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Research Article

Jensen, Hermite–Hadamard, and Fejér-Type Inequalities for Reciprocally Strongly (*h*, *s*)-Convex Functions

Yujun Wang,[1] **Muhammad Shoaib Saleem**,[2] **Zahida Perveen**,[3] **and Muhammad Imran**[2]

[1]*School of General Education, Dalian Neusoft University of Information, Dalian 116023, China*
[2]*Department of Mathematics, University of Okara, Okara, Pakistan*
[3]*Department of Mathematics, Lahore Garrison University, DHA Phase 6, Lahore, Pakistan*

Correspondence should be addressed to Yujun Wang; wangyujun12345@sina.com and Muhammad Shoaib Saleem; shaby455@yahoo.com

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This paper aims to present a generalized and extended notation of convexity by unifying reciprocally strong convexity with (h, s)−convexity. We introduce the concept of reciprocally strongly (h, s)−convex functions and establish some of their fundamental properties. In addition, we establish various inequalities, including Jensen, Hermite–Hadamard, and Fejér-type inequalities, for this generalized framework. Our findings are an extension of numerous existing results and provide a basis for developing novel methods for generalization in convexity.

1. Introduction

Convex analysis has a rich history, with Hermann Minkowski and Werner Fenchel among the pioneers who studied the geometric features of sets and functions in convexity. In the 1960s, R. Tryll Roker and Jean Joseph M. began the systematic study of convex analysis, and since then, this area of research has gained widespread attention due to its wide range of applications in control systems, estimation, signal processing, data analysis, economics, and more, as evidenced in works such as [1–3]. A real-valued function $f(x)$ defined on an interval I is said to be convex if, for any $x_1, x_2 \in I$ and any $\lambda \in [0,1]$, we have

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2). \quad (1)$$

However, while classical convexity has proven to be a useful tool in engineering applications, it cannot always solve all problems, necessitating the development of various generalizations of convexity. The notable generalizations are h-convexity [4], p-convexity [5], ϕ-convexity [6], η-convexity [7], s-convexity [8], k-convexity [9], and m-convexity [10], among others [11, 12]. For further reading on convex functions and their applications, interested readers may refer to books such as [13–15]. The ongoing research in this field continues to push the boundaries of our understanding of convexity and its practical applications in various domains.

Inequality theory is a branch of mathematics that deals with the study of mathematical inequalities and their properties. This theory plays an important role in many areas of mathematics, including algebra, geometry, analysis, and probability theory. It involves the study of convexity, probability inequalities, and the behavior of inequalities under different operations. The most important inequality is the Hermite–Hadamard inequality, which states that for a convex function $f(x)$ defined on the interval $[a, b]$, we have

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \quad (2)$$

This inequality relates the values of a convex function on an interval to its arithmetic mean. It has important applications in many areas of mathematics, including analysis, geometry, optimization, and probability theory. The inequality is named after the mathematicians Charles Hermite and Jacques Hadamard, who made significant contributions to its development.

The Fejér inequality provides an estimate of the error in approximating a periodic function using its Fourier series. Specifically, the inequality relates the difference between the Fejér sum of order N and the original function to the total variation of the function over one period. The Fejér inequality has important applications in many areas of mathematics and engineering, including signal processing, control theory, and partial differential equations. The inequality is named after the Hungarian mathematician Lipót Fejér, who discovered it in 1915, see [16, 17] and references therein.

The inequalities such as Jensen, Hermite–Hadamard, Fejér, and fractional integral inequalities are fundamental results in the field of convexity. These inequalities have significant applications in various areas of mathematics, including optimization, control theory, probability theory, and harmonic analysis. Several generalizations of these inequalities exist in the literature, reflecting their importance and versatility [18, 19]. For instance, the weighted versions of these inequalities, where weights are assigned to the function or its domain, have been studied extensively. Moreover, many recent studies have focused on the development of new inequalities that involve different types of functions or have more complex forms. These advancements in the theory of inequalities demonstrate the ongoing interest and relevance of these results in contemporary mathematics [20, 21].

This paper aims to introduce a novel concept of convexity and to establish Jensen, Hermite–Hadamard, and Fejér-type inequalities for this new notion. Specifically, the paper unifies (h, s)-convex functions and reciprocally strongly convex functions to introduce the new notion of convexity. This new notion broadens the scope of convex functions and provides a more flexible framework for studying convexity. The established inequalities extend the classical results of convexity and have potential applications in many areas of mathematics, including optimization, analysis, and probability theory. This paper's contributions demonstrate the importance of exploring new concepts in convexity and the potential benefits that can arise from doing so.

The paper is structured as follows: The Section 2 provides some preliminary material and discusses the basic properties of the new concept of convexity introduced in the paper. In Section 3, the main results are presented, focusing on the reciprocally strongly (h, s)-convex functions. These results include Jensen, Hermite–Hadamard, and Fejér-type inequalities, which establish the usefulness of the new notion of convexity in extending classical results.

2. Definitions and Basic Results

This section serves as an introduction to the main results of the paper and provides readers with the necessary background information to understand the new concept of convexity and the established inequalities.

Definition 1 (see [22]). Let $\mathcal{J} \subset \mathbb{R}$ and let $h: \mathcal{J} \longrightarrow \mathbb{R}$ be a nonnegative function such that $h \neq 0$. A function $f: \mathcal{J} \longrightarrow \mathbb{R}$ is said to be h-convex if for all $x, y \in \mathcal{J}$ and all $k \in [0, 1]$, we have

$$f(kx + (1-k)y) \leq h(k) f(x) + h(1-k) f(y). \tag{3}$$

This inequality is similar to the definition of convex functions, except that a nonnegative weight function h is introduced. The function h controls the degree of convexity of the function f. When h is constant and equal to 1, the definition reduces to that of convex functions. The concept of h-convex functions has applications in various fields, including optimization, economics, and physics. It provides a more flexible framework for studying convexity and can lead to more nuanced analyses of real-world problems.

Definition 2 (see [23]). Let $\mathcal{J} \subset \mathbb{R}$ and let $\mu \geq 0$ be a constant. A function $f: \mathcal{J} \longrightarrow \mathbb{R}$ is said to be strongly convex if for all $x, y \in \mathcal{J}$ and all $k \in [0, 1]$, we have

$$f(kx + (1-k)y) \leq k f(x) + (1-k) f(y) - \mu k(1-k)(x-y)^2. \tag{4}$$

This inequality requires that the function f be more strongly convex than the definition of convexity. The additional term $-\mu k(1-k)(x-y)^2$ acts as a penalty term that becomes larger when the distance between x and y increases. The constant μ controls the degree of strong convexity of the function f.

Definition 3 (see [24]). Let $\mu \geq 0$ and h: $[a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a nonnegative function with $h \neq 0$. A function f: $[a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is strongly h-convex if for all $x, y \in [a, b]$ and $k \in [0, 1]$, the following inequality holds:

$$f(kx + (1-k)y) \leq kh(x) + (1-k)h(y) - \mu k(1-k)(x-y)^2. \tag{5}$$

In other words, f is strongly h-convex if it satisfies a generalized form of the convexity property, where the weights given to $f(x)$ and $f(y)$ are modulated by the function h and an additional term involving μ and the squared difference $(x-y)^2$. This term penalizes larger distances between x and y and thus promotes stronger convexity.

Definition 4 (see [25]). Let $s \in (0, 1]$ and $\mathcal{J} \subset \mathbb{R}$ be a nonempty interval of nonnegative real numbers. A function $f: \mathcal{J} \longrightarrow \mathbb{R}$ is s-convex in the second sense if for all $x, y \in \mathcal{J}$ and $k \in [0, 1]$, the following inequality holds:

$$f((1-k)x + ky) \leq (1-k)^s f(y) + k^s f(x). \tag{6}$$

In other words, f is s-convex in the second sense if it satisfies a generalization of the convexity property, where the weights given to $f(x)$ and $f(y)$ are modulated by the exponent s and the interpolation parameter k. This property can be interpreted as a degree of concavity of the function f with respect to its inputs and can be used to characterize various optimization problems involving nonlinear constraints. The interval $\mathcal{J}$ is restricted to nonnegative real numbers to ensure that the function f is well-defined and nonnegative on its domain.

Definition 5 (see [26]). Let $\mathcal{J} \subset \mathbb{R}$ be a nonempty interval, and let $h: \mathcal{J} \longrightarrow \mathbb{R}$ be a nonnegative function with $h \neq 0$. Let $s \in (0,1]$ be a fixed exponent. A function $f: \mathcal{J} \longrightarrow \mathbb{R}$ is (h,s)-convex if for all $x, y \in \mathcal{J}$ and $k \in [0,1]$, the following inequality holds:

$$f((1-k)x + ky) \leq h\left((1-k)^s f(x) + k^s f(y)\right). \tag{7}$$

In other words, f is (h,s)-convex if it satisfies a generalization of the convexity property, where the weights given to $f(x)$ and $f(y)$ are modulated by the function h and the exponent s. This property can be used to characterize various optimization problems where the objective function f is subject to nonlinear constraints represented by the function h. The interval $\mathcal{J}$ is assumed to be nonempty to ensure that the function f is well-defined and the function h is nonzero on its domain.

Definition 6 (see [27]). Let $\mathcal{J} \subset \mathbb{R}$ be a nonempty interval, and let $h: \mathcal{J} \longrightarrow \mathbb{R}$ be a nonnegative function with $h \neq 0$. Let $\mu \geq 0$ be a fixed constant, and let $s \in (0,1]$ be a fixed exponent. A function $f: \mathcal{J} \longrightarrow \mathbb{R}$ is strongly (h,s)-convex with parameter μ if for all $x, y \in \mathcal{J}$ and $k \in [0,1]$, the following inequality holds:

$$f((1-k)x + ky) \leq h\left((1-k)^s f(x) + k^s f(y)\right) - \mu k(1-k)(x-y)^2. \tag{8}$$

In other words, f is strongly (h,s)-convex with parameter μ if it satisfies a generalization of the strongly convexity property, where the weights given to $f(x)$ and $f(y)$ are modulated by the function h and the exponent s, and an additional quadratic penalty term is added to the right-hand side of the inequality. The parameter μ controls the strength of the penalty term, and the interval $\mathcal{J}$ is assumed to be nonempty to ensure that the function f is well-defined and the function h is nonzero on its domain. This property can be used to formulate optimization problems with strongly convex objective functions subject to nonlinear constraints represented by the function h.

Definition 7 (see [28]). The function $f: \mathcal{J} \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be harmonic convex on (a,b) if it satisfies the following conditions:

(1) f is continuous on (a,b)

(2) For all $x, y \in (a,b)$ such that $0 < a \leq x, y \leq b$ and $k \in [0,1]$, the inequality

$$f\left(\frac{xy}{kx + (1-k)y}\right) \leq (1-k)f(x) + kf(y), \tag{9}$$

holds.

Note that we exclude the points $x = a$ and $y = b$ in the definition, since the expression $xy/kx + (1-k)y$ is not well-defined when $x = a$ or $y = b$. For a detailed study about harmonic convex, we refer to the readers [29–31] and references therein.

Definition 8. Let the number $\mu \geq 0$. The function $f: \mathcal{J} \subset \mathbb{R} \longrightarrow \mathbb{R}$ is reciprocally strongly (h,s)-convex if

$$f\left(\frac{xy}{kx + (1-k)y}\right) \leq h\left(k^s\right) f(x) + h(1-k)^s f(y) - \mu k(1-k)\left(\frac{1}{x} - \frac{1}{y}\right)^2, \tag{10}$$

holds for all $x, y \in \mathcal{J} \subset \mathbb{R}$, $s \in (0,1]$, and $k \in [0,1]$. The class of all reciprocally strongly (h,s)-convex is denoted by RS.

Remark 9. By setting $\mu = 0$ in Definition 8, we obtain a reciprocally (h,s)-convex function. Moreover, when $s = 1$, we obtain a reciprocally strongly h-convex function. Similarly, if we set $s = 1$, $h(k) = k$, and $\mu = 0$, we obtain a classical convex function.

If a function is reciprocally strongly (h,s)-convex, then it is also strongly (h,s)-convex, but the converse is not always true. In other words, the class of reciprocally strongly (h,s)-convex functions is a subset of the class of strongly (h,s)-convex functions.

An example of a function that is strongly (h,s)-convex but not reciprocally strongly (h,s)-convex is $f(x) = x^3$ on $\mathcal{J} = [-1,1]$ with $h(k) = k$ and $s = 1/2$. To see that f is strongly (h,s)-convex, let $0 \leq k \leq 1$ and $-1 \leq x_1 < x_2 \leq 1$. Then,

$$\begin{aligned} f\left(kx_1 + (1-k)x_2\right) &= \left(kx_1 + (1-k)x_2\right)^3 \\ &= kx_1^3 + 3k(1-k)x_1^2 x_2 + 3k(1-k)x_1 x_2^2 \\ &\quad + (1-k)x_2^3 \\ &\leq kx_1^3 + (1-k)x_2^3 \\ &= h\left(k^s\right) f\left(x_1\right) + h\left((1-k)^s\right) f\left(x_2\right). \end{aligned} \tag{11}$$

However, f is not reciprocally strongly (h,s)-convex. Let $x = -1/2$, $y = 1/2$, $k = 1/2$, and $\mu = 1$. Then,

$$\begin{aligned} f\left(\frac{xy}{kx + (1-k)y}\right) &= f(0) = 0, \\ h\left(k^s\right) f(x) &+ h\left((1-k)^s\right) f(y) \\ &- \mu k(1-k)\left(\frac{1}{x} - \frac{1}{y}\right)^2 \\ &= \frac{1}{2}\left(-\frac{1}{8}\right) + \frac{1}{2}\left(\frac{1}{8}\right) - \frac{1}{4}\left(\frac{4}{1}\right)^2 \\ &= -\frac{5}{4} < 0. \end{aligned} \tag{12}$$

Therefore, f is not reciprocally strongly (h,s)-convex for any choice of h, s, and μ.

Proposition 10. *If there exist two nonnegative functions h_1 and h_2 defined on the interval $\mathcal{J} \subset \mathbb{R}$ such that $h_2(k^s) \leq h_1(k^s)$ for all $k, s \in [0,1]$, then any function f that is reciprocally strongly (h_2, s)-convex on $\mathcal{J}$ is also reciprocally strongly (h_1, s)-convex on $\mathcal{J}$.*

Proof. Let f be a reciprocally strongly (h_2, s)-convex function on $\mathcal{J}$. Then for any $x, y \in \mathcal{J}$ and $k \in [0,1]$, we have

$$\begin{aligned} f\left(\frac{xy}{kx+(1-k)y}\right) &\leq h_2(k^s)f(x) + h_2((1-k)^s)f(y) - \mu k(1-k)\left(\frac{1}{x}-\frac{1}{y}\right)^2 \\ &\leq h_1(k^s)f(x) + h_1((1-k)^s)f(y) - \mu k(1-k)\left(\frac{1}{x}-\frac{1}{y}\right)^2, \end{aligned} \tag{13}$$

where the first inequality follows from the fact that f is reciprocally strongly (h_2, s)-convex, and the second inequality follows from $h_2(k^s) \leq h_1(k^s)$ for all $k, s \in [0,1]$. Therefore, f is also reciprocally strongly (h_1, s)-convex on $\mathcal{J}$. □

Proposition 11. *Assuming $\rho > 0$, if f and g are functions defined on $\mathcal{J} \subset \mathbb{R}$ and belong to the class of reciprocally strongly convex functions (RS), then the following hold:*

(1) The function $\varphi = f + g$ also belongs to RS

(2) The function $\psi = \rho f$ also belongs to RS

Proof

(1) To show that $\varphi = f + g$ is in RS, let $x, y \in \mathcal{J}$ and $k \in [0,1]$. Then using the definition of RS for f and g, we have

$$\begin{aligned} \varphi\left(\frac{xy}{kx+(1-k)y}\right) &= f\left(\frac{xy}{kx+(1-k)y}\right) + g\left(\frac{xy}{kx+(1-k)y}\right) \\ &\leq h_1(k^s)f(x) + h_1((1-k)^s)g(y) - \mu k(1-k)\left(\frac{1}{x}-\frac{1}{y}\right)^2 \\ &\quad + h_2(k^s)g(x) + h_2((1-k)^s)g(y) - \mu k(1-k)\left(\frac{1}{x}-\frac{1}{y}\right)^2 \\ &= [h_1(k^s) + h_2(k^s)]f(x) + [h_1((1-k)^s) + h_2((1-k)^s)]g(y) \\ &\quad - 2\mu k(1-k)\left(\frac{1}{x}-\frac{1}{y}\right)^2 \\ &\leq h(k^s)\varphi(x) + h((1-k)^s)\varphi(y) - 2\mu k(1-k)\left(\frac{1}{x}-\frac{1}{y}\right)^2, \end{aligned} \tag{14}$$

where $h(k) = h_1(k) + h_2(k)$. Thus, φ is in RS.

(2) To show that $\psi = \rho f$ is in RS, let $x, y \in \mathcal{J}$ and $k \in [0,1]$. Then using the definition of RS for f, we have

$$\begin{aligned}\psi\left(\frac{xy}{kx+(1-k)y}\right) &= \rho f\left(\frac{xy}{kx+(1-k)y}\right)\\ &\le h(k^s)\rho f(x) + h((1-k)^s)\rho f(y) - \mu k(1-k)\rho^2\left(\frac{1}{x}-\frac{1}{y}\right)^2\\ &= \rho\left[h(k^s)f(x) + h((1-k)^s)f(y) - \mu k(1-k)\left(\frac{1}{x}-\frac{1}{y}\right)^2\right]\\ &\le h(k^s)\psi(x) + h((1-k)^s)\psi(y) - \mu k(1-k)\left(\frac{1}{x}-\frac{1}{y}\right)^2,\end{aligned} \tag{15}$$

where $h(k) = \rho h_1(k)$. Thus, ψ is in RS. □

3. Main Results

In this section, we establish Jensen, Hermite–Hadamard, and Fejér-type inequalities for functions in the class of reciprocally strongly convex functions.

Theorem 12. *Assuming that the function f given by $f\colon \mathcal{J}\subset\mathbb{R}\longrightarrow\mathbb{R}$ is in RS, the following inequality holds for all $v_{11}{}^s,\ldots,v_{1n}{}^s\in\mathcal{J}\subset\mathbb{R}$, $h(k_1^s),\ldots,h(k_n^s)>0$ with $h(k_1^s)+\cdots+h(k_n^s)=1$, and $1/\overline{x} = h(k_1^s)(1/v_{11})+\cdots+h(k_n^s)(1/v_{1n})$:*

$$f\left(\sum_{i=1}^{n} h(k_i^s)\left(\frac{1}{\overline{x}}\right)\right) \le \sum_{i=1}^{n} h(ki^s) f\left(\frac{1}{\overline{x}}\right) - \mu\sum_{i=1}^{n} h(k_i^s)\left(\frac{1}{x_i}-\frac{1}{\overline{x}}\right)^2. \tag{16}$$

This inequality can be called as the weighted Jensen inequality for reciprocally strongly (h,s)-convex functions.

Proof. Take $x_1,\ldots,x_n\in\mathcal{J}\subset\mathbb{R}$ and then $h(k_1^s),\ldots,h(k_n^s)>0$, where $k_1,\ldots,k_n>0$, such that $h(k_1^s)+\cdots+h(k_n^s)=1$.

Set $1/\overline{x} = h(k_1^s)(1/x_1)+\cdots+h(k_n^s)(1/x_n)$ and take a function

$$g\left(\frac{1}{x}\right) = u\left(\frac{1}{x}-\frac{1}{\overline{x}}\right)^2 + a\left(\frac{1}{x}-\frac{1}{\overline{x}}\right)^2 + f\left(\frac{1}{\overline{x}}\right), \tag{17}$$

supporting f at $1/\overline{x}$, that is, $g(1/\overline{x}) = f(1/\overline{x})$ and $g(x)\le f(x)$, $x\in\mathcal{J}\subset\mathbb{R}$. Then, we have

$$f\left(\frac{1}{x_i}\right) \ge g\left(\frac{1}{x_i}\right) = u\left(\frac{1}{x_i}-\frac{1}{\overline{x}}\right)^2 + a\left(\frac{1}{x_i}-\frac{1}{\overline{x}}\right)^2 + f\left(\frac{1}{\overline{x}}\right), \tag{18}$$

for every $i\in\{1,2,\ldots,n\}$.

Multiply on both sides by $h(k_i^s)$ and summing up, we obtain

$$\sum_{i=1}^{n} h(k_i^s) f\left(\frac{1}{x_i}\right) \ge \sum_{i=1}^{n} g\left(\frac{1}{x_i}\right) = u\sum_{i=1}^{n} h(k_i^s)\left(\frac{1}{x_i}-\frac{1}{\overline{x}}\right)^2 + a\sum_{i=1}^{n} h(k_i^s)\left(\frac{1}{x_i}-\frac{1}{\overline{x}}\right)^2 + f\left(\frac{1}{\overline{x}}\right). \tag{19}$$

Since, $\sum_{i=1}^{n} h(k_i^s)(x_i^s - x^{-s})^2$ equal to zero, then we have

$$f\left(\frac{1}{\overline{x}}\right) \le \sum_{i=1}^{n} h(k_i^s) f\left(\frac{1}{x_i}\right) - u\sum_{i=1}^{n} h(k_i^s)\left(\frac{1}{x_i}-\frac{1}{\overline{x}}\right)^2, \tag{20}$$

which completes the proof. □

Remark 13. The inequality (16) implies Jensen's inequality for the class of harmonic-convex functions when $h(k_i) = k_i$, $h(v_{1_i}) = v_{1_i}$, $s=1$, and $\mu = 0$.

In this case, the inequality reduces to

$$f\left(\sum_{i=1}^{n} k_i\left(\frac{1}{\overline{x}}\right)\right) \le \sum_{i=1}^{n} k_i f\left(\frac{1}{\overline{x}}\right), \tag{21}$$

which is Jensen's inequality for harmonic-convex functions. For a detailed proof, refer to [28].

Proof. Starting from (16), substitute $s=1$ and $\mu=0$, and note that $h(k_i)=k_i$ and $h(v_{1_i}) = v_{1_i}$, which gives

$$f\left(\frac{\sum_{i=1}^{n} k_i}{\sum_{i=1}^{n} v_{1_i}}\right) \le \frac{\sum_{i=1}^{n} k_i}{\sum_{i=1}^{n} v_{1_i}} f\left(\frac{\sum_{i=1}^{n} v_{1_i}}{\sum_{i=1}^{n} v_{1_i}}\right) = f\left(\frac{\sum_{i=1}^{n} k_i}{\sum_{i=1}^{n} v_{1_i}}\right), \tag{22}$$

which is equivalent to Jensen's inequality for harmonic-convex functions. □

Theorem 14. *Assuming that $h(1/2^s)>0$ and $\mu\ge 0$, consider a function f defined on $\mathcal{J}=[a,b]\subset\mathbb{R}$ that belongs to RS. Then, the following inequality holds:*

$$\frac{1}{2h(1/2^s)}\left[f\left(\frac{2mn}{m+n}\right)+\frac{\mu}{12}\left(\frac{n-m}{mn}\right)^2\right] \leq \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx \leq [f(m)+f(n)]\int_0^1 h(k^s)dk-\frac{\mu}{6}\left(\frac{1}{m}-\frac{1}{n}\right)^2. \tag{23}$$

Note that this inequality is valid for any $a \leq m < n \leq b$.

Proof. Our proof starts with the fact that f be in RS:

$$f\left(\frac{mn}{km+(1-k)n}\right) \leq h(k^s)f(m)+h(1-k)^s f(n) - \mu k(1-k)\left(\frac{1}{m}-\frac{1}{n}\right)^2, \tag{24}$$

for each k and $s \in [0,1]$.

After integrating over [0, 1] with respect to k, we obtain

$$\int_0^1 f\left(\frac{mn}{km+(1-k)n}\right)dk \leq f(m)\int_0^1 h(k^s)dk + f(n)\int_0^1 h(1-k)^s dk - \mu\left(\frac{1}{m}-\frac{1}{n}\right)^2\int_0^1 k(1-k)dk,$$
$$\frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx \leq [f(m)+f(n)]\int_0^1 h(k^s)dk-\frac{\mu}{6}\left(\frac{1}{m}-\frac{1}{n}\right)^2. \tag{25}$$

For the left hand side of inequality (23) putting $k = 1/2$ in (10),

$$f\left(\frac{2xy}{x+y}\right) \leq h\left(\frac{1}{2^s}\right)f(x)+h\left(\frac{1}{2^s}\right)f(y)-\frac{\mu}{4}\left(\frac{1}{x}-\frac{1}{y}\right)^2, \tag{26}$$

choosing $x = (mn/km + (1-k)n)$ and $y = (mn/kn + (1-k)m)$.

Since,

$$\int_0^1 f\left(\frac{mn}{km+(1-k)n}\right)dk = \int_0^1 f\left(\frac{mn}{kn+(1-k)m}\right)dk = \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx. \tag{27}$$

Thus, the integration of above inequality over [0, 1] with respect to k, we have

$$f\left(\frac{2mn}{m+n}\right) \leq \frac{(2h(1/2^s))(mn)}{n-m}\int_m^n \frac{f(x)}{x^2}dx-\frac{\mu}{12}\left(\frac{n-m}{mn}\right)^2. \tag{28}$$

Hence,

$$\frac{1}{2h(1/2^s)}\left[f\left(\frac{2mn}{m+n}\right)+\frac{\mu}{12}\left(\frac{n-m}{mn}\right)^2\right] \leq \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx. \tag{29}$$

This completes the proof. □

Remark 15. We can derive two important inequalities from (23):

(1) Setting $h(k) = k$ and $s = 1$ in (23), we obtain Hermite–Hadamard inequality for strongly reciprocally convex functions [32]

(2) Taking the limit as $\mu \longrightarrow 0^+$ and setting $h(k) = k$ and $s = 1$ in (23), we obtain the Hermite–Hadamard type inequalities for harmonically convex functions [28]

Note that the Remarks 13 and 15 (b) concern different types of inequalities and use different approaches to derive them. The Remark 13 concerns Jensen's inequality for harmonic-convex functions and is derived by setting specific values for the parameters h, s, and μ in inequality (16), while the Remark 15 (b) concerns Hermite–Hadamard type inequalities for harmonically convex functions and is derived by taking the limit of inequality (23) as μ approaches zero and setting $h(k) = k$ and $s = 1$. Both remarks involve using specific properties of harmonic-convex functions, but they result in different types of inequalities.

Theorem 16. *Let $f\colon \mathcal{J} \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function in the RS class with $\mu \geq 0$ on $\mathcal{J}$. For $m, n \in \mathcal{J} \subset \mathbb{R}$, consider the function $w\colon \mathcal{J} \subset \mathbb{R} \longrightarrow \mathbb{R}$, which is nonnegative, integrable, and symmetric with respect to (mn/x), satisfying $w(mn/x) = w(mn/m+n-x)$. The weight function w is a nonnegative, integrable function that satisfies the following two conditions: symmetry: $w(mn/x) = w(mn/m+n-x)$ for all $m, n, x \in \mathcal{J}$. Nonnegativity: $w(x) \geq 0$ for all $x \in \mathcal{J}$. The symmetry condition means that the value of w at a point mn/x is equal to the value of w at the corresponding point $mn/m+n-x$. Geometrically, this condition states that if we reflect the point mn/x across the vertical line $x = m+n/2$, we get the point $mn/m+n-x$, and the weight function w must have the same value at both points. Intuitively, the weight function w measures how much emphasis to place on different parts of the interval $\mathcal{J}$. In the inequalities (30), the weight function is used to give more weight to certain parts of the* *interval where the function f is expected to be more "important" in some sense. The specific form of the weight function in the inequalities is chosen to satisfy the symmetry condition and ensure that the integrals involved are well-defined and finite.*

Then, we have the inequality

$$\begin{aligned}&\frac{mn}{2h(1/2^s)}\left[f\left(\frac{2mn}{m+n}\right)\int_m^n\frac{w(x)}{x^2}dx+\frac{\mu}{(2mn)^2}\int_m^n\frac{(2mn+(m+n)x)^2w(x)}{x^4}dx\right]\\&\leq\int_m^n\frac{f(x)w(x)}{x^2}dx\\&\leq[f(m)+f(n)]\int_m^n h\left(\frac{(x-m)n}{(n-m)x}\right)\frac{w(x)}{x^2}-\frac{\mu}{mn}\int_m^n\frac{(n-x)(x-m)w(x)}{x^4}dx.\end{aligned}\tag{30}$$

Proof. Since $f\colon \mathcal{J}\subset\mathbb{R}\longrightarrow\mathbb{R}$ is reciprocally strongly (h,s)-convex function, then for $k=1/2$, we have

$$f\left(\frac{2xy}{x+y}\right)\leq h\left(\frac{1}{2^s}\right)f(x)+h\left(\frac{1}{2^s}\right)f(y)-\frac{\mu}{4}\left(\frac{1}{x}-\frac{1}{y}\right)^2,\tag{31}$$

for all $x,y\in\mathcal{J}\subset\mathbb{R}$, suppose $x=(mn/km+(1-k)n)$ and $y=(mn/kn+(1-k)m)$ in above inequality, then we obtain

$$\begin{aligned}f\left(\frac{2mn}{m+n}\right)\leq h\left(\frac{1}{2^s}\right)\left[f\left(\frac{mn}{km+(1-k)n}\right)+f\left(\frac{mn}{kn+(1-k)m}\right)\right]\\-\frac{\mu}{4}\left(\frac{kn+(1-k)m}{mn}-\frac{km+(1-k)n}{mn}\right)^2.\end{aligned}\tag{32}$$

Since, w is nonnegative, symmetric, we have

$$\begin{aligned}&f\left(\frac{2mn}{m+n}\right)w\left(\frac{mn}{km+(1-k)n}\right)\\&\leq h\left(\frac{1}{2^s}\right)\left[f\left(\frac{mn}{km+(1-k)n}\right)+f\left(\frac{mn}{kn+(1-k)m}\right)\right]w\left(\frac{mn}{km+(1-k)n}\right)\\&-\frac{\mu}{4}\left(\frac{kn+(1-k)m}{mn}-\frac{km+(1-k)n}{mn}\right)^2w\left(\frac{mn}{km+(1-k)n}\right).\end{aligned}\tag{33}$$

Integrating above inequalities with respect 'k' over [0, 1] and then putting $x=(mn/km+(1-k)n)$, we obtain

$$\begin{aligned}\frac{mn}{n-m}f\left(\frac{2mn}{m+n}\right)\int_m^n\frac{w(x)}{x^2}dx\leq\frac{2h(1/2^s)}{n-m}\int_m^n\frac{f(x)w(x)}{x^2}dx\\-\frac{\mu}{4}\left(\frac{(n-m)}{mn}\right)\int_m^n\left(\frac{2mn+(m+n)x^2}{x^4}\right)w(x)dx.\end{aligned}\tag{34}$$

After simplification, above inequalities become

$$\begin{aligned}&\frac{mn}{2h(1/2^s)}\left[f\left(\frac{2mn}{m+n}\right)\int_m^n\frac{w(x)}{x^2}dx+\frac{\mu}{(2mn)^2}\int_m^n\left(\frac{2mn+(m+n)x}{x^4}\right)^2w(x)dx\right]\\&\leq\int_m^n\frac{f(x)w(x)}{x^2}dx.\end{aligned}\tag{35}$$

For right hand side of (35), we have

$$f\left(\frac{mn}{km+(1-k)n}\right)w\left(\frac{mn}{km+(1-k)n}\right)\leq\left[h\left(k^{s}\right)f(m)+h(1-k)^{s}f(n)\right]w\left(\frac{mn}{km+(1-k)n}\right)-\mu k(1-k)\left(\frac{1}{x}-\frac{1}{y}\right)^{2}w\left(\frac{mn}{km+(1-k)n}\right). \tag{36}$$

Integrating the inequality (36) with respect to k over $[0, 1]$ and then putting $x = (mn/km + (1-k)n)$, we obtain

$$\frac{mn}{n-m}\int_{m}^{n}\frac{f(x)w(x)}{x^{2}}dx\leq\frac{mn}{(n-m)}\left[f(m)+f(n)\right]\int_{m}^{n}h\left(\frac{(x-m)n}{(n-m)x}\right)\frac{w(x)}{x^{2}}dx-\frac{\mu}{n-m}\int_{m}^{n}\frac{(n-x)(x-m)w(x)}{x^{4}}dx. \tag{37}$$

After simplification, we have

$$\int_{m}^{n}\frac{f(x)w(x)}{x^{2}}dx\leq\left[f(m)+f(n)\right]\int_{m}^{n}h\left(\frac{(x-m)n}{(n-m)x}\right)\frac{w(x)}{x^{2}}dx-\frac{\mu}{mn}\int_{m}^{n}\frac{(n-x)(x-m)w(x)}{x^{4}}dx. \tag{38}$$

This completes the proof. □

Remark 17. If we set $h(k) = k$ and $s = 1$ in (30), we obtain the Fejér-type inequality for strongly reciprocally convex functions [32].

4. Conclusion

Based on the inequalities presented in this paper, we can conclude that they provide useful tools for analyzing and estimating integrals involving various classes of functions. These inequalities have been derived using different techniques and assumptions, but they all share the common feature of providing bounds on the value of an integral in terms of certain properties of the integrand.

Moreover, the inequalities presented in this paper have a wide range of applications in different fields of mathematics and its applications, including analysis, probability theory, and mathematical physics. They can be used to derive various other results, such as inequalities for derivatives, series expansions, and more.

In addition, the paper highlights the importance of the concept of reciprocal convexity and its various generalizations, as they play a crucial role in deriving and establishing these inequalities. The paper also emphasizes the significance of special functions, such as harmonic and logarithmically convex functions, and their properties in the study of inequalities.

Finally, the paper encourages further research in this area, such as the development of new inequalities, the study of their applications in different fields, and the exploration of their connections with other areas of mathematics.

Data Availability

All data required for this research are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

During the revision process, Yujun Wang was added as an author and contributed to the study's conception, design, data analysis, interpretation, manuscript drafting, and critical revisions. Muhammad Shoaib Saleem contributed to data analysis and supervision, while Zahida Perveen contributed to manuscript writing. Muhammad Imran provided the results. All authors have approved the final manuscript and are accountable for the work.

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