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Research Article

Statistical Prediction Based on Ordered Ranked Set Sampling Using Type-II Censored Data from the Rayleigh Distribution under Progressive-Stress Accelerated Life Tests

Atef F. Hashem [1,2] and Alaa H. Abdel-Hamid [2]

[1] *Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia*
[2] *Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt*

Correspondence should be addressed to Atef F. Hashem; atef011264@science.bsu.edu.eg

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The objective of ranked set sampling is to gather observations from a population that is more likely to cover the population's full range of values. In this paper, the ordered ranked set sample is obtained using the idea of order statistics from independent and nonidentically distributed random variables under progressive-stress accelerated life tests. The lifetime of the item tested under normal conditions is suggested to be subject to the Rayleigh distribution with a scale parameter satisfying the inverse power law such that the applied stress is a nonlinear increasing function of time. Considering the type-II censoring scheme, one-sample prediction for censored lifetimes is discussed. Numerous point predictors including the Bayes point predictor, conditional median predictor, and best unbiased predictor for future order statistics are discussed. Additionally, conditional prediction intervals for future order statistics are also studied. The theoretical findings reported in this work are shown by illustrative examples based on simulated data as well as real data sets. The effectiveness of the prediction methods is then evaluated by a Monte Carlo simulation study.

1. Introduction

Modern devices have been designed and engineered to function flawlessly for an extended period of time under regular operating settings thanks to ongoing advancements in manufacturing technology.

As a result, when conducting traditional life research experiments, manufacturers struggle to provide sufficient information about the failure times for their products. Because of this, accelerated life tests (ALTs) or partial ALTs (PALTs) are used to quickly obtain the required information regarding product failure times and establish the relationship between product life and external stress variables. In ALTs, products are checked under situations that are more stressful than usual to discover early failure times, whereas in PALTs, they are checked in both normal and accelerated situations. Stress levels higher than those used during manufacturing are applied to products during ALTs and PALTs. The observed failures of the products collected from such ALTs or PALTs are used to predict how long they will live under normal conditions of use.

Several techniques, including progressive stress, step stress, and constant stress, can be used to apply stress to ALTs. For further explanation on ALTs and PALTs, one can refer to [1–15].

The researcher or experimenter may not be able to obtain complete data on the failure times of the units in the test trial as the units may be broken or excluded from the test prior to failure or when the unit is canceled. Censored data are those obtained from such situations. Censoring may have the significant benefit of reducing the total cost and duration of the experiment. Type-I and type-II censoring are the two methods that are frequently employed. In type-I censoring, the number of observed failures is a random variable (RV), while the experimental time is fixed. In contrast, the experimental time is an RV in type-II censoring, whereas the observed failure rate is fixed. A number of studies, see [16–18], have covered these two CSs in some detail.

Prediction is viewed as a significant problem in statistical inference. It has numerous uses in reliability, quality control, engineering, business, meteorology, medical sciences, and other fields as well. It is the challenge of predicting the values of unobserved (future) observations or functions of such observations from currently accessible (informative) observations. Two frequently used prediction strategies are the one- and two-sample techniques. An interval predictor and a point predictor are both examples of predictors. It has been discussed by a number of authors, including [5, 7, 19–22].

The ranked set sampling method was proposed in [23] as a more accurate way to compute the mean pasture yield. When determining the population mean, a theoretical base for this sampling method was enhanced and developed in [24]. It could be applied to choose sample units more economically for a test or study. It is frequently recommended when ordering sample units is cheap and simple and measuring sample units is very expensive or complicated. It could be used in a variety of disciplines, including agriculture, biology, ecology, engineering, medicine, and social studies [25]. The steps listed below could be used to obtain a ranked set sample (RSS) with size n from the provided population:

(1) Simple random samples (SRSs), each of the same size n, are created by selecting n^2 items from the provided population.

(2) The items are ranked, according to the variable of interest, for each sample. Several techniques, including expert opinion, readily available information, a person's professional judgment, and other information, may be used in ranking the items.

(3) A single item is measured in each of the ranked samples.

(4) A sample is chosen for actual measurement as follows:

   (i)The smallest item, say X_{11}, is measured in the first sample, and the other items are not measured.

   (ii) The second smallest item, say X_{22}, is measured in the second sample, and the other items are not measured.

   (iii) This approach is repeated until the greatest item of the latest sample, say X_{nn}, is measured.

(5) The procedure described above is referred to as a one cycle RSS with size n, and the data obtained are shown by $\mathbf{X}_{\mathrm{RSS}} = \{X_{11}, X_{22}, \ldots, X_{nn}\}$. It is observed that $X_{11}, X_{22}, \ldots, X_{nn}$ are independent RVs with nonidentical distributions (IRVNIDs).

(6) The preceding steps of $\mathscr{K}$ cycles are repeated to obtain an RSS of size $\mathscr{K}n$ extracted from $\mathscr{K}n^2$ items. The resulting data are denoted as

$$\mathbf{X}_{(\mathscr{K})\mathrm{RSS}} = \left\{X_{1,11}, X_{1,22}, \ldots, X_{1,nn}, \ldots\ldots\ldots\ldots, X_{\mathscr{K},11}, X_{\mathscr{K},22}, \ldots, X_{\mathscr{K},nn}\right\}. \tag{1}$$

The ordered RSS (ORSS) was devised in [26], in which the authors demonstrated how much more effective ORSS is than SRS. It can be achieved by ordering the RSS, $\{X_{11}, X_{22}, \ldots, X_{nn}\}$, in ascending order of magnitude. This proposal was based on the idea of order statistics from IRVNID.

Several researchers have investigated the estimation and prediction problems on the basis of the SRS and ORSS of various distributions. The distribution-free prediction intervals for record values and future order statistics were constructed in [27]. In [28, 29], it was investigated how to predict unobserved data under a type-II censoring scheme (CS) and how to estimate the parameters of Rayleigh and Pareto distributions using Bayesian methods. Based on type-I CS, the step-stress ALT data were used in [30, 31] to estimate the parameters of Rayleigh and exponential distributions. The Bayesian method was explained in [32] to estimate the parameters taken into consideration using progressive-stress ALT (PRSALT) data that are exponentially distributed.

Due to the importance of predictions, ALTs, and RSSs in many areas as mentioned above, many experimenters and engineers would like to obtain the failure times of some items in a short time. Additionally, they may need to predict future failure times for some items that cannot be obtained in the normal state of the experiment. These requirements and their importance motivate us to consider this article in which we apply the PRSALT, with a nonlinear increasing function of time, to items whose lifetimes under normal condition stress are supposed to follow the Rayleigh distribution (RD). ORSSs are obtained using the idea of order statistics from IRVNID under PRSALTs.

Under type-II censoring, numerous point predictors including the Bayes point predictor (BPPRR) (using squared error (SER), linear-exponential (LEX), and general entropy (GEN) loss functions), conditional median predictor (CMPR), and best unbiased predictor (BUPR) for future order statistics are discussed. Furthermore, conditional prediction intervals (CPIs) for future order statistics are also studied.

The remaining sections are arranged as follows: Section 2 discusses the ORSS under the PRSALT. The model and type-II censoring are explained in Section 3. Section 4 discusses some point predictors and CPIs of future order statistics. In

Section 5, representative examples are provided. In Sections 6 and 7, respectively, simulation studies and conclusions are presented.

2. Description of the Model under PRSALT

The RD was originally proposed in [33] in the field of acoustics; since its inception, several researchers have applied the distribution in numerous branches of technology and science. It is extensively applied in communication engineering and oceanography to model wave heights. Furthermore, it has a wide range of applications in lifetime data analysis, particularly in survival analysis and reliability theory. The fact that the RD's failure rate is a linearly increasing function of time at a constant rate makes it a good model for the lifespan of parts and objects that deteriorate quickly over time. As a result, compared to the exponential distribution, the RD's reliability function deteriorates over time at a significantly faster rate.

Assume that an item's lifetime under normal use is represented by the RV X, which is subject to RD with a scale parameter of $\alpha > 0$. Then, the cumulative distribution function (CDF), $F(x)$, of X is represented by

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\alpha}\right)^2\right], \quad x > 0. \tag{2}$$

2.1. Progressive-Stress Model Based on the Rayleigh Distribution. Previous studies of the PRSALT have indicated that the imposed stress is expressed as an increasing linear function of time, see [5, 6, 9]. While in some papers such as [7, 11, 34], the authors suggested PRSALTs taking into account that the imposed stress is represented as a nonlinear increasing function of time. The PRSALT is performed under the following fundamental assumptions.

2.1.1. Assumptions

(1) The lifetime of an item under design stress is governed by RD with CDF (2).

(2) The imposed stress $\zeta(x)$ is a nonlinear increasing function of time x with the form, see Figure 1,

$$\zeta(x) = \sqrt{d}\, x^c, \quad c, d > 0. \tag{3}$$

(3) The relation between the scale parameter α in CDF (2) and the imposed stress ζ is controlled by the inverse power law with two positive parameters θ and λ, i.e.,

$$\alpha(x) = \alpha(\zeta(x)) = \frac{1}{\sqrt{\lambda}[\zeta(x)]^\theta}. \tag{4}$$

(4) The testing process starts by dividing the N testable items into $\mathfrak{B}$ ($> r$bin1) groups, each of which has n items and is administered under PRSALT. Thus,

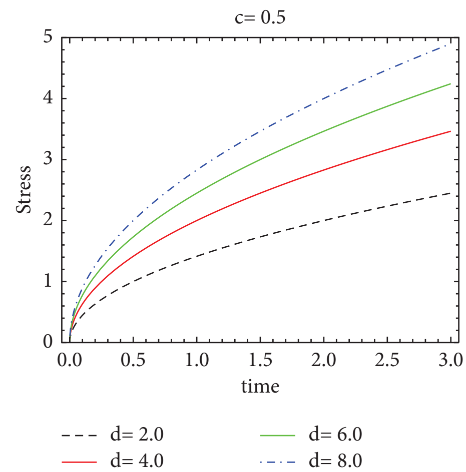

Figure 1: The relation between the stress and time.

$$\zeta_p(x) = \sqrt{d_p x^c}, \quad p = 1, \ldots, \mathfrak{B},\ d_1 < d_2 < \ldots < d_{\mathfrak{B}}. \tag{5}$$

(5) For $p = 1, \ldots, \mathfrak{B}$, the n failure times in group p, indicated by $X_{p,1}, X_{p,2}, \ldots, X_{p,n}$ (with realizations $x_{p,1}, x_{p,2}, \ldots, x_{p,n}$), are statistically independent RVs.

(6) The items' failure mechanisms remain unchanged under any level of stress.

(7) Cumulative exposure model [1] links the distribution under accelerated stress to that under normal stress.

Based on CDF (2) and according to Assumptions 2, 3, and 7, the cumulative exposure model, $\Omega(x)$, can be expressed as

$$\Omega(x) = \int_0^x \frac{dv}{\alpha(\zeta(v))}. \tag{6}$$

The CDF under PRSALT, $G(x)$, takes the form

$$G(x) = F(\zeta(x)), \tag{7}$$

where the function $F(.)$ is the assumed CDF with $\alpha = 1$.

Cumulative exposure model (6), according to Assumptions 3 and 4, becomes

$$\Omega_p(x) = \frac{\sqrt{\lambda} d_p^{\theta/2} x^{c\theta+1}}{c\theta + 1}, \quad p = 1, \ldots, \mathfrak{B}. \tag{8}$$

Using CDFs (2) and (7), the CDF $G_p(x)$ for an item presented in group p under PRSALT takes the form

$$G_p(x) = 1 - \exp\left[-\left(\frac{x}{\vartheta_p}\right)^\beta\right], \quad x > 0, \left(\beta > 2, \vartheta_p > 0\right). \tag{9}$$

One can notice that CDF (9) concerns a Weibull distribution with

$$\left.\begin{aligned}\beta &= 2(c\theta+1)\\ \vartheta_p &= \left(\frac{\beta^2}{4\lambda d_p^{\theta}}\right)^{1/\beta}\end{aligned}\right\}. \tag{10}$$

The corresponding probability density function (PDF), $g_p(x)$, and the hazard rate function (HRF), $\Upsilon_p(x)$, of (9) are given, respectively, by

$$g_p(x) = \frac{\beta}{\vartheta_p^{\beta}} x^{\beta-1} \exp\left[-\left(\frac{x}{\vartheta_p}\right)^{\beta}\right], \quad x>0, \tag{11}$$

$$\Upsilon_p(x) = \frac{\beta}{\vartheta_p^{\beta}} x^{\beta-1}, \quad x>0. \tag{12}$$

PDF (11) and HRF (12) are plotted in Figure 2 for $\theta = 1.5, \lambda = 2.0$, and different values of c and d. It can be noticed that PDF (11) is always unimodal, while HRF (12) is always increasing since $\beta > 2$.

2.2. Ranked Set Sampling with Accelerated Life Tests under Progressive Stress. The next algorithm can be applied to obtain an RSS with size N under PRSALT with $\mathfrak{B}$ (>1) levels of stress:

(1) Fixed values for N, n, and $\mathfrak{B}$ are assigned, such that $N = \mathfrak{B} \times n$.

(2) $\mathfrak{B}n^2$ items are chosen from the provided population, and they are divided into $\mathfrak{B}n$ SRSs, all of the same size n.

(3) $j = 1$ is set.

(4) The N items to be examined are divided into $\mathfrak{B}$ (>1) groups, as is previously indicated in Section 1. Each group is an SRS consisting of n items and is performed under PRSALT with stress levels $\zeta_p(x), p = 1, \ldots, \mathfrak{B}$.

(5) The SRSs in all groups are ordered without practical measurement.

(6) In the p-th ordered SRS, $p = 1, \ldots, \mathfrak{B}$, a single item is measured.

(7) In group p, the j-th smallest item, say $X_{p,jj}, p = 1, \ldots, \mathfrak{B}$, is measured.

(8) $j = j + 1$ is set. If $j = n + 1$, then the previous steps are halted, and it is suggested that we proceed to Step 10. If not, the smallest item in group p, say $X_{p,j+1j+1}, p = 1, \ldots, \mathfrak{B}$, is measured.

(9) Steps 4–8 are iterated.

(10) An RSS of size N is now generated under PRSALT as follows:

$$\left.\begin{array}{cccccccccc}\{\{X_{1,11}< & X_{1,12}< & \ldots< & X_{1,1n}\} & \ldots & \{X_{\mathfrak{B},11}< & X_{\mathfrak{B},12}< & \ldots< & X_{\mathfrak{B},1n}\}\}\\ \{\{X_{1,21}< & X_{1,22}< & \ldots< & X_{1,2n}\} & \ldots & \{X_{\mathfrak{B},21}< & X_{\mathfrak{B},22}< & \ldots< & X_{\mathfrak{B},2n}\}\}\\ \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots & \vdots & \\ \{\{X_{1,n1}< & X_{1,n2}< & \ldots< & X_{1,nn}\} & \underset{\text{Judgment Rank}}{\underset{\square}{\ldots}} & \{X_{\mathfrak{B},n1}< & X_{\mathfrak{B},n2}< & \ldots< & X_{\mathfrak{B},nn}\}\}\end{array}\right\} \Rightarrow \left\{\begin{array}{ccc}\{X_{1,11}, & \ldots, & X_{\mathfrak{B},11},\\ X_{1,22}, & \ldots, & X_{\mathfrak{B},22},\\ \vdots & \vdots & \vdots\\ X_{1,nn}, & \underset{\text{RSS}}{\underset{\square}{\ldots,}} & X_{\mathfrak{B},nn}\}.\end{array}\right. \tag{13}$$

(11) The method described in the previous steps is called a one-cycle RSS of size N under PRSALT, and the outcomes are shown by

$$\mathbf{X}_{RSS} = \begin{array}{cccc}\{X_{1,11}, & X_{2,11} & \ldots, & X_{\mathfrak{B},11},\\ X_{1,22}, & X_{2,22}, & \ldots, & X_{\mathfrak{B},22},\\ \vdots & \vdots & \vdots & \vdots\\ X_{1,nn}, & X_{2,nn}, & \ldots, & X_{\mathfrak{B},nn}\}.\end{array} \tag{14}$$

For instance, $X_{3,55}$ denotes the fifth smallest item in the fifth sample presented in the third group. Additionally, the elements of the RSS are IRVNID.

(12) Steps 2–9 of $\mathscr{K}$ cycles are iterated to obtain an RSS of size $\mathscr{K}N$. The obtained data are shown by

$$\mathbf{X}_{(\mathscr{K})\text{RSS}} = \begin{array}{ccc}\{\{X_{1,1,11}, X_{1,2,11}, \ldots, X_{1,\mathfrak{B},11}\}, & \{X_{1,1,22}, X_{1,2,22}, \ldots, X_{1,\mathfrak{B},22}\}, \ldots, & \{\{X_{1,1,nn}, X_{1,2,nn}, \ldots, X_{1,\mathfrak{B},nn}\},\\ \vdots & \vdots & \vdots\\ \{X_{\mathscr{K},1,11}, X_{\mathscr{K},2,11}, \ldots, X_{\mathscr{K},\mathfrak{B},11}\}, & \{X_{\mathscr{K},1,22}, X_{\mathscr{K},2,22}, \ldots, X_{\mathscr{K},\mathfrak{B},22}\}, \ldots, & \{X_{\mathscr{K},1,nn}, X_{\mathscr{K},2,nn}, \ldots, X_{\mathscr{K},\mathfrak{B},nn}\}\}.\end{array} \tag{15}$$

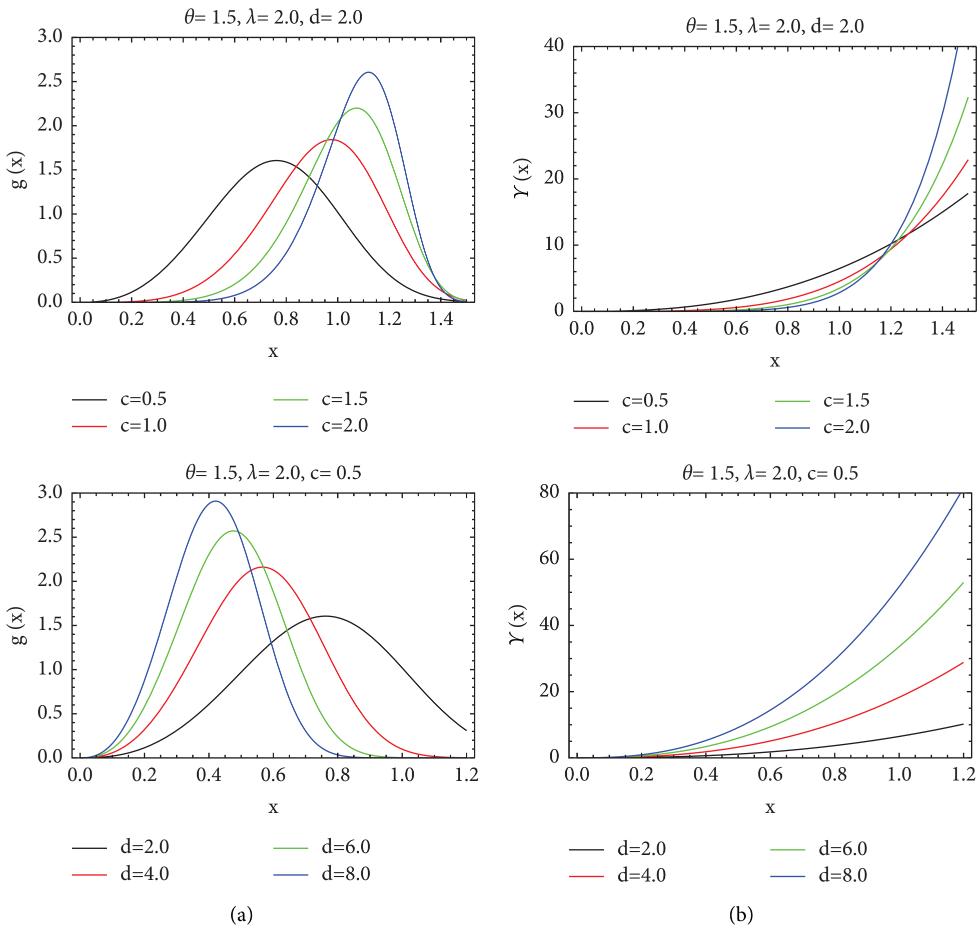

Figure 2: (a, b) The PDFs (HRFs) of the RD under PRSALT for $\theta = 1.5, \lambda = 2.0$, and different values of c and d.

We presume that $\mathbf{X}_{\text{RSS}}$ is a one-cycle RSS from a given population under PRSALT with CDF (9) and PDF (11). The CDF and PDF of $X_{p,rr}$, $p = 1, \ldots, \mathfrak{B}$, denoted by $G_{p,r:n}$ and $g_{p,r:n}$, are then the CDF and PDF of the r-th order statistic of group p, respectively. They can be written as [35, 36]

$$G_{p,r:n}(x) = \sum_{i=r}^{n} \binom{n}{i} \left[G_p(x)\right]^i \left[1 - G_p(x)\right]^{n-i}, \quad (16)$$

$$g_{p,r:n}(x) = r \binom{n}{r} \left[G_p(x)\right]^{r-1} \left[1 - G_p(x)\right]^{n-r} g_p(x), \quad (17)$$

where $G_p(x)$ and $g_p(x)$ are given by (9) and (11), respectively.

It is possible to rewrite CDF (16) and PDF (17) as

$$G_{p,r:n}(x) = 1 - \sum_{i=1}^{r} w_{i,r}^{*}(n) \left[1 - G_p(x)\right]^{n+i-r}, \quad (18)$$

$$g_{p,r:n}(x) = \sum_{i=0}^{r-1} w_{i,r}(n) \left[1 - G_p(x)\right]^{n+i-r} g_p(x), \quad (19)$$

where

$$\left.\begin{aligned} w_{i,r}(n) &= (-1)^i r \binom{r-1}{i} \binom{n}{r} \\ w_{i,r}^{*}(n) &= \frac{w_{i-1,r}(n)}{n+i-r} \end{aligned}\right\}. \quad (20)$$

3. The Model with Type-II Censoring

Type-II CS can be imposed to the ordered one cycle RSS under PRSALT as follows: having determined the RSS for group p, $\{X_{p,11}, X_{p,22}, \ldots, X_{p,nn}\}$, $p = 1, \ldots, \mathfrak{B}$, we order and determine the first m statistics in it, say $\{Z_{p,1} \leq Z_{p,2} \leq \ldots \leq Z_{p,m}\}$. The data collected from this procedure are known as one-cycle type-II censored ORSS and are represented by $\mathbf{Z}_{\text{ORSS}} = \{\{Z_{1,1} \leq Z_{1,2} \leq \ldots \leq Z_{1,m}\}, \ldots, \{Z_{\mathfrak{B},1} \leq Z_{\mathfrak{B},2} \leq \ldots \leq Z_{\mathfrak{B},m}\}\}$. Based on the idea of order statistics from IRVNID which was proposed in [37], it is possible to write the likelihood function for one-cycle ORSS with type-II CS as

$$\mathbb{L}(\theta,\lambda;\mathbf{Z}=\mathbf{z})\propto\prod_{p=1}^{\mathfrak{B}}\left[\sum_{S[j]}\prod_{r=1}^{m}g_{p,j_{p,r}}\left(z_{p,r}\right)\prod_{r=m+1}^{n}\left[1-G_{p,j_{p,r}}\left(z_{p,m}\right)\right]\right], \tag{21}$$

where $\Sigma_{S[j]}$ denote the total of all $n!$ permutations $(j_{p,1},\ldots,j_{p,n})$ of $(1,\ldots,n)$, and $\mathbf{z}=(\mathbf{z}_1,\ldots,\mathbf{z}_{\mathfrak{B}})$, $\mathbf{z}_p=(z_{p,1},\ldots,z_{p,m})$, $p=1,\ldots,\mathfrak{B}$.

Likelihood function (21) can be rewritten as follows:

$$\mathbb{L}(\theta,\lambda;\mathbf{z})\propto\prod_{p=1}^{\mathfrak{B}}\mathrm{Per}\mathbf{U_p}, \tag{22}$$

where $\mathrm{Per}\mathbf{U_p}=\sum_{S[j]}\prod_{r=1}^{n}a_{r,j_{p,r}}$ denotes the permanent of a square real matrix $\mathbf{U_p}=(a_{j,r})$ of size $n\times n$.

$$\mathbf{U_p}=\begin{array}{l}\left(\begin{array}{cccc}g_{p,1}\left(z_{p,1}\right) & g_{p,2}\left(z_{p,1}\right) & \cdots & g_{p,n}\left(z_{p,1}\right)\\ \vdots & \vdots & \ddots & \vdots\\ g_{p,1}\left(z_{p,m}\right) & g_{p,2}\left(z_{p,m}\right) & \cdots & g_{p,n}\left(z_{p,m}\right)\\ 1-G_{p,1}\left(z_{p,m}\right) & 1-G_{p,2}\left(z_{p,m}\right) & \cdots & 1-G_{p,n}\left(z_{p,m}\right)\end{array}\right)_{\}(n-m)\text{rows}}\end{array}. \tag{23}$$

Substitute CDF (18) and PDF (19) into (21), the likelihood function can be expressed as follows:

$$\mathbb{L}(\theta,\lambda;\mathbf{z})\propto\prod_{p=1}^{\mathfrak{B}}\left[\sum_{S[p]}\left(\prod_{r=1}^{m}\sum_{i=0}^{j_{p,r}-1}w_{i,j_{p,r}}(n)\left[1-G_p\left(z_{p,r}\right)\right]^{n+i-j_{p,r}}g_p\left(z_{p,r}\right)\times\prod_{r=m+1}^{n}\sum_{i=1}^{j_{p,r}}w^*_{i,j_{p,r}}(n)\left[1-G_p\left(z_{p,m}\right)\right]^{n+i-j_{p,r}}\right)\right]. \tag{24}$$

Considering equations (9) and (11) and the next relations, we obtain

$$\left.\begin{array}{l}\displaystyle\prod_{r=1}^{m}\sum_{i=0}^{j_{p,r}-1}\Omega_i\left(j_{p,r}\right)=\sum_{\delta_{p,1}=0}^{j_{p,1}-1}\sum_{\delta_{p,2}=0}^{j_{p,2}-1}\cdots\sum_{\delta_{p,m}=0}^{j_{p,m}-1}\prod_{r=1}^{m}\Omega_{\delta_{p,r}}\left(j_{p,r}\right),\\ \displaystyle\prod_{r=m+1}^{n}\sum_{i=1}^{j_{p,r}}\Omega^*_i\left(j_{p,r}\right)=\sum_{\mu_{p,m+1}=1}^{j_{p,m+1}}\sum_{\mu_{p,m+2}=1}^{j_{p,m+2}}\cdots\sum_{\mu_{p,n}=1}^{j_{p,n}}\prod_{r=m+1}^{n}\Omega^*_{\mu_{p,r}}\left(j_{p,r}\right),\end{array}\right\} \tag{25}$$

and it is possible to express the likelihood function as follows:

$$\mathbb{L}(\theta,\lambda;\mathbf{z})\propto\prod_{p=1}^{\mathfrak{B}}\left[\sum_{S[p]}\sum_{\delta_p,\mu_p}^{m,n}\left(D_{\delta_p,\mu_p}\left(\mathbf{j}_p\right)\left[\prod_{r=1}^{m}\beta\vartheta_p^{-\beta}z_{p,r}^{\beta-1}\right]\exp\left[-\vartheta_p^{-\beta}\Psi_{\delta_p,\mu_p}\left(\mathbf{z}_p\right)\right]\right)\right], \tag{26}$$

where $\mathbf{j}_p=(j_{p,1},\ldots,j_{p,m},j_{p,m+1},\ldots,j_{p,n})$, $\delta_p=(\delta_{p,1},\ldots,\delta_{p,m})$, $\mu_p=(\mu_{p,m+1},\ldots,\mu_{p,n})$, and $p=1,\ldots,\mathfrak{B}$, and

$$\sum_{\delta_p,\mu_p}^{m,n} = \sum_{\delta_{p,1}=0}^{j_{p,1}-1}\sum_{\delta_{p,2}=0}^{j_{p,2}-1}\cdots\sum_{\delta_{p,m}=0}^{j_{p,m}-1}\cdot\sum_{\mu_{p,m+1}=1}^{j_{p,m+1}}\sum_{\mu_{p,m+2}=1}^{j_{p,m+2}}\cdots\sum_{\mu_{p,n}=1}^{j_{p,n}}, \tag{27}$$

$$D_{\delta_p,\mu_p}\left(\mathbf{j}_p\right) = \left[\prod_{r=1}^{m} w_{\delta_{p,r},j_{p,r}}(n)\right]\left[\prod_{r=m+1}^{n} w^{*}_{\mu_{p,r},j_{p,r}}(n)\right], \tag{28}$$

$$\Psi_{\delta_p,\mu_p}\left(\mathbf{z}_p\right) = \left[\sum_{r=1}^{m}\left(n+\delta_{p,r}-j_{p,r}+1\right)t_{p,r}^{\beta}\right] + \left[\sum_{r=m+1}^{n}\left(n+\mu_{p,r}-j_{p,r}\right)t_{p,m}^{\beta}\right]. \tag{29}$$

The likelihood function can be modified using the relationships provided in (25) as follows:

$$\mathbb{L}(\theta,\lambda;\mathbf{z}) \propto \sum_{\mathbf{S}^*,\delta^*,\mu^*}^{\mathfrak{B},m,n}\left(\left[\prod_{p=1}^{\mathfrak{B}} D_{\delta_p,\mu_p}\left(\mathbf{j}_p\right)\right]\left[\prod_{p=1}^{\mathfrak{B}}\prod_{r=1}^{m}\beta\vartheta_p^{-\beta}z_{p,r}^{\beta-1}\right]\exp\left[-\sum_{p=1}^{\mathfrak{B}}\vartheta_p^{-\beta}\Psi_{\delta_p,\mu_p}\left(\mathbf{z}_p\right)\right]\right), \tag{30}$$

where β and ϑ_p are as given in (10), $\mathbf{S}^* = (S[1],\ldots,S[\mathfrak{B}])$, $\delta^* = (\delta_1,\ldots,\delta_{\mathfrak{B}})$, $\delta_p = (\delta_{p,1},\ldots,\delta_{p,m})$, $\mu^* = (\mu_1,\ldots,\mu_{\mathfrak{B}})$, $\mu_p = (\mu_{p,m+1},\ldots,\mu_{p,n})$, and $p = 1,\ldots,\mathfrak{B}$, and

$$\sum_{\mathbf{S}^*,\delta^*,\mu^*}^{\mathfrak{B},m,n} = \prod_{p=1}^{\mathfrak{B}}\sum_{S[p]}\sum_{\delta_p,\mu_p}^{m,n} = \sum_{S[1]}\sum_{\delta_1,\mu_1}^{m,n}\cdots\sum_{S[\mathfrak{B}]}\sum_{\delta_{\mathfrak{B}},\mu_{\mathfrak{B}}}^{m,n}, \tag{31}$$

where $\sum_{\delta_p,\mu_p}^{m,n}$ is as defined in (27).

For $\mathscr{K}$-cycle ORSS, the likelihood function under type-II CS can be expressed as

$$\mathbb{L}(\theta,\lambda;\mathbf{z}) \propto \prod_{q=1}^{\mathscr{K}}\left[\sum_{\mathbf{S}_q^*,\delta_q^*,\mu_q^*}^{\mathfrak{B},m,n}\left(\left[\prod_{p=1}^{\mathfrak{B}} D_{\delta_{q,p},\mu_{q,p}}\left(\mathbf{j}_{q,p}\right)\right]\left[\prod_{p=1}^{\mathfrak{B}}\prod_{r=1}^{m}\beta\vartheta_p^{-\beta}z_{q,p,r}^{\beta-1}\right]\times\exp\left[-\sum_{p=1}^{\mathfrak{B}}\vartheta_p^{-\beta}\Psi_{\delta_{q,p},\mu_{q,p}}\left(\mathbf{z}_{q,p}\right)\right]\right)\right]. \tag{32}$$

The likelihood function can be rewritten using the relations provided in (25) as

$$\mathbb{L}(\theta,\lambda;\mathbf{z}) \propto \sum_{\mathbf{S}^{**},\delta^{**},\mu^{**}}^{\mathfrak{B},m,n}\left(\left[\prod_{q=1}^{\mathscr{K}}\prod_{p=1}^{\mathfrak{B}} D_{\delta_{q,p},\mu_{q,p}}\left(\mathbf{j}_{q,p}\right)\right]\left[\prod_{q=1}^{\mathscr{K}}\prod_{p=1}^{\mathfrak{B}}\prod_{r=1}^{m}\beta\vartheta_p^{-\beta},z_{q,p,r}^{\beta-1}\right]\times\exp\left[-\sum_{q=1}^{\mathscr{K}}\sum_{p=1}^{\mathfrak{B}}\vartheta_p^{-\beta}\Psi_{\delta_{q,p},\mu_{q,p}}\left(\mathbf{z}_{q,p}\right)\right]\right), \tag{33}$$

where β and ϑ_p are as given in (10),

$$\begin{aligned}
\sum_{\mathbf{S}^{**},\delta^{**},\mu^{**}}^{\mathfrak{B},m,n} &= \prod_{q=1}^{\mathscr{K}}\sum_{\mathbf{S}_q^*,\delta_q^*,\mu_q^*}^{\mathfrak{B},m,n} = \sum_{\mathbf{S}_1^*,\delta_1^*,\mu_1^*}^{\mathfrak{B},m,n}\cdots\sum_{\mathbf{S}_{\mathscr{K}}^*,\delta_{\mathscr{K}}^*,\mu_{\mathscr{K}}^*}^{\mathfrak{B},m,n},\\
\sum_{\mathbf{S}_q^*,\delta_q^*,\mu_q^*}^{\mathfrak{B},m,n} &= \sum_{S[q,1]}\sum_{\delta_{q,1},\mu_{q,1}}^{m,n}\cdots\sum_{S[\mathbf{q},\mathfrak{B}]}\sum_{\delta_{q,\mathfrak{B}},\mu_{q,\mathfrak{B}}}^{m,n},\\
\sum_{\delta_{q,p},\mu_{q,p}}^{m,n} &= \sum_{\delta_{q,p,1}=0}^{j_{q,p,1}-1}\sum_{\delta_{q,p,2}=0}^{j_{q,p,2}-1}\cdots\sum_{\delta_{q,p,m}=0}^{j_{q,p,m}-1}\cdot\sum_{\mu_{q,p,m+1}=1}^{j_{q,p,m+1}}\sum_{\mu_{q,p,m+2}=1}^{j_{q,p,m+2}}\cdots\sum_{\mu_{q,p,n}=1}^{j_{q,p,n}},
\end{aligned} \tag{34}$$

and $\mathbf{j}_{q,p} = (j_{q,p,1}, \ldots, j_{q,p,m}, j_{q,p,m+1}, \ldots, j_{q,p,n})$, $q = 1, \ldots, \mathscr{K}$, $p = 1, \ldots, \mathfrak{B}$, $\mathbf{S}^{**} = (\mathbf{S}_1^*, \ldots, \mathbf{S}_{\mathscr{K}}^*)$, $\mathbf{S}_q^* = (S[q,1], \ldots, S[q,\mathfrak{B}])$, $\delta^{**} = (\delta_1^*, \ldots, \delta_{\mathscr{K}}^*)$, $\delta_q^* = (\delta_{q,1}, \ldots, \delta_{q,\mathfrak{B}})$, $\delta_{q,p} = (\delta_{q,p,1}, \ldots, \delta_{q,p,m})$, $\mu^{**} = (\mu_1^*, \ldots, \mu_{\mathscr{K}}^*)$, $\mu_q^* = (\mu_{q,1}, \ldots, \mu_{q,\mathfrak{B}})$, and $\mu_{q,p} = (\mu_{q,p,m+1}, \ldots, \mu_{q,p,n})$.

3.1. Formulating the Prior and Posterior Density Functions. It is appropriate to select θ and λ to be dependent since they are merged, as shown in (10). We presume that θ and λ are distributed according to the Lomax distribution. The following is a possible representation of the joint prior density of θ and λ:

$$\pi(\theta,\lambda) = \pi_1(\lambda)\,\pi_2(\theta|\lambda), \tag{35}$$

where

$$\pi_1(\lambda) = e_1 e_2 (1 + e_2\lambda)^{-(e_1+1)}, \quad \lambda > 0,\ (e_1, e_2) > 0, \tag{36}$$

$$\pi_2(\theta|\lambda) = e_3\lambda(1 + \theta\lambda)^{-(e_3+1)}, \quad \theta > 0,\ e_3 > 0. \tag{37}$$

Using (36) and (37), joint prior density (35) takes the following form:

$$\pi(\theta,\lambda) = e_1 e_2 e_3 \lambda (1 + e_2\lambda)^{-(e_1+1)} (1 + \theta\lambda)^{-(e_3+1)}, \quad \theta,\lambda > 0,\ (e_1, e_2, e_3) > 0. \tag{38}$$

The hyperparameter values (e_1, e_2, e_3) can be specified in such a way that the prior means become the approximate expected value of the corresponding parameters.

Using (33) and (38), it is possible to write the joint posterior density function of θ and λ as follows:

$$\begin{aligned}\pi^*(\theta,\lambda|\mathbf{z}) = {} & \mathfrak{I}^{-1}\lambda(1 + e_2\lambda)^{-(e_1+1)}(1 + \theta\lambda)^{-(e_3+1)} \sum_{\mathbf{S}^{**},\delta^{**},\mu^{**}}^{\mathfrak{B},m,n} \left(\left[\prod_{q=1}^{\mathscr{K}} \prod_{p=1}^{\mathfrak{B}} D_{\delta_{q,p},\mu_{q,p}}(\mathbf{j}_{q,p}) \right] \right. \\ & \left. \times \left[\prod_{q=1}^{\mathscr{K}} \prod_{p=1}^{\mathfrak{B}} \prod_{r=1}^{m} \frac{2\lambda d_p^{\theta} z_{q,p,r}^{2c\theta+1}}{c\theta+1} \right] \exp\left[-\sum_{q=1}^{\mathscr{K}} \sum_{p=1}^{\mathfrak{B}} \frac{\lambda d_p^{\theta}}{(c\theta+1)^2} \Psi_{\delta_{q,p},\mu_{q,p}}(\mathbf{z}_{q,p}) \right] \right),\end{aligned} \tag{39}$$

where

$$\begin{aligned}\mathfrak{I} = {} & \sum_{\mathbf{S}^{**},\delta^{**},\mu^{**}}^{\mathfrak{B},m,n} \left(\left[\prod_{q=1}^{\mathscr{K}} \prod_{p=1}^{\mathfrak{B}} D_{\delta_{q,p},\mu_{q,p}}(\mathbf{j}_{q,p}) \right] \int_0^\infty \int_0^\infty \lambda(1 + e_2\lambda)^{-(e_1+1)}(1 + \theta\lambda)^{-(e_3+1)} \right. \\ & \left. \times \left[\prod_{q=1}^{\mathscr{K}} \prod_{p=1}^{\mathfrak{B}} \prod_{r=1}^{m} \frac{2\lambda d_p^{\theta} z_{q,p,r}^{2c\theta+1}}{c\theta+1} \right] \exp\left[-\sum_{q=1}^{\mathscr{K}} \sum_{p=1}^{\mathfrak{B}} \frac{\lambda d_p^{\theta}}{(c\theta+1)^2} \Psi_{\delta_{q,p},\mu_{q,p}}(\mathbf{z}_{q,p}) \right] \mathrm{d}\theta \mathrm{d}\lambda \right).\end{aligned} \tag{40}$$

3.2. Loss Functions. Both Bayes analysis and statistical decision inference rely heavily on the loss function. Its choice must be taken into account for calculating the Bayes estimators for θ and λ and any function of them. Due to its equal weighting of overestimation and underestimation, the SER loss function is one of the most widely used symmetric loss functions for evaluating estimator performance in practice. The following is a formulation of the SER loss function:

$$\mathscr{L}(\hat{\varrho},\varrho) \propto (\hat{\varrho} - \varrho)^2, \tag{41}$$

where $\hat{\varrho}$ indicates the estimator of ϱ.

Considering the SER loss function, the Bayes estimate (BE) of ϱ is provided by

$$\hat{\varrho} = E[\varrho|\mathbf{z}]. \tag{42}$$

In some circumstances, overestimating or underestimating might have different effects. Engineering, medicinal, and biomedical sciences frequently encounter such circumstances. Overestimation is typically more harmful than underestimation. For instance, when we estimate the average dependable working life of components, an asymmetric loss function may be more suitable in this case. There are many asymmetric loss functions proposed for use, including the LEX and GEN loss functions.

The following formula for the LEX loss function was provided in [38]:

$$\mathscr{L}(\varpi) \propto e^{\xi\varpi} - \xi\varpi - 1, \quad \xi \neq 0, \tag{43}$$

where $\varpi = \tilde{\varrho} - \varrho$ and $\tilde{\varrho}$ is the LEX estimator of ϱ.

Considering the LEX loss function, the BE of γ is provided by

$$\tilde{\varrho} = \frac{-1}{\xi}\ln\left[E\left(e^{-\xi\varrho}|\mathbf{z}\right)\right]. \tag{44}$$

The following formula for the GEN loss function was provided in [39]:

$$\mathscr{L}(\ddot{\varrho},\varrho) \propto \left(\frac{\ddot{\varrho}}{\varrho}\right)^{\xi} - \xi\ln\left[\frac{\ddot{\varrho}}{\varrho}\right] - 1, \quad \xi \neq 0. \tag{45}$$

Considering the GEN loss function, the BE of ϱ is provided by

$$\ddot{\varrho} = \left[E\left(\varrho^{-\xi}|\mathbf{z}\right)\right]^{-1/\xi}. \tag{46}$$

The methods for obtaining point predictors and prediction intervals for future order statistics are covered in the section that follows.

4. One-Sample Prediction Procedure

The following is how a one-sample prediction scheme is carried out: Suppose that, for $q = 1,\ldots,\mathscr{K}$ and $p = 1,\ldots,\mathfrak{B}$, $Z_{q,p,1} \leq Z_{q,p,2} \leq \ldots \leq Z_{q,p,m}$ is an informative type-II q-cycle ORSS of size m taken from a sample of size n. Suppose that $Z_{q,p,m+1} \leq Z_{q,p,m+2} \leq \ldots \leq Z_{q,p,n}$ be the unobserved future order statistics from the same sample, which is yet to observe. Let $T_{q,p,s} = Z_{q,p,m+s}$, $s = 1,\ldots,n-m$. Predicting the remaining order statistics $T_{q,p,s}$, $s = 1,\ldots,n-m$, $q = 1,\ldots,\mathscr{K}$, $p = 1,\ldots,\mathfrak{B}$ is our current goal.

The conditional PDF of $T_{q,p,s}$, with realization $t_{q,p,s}$, can take the following form [35, 36, 40]:

$$h_{q,p}\left(t_{q,p,s}|\theta,\lambda\right) = \frac{1}{(s-1)!\,(n^*-s)!}\sum_{D[n^*]}\left(\prod_{r=1}^{s-1} G^*_{p,j_{q,p,r}}\left(t_{q,p,s}\right) g^*_{p,j_{q,p,s}}\left(t_{q,p,s}\right)\prod_{r=s+1}^{n^*}\left[1 - G^*_{p,j_{q,p,r}}\left(t_{q,p,s}\right)\right]\right), \tag{47}$$

where $T_{q,p,s} > Z_{q,p,m}$ and $n^* = n-m$, and

$$\begin{aligned}
g^*_{p,l}\left(t_{q,p,s}\right) &= l\binom{n^*}{l}\left[R_p\left(z_{q,p,m}\right) - R_p\left(t_{q,p,s}\right)\right]^{l-1}\left[R_p\left(t_{q,p,s}\right)\right]^{n^*-l}\left[R_p\left(z_{q,p,m}\right)\right]^{-n^*} g_p\left(t_{q,p,s}\right)\\
&= l\binom{n^*}{l}\sum_{k_1=0}^{l-1}(-1)^{k_1}\binom{l-1}{k_1}\left[R_p\left(z_{q,p,m}\right)\right]^{-\varepsilon-1}\left[R_p\left(t_{q,p,s}\right)\right]^{\varepsilon} g_p\left(t_{q,p,s}\right)\\
&= l\binom{n^*}{l}\sum_{k_1=0}^{l-1}(-1)^{k_1}\binom{l-1}{k_1}\frac{2\lambda d_p^{\theta} t_{q,p,s}^{2c\theta+1}}{c\theta+1}\exp\left[-\frac{\lambda d_p^{\theta}(\varepsilon+1)}{(c\theta+1)^2}\left(t_{q,p,s}^{2(c\theta+1)} - z_{q,p,m}^{2(c\theta+1)}\right)\right],
\end{aligned} \tag{48}$$

$$G^*_{p,l}\left(t_{q,p,s}\right) = l\binom{n^*}{l}\sum_{k_1=0}^{l-1}(-1)^{k_1}\binom{l-1}{k_1}\frac{1}{\varepsilon+1}\left(1 - \exp\left[-\frac{\lambda d_p^{\theta}(\varepsilon+1)}{(c\theta+1)^2}\left(t_{q,p,s}^{2(c\theta+1)} - z_{q,p,m}^{2(c\theta+1)}\right)\right]\right), \tag{49}$$

where $\varepsilon = n^* - l + k_1$.

4.1. Bayesian Prediction by a Point. In the following manner, based on different loss functions, the BPPRs of the s-th order statistic, $T_{q,p,s}$, in the future sample, will be obtained.

The predictive PDF of $T_{q,p,s}, s = 1,\ldots,n^*, q = 1,\ldots,\mathscr{K}, p = 1,\ldots,\mathfrak{B}$ can be formulated as follows:

$$h^*_{q,p}\left(t_{q,p,s}|\mathbf{z}\right) = \int_0^{\infty}\int_0^{\infty} h_{q,p}\left(t_{q,p,s}|\theta,\lambda\right)\pi^*(\theta,\lambda|\mathbf{z})\mathrm{d}\theta\mathrm{d}\lambda. \tag{50}$$

Considering the SER, LEX, and GEN loss functions, the BPPRs of $T_{q,p,s}, s = 1,\ldots,n^*, q = 1,\ldots,\mathscr{K}, p = 1,\ldots,\mathfrak{B}$ are given, respectively, by

$$\begin{aligned}
T^{\mathrm{SER}}_{q,p,s} &= \int_{z_{q,p,m}}^{\infty} t_{q,p,s} h^*_{q,p}\left(t_{q,p,s}|z\right)\mathrm{d}t_{q,p,s},\\
T^{\mathrm{LEX}}_{q,p,s} &= \frac{-1}{\upsilon}\log\left[\int_{z_{q,p,m}}^{\infty} e^{-\upsilon t_{q,p,s}} h^*_{q,p}\left(t_{q,p,s}|z\right)\mathrm{d}t_{q,p,s}\right],\\
T^{\mathrm{GEN}}_{q,p,s} &= \left[\int_{z_{q,p,m}}^{\infty} t_{q,p,s}^{-\upsilon} h^*_{q,p}\left(t_{q,p,s}|z\right)\mathrm{d}t_{q,p,s}\right]^{-1/\upsilon}.
\end{aligned} \tag{51}$$

4.2. Best Unbiased Predictors, Conditional Median Predictors, and Conditional Prediction Intervals. A median unbiased predictor is defined according to the concept of median unbiasedness. Several characteristics of the median unbiased predictor, in the context of traditional type-II CS, were investigated by Takada [41]. The CMPR was introduced in [42] as a specific kind of median unbiased predictor. If the statistic $T^{C}_{q,p,s}$ is the median of the conditional distribution of $T_{q,p,s}$, it is named the CMPR of $T_{q,p,s}$.

The conditional CDF of $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$ that corresponds to PDF (47) takes the following form:

$$H_{q,p}\left(t_{q,p,s}|\theta,\lambda\right) = \sum_{k=s}^{n^*} \frac{1}{k!\,(n^*-k)!} \sum_{D[n^*]} \left(\prod_{r=1}^{k} G^{*}_{p,j_{q,p,r}}\left(t_{q,p,s}\right) \prod_{r=k+1}^{n^*} \left[1 - G^{*}_{p,j_{q,p,r}}\left(t_{q,p,s}\right)\right] \right), \tag{52}$$

where $g^{*}_{p,j_{q,p,r}}(t_{q,p,s})$ and $G^{*}_{p,j_{q,p,r}}(t_{q,p,s})$ are given, respectively, by (48) and (49).

Replacing (θ,λ) by their BEs $(\widehat{\theta},\widehat{\lambda})$, the CMPR $T^{C}_{q,p,s}$ of $T_{q,p,s}$ can be achieved by solving the next equation with respect to $t_{q,p,s}$:

$$\sum_{k=s}^{n^*} \frac{1}{k!\,(n^*-k)!} \sum_{D[n^*]} \left(\prod_{r=1}^{k} G^{*}_{p,j_{q,p,r}}\left(t_{q,p,s}\right) \prod_{r=k+1}^{n^*} \left[1 - G^{*}_{p,j_{q,p,r}}\left(t_{q,p,s}\right)\right] \right) = 0.5. \tag{53}$$

The following two equations should be simultaneously solved to calculate the bounds of $100\tau\%$ CPI $(T^{\mathrm{LB}}_{q,p,s}, T^{\mathrm{UB}}_{q,p,s})$ of $T_{q,p,s}$:

$$\begin{aligned}
&\sum_{k=s}^{n^*} \frac{1}{k!\,(n^*-k)!} \sum_{D[n^*]} \left(\prod_{r=1}^{k} G^{*}_{p,j_{q,p,r}}\left(t^{\mathrm{LB}}_{q,p,s}\right) \prod_{r=k+1}^{n^*} \left[1 - G^{*}_{p,j_{q,p,r}}\left(t^{\mathrm{LB}}_{q,p,s}\right)\right] \right) = \frac{1-\tau}{2},\\
&\sum_{k=s}^{n^*} \frac{1}{k!\,(n^*-k)!} \sum_{D[n^*]} \left(\prod_{r=1}^{k} G^{*}_{p,j_{q,p,r}}\left(t^{\mathrm{UB}}_{q,p,s}\right) \prod_{r=k+1}^{n^*} \left[1 - G^{*}_{p,j_{q,p,r}}\left(t^{\mathrm{UB}}_{q,p,s}\right)\right] \right) = \frac{1+\tau}{2}.
\end{aligned} \tag{54}$$

Here, $g^{*}_{p,j_{q,p,r}}(t_{q,p,s})$ and $G^{*}_{p,j_{q,p,r}}(t_{q,p,s})$ are given, respectively, by (48) and (49).

The predictor $T^{B}_{q,p,s}$ is called BUPR of $T_{q,p,s}$ if the predictor error $(T^{B}_{q,p,s} - T_{q,p,s})$ has a mean of zero and a variance that is smaller than or equal to that of any other unbiased predictors of $T_{q,p,s}$.

The following integral gives the BUPR $T^{B}_{q,p,s}$ of $T_{q,p,s}$:

$$T^{B}_{q,p,s} = \int_{z_{q,p,m}}^{\infty} t_{q,p,s} h_{q,p}\left(t_{q,p,s}|\theta,\lambda\right) \mathrm{d}t_{q,p,s}, \tag{55}$$

where $h_{q,p}(t_{q,p,s}|\theta,\lambda)$ is given by (47).

If the parameters (θ,λ) are unknown, then they can be replaced by their BEs $(\widehat{\theta},\widehat{\lambda})$. Further explanation about BUPR, CMPR, and CPI can be found in [20, 22] and [43].

5. Illustrative Examples

Simulated data as well as real data sets are used in this section to demonstrate the point predictor methods described in this article.

5.1. Simulated Data Set. The hyperparameter values ($e_1 = 1.5, e_2 = 1.2$, and $e_3 = 3.8$) are selected to produce the population parameter values ($\theta = 0.21$ and $\lambda = 1.67$). Under PRSALT with two groups, we generate five SRSs, each of size 10, and divide each SRS into two groups, each of size 5, see the third column of Table 1. In each SRS, the first and second groups are generated using CDF (9) with $c = 0.5, d_1 = 2$, and $d_2 = 4$. We apply the technique of RSS to these SRSs to obtain a one-cycle RSS and then arrange it to obtain the ORSS. This is shown in the last two columns of Table 1. We apply the type-II censoring procedure to the values of the ORSS, listed in the last two columns of Table 1, by selecting the first m values of them.

Based on the ORSS under PRSALT listed in Table 1, the BPPRs, CMPRs, BUPRs, and 95% CPIs for $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, p = 1, 2$ are computed and presented in Table 2.

5.2. Application to Real Data. Our goal now is to demonstrate the point predictor methods discussed in this article using a real data set examined in [44]. The data represent the

Table 1: One-cycle SRSs, RSSs, and ORSSs.

SN	p	SRSs					RSS $p=1$	RSS $p=2$	ORSS $p=1$	ORSS $p=2$
1	1	0.21815	0.63088	0.73351	1.19821	1.26813	0.21815	0.47232	0.21815	0.47232
	2	0.47232	0.98982	1.08843	1.13972	1.37275				
2	1	0.36888	0.42803	0.45835	0.82731	0.94087	0.42803	0.82936	0.38844	0.78784
	2	0.30826	0.82936	0.93731	0.94969	1.26175				
3	1	0.27344	0.29444	0.38844	0.71519	1.02443	0.38844	0.93447	0.42803	0.82936
	2	0.36015	0.71651	0.93447	1.21126	1.21455				
4	1	0.26581	0.37042	0.60568	0.86605	1.15941	0.86605	0.78784	0.86605	0.93447
	2	0.08036	0.64262	0.68475	0.78784	1.20562				
5	1	0.18663	0.51645	0.52308	0.69537	1.17450	1.17450	1.55844	1.17450	1.55844
	2	0.19259	0.24219	0.70441	1.07971	1.55844				

SN: sample number.

Table 2: Based on the data given in Table 1 BPPRs, CMPRs, BUPRs, and 95% CPIs for $T_{q,p,s}$, $s = 1, \ldots, n^*$, $q = 1$, $p = 1, 2$.

n	m	p	$T_{q,p,s}$	SE BPPR	LEX $\xi=-0.5$ BPPR	LEX $\xi=0.5$ BPPR	GE $\xi=-0.5$ BPPR	GE $\xi=0.8$ BPPR	BUPR	CMPR	CPI
5	2	1	$T_{1,1,1}$	0.64955	0.65928	0.64050	0.63636	0.60526	0.63209	0.60669	(0.40050, 1.01021)
			$T_{1,1,2}$	0.98051	0.99959	0.96289	0.96275	0.91841	0.94805	0.93201	(0.57430, 1.41472)
			$T_{1,1,3}$	1.44297	1.48920	1.40278	1.41507	1.34626	1.39137	1.35427	(0.85757, 2.13209)
		2	$T_{1,2,1}$	0.92879	0.93277	0.92508	0.92500	0.91603	0.91593	0.89308	(0.79259, 1.16415)
			$T_{1,2,2}$	1.15535	1.16676	1.14492	1.14663	1.12569	1.12584	1.10623	(0.87512, 1.48812)
			$T_{1,2,3}$	1.52872	1.56421	1.49826	1.50912	1.46241	1.47624	1.43773	(1.05310, 2.11455)
	3	1	$T_{1,1,1}$	0.69406	0.70439	0.68445	0.68091	0.64992	0.68295	0.65035	(0.43954, 1.11073)
			$T_{1,1,2}$	1.14105	1.17187	1.11325	1.11679	1.05663	1.11699	1.07978	(0.63119, 1.81188)
		2	$T_{1,2,1}$	0.98490	0.98962	0.98049	0.98064	0.97051	0.97628	0.94738	(0.83451, 1.27612)
			$T_{1,2,2}$	1.31479	1.33567	1.29612	1.30097	1.26795	1.29195	1.25172	(0.93584, 1.87304)
	4	1	$T_{1,1,1}$	1.17813	1.1967	1.16166	1.16477	1.13378	1.1684	1.10774	(0.87620, 1.78799)
		2	$T_{1,2,1}$	1.21192	1.22752	1.19808	1.20101	1.1757	1.20189	1.14481	(0.94308, 1.76818)

failure times (in hours) of electrolytic capacitors with a size of 32 volts and 22 microfarads put under two groups of PRSALT. There are 30 units in each testing group. The failure times (in hours) are as follows:

First group $(c = 1.0, d_1 = 5.0417)$: 7.21, 10.24, 10.26, 10.37, 10.51, 10.56, 11.25, 11.28, 11.29, 11.35, 12.23, 12.25, 12.36, 12.57, 13.03, 13.04, 13.05, 13.27, 13.46, 13.49, 14.23, 14.45, 15.00, 15.43, 15.47, 16.55, 17.07, 17.21, 17.23, 18.49.

Second group $(c = 1.0, d_2 = 5.833)$: 7.36, 7.55, 7.57, 8.00, 8.23, 8.46, 9.02, 9.03, 9.04, 9.22, 9.32, 9.34, 9.49, 10.28, 10.53, 11.33, 11.34, 11.54, 12.16, 12.53, 12.55, 13.20, 14.06, 14.21, 14.21, 14.21, 16.24, 16.41, 17.53, 21.26.

Before moving on, the statistical test of Kolmogorov–Smirnov (K–S) and its accompanying p value are used for each group to determine whether Weibull distribution with CDF (9) is valid for fitting the aforementioned data. It can be shown from the aforementioned data and CDF (9) that the estimates $(\widehat{\theta} = 1.12251$ and $\widehat{\lambda} = 0.0000107)$ maximize the likelihood function of θ and λ. The K–S test statistic and the p value are given, respectively, by

First group: K–S statistic = 0.18113, and p value = 0.27858

Second group: K–S statistic = 0.21773 and p value = 0.11632

As can be seen, the Weibull distribution with CDF (9) matches the provided real data set well because all of the p values are higher than 0.050. This is further demonstrated by depicting the empirical CDF of the provided real data set along with CDF (4) for each group, as shown in Figure 3. We select the hyperparameter values $(e_1 = 350, e_2 = 270$, and $e_3 = 82700)$ to produce the population parameter values $(\widehat{\theta} = 1.12251$ and $\widehat{\lambda} = 0.0000107)$ using (21) and (22).

Under PRSALT with two groups, we choose five SRSs of size 10 each and divide each SRS into two groups of size 5 each, see the third column of Table 3. In each SRS, the first and second groups are drawn, respectively, from the above data under the first and second levels of stress. The technique of RSS is applied to these SRSs to obtain a one-cycle RSS and then order it to obtain the ORSS presented in the last two columns of Table 3. The type-II censoring procedure is applied to the values of the ORSS, listed in the last two columns of Table 3, by selecting the first m values of them.

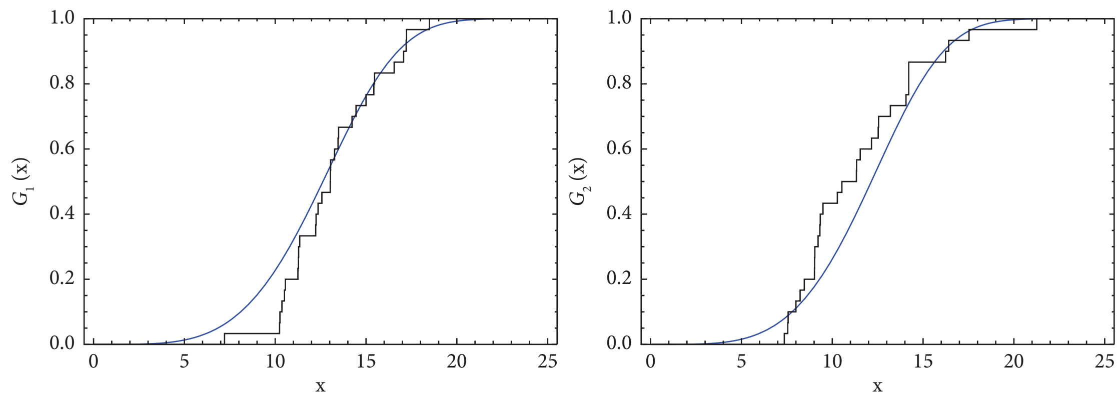

FIGURE 3: Empirical CDFs versus CDFs of Weibull CDF (4) for the given data.

TABLE 3: One-cycle real SRSs, RSSs, and ORSSs.

SN	p	SRSs					RSS		ORSS	
							$p=1$	$p=2$	$p=1$	$p=2$
1	1	11.29	12.57	13.03	13.49	17.23	11.29	9.32	10.37	8.23
	2	9.32	11.34	14.21	16.24	21.26				
2	1	11.25	14.23	15	17.07	17.21	14.23	8.23	11.29	9.32
	2	7.55	8.23	9.03	9.49	14.21				
3	1	10.56	13.04	13.05	13.46	14.45	13.05	10.53	13.05	10.53
	2	9.04	9.34	10.53	11.54	14.21				
4	1	7.21	10.24	10.26	10.37	13.27	10.37	16.41	14.23	12.53
	2	7.57	10.28	11.33	16.41	17.53				
5	1	11.28	12.23	12.25	12.36	16.55	16.55	12.53	16.55	16.41
	2	7.36	9.02	9.22	12.16	12.53				

TABLE 4: Based on the data given in Table 3 BPPRs, CMPRs, BUPRs, and 95% CPIs for $T_{q,p,s}$, $s=1,\ldots,n^*$, $q=1$, $p=1,2$.

n	m	p	$T_{q,p,s}$	SER	LEX		GEN		BUPR	CMPR	CPI
					$\xi=-0.5$	$\xi=0.5$	$\xi=-0.5$	$\xi=0.8$			
				BPPR	BPPR	BPPR	BPPR	BPPR			
5	2	1	$T_{1,1,1}$	13.1865	14.1484	12.7596	13.1458	13.0476	12.3492	12.1972	(11.3333, 14.1979)
			$T_{1,1,2}$	15.6487	17.5133	14.802	15.5776	15.4166	13.2094	13.1281	(11.8172, 15.0661)
			$T_{1,1,3}$	18.7506	21.6111	17.3185	18.5638	18.5635	14.9561	14.8243	(12.8370, 17.8024)
		2	$T_{1,2,1}$	11.7342	12.9795	11.1513	11.6724	11.5222	10.4052	10.2705	(9.36721, 12.1893)
			$T_{1,2,2}$	14.5198	16.5926	13.5295	14.4338	14.2282	11.8825	11.8207	(10.1202, 14.0072)
			$T_{1,2,3}$	17.8064	20.8585	16.2416	17.6389	17.5063	14.7631	14.6474	(12.0003, 18.162)
	3	1	$T_{1,1,1}$	14.6695	15.245	14.3389	14.6422	14.5746	13.7987	13.6567	(13.0766, 15.2919)
			$T_{1,1,2}$	17.5558	19.2984	16.6092	17.4859	17.3223	15.3374	15.1741	(13.5977, 17.9807)
		2	$T_{1,2,1}$	12.7421	13.6072	12.2319	12.6934	12.5734	11.681	11.5134	(10.5766, 13.7035)
			$T_{1,2,2}$	16.1307	18.1305	14.9589	16.0404	15.817	13.6973	13.5589	(11.4237, 16.744)
	4	1	$T_{1,1,1}$	16.5513	17.7936	15.9271	16.5021	16.382	15.4482	15.1992	(14.2696, 17.9503)
		2	$T_{1,2,1}$	15.1851	16.6127	14.4419	15.1217	14.9662	13.9897	13.7301	(12.5813, 16.7807)

Table 5: BPPRs using SER and LEX loss functions for $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$ (one-cycle).

$\mathscr{K}$	n	m	p	$T_{q,p,s}$	Exact value	SER			LEX $\xi = -0.5$			LEX $\xi = 0.5$		
						ABPPR	BIAS	MSPER	ABPPR	BIAS	MSPER	ABPPR	BIAS	MSPER
1	5	2	1	$T_{1,1,1}$	0.68737	0.58745	0.0068	0.00366	0.58882	0.00816	0.00368	0.58616	0.0055	0.00365
				$T_{1,1,2}$	0.88572	0.88178	0.02579	0.01732	0.88973	0.03375	0.01781	0.87432	0.01833	0.01699
				$T_{1,1,3}$	1.18274	1.42355	0.05562	0.05748	1.45422	0.08629	0.06198	1.39642	0.02849	0.05515
			2	$T_{1,2,1}$	0.64631	0.54412	0.00413	0.00346	0.53074	−0.00924	0.01951	0.5632	0.02322	0.02047
				$T_{1,2,2}$	0.83262	0.82655	0.029	0.01579	0.83486	0.03731	0.01739	0.81612	0.01857	0.0174
				$T_{1,2,3}$	1.11576	1.33896	0.06387	0.05161	1.36932	0.09423	0.05678	1.3129	0.0378	0.04904
		3	1	$T_{1,1,1}$	0.88572	0.7929	0.00537	0.00804	0.79544	0.00792	0.00808	0.79051	0.00298	0.00802
				$T_{1,1,2}$	1.18274	1.29558	0.0251	0.05727	1.3164	0.04592	0.05878	1.27658	0.0061	0.05668
			2	$T_{1,2,1}$	0.83262	0.74541	0.00617	0.00687	0.74763	0.00839	0.00691	0.74341	0.00417	0.00685
				$T_{1,2,2}$	1.11576	1.22012	0.03164	0.05013	1.2405	0.05201	0.05192	1.20161	0.01312	0.04926
		4	1	$T_{1,1,1}$	1.18274	1.15293	0.00895	0.04607	1.167	0.02302	0.04654	1.14033	−0.00365	0.04601
			2	$T_{1,2,1}$	1.11576	1.08449	0.01179	0.03977	1.09754	0.02483	0.04026	1.07284	0.00013	0.03963

Table 6: BPPRs using GEN loss functions for $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$ (one-cycle).

$\mathscr{K}$	n	m	p	$T_{q,p,s}$	GEN					
					$\xi = -0.5$			$\xi = 0.8$		
					ABPPR	BIAS	MSPER	ABPPR	BIAS	MSPER
1	5	2	1	$T_{1,1,1}$	0.58531	0.00465	0.00364	0.58032	−0.00033	0.00362
				$T_{1,1,2}$	0.8736	0.01762	0.01696	0.85367	−0.00231	0.01668
				$T_{1,1,3}$	1.40487	0.03694	0.05573	1.35922	−0.00871	0.05443
			2	$T_{1,2,1}$	0.54003	0.00005	0.00386	0.5473	0.00732	0.00768
				$T_{1,2,2}$	0.82045	0.0229	0.02069	0.79632	−0.00123	0.015
				$T_{1,2,3}$	1.31894	0.04385	0.04939	1.27552	0.00042	0.05836
		3	1	$T_{1,1,1}$	0.78996	0.00243	0.00802	0.78305	−0.00448	0.00804
				$T_{1,1,2}$	1.28118	0.01069	0.05675	1.24578	−0.0247	0.05729
			2	$T_{1,2,1}$	0.74256	0.00332	0.00684	0.73605	−0.00319	0.00685
				$T_{1,2,2}$	1.20523	0.01674	0.04937	1.16892	−0.01957	0.04941
		4	1	$T_{1,1,1}$	1.14235	−0.00162	0.046	1.11775	−0.02623	0.04676
			2	$T_{1,2,1}$	1.07419	0.00148	0.03963	1.05042	−0.02229	0.04015

Table 7: CMPRs, BUPRs, and 95% CPIs for $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$ (one-cycle).

$\mathscr{K}$	n	m	p	$T_{q,p,s}$	BUPR			CMPR			CPI		
					ABUPR	BIAS	MSPER	ACMPR	BIAS	MSPER	PI	AIL	CPR
1	5	2	1	$T_{1,1,1}$	0.58066	0.000	0.00361	0.56436	−0.0163	0.00389	(0.51627, 0.73323)	0.21695	0.672
				$T_{1,1,2}$	0.85599	0.000	0.01665	0.84527	−0.01072	0.01677	(0.63901, 1.13427)	0.49525	0.81
				$T_{1,1,3}$	1.36793	0.000	0.05429	1.34648	−0.02145	0.05476	(0.97648, 1.88095)	0.90447	0.752
			2	$T_{1,2,1}$	0.53999	0.000	0.00317	0.52478	−0.0152	0.00341	(0.47925, 0.68301)	0.20376	0.651
				$T_{1,2,2}$	0.79755	0.000	0.0144	0.78759	−0.00996	0.0145	(0.5949, 1.05713)	0.46223	0.816
				$T_{1,2,3}$	1.2751	0.000	0.0471	1.25506	−0.02004	0.04751	(0.90998, 1.72642)	0.81645	0.783
		3	1	$T_{1,1,1}$	0.78752	0.000	0.00801	0.76331	−0.02422	0.00861	(0.69035, 1.01551)	0.32516	0.802
				$T_{1,1,2}$	1.27048	0.000	0.05662	1.24438	−0.0261	0.05731	(0.88693, 1.80228)	0.91535	0.876
			2	$T_{1,2,1}$	0.73924	0.000	0.00683	0.71674	−0.0225	0.00734	(0.64907, 0.951)	0.30193	0.774
				$T_{1,2,2}$	1.18849	0.000	0.04897	1.16404	−0.02445	0.04959	(0.83149, 1.68434)	0.85285	0.877
		4	1	$T_{1,1,1}$	1.14398	0.000	0.04599	1.09181	−0.05217	0.04874	(0.89436, 1.675)	0.78064	0.948
			2	$T_{1,2,1}$	1.07271	0.000	0.03961	1.02398	−0.04873	0.04202	(0.84063, 1.56759)	0.72696	0.935

Based on the ORSS under PRSALT listed in Table 3, the BPPRs, CMPRs, BUPRs, and 95% CPIs for $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, p = 1, 2$ are computed and presented in Table 4.

6. Simulation Study

A Monte Carlo simulation study is executed in this section to determine BPPRs, CMPRs, BUPRs, and CPIs for the s-th order statistic in group p, $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$. The next steps can be followed to perform a Monte Carlo simulation:

(1) Using (36) and (37), we choose the hyperparameter values ($e_1 = 1.5, e_2 = 1.2$, and $e_3 = 3.8$) to produce the population parameter values ($\theta = 0.21$ and $\lambda = 1.67$). The hyperparameter values were determined to meet the unbiasedness requirements [3, 45] as follows:

$$\begin{aligned} E[\hat{\lambda}] &= \frac{1}{e_2(e_1 - 1)} = \lambda, \\ E[\hat{\theta}] &= \frac{1}{\lambda(e_3 - 1)} = \theta, \end{aligned} \tag{56}$$

where E denotes the expectation.

(2) Assume $\mathfrak{B} = 2$ (two groups), we generate five SRSs of size 10 each and divide each SRS into two groups of size 5 each. In each SRS, the two groups are generated using CDF (9) with $c = 0.5, d_1 = 2$, and $d_2 = 4$, respectively.

(3) As described in Subsection 2.2, we apply the RSS method to the SRSs generated in Step 2 to get a one-cycle RSS, which is then ordered to get the ORSS.

(4) We apply the type-II censoring procedure to the ORSS values that are acquired in Step 3 by selecting the first $m (= 2, 3, 4)$ values of them.

Table 8: BPPRs using SER and LEX loss functions for $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$ (two-cycle).

$\mathscr{K}$	n	m	p	$T_{q,p,s}$	Exact value	SER			LEX					
									$\xi = -0.5$			$\xi = 0.5$		
						ABPPR	BIAS	MSPER	ABPPR	BIAS	MSPER	ABPPR	BIAS	MSPER
1	5	2	1	$T_{1,1,1}$	0.69438	0.58506	0.00328	0.00353	0.58615	0.00437	0.00354	0.58402	0.00224	0.00352
				$T_{1,1,2}$	0.88627	0.86742	0.01317	0.01658	0.87328	0.01903	0.01678	0.86179	0.00754	0.01647
				$T_{1,1,3}$	1.18939	1.39627	0.03011	0.05564	1.4186	0.05243	0.05752	1.37576	0.0096	0.05482
			2	$T_{1,2,1}$	0.65358	0.5447	0.00261	0.00323	0.54142	−0.00066	0.00544	0.54992	0.00783	0.00556
				$T_{1,2,2}$	0.82822	0.81225	0.0154	0.01493	0.81795	0.02111	0.01515	0.80681	0.00996	0.01479
				$T_{1,2,3}$	1.12003	1.31911	0.03713	0.04995	1.34182	0.05983	0.05225	1.29829	0.01631	0.04879
		3	1	$T_{1,1,1}$	0.88627	0.79665	0.00267	0.00796	0.79893	0.00494	0.00797	0.7945	0.00051	0.00795
				$T_{1,1,2}$	1.18939	1.29083	0.01329	0.05736	1.30881	0.03127	0.05817	1.27415	−0.00339	0.0572
			2	$T_{1,2,1}$	0.83017	0.74382	0.00334	0.007	0.7457	0.00522	0.00703	0.74218	0.00169	0.007
				$T_{1,2,2}$	1.11609	1.21342	0.0182	0.05068	1.23093	0.03571	0.05164	1.19726	0.00204	0.05035
		4	1	$T_{1,1,1}$	1.18939	1.15677	0.0046	0.04579	1.16982	0.01764	0.04609	1.14498	−0.00719	0.04583
			2	$T_{1,2,1}$	1.11609	1.07954	0.00668	0.04145	1.09196	0.0191	0.04178	1.06833	−0.00453	0.04144
2	5	2	1	$T_{2,1,1}$	0.68266	0.58088	0.00331	0.00362	0.58199	0.00442	0.00363	0.57981	0.00225	0.00361
				$T_{2,1,2}$	0.88878	0.86386	0.01319	0.01676	0.86978	0.01911	0.01696	0.85818	0.00751	0.01665
				$T_{2,1,3}$	1.18128	1.39424	0.0301	0.05583	1.41664	0.05249	0.05773	1.37367	0.00953	0.05501
			2	$T_{2,2,1}$	0.64523	0.54401	0.00276	0.00329	0.54141	0.00016	0.00443	0.54828	0.00702	0.00452
				$T_{2,2,2}$	0.83682	0.81613	0.01538	0.01487	0.82182	0.02107	0.01509	0.81069	0.00995	0.01473
				$T_{2,2,3}$	1.11542	1.31996	0.03715	0.05001	1.34268	0.05987	0.05231	1.29913	0.01632	0.04885
		3	1	$T_{2,1,1}$	0.88878	0.78993	0.00268	0.00811	0.79224	0.00499	0.00813	0.78773	0.00048	0.0081
				$T_{2,1,2}$	1.18128	1.28221	0.01328	0.05772	1.30028	0.03135	0.05854	1.26544	−0.00349	0.05757
			2	$T_{2,2,1}$	0.83786	0.74267	0.00336	0.00707	0.74474	0.00542	0.00709	0.74075	0.00143	0.00706
				$T_{2,2,2}$	1.11542	1.21218	0.01821	0.05081	1.22972	0.03576	0.05178	1.19598	0.00202	0.05049
		4	1	$T_{2,1,1}$	1.18128	1.1492	0.0046	0.04641	1.16242	0.01782	0.04672	1.13726	−0.00734	0.04645
			2	$T_{2,2,1}$	1.11542	1.08764	0.00668	0.04086	1.0999	0.01894	0.04118	1.07658	−0.00439	0.04084

(5) We iterate Steps 2–4 $\mathscr{K}$ times to obtain $\mathscr{K}$-cycle type-II censored ORSSs.

(6) The BPPRs, CMPRs, BUPRs, and 95% CPIs for $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$ are computed, as indicated earlier in Section 4.

(7) If $\widehat{T}_{q,p,s}$ is a prediction of $T_{q,p,s}$, then the mean squared prediction errors (MSPERs) and biases of $\widehat{T}_{q,p,s}$ are given by

$$\begin{aligned} \mathrm{MSPER}\left(\widehat{T}_{q,p,s}\right) &= E\left(\widehat{T}_{q,p,s} - T_{q,p,s}\right)^2, \\ \mathrm{BIAS}\left(\widehat{T}_{q,p,s}\right) &= E\left(\widehat{T}_{q,p,s} - T_{q,p,s}\right). \end{aligned} \tag{57}$$

(8) We iterate the above steps 1000 times.

(9) The coverage probabilities (CPRs) of the CPIs are computed according to the following relation:

$$\mathrm{CPR} = \frac{\text{Number of CPIs that include} T_{q,p,s}}{1000}. \tag{58}$$

(10) We compute the average of BPPRs (ABPPR), CMPRs (ACMPR), and BUPRs (ABUPR).

Tables 5–10 present the obtained numerical results.

6.1. Simulation Results. The results presented in Tables 5–10 indicate the following:

(1) Through MSPERs, BUPRs are the most accurate point predictors.

(2) Through bias and MSPERs, the BPPRs based on LEX (at $\xi = 0.5$) and GEN (at $\xi = -0.5$) loss functions perform better than the BPPRs based on SER loss functions.

(3) The MSPERs and bias of BPPR, BUPR, and CMPR of $T_{q,p,s}$ increase as the index s increases for $q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$.

(4) By increasing the stress level, the MSPERs (bias) of BPPR, BUPR, and CMPR of $T_{q,p,s}$ decrease (increases).

(5) The CPRs of the CPIs are right near the 95% actual confidence levels by increasing m.

(6) By increasing the stress level, the AILs decrease since by increasing the stress level, the failure times decrease.

(7) The AILs of CPI of $T_{q,p,s}$ increase as the index s increases for $q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$.

Except for a few unusual cases, the results above are accurate, and this could be because of data fluctuations.

TABLE 9: BPPRs using GEN loss functions for $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$ (two-cycle).

$\mathscr{K}$	n	m	p	$T_{q,p,s}$	GEN					
					$\xi = -0.5$			$\xi = 0.5$		
					ABPPR	BIAS	MSPER	ABPPR	BIAS	MSPER
1	5	2	1	$T_{1,1,1}$	0.58326	0.00148	0.00352	0.579	−0.00278	0.00353
				$T_{1,1,2}$	0.86102	0.00677	0.01646	0.84506	−0.00919	0.01652
				$T_{1,1,3}$	1.38176	0.0156	0.05497	1.34565	−0.02052	0.05517
			2	$T_{1,2,1}$	0.54212	0.00004	0.00335	0.54113	−0.00095	0.00341
				$T_{1,2,2}$	0.80563	0.00878	0.01477	0.78923	−0.00761	0.01477
				$T_{1,2,3}$	1.30352	0.02154	0.04899	1.26476	−0.01722	0.04877
		3	1	$T_{1,1,1}$	0.79397	−0.00002	0.00795	0.78758	−0.00641	0.008
				$T_{1,1,2}$	1.27802	0.00048	0.05719	1.24619	−0.03135	0.05823
			2	$T_{1,2,1}$	0.74108	0.0006	0.00699	0.73508	−0.0054	0.00703
				$T_{1,2,2}$	1.20022	0.00501	0.05037	1.16776	−0.02745	0.05112
		4	1	$T_{1,1,1}$	1.1468	−0.00538	0.04581	1.12342	−0.02875	0.04668
			2	$T_{1,2,1}$	1.06948	−0.00338	0.04143	1.04608	−0.02678	0.0422
2	5	2	1	$T_{2,1,1}$	0.579	0.00144	0.00361	0.57458	−0.00298	0.00363
				$T_{2,1,2}$	0.85737	0.00669	0.01664	0.84114	−0.00953	0.01671
				$T_{2,1,3}$	1.37967	0.01553	0.05517	1.34337	−0.02077	0.05538
			2	$T_{2,2,1}$	0.54144	0.00019	0.00337	0.53972	−0.00154	0.00335
				$T_{2,2,2}$	0.80955	0.00881	0.01471	0.79325	−0.0075	0.01471
				$T_{2,2,3}$	1.30436	0.02155	0.04905	1.26555	−0.01726	0.04883
		3	1	$T_{2,1,1}$	0.78717	−0.00007	0.0081	0.78063	−0.00662	0.00816
				$T_{2,1,2}$	1.26925	0.00031	0.05755	1.23702	−0.03191	0.05863
			2	$T_{2,2,1}$	0.7400	0.00068	0.00706	0.73376	−0.00556	0.0071
				$T_{2,2,2}$	1.19894	0.00498	0.0505	1.1664	−0.02757	0.05127
		4	1	$T_{2,1,1}$	1.13906	−0.00554	0.04643	1.11528	−0.02932	0.04734
			2	$T_{2,2,1}$	1.07778	−0.00319	0.04084	1.05482	−0.02615	0.04157

Table 10: CMPRs, BUPRs, and 95% CPIs for $T_{q,p,s}, s = 1, \ldots, n^*, q = 1, \ldots, \mathscr{K}, p = 1, \ldots, \mathfrak{B}$ (two-cycle).

$\mathscr{K}$	n	m	p	$T_{q,p,s}$	BUPR			CMPR			CPI		
					ABUPR	BIAS	MSPER	ACMPR	BIAS	MSPER	PI	AIL	CPR
1	5	2	1	$T_{1,1,1}$	0.58178	0.000	0.00352	0.5657	−0.01608	0.00378	(0.51837, 0.73229)	0.21392	0.661
				$T_{1,1,2}$	0.85425	0.000	0.01641	0.84336	−0.0109	0.01653	(0.63923, 1.13152)	0.49229	0.839
				$T_{1,1,3}$	1.36616	0.000	0.0547	1.34424	−0.02193	0.05519	(0.97392, 1.85339)	0.87946	0.761
			2	$T_{1,2,1}$	0.54209	0.000	0.00316	0.5269	−0.01518	0.0034	(0.48167, 0.68474)	0.20307	0.650
				$T_{1,2,2}$	0.79684	0.000	0.01468	0.78669	−0.01015	0.01479	(0.59263, 1.05908)	0.46645	0.843
				$T_{1,2,3}$	1.28198	0.000	0.04844	1.26138	−0.02061	0.04887	(0.91197, 1.75831)	0.84635	0.781
		3	1	$T_{1,1,1}$	0.79399	0.000	0.00795	0.76982	−0.02417	0.00854	(0.69733, 1.0213)	0.32397	0.817
				$T_{1,1,2}$	1.27754	0.000	0.05718	1.25083	−0.02671	0.0579	(0.89288, 1.81384)	0.92096	0.873
			2	$T_{1,2,1}$	0.74048	0.000	0.00699	0.71785	−0.02263	0.00751	(0.64962, 0.95369)	0.30407	0.801
				$T_{1,2,2}$	1.19521	0.000	0.05031	1.17022	−0.02499	0.05095	(0.8343, 1.69803)	0.86373	0.878
		4	1	$T_{1,1,1}$	1.15218	0.000	0.04577	1.0997	−0.05247	0.04855	(0.90365, 1.68382)	0.78017	0.938
			2	$T_{1,2,1}$	1.07286	0.000	0.04141	1.02321	−0.04965	0.04389	(0.83561, 1.57811)	0.7425	0.952
2	5	2	1	$T_{2,1,1}$	0.57756	0.000	0.00361	0.56136	−0.0162	0.00388	(0.51323, 0.72964)	0.21641	0.681
				$T_{2,1,2}$	0.85067	0.000	0.01659	0.83984	−0.01083	0.01671	(0.63418, 1.12908)	0.4949	0.818
				$T_{2,1,3}$	1.36414	0.000	0.0549	1.34225	−0.02189	0.05538	(0.97099, 1.85103)	0.88004	0.778
			2	$T_{2,2,1}$	0.54125	0.000	0.00325	0.52603	−0.01522	0.00348	(0.48004, 0.68493)	0.20489	0.650
				$T_{2,2,2}$	0.80075	0.000	0.01463	0.79058	−0.01017	0.01473	(0.59717, 1.02596)	0.42879	0.838
				$T_{2,2,3}$	1.28281	0.000	0.04849	1.2622	−0.02061	0.04892	(0.91327, 1.70431)	0.79104	0.812
		3	1	$T_{2,1,1}$	0.78724	0.000	0.0081	0.76293	−0.02432	0.0087	(0.68955, 1.01637)	0.32681	0.801
				$T_{2,1,2}$	1.26893	0.000	0.05754	1.2424	−0.02653	0.05826	(0.88244, 1.80611)	0.92367	0.883
			2	$T_{2,2,1}$	0.73932	0.000	0.00706	0.71661	−0.02271	0.00758	(0.64804, 0.95339)	0.30536	0.798
				$T_{2,2,2}$	1.19396	0.000	0.05045	1.16898	−0.02498	0.05109	(0.8326, 1.69718)	0.86457	0.886
		4	1	$T_{2,1,1}$	1.1446	0.000	0.04639	1.09201	−0.05259	0.04918	(0.89395, 1.67904)	0.78509	0.969
			2	$T_{2,2,1}$	1.08097	0.000	0.04082	1.03145	−0.04951	0.04329	(0.84597, 1.58317)	0.7372	0.944

7. Conclusion

The ORSS method has received increasing attention in recent years due to its effectiveness in estimation. This fact has been demonstrated in this article since it has been observed that the estimates calculated under ORSSs are more effective than those calculated under SRSs. Based on type-II censoring, the ORSS method under PRSALTs has been applied to items to be tested. The lifetime of an item under normal use was supposed to follow RD with a scale parameter satisfying the inverse power law such that the imposed stress is expressed by a nonlinear increasing function of time. A one-sample prediction procedure for the unobserved failure times under type-II censoring has been investigated. Some point predictors such as the BPPR, CMPR, and BUPR as well as CPI for future order statistics have been discussed. The performance and effectiveness of the prediction methods described in the article have been demonstrated through Monte Carlo simulations as well as real data. The numerical results have shown that the prediction methods have perfect performance.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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