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Research Article

A New Proof of Rational Cycles for Collatz-Like Functions Using a Coprime Condition

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In this paper, we study the bounded trajectories of Collatz-like functions. Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β are coprime. Let $\bar{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \leq i \leq \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod{\beta}$. We define the function $C_{(\alpha,\beta,\bar{k})}\colon \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$ and the sequence $\left\{n, C_{(\alpha,\beta,\bar{k})}(n), C^2_{(\alpha,\beta,\bar{k})}(n), \cdots\right\}$ a trajectory of n. We say that the trajectory of n is an integral loop if there exists some N in $\mathbb{Z}_{>0}$ so that $C^N_{(\alpha,\beta,\bar{k})}(n) = n$. We define the characteristic mapping $\chi_{(\alpha,\beta,\bar{k})}\colon \mathbb{Z}_{>0} \longrightarrow \{0, 1, \ldots, \beta - 1\}$ and the sequence $\left\{n, \chi_{(\alpha,\beta,\bar{k})}(n), \chi^2_{(\alpha,\beta,\bar{k})}(n), \cdots\right\}$ the characteristic trajectory of n. Let $B \in \mathbb{Z}_\beta$ be a β-adic sequence so that $B = (\chi^i_{(\alpha,\beta,\bar{k})}(n))_{i\geq 0}$. We say that B is eventually periodic if it eventually has a purely β-adic expansion. We show that the trajectory of n eventually enters an integral loop if and only if B is eventually periodic.

1. Introduction

The $3x + 1$ problem, also known as the Collatz conjecture, had its beginnings in the University of Berlin in 1930 [1]. In particular, Dr. Collatz was interested in cycles that could be generated with number-theoretic functions. This study later evolved to what we know today as the $3x + 1$ problem or Collatz function. It was not until the International Congress of Mathematics of 1950 that the problem began to spread [2]. Eventually, the problem gained sufficient attention and became something akin to American folklore.

Definition 1

$$C(n) := \begin{cases} 3n + 1, & n \equiv 1 \pmod{2}, \\ \dfrac{n}{2}, & n \equiv 0 \pmod{2}, \end{cases} \qquad T(n) := \begin{cases} \dfrac{3n + 1}{2}, & n \equiv 1 \pmod{2}, \\ \dfrac{n}{2}, & n \equiv 0 \pmod{2}. \end{cases} \tag{1}$$

What makes this problem initially interesting is the fact that the conjecture itself is so difficult to prove. The problem has proven to be quite evasive to proof by all modern mathematical techniques tried thus far. Insomuch that particular generalizations of the $3x + 1$ problem have been demonstrated to be arithmetically undecidable [3–5]. The $3x + 1$ problem stands on the same footing as the Riemann hypothesis being a Π_2 problem with regard to complexity

[4]. It is even attributed to Dr. Erdős to have said, “Mathematics is not yet ripe enough for such questions.”

Conjecture 2 *(Collatz). For all $n \in \mathbb{Z}_{>0}$, there exists $i \in \mathbb{Z}_{\geq 0}$ such that $C^i(n) = 1$.*

A resolution to the Collatz conjecture in itself would arguably not be of much practical value. Rather more important would be the method of proof. From a pessimistic standpoint, there may be a solution that only resolves the $3x + 1$ problem uniquely. This would be a pity, as the $3x + 1$ problem is only one of a vast collection of number-theoretical functions of its kind. More optimistically, a hypothetical method of proof could be generalized yielding new knowledge and tools to study other topics of mathematics. Such mathematical topics include but are not limited to exponential diophantine equations, discrete dynamical systems, mathematical logic, and Turing machines, among others [6]. The elusiveness of an answer belies the fact the Collatz conjecture touches nearly every branch of mathematics.

In this paper, we study properties of integral loops in a generalized Collatz-like mapping. We make an improvement on the work performed regarding the set of rational cycles for the $3x + 1$ problem. It has been proven that the parity sequence (characteristic sequence) associated with the trajectory of a starting integer is eventually periodic if and only if the trajectory eventually enters an integral loop [7, 8]. Previous proofs of this fact depended on the 2-adic nature of the parity sequence. Here, we present an alternative proof that is 2-adic invariant. We also provide examples.

2. Preliminary Definitions

2.1. Integral Loops

Definition 3. Fix $k \in \mathbb{Z}_{>0}^{odd}$. Let $C_k\colon \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$ be defined as

$$C_k(n) = \begin{cases} 3n + k, & \text{if } n \equiv 1 \pmod 2, \\ \dfrac{n}{2}, & \text{if } n \equiv 0 \pmod 2. \end{cases} \tag{2}$$

Let $C_k^0(n) = n$. Furthermore, for $s \geq 0$, we denote function composition as $C_k^{s+1}(n) = C_k^s \circ C_k(n)$. Finally, if $k = 1$, then we write C instead of C_1.

Definition 4. Fix $k \in \mathbb{Z}_{>0}^{odd}$. Let $\chi_k\colon \mathbb{Z}_{>0} \longrightarrow \{0, 1\}$ be defined as

$$\chi_k(n) = \begin{cases} 1, & \text{if } C_k(n) \equiv 1 \pmod 2, \\ 0, & \text{if } C_k(n) \equiv 0 \pmod 2. \end{cases} \tag{3}$$

Let $\chi_k^0(n) \equiv n \pmod 2$. Furthermore, for $s \geq 0$, we denote function composition as $\chi_k^{s+1}(n) = \chi_k \circ C_k^{s+1}(n)$. Finally, if $k = 1$, then we write χ instead of χ_1.

Definition 5. Fix $k \in \mathbb{Z}_{>0}^{odd}$ and $n \in \mathbb{Z}_{>0}$. Let $\{C_k^0(n), C_k^1(n), \ldots\}$ denote the trajectory of n.

Definition 6. Fix $k \in \mathbb{Z}_{>0}^{odd}$ and $n \in \mathbb{Z}_{>0}$. Let $\{\chi_k^0(n), \chi_k^1(n), \ldots\}$ denote the characteristic trajectory of n.

Let $\{0, 1\}^{**}$ denote the set of binary sequences of infinite length.

Definition 7. Fix $k \in \mathbb{Z}_{>0}^{odd}$ and $n \in \mathbb{Z}_{>0}$. We say $B \in \{0, 1\}^{**}$ is associated with the characteristic trajectory of n if $B = (\chi_k^i(n))_{i \geq 0}$.

Proposition 8. *Fix $k \in \mathbb{Z}_{>0}^{odd}$, $n \in \mathbb{Z}_{>0}$, and $N \in \mathbb{Z}_{>0}$. Let $B \in \{0, 1\}^{**}$ such that $B = (\chi_k^i(n))_{i \geq 0}$. Let ρ and ν be the count of the number of zeros and ones, respectively, in B for all i between zero and $N - 1$ inclusive. For $1 \leq j \leq \nu - 1$, let m_j denote the number of zeros between $(j - 1)$th and jth one.*

(1) If n is odd, let $m_0 = 0$ and m_ν be the remaining zeros after the last one.

(2) If n is even, let m_0 be the number of leading zeros. Observe that $n = 2^{m_0}\tilde{n}$ for $\tilde{n}$ being odd. So, we obtain $m_1, \ldots, m_{\nu-1}$ by (1) and $\tilde{n}$.

Then, n satisfies the following equality:

$$C_k^N(n) = \frac{3^\nu n + k2^{m_0}\left(3^{\nu-1} + \sum_{r=1}^{\nu-1} 3^{\nu-1-r} 2^{\sum_{j=1}^{r} m_j}\right)}{2^\rho}. \tag{4}$$

Proof. Let us assume first that n is odd, which implies that $m_0 = 0$. Therefore, equation (4) reduces to the form,

$$C_k^N(n) = \frac{3^\nu n + k\left(3^{\nu-1} + \sum_{r=1}^{\nu-1} 3^{\nu-1-r} 2^{\sum_{j=1}^{r} m_j}\right)}{2^\rho}. \tag{5}$$

By construction of ρ and ν, we make the substitution $N = \rho + \nu$. We know that $C_k^N(n)$ represents N compositions of the C_k mapping with starting integer n. Therefore, we can advance the trajectory of n by one and subtract one step from N in the following manner:

$$C_k^N(n) = C_k^{\rho+\nu}(n) = C_k^{\rho+\nu-1} \circ C_k(n) = C_k^{\rho+\nu-1}(3n + k). \tag{6}$$

Both the terms $3n$ and k are odd. Therefore, their sum is even. We can divide by two, m_1 number of times yielding the following expression:

$$C_k^N(n) = C_k^{\rho-m_1+\nu-1} \circ C_k^{m_1}(3n + k) = C_k^{\rho-m_1+\nu-1}\left(\frac{3n + k}{2^{m_1}}\right). \tag{7}$$

By hypothesis, this expression is again odd. Therefore, we can apply the C_k map for one more step.

$$C_k^N(n) = C_k^{\rho-m_1+\nu-2} \circ C_k\left(\frac{3n + k}{2^{m_1}}\right) = C_k^{\rho-m_1+\nu-2}\left(\frac{3^2 n + 3k + 2^{m_1}k}{2^{m_1}}\right). \tag{8}$$

In turn, this expression is again even. Therefore, we can divide by m_2 powers of two. Following this, we can reduce the expression again,

$$C_k^N(n) = C_k^{\rho-m_1-m_2+\nu-2} \circ C_k^{m_2}\left(\frac{3n+k}{2^{m_1}}\right) = C_k^{\rho-m_1-m_2+\nu-2}\left(\frac{3^2n+3k+2^{m_1}k}{2^{m_1+m_2}}\right). \tag{9}$$

We proceed to follow this process inductively until the length of the entire finite subsequence of B is exhausted.

$$\begin{aligned} C_k^N(n) &= C_k^{\rho-m_1-\ldots-m_\nu+\nu-1-\ldots-1} \circ C_k^{m_{\nu-1}}\left(\frac{3^\nu n+3^{\nu-1}k+3^{\nu-2}2^{m_1}k+3^{\nu-3}2^{m_1+m_2}k+\ldots}{2^{m_1+m_2+\ldots}}\right),\\ &= C_k^0\left(\frac{3^\nu n+3^{\nu-1}k+3^{\nu-2}2^{m_1}k+3^{\nu-3}2^{m_1+m_2}k+\ldots}{2^\rho}\right),\\ &= \frac{3^\nu n+k2^{m_0}\left(3^{\nu-1}+\sum_{r=1}^{\nu-1}3^{\nu-1-r}2^{\sum_{j=1}^r m_j}\right)}{2^\rho}. \end{aligned} \tag{10}$$

On the other hand, we suppose that n is even. Then, $m_0>0$ and by hypothesis $n=2^{m_0}\tilde{n}$ for an odd integer $\tilde{n}$. Starting from equation (4), we find by taking m_0 steps,

$$C_k^N(n) = C_k^{\rho+\nu}(n) = C_k^{\rho-m_0+\nu}\circ C_k^{m_0}(n) = C_k^{\rho-m_0+\nu}\left(\frac{n}{2^{m_0}}\right). \tag{11}$$

Recalling by hypothesis $n=2^{m_0}\tilde{n}$ implies that $n/2^{m_0}=\tilde{n}$. Substituting this quantity in equation (11) yields

$$C_k^N(n) = C_k^{\rho-m_0+\nu}\left(\frac{n}{2^{m_0}}\right) = C_k^{\rho-m_0+\nu}(\tilde{n}). \tag{12}$$

We then consider the previous argument starting with $\tilde{n}$, $\tilde{\rho}=\rho-m_0$, and ν unchanged.

$$\begin{aligned} C_k^{N-m_0}(\tilde{n}) &= \frac{3^\nu\tilde{n}+k\left(3^{\nu-1}+\sum_{r=1}^{\nu-1}3^{\nu-1-r}2^{\sum_{j=1}^r m_j}\right)}{2^{\tilde{\rho}}},\\ &= \frac{3^\nu n/2^{m_0}+k\left(3^{\nu-1}+\sum_{r=1}^{\nu-1}3^{\nu-1-r}2^{\sum_{j=1}^r m_j}\right)}{2^{\rho-m_0}},\\ &= \left(\frac{2^{m_0}}{2^{m_0}}\right)\frac{3^\nu n/2^{m_0}+k\left(3^{\nu-1}+\sum_{r=1}^{\nu-1}3^{\nu-1-r}2^{\sum_{j=1}^r m_j}\right)}{2^{\rho-m_0}},\\ &= \frac{3^\nu n+k2^{m_0}\left(3^{\nu-1}+\sum_{r=1}^{\nu-1}3^{\nu-1-r}2^{\sum_{j=1}^r m_j}\right)}{2^\rho}. \end{aligned} \tag{13}$$

Therefore, we find that

$$C_k^N(n) = C_k^{N-m_0}(\tilde{n}) = \frac{3^\nu n+k2^{m_0}\left(3^{\nu-1}+\sum_{r=1}^{\nu-1}3^{\nu-1-r}2^{\sum_{j=1}^r m_j}\right)}{2^\rho}. \tag{14}$$

□

Proposition 9. *Fix $k\in\mathbb{Z}_{>0}^{odd}$ and $n\in\mathbb{Z}_{>0}$. Let $B\in\{0,1\}^{**}$ such that $B=(\chi_k^i(n))_{i\geq 0}$. We suppose that N is the smallest positive integer such that $C_k^N(n)=n$. Let ρ and ν be the count of the number of zeros and ones, respectively, in B for all i between 0 and N inclusive. For $1\leq j\leq\nu-1$, let m_j denote the number of zeros between $(j-1)$th and jth one.*

(1) If n is odd, let $m_0=0$ and m_ν be the remaining zeros after the last one.

(2) If n is even, let m_0 be the number of leading zeros. Observe that $n=2^{m_0}\tilde{n}$ for $\tilde{n}$ being odd. So, we obtain $m_1,\ldots,m_{\nu-1}$ by (1) and $\tilde{n}$.

Then, n satisfies the following equality:

$$n = \frac{k2^{m_0}\left(3^{\nu-1}+\sum_{r=1}^{\nu-1}3^{\nu-1-r}2^{\sum_{j=1}^r m_j}\right)}{2^\rho-3^\nu}. \tag{15}$$

Proof. Equation (15) can be derived from equation (4) by setting $C_k^N(n) = n$.

We will refer to equation (15) as the integral loop formula. □

Example 1. We consider the integral loop induced by $n = 157$ and $k = 175$. With respect to equation (15), we find that

$$\begin{aligned} n &= \frac{175\left(3^{4-1} + 3^{4-2} \times 2^1 + 3^{4-3} \times 2^{1+3} + 3^{4-4} \times 2^{1+3+2}\right)}{2^8 - 3^4}, \\ &= \frac{175\,(27 + 9 \times 2 + 3 \times 16 + 1 \times 64)}{175} = 157. \end{aligned} \tag{16}$$

2.2. Rational Cycles

Definition 10. Let $B \in \{0,1\}^{**}$. We say that B is a rational 2-adic if it can be written $a = p/q$ for some integers p, q with q odd [8].

Definition 11. Let $B \in \{0,1\}^{**}$. We say that B is eventually periodic if there exists a positive integer K such that $B^{K+i} = B^i$ for all sufficiently large i. If $B^{K+i} = B^i$ for all i, then B is called periodic [8].

Theorem 12. *There is a one-to-one correspondence between rational numbers $a = p/q$, where q is odd, and eventually periodic sequences B, which associates with each such rational number a the bit sequence $(B^0, B^1, \ldots)$ of its 2-adic expansion. The sequence B is strictly periodic if and only if $a \leq 0$ and $|a| < 1$. [9].*

Proof. Let $B = (B^0, B^1, \ldots)$ be a strictly periodic sequence of period K. Set $a = \sum_{i=0}^{\infty} B^i 2^i$. Computing in $\mathbb{Z}_2$, we find

$$2^K a = \sum_{i=0}^{\infty} B^i 2^{i+K} = \sum_{i=0}^{\infty} B^{i+K} 2^{i+K} = \sum_{i=T}^{\infty} B^i 2^i = a - \sum_{i=0}^{K-1} B^i 2^i. \tag{17}$$

Hence,

$$a = -\frac{\sum_{i=0}^{K-1} B^i 2^i}{2^K - 1}. \tag{18}$$

is a negative rational number. We write $a = p/q$ as a fraction reduced to the lowest terms with q positive. Then, q is odd, $p \leq 0$, and $|p| < q$.

Conversely, suppose $a = p/q$ is given in the lowest terms with q an odd positive integer, $p \leq 0$, and $|p| < q$. Let $K = \mathrm{ord}_q(2)$ be the smallest integer such that $2^K \equiv 1 \pmod{q}$. Then, $2^K - 1$ is divisible by q, so set $s = (2^K - 1)/q$ and write $s \times (-p) = \sum_{i=0}^{K-1} B^i 2^i$. Thus, $a = sp/(2^K - 1)$. The calculations leading to equation (18) may be run backward to see that the segment $B^0, B^1, \ldots, B^{K-1}$ is a single period of a strictly periodic sequence.

Now, we suppose that $a = p/q$ is an arbitrary rational number. Let $M = \lceil a \rceil$ be the next largest integer. If $M \geq 0$, then its 2-adic expansion is finite, ending in a string of zeros. If $M < 0$, then its 2-adic expansion ends in an infinite string of ones. However, $a = M + p'/q'$, where $p' \leq 0$ and $|p'| < q$. Thus, the 2-adic expansion for p'/q' is periodic. It follows that the 2-adic expansion for the sum $a = M + p'/q'$ is eventually periodic.

Conversely, an eventually periodic sequence $B = (B^0, B^1, \ldots)$ corresponds to a rational number $a = \sum_{i=0}^{\infty} B^i 2^i$ because it is given by a finite transient term $\sum_{i=0}^{K-1} B^i 2^i$ for some nonnegative integer K plus a periodic term, $\sum_{i=K}^{\infty} B^i 2^i = 2^K \sum_{j=0}^{\infty} B^{j+K} 2^j$, both of which are rational numbers. □

Proposition 13. *Fix $k \in \mathbb{Z}_{>0}^{odd}$ and $n \in \mathbb{Z}_{>0}$. Let $B^{**} \in \{0,1\}^{**}$ such that $B = (\chi_k^i(n))_{i \geq 0}$. We assume that the trajectory of n is bounded above. Then, there exists some n' contained in the trajectory of n and the smallest N in $\mathbb{Z}_{>0}$ such that $C_k^N(n') = n'$ if and only if B is a rational 2-adic.*

Proof. We suppose that first in the trajectory of n, there exists an $n', N \in \mathbb{Z}_{>0}$ so that $C_k^N(n') = n'$. We assume further that N is the smallest positive integer which satisfies this equality. Then, B has a repeating period of length N and is rational.

Now, we assume that B is a rational 2-adic. Let x_0 be the starting value whose characteristic trajectory generates the repeating period of B. Let us define the symbol σ as follows:

$$\sigma = 3^{\widetilde{\nu}-1} + \sum_{r=1}^{\widetilde{\nu}-1} 3^{\widetilde{\nu}-1-r} 2^{\sum_{j=1}^{r} m_j}, \tag{19}$$

where $\Sigma m_j = \widetilde{\rho}$. We observe immediately that σ is finite as both $\widetilde{\rho}$ and $\widetilde{\nu}$ are finite. We have the following sequence of values by continuous application of the Collatz map $C_k^{\widetilde{\rho}+\widetilde{\nu}}$ generating values $x_1, x_2, \ldots x_n$

$$\begin{aligned} 2^{\widetilde{\rho}} x_1 &= 3^{\widetilde{\nu}} x_0 + k\sigma, \\ 2^{\widetilde{\rho}} x_2 &= 3^{\widetilde{\nu}} x_1 + k\sigma, \\ &\cdots \\ 2^{\widetilde{\rho}} x_n &= 3^{\widetilde{\nu}} x_{n-1} + k\sigma. \end{aligned} \tag{20}$$

We can consolidate the terms into an expression that includes only x_n and x_0 as follows:

$$2^{\widetilde{\rho}n}x_n = 3^{\widetilde{\nu}n}x_0 + k\sigma\sum_{i=0}^{n-1}3^{(n-1-i)\widetilde{\nu}}2^{i\widetilde{\rho}}. \quad (21)$$

Thus, solving for x_0 yields

$$x_0 = \left(\frac{2^{\widetilde{\rho}}}{3^{\widetilde{\nu}}}\right)^n\left(x_n - \left(\frac{k\sigma}{2^{\widetilde{\rho}}}\right)\left(\sum_{i=0}^{n-1}\left(\frac{3^{\widetilde{\nu}}}{2^{\widetilde{\rho}}}\right)^{i-(n-1)}\right)\right),$$
$$x_0 = \left(\frac{2^{\widetilde{\rho}}}{3^{\widetilde{\nu}}}\right)^n\left(x_n - \left(\frac{k\sigma}{2^{\widetilde{\rho}}}\right)\left(\frac{1-\left(3^{\widetilde{\nu}}/2^{\widetilde{\rho}}\right)^n}{1-3^{\widetilde{\nu}}/2^{\widetilde{\rho}}}\right)\right), \quad (22)$$
$$x_0 = \left(\frac{2^{\widetilde{\rho}}}{3^{\widetilde{\nu}}}\right)^n\left(x_n - k\sigma\left(\frac{1-\left(3^{\widetilde{\nu}}/2^{\widetilde{\rho}}\right)^n}{2^{\widetilde{\rho}}-3^{\widetilde{\nu}}}\right)\right).$$

We suppose first that $2^{\widetilde{\rho}}/3^{\widetilde{\nu}} > 1$. Following from equation (21), we distribute the first term, yielding the following:

$$x_0 = \left(\frac{2^{\widetilde{\rho}}}{3^{\widetilde{\nu}}}\right)^n\left(x_n - \left(\frac{k\sigma}{2^{\widetilde{\rho}}-3^{\widetilde{\nu}}}\right)\right) + \frac{k\sigma}{2^{\widetilde{\rho}}-3^{\widetilde{\nu}}}. \quad (23)$$

Then, there exists an N such that for all $n > N$, $x_n = k\sigma/2^{\widetilde{\rho}} - 3^{\widetilde{\nu}}$. We find a new x_N, which maybe different from x_0 such that x_N generates an integral loop coinciding with the original rational loop.

If conversely we suppose $2^{\widetilde{\rho}}/3^{\widetilde{\nu}} < 1$ we simply reverse equation (21) to solve for x_n in terms of x_0:

$$x_n = \left(\frac{3^{\widetilde{\nu}}}{2^{\widetilde{\rho}}}\right)^n\left(x_0 - \left(\frac{k\sigma}{2^{\widetilde{\rho}}-3^{\widetilde{\nu}}}\right)\right) + \frac{k\sigma}{2^{\widetilde{\rho}}-3^{\widetilde{\nu}}}. \quad (24)$$

We suppose first that $x_0 = k\sigma/2^{\widetilde{\rho}} - 3^{\widetilde{\nu}}$, then it follows that $x_1 = k\sigma/2^{\widetilde{\rho}} - 3^{\widetilde{\nu}}$. This equality is true for all, and thus, we have an integral loop. So, assume instead that $x_0 \neq k\sigma/2^{\widetilde{\rho}} - 3^{\widetilde{\nu}}$. Then, we can write Equation (22) in the form of a ratio exhibited in equation:

$$\frac{x_n - k\sigma/2^{\widetilde{\rho}} - 3^{\widetilde{\nu}}}{x_0 - k\sigma/2^{\widetilde{\rho}} - 3^{\widetilde{\nu}}} = \left(\frac{3^{\widetilde{\nu}}}{2^{\widetilde{\rho}}}\right)^n. \quad (25)$$

However, recall x_0 and $k\sigma/2^{\widetilde{\rho}} - 3^{\widetilde{\nu}}$ are both fixed values. Thus, the left hand side has a fixed denominator. However, as the right hand side is the quotient of two primes, the denominator expands with irreducible powers of two. If x_n is a trajectory of integers, then there would be some index N of which the denominator on the right hand side would be larger than the left hand side. This would be a contradiction.

Finally, as $\widetilde{\nu} = \ln(2)/\ln(3)\widetilde{\rho}$, as a irrational ratio cannot be expressed with a finite binary sequence.

Thus, we have shown that no rational cycle apart from an integral loop exists for any $3x + k$ mapping. □

3. Generalizations

3.1. Two-State System with Halting Condition. We would like to consider first a generalization of the two-state system given in Definition 3. To begin, we need the following lemma:

Lemma 14. *Fix $\alpha, \beta, k \in \mathbb{Z}_{>0}$ so that α, β, and k are pairwise coprimes. Then, there exists a pair of integers $x, y \in \mathbb{Z}$ satisfying the following linear Diophantine equation:*

$$\alpha x + \beta y = -k. \quad (26)$$

Proof. A necessary and sufficient condition that a linear Diophantine equation is solvable is that k is a multiple of the greatest common denominator of α and β [10]. By hypothesis, $\gcd(\alpha, \beta) = 1$. Thus, k is necessarily a multiple. We conclude that the linear Diophantine equation is always solvable. □

Proposition 15. *Fix $\alpha, \beta, k \in \mathbb{Z}_{>0}$ so that α, β and k are pairwise coprimes. Then, there exists a $t \in \{1, \ldots, \beta - 1\}$ satisfying the following equation:*

$$\alpha t + k \equiv 0 \pmod{\beta}. \quad (27)$$

Proof. We have shown from Lemma 14 that equation (26) is always solvable for a pair of integers x and y. Thus, let $t = x \pmod{\beta}$ so that t is an integer in $\{1, \ldots, \beta - 1\}$. □

Definition 16. Fix $\alpha, \beta, k \in \mathbb{Z}_{>0}$ so that α, β, and k are pairwise coprimes. Let $t \in \{1, \ldots, \beta - 1\}$ satisfying $\alpha t + k \equiv 0 \pmod{\beta}$. Let $C_{(\alpha,\beta,k)}\colon \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$ be defined as

$$C_{(\alpha,\beta,k)}(n) = \begin{cases} \alpha n + k, & \text{if } n \equiv t \pmod{\beta}, \\ \dfrac{n}{\beta}, & \text{if } n \equiv 0 \pmod{\beta}, \\ \text{HALT}, & \text{else}. \end{cases} \quad (28)$$

Let $C^0_{(\alpha,\beta,k)}(n) = n$. Furthermore, for $s \geq 0$, we denote function composition as $C^{s+1}_{(\alpha,\beta,k)}(n) = C^s_{(\alpha,\beta,k)} \circ C_{(\alpha,\beta,k)}(n)$.

For what follows let ϵ be the empty character.

Definition 17. Fix $\alpha, \beta, k \in \mathbb{Z}_{>0}$ so that α, β and k are pairwise coprimes. Let $t \in \{1, \ldots, \beta - 1\}$ satisfying $\alpha t + k \equiv 0 \pmod{\beta}$. Let $\chi_{(\alpha,\beta,k)}\colon \mathbb{Z}_{>0} \longrightarrow \{0, 1\}$ be defined as

$$\chi_{(\alpha,\beta,k)}(n) = \begin{cases} 1, & \text{if } C_{(\alpha,\beta,k)}(n) \equiv t \pmod{\beta}, \\ 0, & \text{if } C_{(\alpha,\beta,k)}(n) \equiv 0 \pmod{\beta}, \\ \epsilon, & \text{else}. \end{cases} \quad (29)$$

Let $\chi^0_{(\alpha,\beta,k)}(n) \equiv n \pmod{\beta}$. Furthermore, for $s \geq 0$, we denote function composition as $\chi^{s+1}_{(\alpha,\beta,k)}(n) = \chi_{(\alpha,\beta,k)}{}^{\circ}C^{s+1}_{(\alpha,\beta,k)}(n)$.

We define the trajectory and characteristic trajectory of n in a similar manner to Definitions 5 and 6, respectively. Furthermore, we associate $B \in \{0,1\}^{**}$ with n in a similar manner to Definition 7.

Proposition 18. *Fix $\alpha, \beta, k \in \mathbb{Z}_{>0}$ so that α, β and k are pairwise coprimes. Let $t \in \{1, \ldots, \beta - 1\}$ satisfying $\alpha t + k \equiv 0 \pmod{\beta}$. Fix $N \in \mathbb{Z}_{>0}$. Let $B \in \{0,1\}^{**}$ such that $B = (\chi^i_{(\alpha,\beta,k)}(n))_{i\geq 0}$. Let ρ and ν be the count of the number of* *zeros and ones, respectively, in B for all i between zero and $N-1$ inclusive. For $1 \le j \le \nu - 1$, let m_j denote the number of zeros between $(j-1)$th and jth one. We assume that the halting condition for $C_{(\alpha,\beta,k)}$ is not met for all $0 \le i \le N$.*

(1) If $n \equiv t \pmod{\beta}$, let $m_0 = 0$ and m_ν be the remaining zeros after the last one.

(2) If $n \equiv 0 \pmod{\beta}$, let m_0 be the number of leading zeros. We observe that $n = \beta^{m_0}\tilde{n}$ for $\tilde{n} \equiv t \pmod{\beta}$. So, we obtain $m_1, \ldots, m_{\nu-1}$ by (1) and $\tilde{n}$.

Then, n satisfies the following equality:

$$C^N_{(\alpha,\beta,k)}(n) = \frac{\alpha^\nu n + k\beta^{m_0}\left(\alpha^{\nu-1} + \sum_{r=1}^{\nu-1}\alpha^{\nu-1-r}\beta^{\sum_{j=1}^{r} m_j}\right)}{\beta^\rho}. \quad (30)$$

Proof. By hypothesis, the halting condition is not met. Thus, every step of the characteristic trajectory associated with n is nonempty. In that context, the proof is essentially identical to Proposition 8. □

Proposition 19. *Fix $\alpha, \beta, k \in \mathbb{Z}_{>0}$ so that α, β, and k are pairwise coprimes. Let $t \in \{1, \ldots, \beta - 1\}$ satisfying $\alpha t + k \equiv 0 \pmod{\beta}$. Let $B \in \{0,1\}^{**}$ such that $B = (\chi^i_{(\alpha,\beta,k)}(n))_{i \ge 0}$. Suppose that N is the smallest positive integer such that $C^{N+1}_{(\alpha,\beta,k)}(n) = n$. Let ρ and ν be the count of the number of zeros and ones, respectively, in B for all i between 0 and N inclusive. For $1 \le j \le \nu - 1$, let m_j denote the number of zeros between $(j-1)$th and jth one. We assume that the halting condition for $C_{(\alpha,\beta,k)}$ is not met for all $0 \le i \le N$.*

(1) If $n \equiv t \pmod{\beta}$, let $m_0 = 0$ and m_ν be the remaining zeros after the last one.

(2) If $n \equiv 0 \pmod{\beta}$, let m_0 be the number of leading zeros. We observe that for $\tilde{n} \equiv t \pmod{\beta}$. So, we obtain $m_1, \ldots, m_{\nu-1}$ by (1) and $\tilde{n}$.

Then, n satisfies the following equality:

$$n = \frac{k\beta^{m_0}\left(\alpha^{\nu-1} + \sum_{r=1}^{\nu-1}\alpha^{\nu-1-r}\beta^{\sum_{j=1}^{r} m_j}\right)}{\beta^\rho - \alpha^\nu}. \quad (31)$$

Proof. This follows from Proposition 18, where we derive equation (31) from equation (30) by setting $C^N_{(\alpha,\beta,k)}(n) = n$. □

Example 2. Fix $\alpha = 7$ and $\beta = 5$. We consider that the integral loop is induced by $k = 282$ and $n = 109$. The mapping $C_{(7,5,282)}$ appears as

$$C_{(7,5,282)}(n) = \begin{cases} 7n + 282, & \text{if } n \equiv 4 \pmod{5}, \\ \dfrac{n}{5}, & \text{if } n \equiv 0 \pmod{5}, \\ \text{HALT}, & \text{else}. \end{cases} \quad (32)$$

We find that the trajectory and characteristic trajectory of 109 appear as

$$\begin{aligned} &\{109, 1045, 209, 1745, 349, 2725, 545\}, \\ &\{1, 0, 1, 0, 1, 0, 0\}. \end{aligned} \quad (33)$$

Proposition 20. *Fix $\alpha, \beta, k \in \mathbb{Z}_{>0}$ so that α, β, and k are pairwise coprimes. Let $t \in \{1, \ldots, \beta - 1\}$ satisfying $\alpha t + k \equiv 0 \pmod{\beta}$. Let $B \in \{0,1\}^{**}$ such that $B = (\chi^i_{(\alpha,\beta,k)}(n))_{i \ge 0}$. We assume that the trajectory of n is bounded above. Then, there exists some n' contained in the trajectory of n and the smallest N in $\mathbb{Z}_{>0}$ such that $C^N_{(\alpha,\beta,k)}(n') = n'$ if and only if B is a rational 2-adic.*

Proof. The proof is similar to the proof of Proposition 13. Here, we replace 3 with α and 2 with β. Since α and β are coprimes, then $\beta^\rho - \alpha^\nu$ does not vanish and the quotient $(\widetilde{\alpha^\nu}/\widetilde{\beta^\rho})^n$ again has a denominator that expands with higher powers of n. Thus, we arrive at the same contradiction. □

3.2. Multistate System with No Halting Condition

Definition 21. Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β coprime. Let $\bar{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \le i \le \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod{\beta}$. Let $C_{(\alpha,\beta,\bar{k})}\colon \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{>0}$ be defined as

$$C_{(\alpha,\beta,\bar{k})}(n) = \begin{cases} \alpha n + k_i, & \text{if } \alpha n + k_i \equiv 0 \pmod{\beta}, \\ \dfrac{n}{\beta}, & \text{if } n \equiv 0 \pmod{\beta}. \end{cases} \quad (34)$$

Let $C^0_{(\alpha,\beta,\bar{k})}(n) = n$. Furthermore, for $s \ge 0$, we denote function composition as $C^{s+1}_{(\alpha,\beta,\bar{k})}(n) = C^s_{(\alpha,\beta,\bar{k})} \circ C_{(\alpha,\beta,\bar{k})}(n)$.

Definition 22. Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β are coprimes. Let $\bar{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \le i \le \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod{\beta}$. Let $\chi_{(\alpha,\beta,\bar{k})}\colon \mathbb{Z}_{>0} \longrightarrow \{0, 1, 2, \ldots, \beta - 1\}$ be defined as

$$\chi_{(\alpha,\beta,\bar{k})}(n) = \begin{cases} i, & \text{if } -\alpha \times C_{(\alpha,\beta,\bar{k})}(n) \equiv i \pmod{\beta}, \\ 0, & \text{if } C_{(\alpha,\beta,\bar{k})}(n) \equiv 0 \pmod{\beta}. \end{cases} \quad (35)$$

Let $\chi^0_{(\alpha,\beta,\bar{k})}(n) \equiv n \pmod{\beta}$. Furthermore, for $s \ge 0$, we denote function composition as $\chi^{s+1}_{(\alpha,\beta,\bar{k})}(n) = \chi_{(\alpha,\beta,\bar{k})} \circ C^{s+1}_{(\alpha,\beta,\bar{k})}(n)$.

Definition 23. Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β are coprimes. Let $\bar{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \le i \le \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod{\beta}$. Fix $n \in \mathbb{Z}_{>0}$. Let $\left\{C^0_{(\alpha,\beta,\bar{k})}(n), C^1_{(\alpha,\beta,\bar{k})}(n), \ldots\right\}$ denote the trajectory of n.

Definition 24. Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β are coprimes. Let $\bar{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \le i \le \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod{\beta}$. Fix $n \in \mathbb{Z}_{>0}$. Let

$\left\{\chi^0_{(\alpha,\beta,\overline{k})}(n), \chi^1_{(\alpha,\beta,\overline{k})}(n), \ldots\right\}$ denote the characteristic trajectory of n.

Fix $\beta \in \mathbb{Z}_{>1}$. Let $\{0,1,2,\ldots,\beta-1\}^{**}$ denote the set of β-adic sequences of infinite length.

Definition 25. Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β are coprimes. Let $\overline{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \le i \le \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod{\beta}$. Fix $n \in \mathbb{Z}_{>0}$. We say $B \in \{0,1,2,\ldots,\beta-1\}^{**}$ is associated with the characteristic trajectory of n if $B = (\chi^i_{(\alpha,\beta,\overline{k})}(n))_{i\ge 0}$.

Definition 26. Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β are coprimes. Let $\overline{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \le i \le \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod{\beta}$. Fix $n \in \mathbb{Z}_{>0}$. Fix $n \in \mathbb{Z}^{odd}_{>0}$, and $N \in \mathbb{Z}_{>0}$. We suppose further that n starts an integral loop of length N. Let $B \in \{0,1,2,\ldots,\beta-1\}^{**}$ such that $B = (\chi^i_{(\alpha,\beta,\overline{k})}(n))_{i\ge 0}$. Let ρ and ν be the count of the number of zeros and nonzero coefficients in a single period of B. Let i_j be the subindexing of i so that for all $0 \le j \le \nu - 1$, we have $\chi^{i_j}_{(\alpha,\beta,\overline{k})}(n) \equiv 0 \pmod{\beta}$. We call $(i_j)_{j=0}^{\nu-1}$ the indexing sequence associated with n.

Proposition 27. *Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β are coprimes. Let $\overline{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \le i \le \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod{\beta}$. Fix $N \in \mathbb{Z}_{>0}$. Let $B \in \{0,1,2,\ldots,\beta-1\}^{**}$ such that $B = (\chi^i_{(\alpha,\beta,\overline{k})}(n))_{i\ge 0}$. Let ρ and ν be the count of the number of zero and nonzero coefficients, respectively, in B for all i between zero and $N-1$ inclusive. For $1 \le j \le \nu - 1$, let m_j denote the number of zeros between the $(j-1)$th and jth nonzero coefficient. Let $(i_j)_{j=0}^{\nu-1}$ be the indexing sequence associated with n.*

(1) If $\alpha n \equiv -i \pmod{\beta}$, let $m_0 = 0$ and m_ν be the remaining zeros after the last nonzero coefficient.

(2) If $n \equiv 0 \pmod{\beta}$, let m_0 be the number of leading zeros. We observe that $n = \beta^{m_0}\tilde{n}$ for $\tilde{n} \equiv 0 \pmod{\beta}$. So, we obtain $m_1, \ldots, m_{\nu-1}$ by (1) and $\tilde{n}$.

Then, n satisfies the following equality:

$$C^N_{(\alpha,\beta,\overline{k})}(n) = \frac{\alpha^\nu n + \beta^{m_0}\left(\alpha^{\nu-1}k_{B^{i_0}} + \sum_{r=1}^{\nu-1}\alpha^{\nu-1-r}k_{B^{i_r}}\beta^{\sum_{j=1}^{r} m_j}\right)}{\beta^\rho}, \tag{36}$$

where $k_{B^{i_r}} \equiv B^{i_r} \pmod{\beta}$ for the r-th nonzero coefficient in a single period of B.

Because Proposition 27 is logically distinct from Proposition 8, it is necessary to restate the proof again.

Proof. Let us assume first that $n \equiv 0 \pmod{\beta}$, which implies that $m_0 = 0$. Therefore, equation (36) reduces to the form

$$C^N_{(\alpha,\beta,\overline{k})}(n) = \frac{\alpha^\nu n + \alpha^{\nu-1}k_{B^{i_0}} + \sum_{r=1}^{\nu-1}\alpha^{\nu-1-r}k_{B^{i_r}}\beta^{\sum_{j=1}^{r} m_j}}{\beta^\rho}. \tag{37}$$

By construction of ρ and ν, we make the substitution $N = \rho + \nu$. We know that $C^N_{(\alpha,\beta,\overline{k})}(n)$ represents N compositions of the $C_{(\alpha,\beta,\overline{k})}$ mapping with starting integer n. Therefore, we can advance the trajectory of n by one and subtract one step from N in the following manner:

$$C^N_{(\alpha,\beta,\overline{k})}(n) = C^{\rho+\nu}_{(\alpha,\beta,\overline{k})}(n) = C^{\rho+\nu-1}_{(\alpha,\beta,\overline{k})} \circ C_{(\alpha,\beta,\overline{k})}(n) = C^{\rho+\nu-1}_{(\alpha,\beta,\overline{k})}(\alpha n + k_{B^{i_0}}). \tag{38}$$

We recall that $\alpha n + k_{B^{i_0}} \equiv 0 \pmod{\beta}$. We can divide by β, m_1 number of times yielding the following expression:

$$C^N_{(\alpha,\beta,\overline{k})}(n) = C^{\rho-m_1+\nu-1}_{(\alpha,\beta,\overline{k})} \circ C^{m_1}_{(\alpha,\beta,\overline{k})}(\alpha n + k_{B^{i_0}}) = C^{\rho-m_1+\nu-1}_{(\alpha,\beta,\overline{k})}\left(\frac{\alpha n + k_{B^{i_0}}}{\beta^{m_1}}\right). \tag{39}$$

By hypothesis, this expression is again nonzero modulo β. Therefore, we can apply the $C_{(\alpha,\beta,\overline{k})}$ map for one more step.

$$C^N_{(\alpha,\beta,\overline{k})}(n) = C^{\rho-m_1+\nu-2}_{(\alpha,\beta,\overline{k})} \circ C_{(\alpha,\beta,\overline{k})}\left(\frac{\alpha n + k_{B^{i_0}}}{2^{m_1}}\right) = C^{\rho-m_1+\nu-2}_{(\alpha,\beta,\overline{k})}\left(\frac{\alpha^2 n + \alpha k_{B^{i_0}} + \beta^{m_1}k_{B^{i_1}}}{\beta^{m_1}}\right). \tag{40}$$

In turn, this expression is again zero modulo β. Therefore, we can divide by m_2 powers of β. Following this, we can reduce the expression again

$$C^{N}_{(\alpha,\beta,\overline{k})}(n) = C^{\rho-m_1-m_2+\nu-2}_{(\alpha,\beta,\overline{k})} \circ C^{m_2}_{(\alpha,\beta,\overline{k})}\left(\frac{\alpha n + k_{B^{i_0}}}{\beta^{m_1}}\right) = C^{\rho-m_1-m_2+\nu-2}_{(\alpha,\beta,\overline{k})}\left(\frac{\alpha^2 n + \alpha k_{B^{i_0}} + \beta^{m_1} k_{B^{i_1}}}{\beta^{m_1+m_2}}\right). \tag{41}$$

We proceed to follow this process inductively until the length of the entire finite subsequence of B is exhausted.

$$\begin{aligned} C^{N}_{(\alpha,\beta,\overline{k})}(n) &= C^{\rho-m_1-\ldots-m_\nu+\nu-1-\ldots-1}_{(\alpha,\beta,\overline{k})} \circ C^{m_{\nu-1}}_{(\alpha,\beta,\overline{k})}\left(\frac{\alpha^\nu n + \alpha^{\nu-1} k_{B^{i_0}} + \alpha^{\nu-2}\beta^{m_1} k_{B^{i_1}} + \alpha^{\nu-3}\beta^{m_1+m_2} k_{B^{i_2}} + \ldots}{\beta^{m_1+m_2+\ldots}}\right), \\ &= C^{0}_{(\alpha,\beta,\overline{k})}\left(\frac{\alpha^\nu n + \alpha^{\nu-1} k_{B^{i_0}} + \alpha^{\nu-2}\beta^{m_1} k_{B^{i_1}} + \alpha^{\nu-3}\beta^{m_1+m_2} k_{B^{i_2}} + \ldots}{\beta^{\rho}}\right), \\ &= \frac{\alpha^\nu n + \alpha^{\nu-1} k_{B^{i_0}} + \sum_{r=1}^{\nu-1} \alpha^{\nu-1-r} k_{B^{i_r}} \beta^{\sum_{j=1}^{r} m_j}}{\beta^{\rho}}. \end{aligned} \tag{42}$$

On the other hand, we suppose that $n \equiv 0 \pmod{\beta}$. Then, $m_0 > 0$, and by hypothesis, $n = \beta^{m_0}\widetilde{n}$ for $\widetilde{n} \equiv 0 \pmod{\beta}$. Starting from equation (36), we find by taking m_0 steps, we have

$$\begin{aligned} C^{N}_{(\alpha,\beta,\overline{k})}(n) &= C^{\rho+\nu}_{(\alpha,\beta,\overline{k})}(n) = C^{\rho-m_0+\nu}_{(\alpha,\beta,\overline{k})} \circ \\ C^{m_0}_{(\alpha,\beta,\overline{k})}(n) &= C^{\rho-m_0+\nu}_{(\alpha,\beta,\overline{k})}\left(\frac{n}{\beta^{m_0}}\right). \end{aligned} \tag{43}$$

Recalling by hypothesis $n = \beta^{m_0}\widetilde{n}$ implies $n/\beta^{m_0} = \widetilde{n}$. Substituting this quantity in equation (43) yields

$$C^{N}_{(\alpha,\beta,\overline{k})}(n) = C^{\rho-m_0+\nu}_{(\alpha,\beta,\overline{k})}\left(\frac{n}{\beta^{m_0}}\right) = C^{\rho-m_0+\nu}_{(\alpha,\beta,\overline{k})}(\widetilde{n}). \tag{44}$$

We then consider the previous argument starting with $\widetilde{n}$, $\widetilde{\rho} = \rho - m_0$, and ν unchanged.

$$\begin{aligned} C^{N-m_0}_{(\alpha,\beta,\overline{k})}(\widetilde{n}) &= \frac{\alpha^\nu \widetilde{n} + \alpha^{\nu-1} k_{B^{i_0}} + \sum_{r=1}^{\nu-1} \alpha^{\nu-1-r} k_{B^{i_r}} \beta^{\sum_{j=1}^{r} m_j}}{\beta^{\rho}}, \\ &= \frac{\alpha^\nu n/2^{m_0} + \alpha^{\nu-1} k_{B^{i_0}} + \sum_{r=1}^{\nu-1} \alpha^{\nu-1-r} k_{B^{i_r}} \beta^{\sum_{j=1}^{r} m_j}}{\beta^{\rho-m_0}}, \\ &= \left(\frac{\beta^{m_0}}{\beta^{m_0}}\right) \frac{\alpha^\nu n/\beta^{m_0} + \alpha^{\nu-1} k_{B^{i_0}} + \sum_{r=1}^{\nu-1} \alpha^{\nu-1-r} k_{B^{i_r}} \beta^{\sum_{j=1}^{r} m_j}}{\beta^{\rho-m_0}}, \\ &= \frac{\alpha^\nu n + \beta^{m_0}\left(\alpha^{\nu-1} k_{B^{i_0}} + \sum_{r=1}^{\nu-1} \alpha^{\nu-1-r} k_{B^{i_r}} \beta^{\sum_{j=1}^{r} m_j}\right)}{\beta^{\rho}}. \end{aligned} \tag{45}$$

Therefore, we find that

$$C^{N}_{(\alpha,\beta,\overline{k})}(n) = C^{N-m_0}_{(\alpha,\beta,\overline{k})}(\widetilde{n}) = \frac{\alpha^\nu n + \beta^{m_0}\left(\alpha^{\nu-1} k_{B^{i_0}} + \sum_{r=1}^{\nu-1} \alpha^{\nu-1-r} k_{B^{i_r}} \beta^{\sum_{j=1}^{r} m_j}\right)}{\beta^{\rho}}. \tag{46}$$

□

Example 3. Fix $\alpha = 7$, $\beta = 4$, $k_1 = 1$, $k_2 = 2$, and $k_3 = 3$. The mapping $C_{(7,4,\overline{k})}$ appears as

$$C_{(7,4,\overline{k})}(n) = \begin{cases} 7n+1, & \text{if } n \equiv 1 \pmod 4, \\ 7n+2, & \text{if } n \equiv 2 \pmod 4, \\ 7n+3, & \text{if } n \equiv 3 \pmod 4, \\ \dfrac{n}{4}, & \text{if } n \equiv 0 \pmod 4. \end{cases} \tag{47}$$

Under the same hypothesis, we consider the first nine steps of the characteristic trajectory induced by $n = 3713$.

$$\{1,0,2,0,0,3,0,0,0\}. \tag{48}$$

We verify equation (36) for $C^9_{(7,4,\overline{k})}(3713)$.

$$\begin{aligned} C^9_{(7,4,\overline{k})}(3713) &= \frac{7^3 \times 3713 + k_1 \times 7^2 \times 4^0 + k_2 \times 7^1 \times 4^1 + k_3 \times 7^0 \times 4^{1+2}}{4^6}, \\ &= \frac{7^3 \times 3713 + 1 \times 7^2 \times 4^0 + 2 \times 7^1 \times 4^1 + 3 \times 7^0 \times 4^3}{4^6}, \\ &= \frac{1273559 + 49 + 56 + 192}{4096}, \\ &= \frac{1273856}{4096} = 311. \end{aligned} \tag{49}$$

Proposition 28. *Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β are coprimes. Let $\overline{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \le i \le \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod \beta$. Fix $N \in \mathbb{Z}_{>0}$. Let $B \in \{0,1,2,\ldots,\beta-1\}^{**}$ such that $B = (\chi^i_{(\alpha,\beta,\overline{k})}(n))_{i\ge 0}$. We suppose that N is the smallest positive integer such that $C^{N+1}_{(\alpha,\beta,\overline{k})}(n) = n$. Let ρ and ν be the count of the number of zeros and ones, respectively, in B for all i between 0 and N inclusive. For $1 \le j \le \nu - 1$, let m_j denote the number of zeros between the $(j-1)$th and jth nonzero coefficient. Let $(i_j)_{j=0}^{\nu-1}$ be the indexing sequence associated with n.*

(1) If $\alpha n \equiv -i \pmod \beta$, let $m_0 = 0$ and m_ν be the remaining zeros after the last nonzero coefficient.
(2) If $n \equiv 0 \pmod \beta$, let m_0 be the number of leading zeros. We observe that $n = \beta^{m_0}\tilde{n}$ for $\tilde{n} \equiv 0 \pmod \beta$. So, we obtain $m_1, \ldots, m_{\nu-1}$ by (1) and $\tilde{n}$.

Then, n satisfies the following equality:

$$n = \frac{\beta^{m_0}\left(\alpha^{\nu-1}k_{B^{i_0}} + \sum_{r=1}^{\nu-1}\alpha^{\nu-1-r}k_{B^{i_r}}\beta^{\sum_{j=1}^{r} m_j}\right)}{\beta^{\rho} - \alpha^{\nu}}. \tag{50}$$

Proof. Equation (50) can be derived from equation (36) by setting $C^N_{(\alpha,\beta,\overline{k})}(n) = n$. ☐

Example 4. Fix $\alpha = 5$ and $\beta = 3$. We consider the integral loop induced by $k_1 = 19558$, $k_2 = 39116$, and $n = 358$. The mapping $C_{(5,3,\overline{k})}$ appears as

$$C_{(5,3,\overline{k})}(n) = \begin{cases} 5n + 19558, & \text{if } n \equiv 1 \pmod 3, \\ 5n + 39116, & \text{if } n \equiv 2 \pmod 3, \\ \dfrac{n}{3}, & \text{if } n \equiv 0 \pmod 3. \end{cases} \tag{51}$$

We find that the trajectory and characteristic trajectory of 358 appear as

$$\begin{aligned} &\{358, 21348, 7116, 2372, 50976, 16992, 5664, 1888, 28998, 9666, 3222, 1074\}, \\ &\{1,0,0,2,0,0,0,1,0,0,0,0\}. \end{aligned} \tag{52}$$

Definition 29. Fix $\beta \in \mathbb{Z}_{>1}$. Let $B \in \{0, 1, \ldots, \beta - 1\}^{**}$. We say that B is *rational β-adic* if B eventually has a purely periodic β-adic expansion.

Proposition 30. *Fix $\alpha, \beta \in \mathbb{Z}_{>0}$ so that α and β are coprimes. Let $\overline{k} = (k_1, \ldots, k_{\beta-1})$ so that for each $1 \le i \le \beta - 1$, $k_i \in \mathbb{Z}_{>0}$, k_i is coprime to α and β, and $k_i \equiv i \pmod{\beta}$. Let $B \in \{0, 1, 2, \ldots, \beta - 1\}^{**}$ such that $B = (\chi^i_{(\alpha,\beta,\overline{k})}(n))_{i\ge 0}$. Let $(i_j)_{j=0}^{\nu-1}$ be the indexing sequence associated to n. We assume that the trajectory of n is bounded above. Then, there exists some n' contained in the trajectory of n and the smallest N in $\mathbb{Z}_{>0}$ such that $C^N_{\overline{k}}(n') = n'$ if and only if B is a rational β-adic.*

Proof. We suppose first that in the trajectory of n, there exists an $n', N \in \mathbb{Z}_{>0}$ so that $C^N_{(\alpha,\beta,\overline{k})}(n') = n'$. We assume further that N is the smallest positive integer which satisfies this equality. Then, B has a repeating period of length N and is rational.

Now, we assume that B is a rational β-adic. Let x_0 be the starting value whose characteristic trajectory generates the repeating period of B. Let us define the symbol σ as follows:

$$\sigma = \alpha^{\nu-1} k_{B^{i_0}} + \sum_{r=1}^{\nu-1} \alpha^{\nu-1-r} k_{B^{i_r}} \beta^{\sum_{j=1}^{r} m_j}, \tag{53}$$

where $\Sigma m_j = \widetilde{\rho}$. We observe immediately that σ is finite as both $\widetilde{\rho}$ and $\widetilde{\nu}$ are finite. We have the following sequence of values by continuous application of the map $C^{\widetilde{\rho}+\widetilde{\nu}}_{(\alpha,\beta,\overline{k})}$ generating values $x_1, x_2, \ldots x_n$

$$\begin{aligned} \beta^{\widetilde{\rho}} x_1 &= \alpha^{\widetilde{\nu}} x_0 + \sigma, \\ \beta^{\widetilde{\rho}} x_2 &= \alpha^{\widetilde{\nu}} x_1 + \sigma, \\ &\cdots \\ \beta^{\widetilde{\rho}} x_n &= \alpha^{\widetilde{\nu}} x_{n-1} + \sigma. \end{aligned} \tag{54}$$

We can consolidate the terms into an expression that includes only x_n and x_0 as follows:

$$\beta^{\widetilde{\rho}n} x_n = \alpha^{\widetilde{\nu}n} x_0 + \sigma \sum_{i=0}^{n-1} \alpha^{(n-1-i)\widetilde{\nu}} \beta^{i\widetilde{\rho}}. \tag{55}$$

Thus, solving for x_0 yields

$$\begin{aligned} x_0 &= \left(\frac{\beta^{\widetilde{\rho}}}{\alpha^{\widetilde{\nu}}}\right)^n \left(x_n - \left(\frac{\sigma}{\beta^{\widetilde{\rho}}}\right)\left(\sum_{i=0}^{n-1}\left(\frac{\alpha^{\widetilde{\nu}}}{\beta^{\widetilde{\rho}}}\right)^{i-(n-1)}\right)\right), \\ x_0 &= \left(\frac{\beta^{\widetilde{\rho}}}{\alpha^{\widetilde{\nu}}}\right)^n \left(x_n - \left(\frac{\sigma}{2^{\widetilde{\rho}}}\right)\left(\frac{1-\left(\alpha^{\widetilde{\nu}}/\beta^{\widetilde{\rho}}\right)^n}{1-\alpha^{\widetilde{\nu}}/\beta^{\widetilde{\rho}}}\right)\right), \\ x_0 &= \left(\frac{\beta^{\widetilde{\rho}}}{\alpha^{\widetilde{\nu}}}\right)^n \left(x_n - \sigma\left(\frac{1-\left(\alpha^{\widetilde{\nu}}/\beta^{\widetilde{\rho}}\right)^n}{\beta^{\widetilde{\rho}}-\alpha^{\widetilde{\nu}}}\right)\right). \end{aligned} \tag{56}$$

We suppose first that $\widetilde{\beta^{\rho}}/\widetilde{\alpha^{\nu}} > 1$. Following from equation (55), we distribute the first term, yielding the following:

$$x_0 = \left(\frac{\beta^{\widetilde{\rho}}}{\alpha^{\widetilde{\nu}}}\right)^n \left(x_n - \left(\frac{\sigma}{\beta^{\widetilde{\rho}}-\alpha^{\widetilde{\nu}}}\right)\right) + \frac{\sigma}{\beta^{\widetilde{\rho}}-\alpha^{\widetilde{\nu}}}. \tag{57}$$

Then, there exists an N such that for all $n > N$, $x_n = \sigma/\beta^{\widetilde{\rho}} - \alpha^{\widetilde{\nu}}$. We find a new x_N, which maybe different from x_0 such that x_N generates an integral loop coinciding with the original rational loop.

If conversely we suppose $\widetilde{\beta^{\rho}}/\widetilde{\alpha^{\nu}} < 1$, then we simply reverse equation (55) to solve for x_n in terms of x_0:

$$x_n = \left(\frac{\alpha^{\widetilde{\nu}}}{\beta^{\widetilde{\rho}}}\right)^n \left(x_0 - \left(\frac{\sigma}{\beta^{\widetilde{\rho}}-\alpha^{\widetilde{\nu}}}\right)\right) + \frac{\sigma}{\beta^{\widetilde{\rho}}-\alpha^{\widetilde{\nu}}}. \tag{58}$$

Suppose first that $x_0 = \sigma/\beta^{\widetilde{\rho}} - \alpha^{\widetilde{\nu}}$, then it follows that $x_1 = \sigma/\beta^{\widetilde{\rho}} - \alpha^{\widetilde{\nu}}$. This equality is true for all x, and thus, we have an integral loop. So, we assume instead that $x_0 \neq \sigma/\beta^{\widetilde{\rho}} - \alpha^{\widetilde{\nu}}$. Then, we can write equation (56) in the form of a ratio exhibited in equation:

$$\frac{x_n - \sigma/\beta^{\widetilde{\rho}} - \alpha^{\widetilde{\nu}}}{x_0 - \sigma/\beta^{\widetilde{\rho}} - \alpha^{\widetilde{\nu}}} = \left(\frac{\alpha^{\widetilde{\nu}}}{\beta^{\widetilde{\rho}}}\right)^n. \tag{59}$$

However, we recall that x_0 and $\sigma/\beta^{\widetilde{\rho}} - \alpha^{\widetilde{\nu}}$ are both fixed values. Thus, the left hand side has a fixed denominator. However, as the right hand side is the quotient of two coprime integers, the denominator expands with irreducible powers of two. If x_n is a trajectory of integers, then there would be some index N of which the denominator on the right hand side would be larger than the left hand side. This would be a contradiction. Finally, as $\widetilde{\nu} = \ln(\beta)/\ln(\alpha)\widetilde{\rho}$, as a irrational ratio cannot be expressed with a finite β-adic sequence. Thus, we have shown that no rational cycle apart from an integral loop exists for any $C_{(\alpha,\beta,\overline{k})}$ mapping. □

Example 5. Fix $\alpha = 5$, $\beta = 3$, $k_1 = 1$, and $k_2 = 155$. The mapping $C_{(5,3,\overline{k})}$ appears as

$$C_{(5,3,\overline{k})}(n) = \begin{cases} 5n + 1, & if\ \mathrm{n} \equiv 1 \pmod{3}, \\ 5n + 155, & if\ \mathrm{n} \equiv 2 \pmod{3}, \\ \dfrac{n}{3}, & if\ \mathrm{n} \equiv 0 \pmod{3}. \end{cases} \tag{60}$$

We find that the trajectory and characteristic trajectory starting at $n = 25$ appears as

$$\begin{aligned} &\{25, 126, 42, 14, 225, 75, \ldots\}, \\ &\{1, 0, 0, 2, 0, 0, \ldots\}. \end{aligned} \tag{61}$$

That is to say, 25 starts an integral loop which induces a rational 3-adic sequence. Let us attempt to construct a trajectory, different from the integral loop starting at 25, based on the 3-adic sequence $\{1, 0, 0, 2, 0, 0, \ldots\}$. We will proceed inductively by considering the smallest starting

Table 1: Sequence of starting integers n having s repetitions of the finite 3-adic sequence $\{1, 0, 0, 2, 0, 0\}$.

s	n	B
1	106	$\{1, 0, 0, 2, 0, 0, 2, \ldots\}$
2	6586	$\{1, 0, 0, 2, 0, 0, 1, 0, 0, 2, 0, 0, 2, \ldots\}$
3	531466	$\{1, 0, 0, 2, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 2, 0, 0, 2, \ldots\}$
4	43046746	$\{1, 0, 0, 2, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 2, 0, 0, 2, \ldots\}$
...	...	...
s	$n_s = 25 + 81^s$	$\{1, 0, 0, 2, 0, 0, \ldots, 1, 0, 0, 2, 0, 0, 2\}$

integer corresponding to s repetitions of the finite sequence (see Table 1) $\{1, 0, 0, 2, 0, 0\}$.

We find that the sequence of starting integers $\{n_s\}_{s\geq 1}$ diverges. We conclude that the only trajectory corresponding to the rational 3-adic sequence $\{1, 0, 0, 2, 0, 0, \ldots\}$ for all iterations is the integral loop starting at $n = 25$.

4. Conclusion

We have shown that the characteristic trajectory of a starting integer is eventually periodic if and only if the trajectory eventually enters an integral loop. This is not a new idea, but the proof presented here is new. The proof contained in Corollary 2.1a [7] employs a direct proof comparing parity vectors (characteristic trajectories). That proof would also apply to any two state system with halting condition (see Section 3.1). In that case, the nonzero symbol t is replaced with one. The proof contained in Proposition 4 [8] demonstrates that if a characteristic trajectory is terminally periodic, then it represents a 2-adic number. The proof here does not require detailed knowledge of the 2-adics. Apart from the previous two proofs, this new proof demonstrates the claim is 2-adic invariant. In this way, we have presented a generalization of a known result.

Data Availability

The data used to support the findings of this study can be obtained from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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