widetilde{X}_{k+1}\Big)\Big) \\ & + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \widetilde{\theta}\Big(\big[\widetilde{X}_i, \widetilde{X}_j\big], \widetilde{X}_1, \ldots, \widehat{\widetilde{X}}_i, \ldots, \widehat{\widetilde{X}}_j, \ldots, \widetilde{X}_{k+1}\Big).\end{aligned} \tag{75}$$

□

Remark 7 (Conclusion). The previous result shows that $\widetilde{d}$ is the exterior derivative in M^A and satisfies $\widetilde{d}^2 = 0$, which makes the sequence $(\Omega^*(M^A, \mathbb{R}), \widetilde{d})$ to a complex of differential forms on M^A, and we write $H_{DR}(M^A)$ for the de Rham cohomology on M^A and denote by $H(M^A, \mathbb{R})$ the cohomology associated to the complex $(\Omega^*(M^A, \mathbb{R}), \widetilde{d})$. This gives the possibility to extend this area in different directions of differential geometry with applications.

Data Availability

The data supporting the current study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. Weil, "Théorie des points proches sur les variétés différentiables," *Colloque Geom. diff.*, pp. 111–117, 1953.

[2] I. Kolar, "Handbook of global analysis," *Weil Bundles as Generalized Jet Space*, pp. 625–664, Elsevier, Amsterdam, The Netherlands, 2008.

[3] A. Morimoto, "Prolongation of connections to bundles of infinitely near points," *Journal of Differential Geometry*, vol. 11, no. 4, pp. 479–498, 1976.

[4] I. Kolar and P. Michor, *J. Slovak Natural Operations in Differential Geometry*, Springer, Berlin, Germany, 1993.

[5] B. G. R. Bossoto and E. Okassa, "Champs de vecteurs et formes differentielles sur une variété des points proches," *Archivum Mathematicum*, vol. 44, pp. 1959–171, 2008.

[6] V. Nkou, B. G. R. Bossoto, and E. Okassa, "New properties of prolongations of Linear connections on Weil bundles," *Acta Mathematica Universitatis Comenianae*, vol. 85, no. 1, pp. 69–80, 2016.

[7] V. Nkou, B. G. R. Bossoto, and E. Okassa, "New characterizations of vector fields on Weil bundles," *Theoretical Mathematics and Applications*, vol. 5, no. 2, pp. 1792–9709, 2015.

[8] E. Okassa, "Prolongements des champs de vecteurs à des variétés des points proches," *C. R. Acad. Sci. Paris*, vol. 300, no. 6, pp. 173–176, 1985.

[9] E. Okassa, "Prolongements des champs de vecteurs à des variétés des points proches," *Annales de la Faculté des Sciences de Toulouse, Mathématiques*, vol. 8, no. 3, pp. 349–366, 1986-1987.

[10] E. Okassa, "Relèvements des structures symplectiques et pseudo-riemanniennes à des variétés de points proches," *Nagoya Mathematical Journal*, vol. 115, pp. 63–71, 1989.

[11] R. Alonso, "Jet manifold associated to Weil bundle Archivum Mathematicum," *Archivum Mathematicum*, vol. 36, pp. 195–199, 2000.

[12] D. R. Grigore and D. Krupka, "Invariants of velocities and higher-order Grassmann bundles," *Journal of Geometry and Physics*, vol. 24, no. 3, pp. 244–264, 1998.

[13] A. Mialebama Bouesso and B. G. R. Bossoto, *Some Geometric Properties on Weil Bundles.*

[14] I. Kolar, "Affine structure on Weil bundles," *Nagoya Mathematical Journal*, vol. 158, pp. 99–106, 2000.