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Retraction

Retracted: On Some Classes of Estimators Derived from the Positive Part of James–Stein Estimator

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] A. Hamdaoui, A. Benkhaled, M. Alshahrani, M. Terbeche, W. Almutiry, and A. Alahmadi, "On Some Classes of Estimators Derived from the Positive Part of James–Stein Estimator," *Journal of Mathematics*, vol. 2023, Article ID 5221061, 12 pages, 2023.

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Research Article

On Some Classes of Estimators Derived from the Positive Part of James–Stein Estimator

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This work consists of developing shrinkage estimation strategies for the multivariate normal mean when the covariance matrix is diagonal and known. The domination of the positive part of James–Stein estimator (PPJSE) over James–Stein estimator (JSE) relative to the balanced loss function (BLF) is analytically proved. We introduce a new class of shrinkage estimators which ameliorate the PPJSE, and then we construct a series of polynomial shrinkage estimators which improve the PPJSE; also, any estimator of this series can be ameliorated by adding to it a new term of higher degree. We end this paper by simulation studies which confirm the performance of the suggested estimators.

1. Introduction

The minimax approach has received the most extensive development in the estimation of the mean parameter of a random vector $Y \sim N_d(\nu, \sigma^2 I_d)$. It has been known since Stein [1] that if $d < 3$, the maximum likelihood estimator (MLE) Y is minimax and admissible. Namely, the MLE is minimax and it is considered to bethe best estimator of the mean δ under the quadratic loss function. However, when $d > 2$, Stein [1] and James and Stein [2] showed that the shrinkage estimator $\delta_a = (1 - (a/\|Y\|^2))Y$ with the shrinkage function $\phi_a = (1 - (a/\|Y\|^2))$ which shrinks the components of the vector Y to zero has a quadratic risk inferior to the MLE for specific values of the real parameter a. This explains the inadmissibility of the MLE for $d > 2$. The better estimator in the class of estimator δ_a is called the JSE.

Several studies have been interested in constructing new shrinkage estimators that improve both the MLE and the JSE, for example, Lindley [3], Bhattacharya [4], Berger [5], Stein [6], Norouzirad and Arashi [7], Cheng and Chaturvedi [8], and Kashani et al. [9]. Other studies developed the shrinkage estimators under the Bayesian framework, and we cite, for example, Strawderman [10], Lindley [11], Efron and Morris [12], Hudson [13], and Hamdaoui et al. [14].

As the shrinkage real function can take negative values which can affect it by losing its target of reducing the compounds of the MLE to 0, Baranchik [15] introduced the PPJSE estimator $\delta_a^+ = (1 - (a/\|Y\|^2))^+ Y$ which can take only positive values, where $(1-(a/\|Y\|^2))^+ = \max(0; 1-(a/\|Y\|^2))$. Baranchik [15] shows that under the quadratic loss function, the PPJSE dominates the MLE and it also ameliorates the JSE. The shrinkage estimators in all of the above cited studies were based on the quadratic loss function.

Zellner [16] extended the problem of estimating the multivariate normal mean in large dimension, and then he suggested the BLF that generalizes the quadratic loss function. The published papers in this direction include Sanjari Farsipour and Asgharzadeh [17], Selahattin and Issam [18], Nimet and Selahattin [19], Lahoucine et al. [20], Karamikabir and Afsahri [21], and Karamikabir et al. [22].

PPJSE is one of the best estimators that significantly improves the JSE under the quadratic loss function. Benmansour and Hamdaoui [23] and Hamdaoui and Benmansour [24] have proved this in the simulation section in their studies. Hamdaoui [25] also proposed a class of shrinkage estimators derived from the MLE and improved the PPJSE under the quadratic loss function. Therefore, in this work, we generalize the results obtained in Hamdaoui [25] by using the BLF instead of the quadratic loss function in the comparison between two different estimators. That is, we deal with the model $Y \sim N_d(\nu, I_d)$. The main goal is to estimate the parameter ν by shrinkage estimators derived from the MLE. To determine the quality of each considered estimator, we use the risk function that is based on the BLF.

This paper is arranged as follows. In Section 2, we give details of the shrinkage estimators and recall some important published results. Also, we introduce a class of estimators that improve the PPJSE. In Section 3, we construct a series of shrinkage polynomial type estimators derived from the PPJSE and prove the domination and performance properties of these estimators between them. We end this work by simulation results followed by the conclusion.

2. A New Class of Estimators That Improve the PPJSE

First, we consider the model that has the random variable Y to follow the multivariate normal distribution with a mean vector ν and identity covariance matrix I_d. In this model, we will focus on estimating the mean parameters ν using the shrinkage estimators that are based on the BLF. For the quality comparison of any estimator T of ν, we incorporate the BLF in the calculation of its risk function as defined in Hamdaoui et al. [26].

$$\ell_\omega(T,\nu) = \omega\left\|T - T_0\right\|^2 + (1-\omega)\|T-\nu\|^2, \quad 0 \le \omega < 1. \tag{1}$$

Then, based on equation (1), the risk function is defined as

$$R_\omega(T,\nu) = \mathbb{E}\left(\ell_\omega(T,\nu)\right). \tag{2}$$

In this case, the MLE is $Y: = T_0$, its risk function is equal to $(1-\omega)d$, and the classical estimator that dominates the MLE under the BLF given in equation (1) is the following JSE:

$$T_{JS}(Y) = \left(1 - \frac{\alpha}{\|Y\|^2}\right)Y, \tag{3}$$

where $\alpha = (1-w)(d-2)$. Its risk function under the BLF is

$$R_\omega\left(T_{JS}(Y),\nu\right) = (1-\omega)d - (1-w)^2(d-2)^2\mathbb{E}\left(\frac{1}{\|Y\|^2}\right). \tag{4}$$

Also, the classical estimator that improves the JSE is the PPJSE defined as

$$T_{JS+}(Y) = \left(1 - \alpha\frac{1}{\|Y\|^2}\right)^+ Y = \left(1 - \alpha\frac{1}{\|Y\|^2}\right)\mathbb{1}_{(\alpha/\|Y\|^2)\le 1}Y, \tag{5}$$

where $(1-\alpha(1/\|Y\|^2))^+ = \max(0, 1-\alpha(1/\|Y\|^2))$ and $\mathbb{1}_{(\alpha/\|Y\|^2)\le 1}$ is the indicator function of $\left\{(\alpha/\|Y\|^2)\le 1\right\}$. Hamdaoui et al. [26] demonstrated that its risk function is defined as

$$R_\omega\left(T_{JS+}(Y),\nu\right) = R_\omega\left(T_{JS}(Y),\nu\right) + \mathbb{E}\left[\left(\|Y\|^2 + \frac{(1-\omega)^2(d-2)^2}{\|Y\|^2} - 2(1-\omega)d\right)\mathbb{1}_{(\alpha/\|Y\|^2)\ge 1}\right]. \tag{6}$$

They also proved that, based on the BLF, $T_{JS+}(Y)$ dominates $T_{JS}(Y)$.

Now, we will construct a simple class of estimators that improves $T_{JS+}(Y)$ under the BLF. We add a term of the form $\beta(1/\|Y\|^2)^2\mathbb{1}_{(\alpha/\|Y\|^2)\le 1}Y$ to the PPJSE estimator $T_{JS+}(Y)$. That is, we consider the following estimator:

$$T^{(2)}_{\beta,JS+}(Y) = T_{JS+}(Y) + \beta\left(\frac{1}{\|Y\|^2}\right)^2\mathbb{1}_{(\alpha/\|Y\|^2)\le 1}Y, \tag{7}$$

where the constant β can be related to d and ω.

Proposition 1. *Based on the BLF, the risk function of the estimator $T^{(2)}_{\beta,JS+}(Y)$ given in equation (7) can be expressed as*

$$R_\omega\left(T^{(2)}_{\beta,JS+}(Y),\nu\right) = R_\omega\left(T_{JS+}(Y),\nu\right) + \beta^2\mathrm{IE}\left(\frac{1}{\|Y\|^6}\mathbb{1}_{(\alpha/\|Y\|^2)\le 1}\right) - 4\beta(1-\omega)\mathrm{IE}\left(\frac{1}{\|Y\|^4}\mathbb{1}_{(\alpha/\|Y\|^2)\le 1}\right). \tag{8}$$

Proof. As

$$
\begin{aligned}
R_\omega\left(T^{(2)}_{\beta,JS+}(Y),\nu\right) &= \omega\mathbb{E}\left(\left\| T_{JS+}(Y) + \beta\frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y - Y\right\|^2\right)\\
&\quad + (1-\omega)\mathbb{E}\left(\left\| T_{JS+}(Y) + \beta\frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y - \nu\right\|^2\right)\\
&= R_\omega\left(T_{JS+}(Y),\nu\right) + \beta^2\mathbb{E}\left(\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)\\
&\quad + 2\omega\beta\mathbb{E}\left(\left\langle T_{JS+}(Y) - Y, \frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y\right\rangle\right)\\
&\quad + 2(1-\omega)\beta\mathbb{E}\left(\left\langle T_{JS+}(Y) - \nu, \frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y\right\rangle\right),
\end{aligned}
\tag{9}
$$

then

$$
\begin{aligned}
R_\omega\left(T^{(2)}_{\beta,JS+}(Y),\nu\right) &= R_\omega\left(T_{JS+}(Y),\nu\right) + \beta^2\mathbb{E}\left(\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)\\
&\quad + 2\omega\beta\mathbb{E}\left(\left\langle T_{JS+}(Y) - Y, \frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y\right\rangle\right)\\
&\quad + 2(1-\omega)\beta E\left(\left\langle T_{JS+}(Y) - Y + Y - \nu, \frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y\right\rangle\right).
\end{aligned}
\tag{10}
$$

Thus,

$$
\begin{aligned}
R_\omega\left(T^{(2)}_{\beta,JS+}(Y),\nu\right) &= R_\omega\left(T_{JS+}(Y),\nu\right) + \beta^2\mathbb{E}\left(\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)\\
&\quad + 2\beta\mathbb{E}\left(\left\langle T_{JS+}(Y) - Y, \frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)}Y\right\rangle\right)\\
&\quad + 2(1-\omega)\beta\mathbb{E}\left(\left\langle Y - \nu, \frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)}Y\right\rangle\right).
\end{aligned}
\tag{11}
$$

The second expectation of equation (11) can be expressed as

$$\mathbb{E}\left(\langle T_{JS+}(Y)-Y,\frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y\rangle\right)=\mathbb{E}\left(\langle -\frac{\alpha}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y,\frac{1}{\left(\alpha/\|Y\|^2\right)\leq 1}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y\rangle\right)=-\alpha\mathbb{E}\left(\frac{1}{\|Y\|^4}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right). \tag{12}$$

Also, based on Lemma 2.1 of Shao and Strawderman [27], the third expectation of equation (11) can be expressed as

$$\begin{aligned}&\mathbb{E}\left(\langle Y-\nu,\frac{1}{\left(\|Y\|^2\right)^2}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y\rangle\right)\\&=(d-4)\mathbb{E}\left(\frac{1}{\|Y\|^4}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right).\end{aligned} \tag{13}$$

Then, according to equations (11), (12), and (13), we obtain the desired result. □

Theorem 1. *For $d>4$ and based on the BLF, a sufficient condition for which the estimator $T^{(2)}_{\beta,JS+}(Y)$ dominates $T_{JS+}(Y)$ is*

$$0\leq\beta\leq 4(1-\omega)^2(d-2). \tag{14}$$

Proof. As

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)&=\mathbb{E}\left(\frac{1}{\|Y\|^2}\frac{1}{\|Y\|^4}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)\\&\leq\mathbb{E}\left(\frac{1}{\alpha}\frac{1}{\|Y\|^4}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)\\&=\frac{1}{(1-\omega)(d-2)}\mathbb{E}\left(\frac{1}{\|Y\|^4}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right),\end{aligned} \tag{15}$$

we can deduce from Proposition 1 that

$$\begin{aligned}R_\omega\left(T^{(2)}_{\beta,JS+}(Y),\nu\right)\leq{}&R_\omega\left(T_{JS+}(Y),\nu\right)\\&+\beta\left(\frac{\beta}{(1-\omega)(d-2)}-4(1-\omega)\right)\\&\mathbb{E}\left(\frac{1}{\|Y\|^4}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right).\end{aligned} \tag{16}$$

Consequently, a sufficient condition for which the estimator $T^{(2)}_{\beta,JS+}(Y)$ dominates $T_{JS+}(Y)$ is

$$\frac{\beta}{(1-\omega)(d-2)}-4(1-\omega)\leq 0, \tag{17}$$

which is equivalent to

$$0\leq\beta\leq 4(1-\omega)^2(d-2). \tag{18}$$

From the convexity of the right hand side of inequality (16) with respect to β and taking its first derivative, we can deduce that this term takes its minimum value when

$$\widehat{\beta}=2(1-\omega)^2(d-2), \tag{19}$$

and if we substitute β by $\widehat{\beta}$, we obtain the domination of $T^{(2)}_{\widehat{\beta},JS+}(Y)$ over $T_{JS+}(Y)$, as shown below:

$$\begin{aligned}R_\omega\left(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu\right)&\leq R_\omega\left(T_{JS+}(Y),\nu\right)\\&\quad-4(1-\omega)^3(d-2)\mathbb{E}\left(\frac{1}{\|Y\|^4}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)\\&\leq R_\omega\left(T_{JS+}(Y),\nu\right).\end{aligned} \tag{20}$$

□

3. The Performance of Some Derived Shrinkage Estimators from the PPJSE

In Section 2, we note that when a term of the form $\beta(1/\|Y\|^2)^2\mathbb{I}_{(\alpha/\|Y\|^2)\leq 1}Y$ is added to the $T_{JS+}(Y)$, we obtain estimators that have smaller risk than the risk of $T_{JS+}(Y)$. Therefore, following this effect, the main idea of this section is to construct new classes of estimators deduced by modifying $T_{JS+}(Y)$. We add recursively a term of the form $c(1/\|Y\|^2)^m\mathbb{I}_{(\alpha/\|Y\|^2)\leq 1}Y$, where m is an integer parameter and c is a constant that can be related to d and ω. Consequently, we build a series of estimators of polynomial type with the indeterminate $(1/\|Y\|^2)\mathbb{I}_{(\alpha/\|Y\|^2)\leq 1}Y$ such as if we increase the degree of the polynomial, we obtain a best estimator. Now, consider the estimator

$$\begin{aligned}T^{(3)}_{\gamma,JS+}(Y)&=T^{(2)}_{\widehat{\beta},JS+}(Y)+\gamma\left(\frac{1}{\|Y\|^2}\right)^3\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y\\&=\left(1-\alpha\frac{1}{\|Y\|^2}+\widehat{\beta}\left(\frac{1}{\|Y\|^2}\right)^2+\gamma\left(\frac{1}{\|Y\|^2}\right)^3\right)\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y,\end{aligned} \tag{21}$$

where $\widehat{\beta}$ is defined in equation (19) and the positive real parameter γ can be related to d and ω.

Proposition 2. *Based on the BLF ℓ_ω, the risk function of $T^{(3)}_{\gamma,JS+}(Y)$ given in equation (21) is*

$$\begin{aligned}R_\omega\left(T^{(3)}_{\gamma,JS+}(Y),\nu\right)={}&R_\omega\left(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu\right)\\&+\gamma^2\mathbb{E}\left(\frac{1}{\|Y\|^{10}}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)\\&+4\gamma(1-\omega)(d-6)\mathbb{E}\left(\frac{1}{\|Y\|^8}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)\\&-8\gamma(1-\omega)\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right).\end{aligned} \tag{22}$$

Proof.

$$\begin{aligned}
R_\omega\left(T^{(3)}_{\gamma,JS+}(Y),\nu\right) &= \omega\mathbb{E}\left(\left\|T^{(2)}_{\widehat{\beta},JS+}(Y)+\gamma\left(\frac{1}{\|Y\|^2}\right)^3\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y-Y\right\|^2\right)\\
&\quad+(1-\omega)\mathbb{E}\left(\left\|T^{(2)}_{\widehat{\beta},JS+}(Y)+\gamma\left(\frac{1}{\|Y\|^2}\right)^3\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y-\nu\right\|^2\right)\\
&= R_\omega\left(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu\right)+\gamma^2\mathbb{E}\left(\frac{1}{\left(\|Y\|^2\right)^5}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}\right)+2\gamma\omega\mathbb{E}\left(\left\langle T^{(2)}_{\widehat{\beta},JS+}(Y)-Y,\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y\right\rangle\right)\\
&\quad+2c(1-\omega)\mathbb{E}\left(\left\langle T^{(2)}_{\widehat{\beta},JS+}(Y)-\nu,\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y\right\rangle\right)\\
&= R_\omega\left(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu\right)+\gamma^2\mathbb{E}\left(\frac{1}{\left(\|Y\|^2\right)^5}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}\right)+2\gamma\omega\mathbb{E}\left(\left\langle T^{(2)}_{\widehat{\beta},JS+}(Y)-Y,\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y\right\rangle\right)\\
&\quad+2\gamma(1-\omega)\mathbb{E}\left(\left\langle T^{(2)}_{\widehat{\beta},JS+}(Y)-Y+Y-\nu,\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y\right\rangle\right)\\
&= R_\omega\left(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu\right)+\gamma^2 IE\left(\frac{1}{\left(\|Y\|^2\right)^5}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}\right)+2\gamma\mathbb{E}\left(\left\langle T^{(2)}_{\widehat{\beta},JS+}(Y)-Y,\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y\right\rangle\right)\\
&\quad+2\gamma(1-\omega)\mathbb{E}\left(\left\langle Y-\nu,\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y\right\rangle\right).
\end{aligned}\tag{23}$$

As

$$\begin{aligned}
\mathbb{E}\left(\left\langle \mathbb{T}^{(2)}_{\widehat{\beta},JS+}(Y)-Y,\frac{1}{\left(\|Y\|^2\right)^3}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y\right\rangle\right) &= \mathbb{E}\left(\left\langle\left(1-\frac{\alpha}{\|Y\|^2}+\frac{\widehat{\beta}}{\|Y\|^4}\right)\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y-Y,\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y\right\rangle\right)\\
&= \mathbb{E}\left(-\alpha\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}+\widehat{\beta}\frac{1}{\|Y\|^8}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}\right),
\end{aligned}\tag{24}$$

by applying Lemma 2.1 of Shao and Strawderman [27], we obtain

$$\begin{aligned}
&\mathbb{E}\left\langle Y-\nu,\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}Y\right\rangle\\
&\quad=(d-6)\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\le 1}\right).
\end{aligned}\tag{25}$$

From equations (23), (24), and (25), we get the desired result. □

Theorem 2. *For $d>6$ and based on the BLF, a sufficient condition for which the estimator $T^{(3)}_{\gamma,JS+}(Y)$ dominates $T^{(2)}_{\widehat{\beta},JS+}(Y)$ is*

$$0\le\gamma\le 4(1-\omega)^3(d-2)^2. \tag{26}$$

Proof. As

$$\mathbb{E}\left(\frac{1}{\|Y\|^{10}}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right) = \mathbb{E}\left(\frac{1}{\|Y\|^4}\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)$$
$$\leq \mathbb{E}\left(\frac{1}{\alpha^2}\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right) \quad (27)$$
$$= \frac{1}{\alpha^2}\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right),$$

and

$$\mathbb{E}\left(\frac{1}{\|Y\|^{8}}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right) = \mathbb{E}\left(\frac{1}{\|Y\|^2}\frac{1}{\|Y\|^6}\mathbb{I}_{\left(d-2/\|Y\|^2\right)\leq 1}\right)$$
$$\leq \mathbb{E}\left(\frac{1}{\alpha}\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right) \quad (28)$$
$$= \frac{1}{\alpha}\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right),$$

by using equations (27) and (28) and Proposition 2, we obtain

$$R_\omega\left(T^{(3)}_{\gamma,JS+}(Y),\nu\right) \leq R_\omega\left(T^{(2)}_{\widehat{\beta},JS+},\nu\right)$$
$$+\gamma^2\frac{1}{\alpha^2}\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)$$
$$+2\gamma\frac{\widehat{\beta}}{\alpha}\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)$$
$$-8\gamma(1-\omega)\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right) \quad (29)$$
$$= R_\omega\left(T^{(2)}_{\widehat{\beta}}(Y),\nu\right)$$
$$+\gamma\left(\frac{\gamma}{(1-\omega)^2(d-2)^2}-4(1-\omega)\right)$$
$$\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right).$$

Then, a sufficient condition for which the estimator $T^{(3)}_{\gamma,JS+}(\|Y\|^2)$ dominates $\delta^{(2)}_{\widehat{\beta},JS+}$ is

$$\frac{\gamma}{(1-\omega)^2(d-2)^2}-4(1-\omega)\leq 0, \quad (30)$$

which can be expressed as

$$0\leq\gamma\leq 4(1-\omega)^3(d-2)^2. \quad (31)$$

The value of γ that minimizes the right hand side of inequality (29) is

$$\widehat{\gamma} = 2(1-\omega)^3(d-2)^2. \quad (32)$$

Then, by substituting $\gamma=\widehat{\gamma}$ in inequality (29), we get

$$R_\omega\left(T^{(3)}_{\gamma,JS+}(Y),\nu\right) \leq R_\omega\left(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu\right)$$
$$-4(1-\omega)^4(d-2)^2\mathbb{E}\left(\frac{1}{\|Y\|^6}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right) \quad (33)$$
$$\leq R_\omega\left(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu\right).$$

Now, we consider the new estimator that dominates $T^{(3)}_{\gamma,JS+}(Y)$ that is defined as

$$T^{(4)}_{\delta,JS+}\left(\|Y\|^2\right) = T^{(3)}_{\widehat{\gamma},JS+}(Y)+\delta\left(\frac{1}{\|Y\|^2}\right)^4\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y$$
$$=\left(1-\alpha\frac{1}{\|Y\|^2}+\widehat{\beta}\left(\frac{1}{\|Y\|^2}\right)^2+\widehat{\gamma}\left(\frac{1}{\|Y\|^2}\right)^3\right. \quad (34)$$
$$\left.+\delta\left(\frac{1}{\|Y\|^2}\right)^4\right)\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}Y,$$

where $\widehat{\beta}$ and $\widehat{\gamma}$ are defined in equations (19) and (32), respectively, and the parameter δ behaves like γ in equation (21). The analogous technique used in the proof of Proposition 2 leads to the following proposition. □

Proposition 3. *Based on the BLF ℓ_ω, the risk function of $T^{(4)}_{\delta,JS+}$ given in equation (34) is*

$$R_\omega\left(T^{(4)}_{\delta,JS+}(Y),\nu\right) = R_\omega\left(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu\right)$$
$$+\delta^2\mathbb{E}\left(\frac{1}{\|Y\|^{14}}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)$$
$$+2\delta\widehat{\gamma}\mathbb{E}\left(\frac{1}{\|Y\|^{12}}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right) \quad (35)$$
$$+2\delta\widehat{\beta}\mathbb{E}\left(\frac{1}{\|Y\|^{10}}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right)$$
$$-12\delta(1-\omega)\mathbb{E}\left(\frac{1}{\|Y\|^{8}}\mathbb{I}_{\left(\alpha/\|Y\|^2\right)\leq 1}\right).$$

Theorem 3. *For $d>8$ and based on the BLF ℓ_ω, a sufficient condition for which the estimator $T^{(4)}_{\delta,JS+}(Y)$ dominates $T^{(3)}_{\widehat{\gamma},JS+}(Y)$ is*

$$0\leq\delta\leq 4(1-\omega)^4(d-2)^3. \quad (36)$$

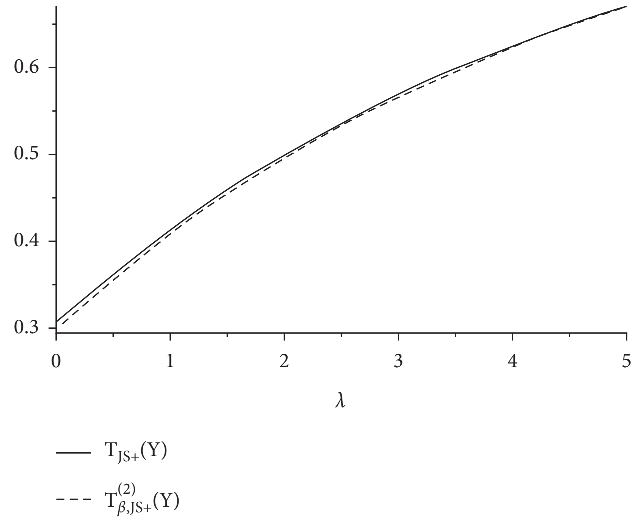

FIGURE 1: Curves of $R_{\omega}(T_{JS+}(Y), \nu)/R_{\omega}(Y, \nu)$ and $R_{\omega}(T^{(2)}_{\widehat{\beta},JS+}(Y), \nu)/R_{\omega}(Y, \nu)$ as functions of λ for $d = 6$ and $\omega = 0.1$.

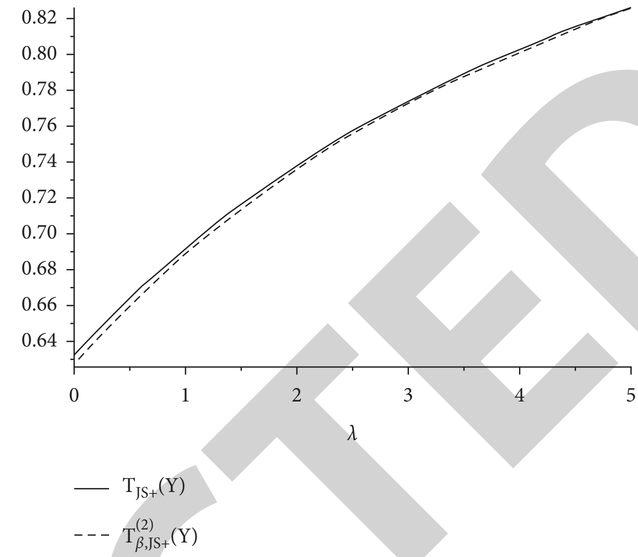

FIGURE 2: Curves of $R_{\omega}(T_{JS+}(Y), \nu)/R_{\omega}(Y, \nu)$ and $R_{\omega}(T^{(2)}_{\widehat{\beta},JS+}(Y), \nu)/R_{\omega}(Y, \nu)$ as functions of λ for $d = 6$ and $\omega = 0.5$.

Proof. As

$$\mathbb{E}\left(\frac{1}{\|Y\|^{14}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right) = \mathbb{E}\left(\frac{1}{\|Y\|^{6}}\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right) \leq \mathbb{E}\left(\frac{1}{\alpha^{3}}\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right) = \frac{1}{\alpha^{3}}\mathbb{E}\left(\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right), \tag{37}$$

$$\mathbb{E}\left(\frac{1}{\|Y\|^{12}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right) = \mathbb{E}\left(\frac{1}{\|Y\|^{4}}\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right) \leq \mathbb{E}\left(\frac{1}{\alpha^{2}}\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right) = \frac{1}{\alpha^{2}}\mathbb{E}\left(\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right), \tag{38}$$

and

$$\mathbb{E}\left(\frac{1}{\|Y\|^{10}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right) = \mathbb{E}\left(\frac{1}{\|Y\|^{2}}\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right) \leq \mathbb{E}\left(\frac{1}{\alpha}\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right) = \frac{1}{\alpha}\mathbb{E}\left(\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right), \tag{39}$$

by using equations (37), (38), and (39) and Proposition 3, we have

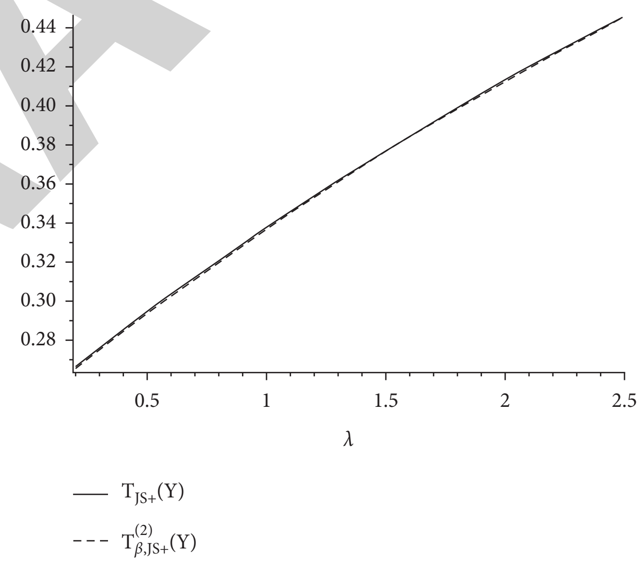

FIGURE 3: Curves of $R_{\omega}(T_{JS+}(Y), \nu)/R_{\omega}(Y, \nu)$ and $R_{\omega}(T^{(2)}_{\widehat{\beta},JS+}(Y), \nu)/R_{\omega}(Y, \nu)$ as functions of λ for $d = 8$ and $\omega = 0.1$.

$$R_{\omega}\left(T^{(4)}_{\delta,JS+}(Y), \nu\right) \leq R_{\omega}\left(T^{(3)}_{\widehat{\gamma},JS+}(Y), \nu\right) + \delta\left(\frac{\gamma}{\alpha^{3}} - 4(1-\omega)\right)\mathbb{E}\left(\frac{1}{\|Y\|^{8}}\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}\right). \tag{40}$$

Then, a sufficient condition for which the estimator $T^{(4)}_{\delta,JS+}(Y)$ dominates $T^{(3)}_{\widehat{\gamma},JS+}(Y)$ is

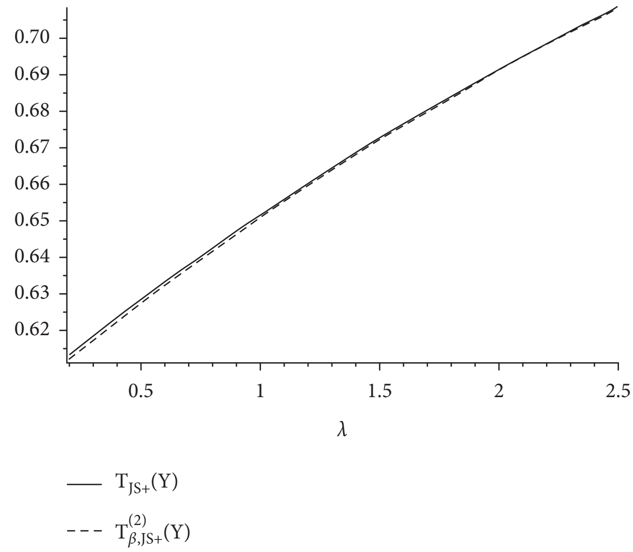

FIGURE 4: Curves of $R_\omega(T_{JS+}(Y),\nu)/R_\omega(Y,\nu)$ and $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$ as functions of λ for $d=8$ and $\omega=0.5$.

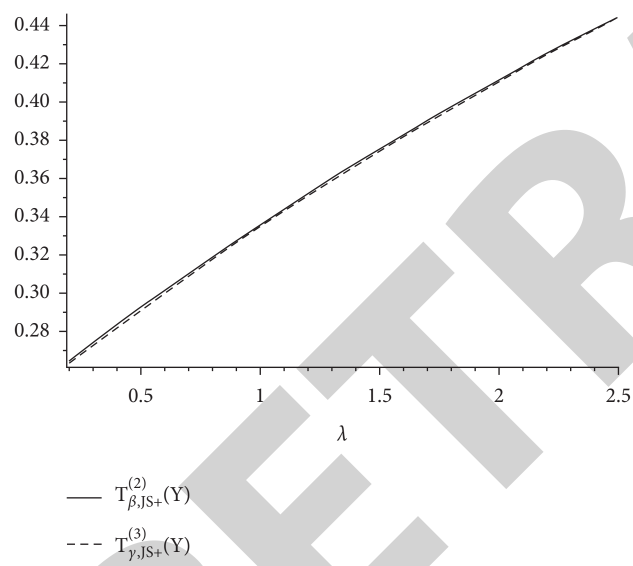

FIGURE 5: Curves of $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$ and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ as functions of λ for $d=8$ and $\omega=0.1$.

$$0\le\delta\le 4(1-\omega)^4(d-2)^3, \tag{41}$$

and the optimal value for δ that minimizes the right hand side of equation (40) is

$$\widehat{\delta}=2(1-\omega)^4(d-2)^3. \tag{42}$$

If we take $\delta=\widehat{\delta}$, the inequality in equation (40) becomes

$$\begin{aligned}R_\omega\left(T^{(4)}_{\delta,JS+}(Y),\nu\right)&\le R_\omega\left(\delta^{(3)}_{\widehat{\gamma},JS+}(Y),\nu\right)\\&\quad-4(1-\omega)^4(d-2)^3\mathbb{E}\left(\frac{1}{\|Y\|^8}\right)\\&\le R\left(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu\right).\end{aligned} \tag{43}$$

□

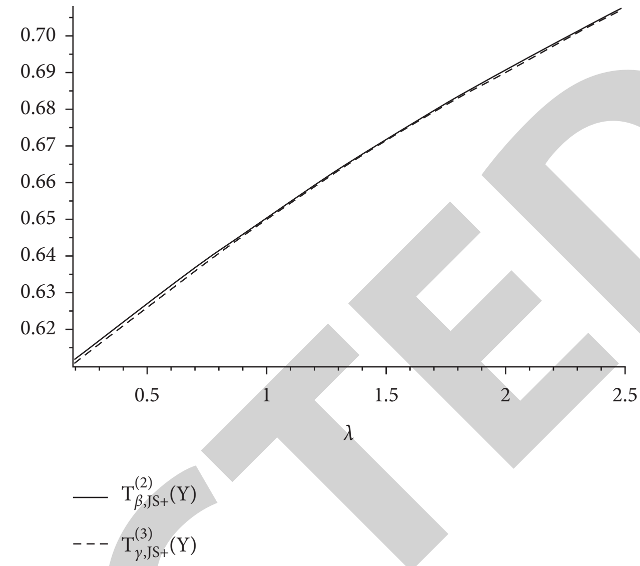

FIGURE 6: Curves of $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$ and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ as functions of λ for $d=8$ and $\omega=0.5$.

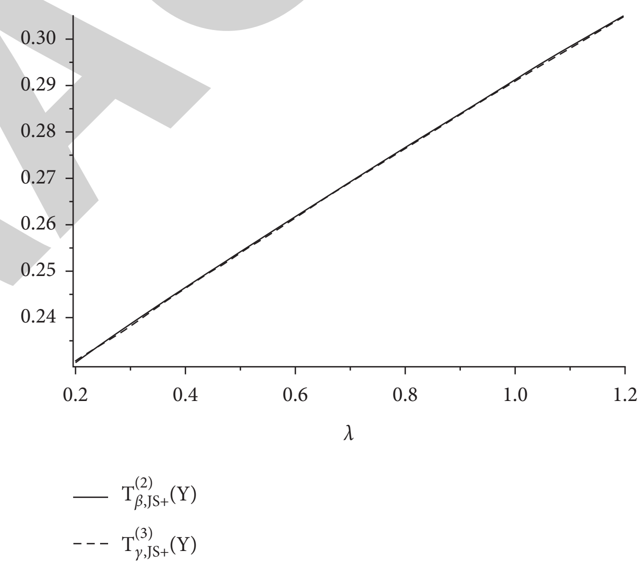

FIGURE 7: Curves of $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$ and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ as functions of λ for $d=10$ and $\omega=0.1$.

4. Simulation Studies

In this section, we present figures and tables that show the values of the risk ratios of the estimators $T_{JS+}(Y)$, $T^{(2)}_{\widehat{\beta},JS+}(Y)$, $T^{(3)}_{\widehat{\gamma},JS+}(Y)$, and $T^{(4)}_{\widehat{\delta},JS+}(Y)$, to the MLE. We recall that $T_{JS+}(Y)$ is defined in equation (5) and its risk function under the BLF ℓ_ω is given in equation (6), and the estimators $T^{(2)}_{\widehat{\beta},JS+}(Y)$, $T^{(3)}_{\widehat{\gamma},JS+}(Y)$, and $T^{(4)}_{\widehat{\delta},JS+}(Y)$ are defined, respectively, in equations (7), (21), and (34) with $\beta=\widehat{\beta}=(1-\omega)(d-6)$, $\gamma=\widehat{\gamma}=(1-\omega)(d-10)^2$, and $\delta=\widehat{\delta}=2(1-\omega)\ (d^2-28p+188)(d-14)$. Their risk functions under ℓ_ω are obtained by

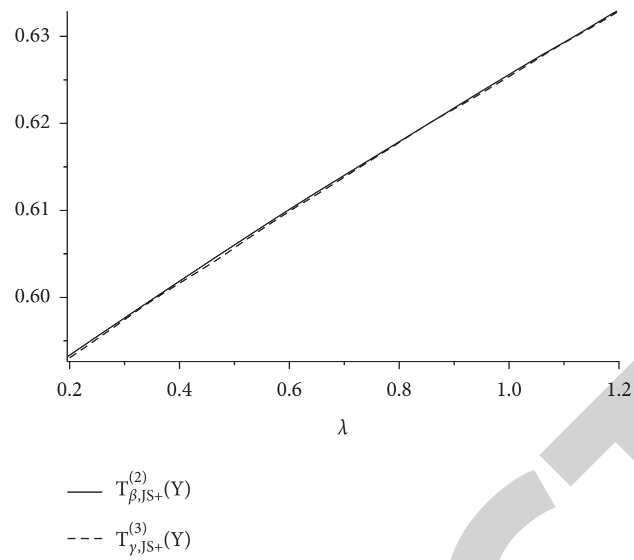

Figure 8: Curves of $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$ and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ as functions of λ for $d = 10$ and $\omega = 0.5$.

Table 1: Values of risk ratios $R_\omega(T_{JS+}(Y),\nu)/R_\omega(Y,\nu)$ (top), $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$ (middle), and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ (bottom) for $d = 8$ and different values of ω and $\lambda = \|\nu\|^2$.

λ	ω					
	0.0	0.1	0.2	0.5	0.7	0.9
	0.2800	0.3567	0.4341	0.6619	0.8030	0.9353
1.2418	0.2785	0.3552	0.4327	0.6608	0.8024	0.9352
	0.2769	0.3537	0.4312	0.6599	0.8020	0.9351
	0.3775	0.4455	0.5131	0.7083	0.8289	0.9435
2.4948	0.3764	0.4445	0.5121	0.7076	0.8286	0.9435
	0.3753	0.4433	0.5111	0.7069	0.8283	0.9435
	0.5221	0.5749	0.6266	0.7739	0.8661	0.9556
5.0019	0.5215	0.5743	0.6261	0.7735	0.8659	0.9555
	0.5209	0.5738	0.6256	0.7733	0.8658	0.9555
	0.6944	0.7267	0.7585	0.8507	0.9107	0.9702
10.4311	0.6942	0.7266	0.7584	0.8506	0.9106	0.9702
	0.6941	0.7265	0.7583	0.8506	0.9106	0.9702
	0.7721	0.7954	0.8185	0.8869	0.9322	0.9774
15.4110	0.7720	0.7954	0.8184	0.8869	0.9322	0.9774
	0.7720	0.7953	0.8184	0.8869	0.9322	0.9774
	0.8150	0.8337	0.8522	0.9077	0.9446	0.9815
20.0000	0.8150	0.8337	0.8522	0.9077	0.9446	0.9815
	0.8150	0.8337	0.8522	0.9077	0.9446	0.9815

substituting β by $\widehat{\beta}$, γ by $\widehat{\gamma}$, and δ by $\widehat{\delta}$ in Propositions 1, 2, and 3, respectively. We denote the risk ratios of the above estimators as $R_\omega(T_{JS+}(Y),\nu)/R_\omega(Y,\nu)$, $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$, $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$, and $R_\omega(T^{(4)}_{\widehat{\delta},JS+}(Y),\nu)/R_\omega(Y,\nu)$, respectively. First, for selected values of d and ω, we graph $R_\omega(T_{JS+}(Y),\nu)/R_\omega(Y,\nu)$, $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$, and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ as functions of $\lambda = \|\nu\|^2$. In the second part, we give two types of tables. The first one includes the values of $R_\omega(T_{JS+}(Y),\nu)/R_\omega(Y,\nu)$, $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$, and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ for fixes values of d and ω at different values of $\lambda = \|\nu\|^2$. The second table shows the values of $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ and $R_\omega(T^{(4)}_{\widehat{\delta},JS+}(Y),\nu)/R_\omega(Y,\nu)$ for fixed values of d and ω at different values of $\lambda = \|\nu\|^2$.

Figures 1–8 show that $R_\omega(T_{JS+}(Y),\nu)/R_\omega(Y,\nu)$, $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$, and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ are less than one which indicate that the estimators $T_{JS+}(Y)$, $T^{(2)}_{\widehat{\beta},JS+}(Y)$, and $T^{(3)}_{\widehat{\gamma},JS+}(\|Y\|^2)$ are better than the MLE

Table 2: Values of risk ratios $R_\omega(T_{JS+}(Y),\nu)/R_\omega(Y,\nu)$ (top), $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$ (middle), and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ (bottom) for $d=10$ and different values of ω and $\lambda=\|\nu\|^2$.

λ	ω					
	0.0	0.1	0.2	0.5	0.7	0.9
1.2418	0.2261	0.3083	0.3914	0.6349	0.7855	0.9290
	0.2257	0.3078	0.3910	0.6346	0.7853	0.9289
	0.2252	0.3073	0.3905	0.6342	0.7852	0.9289
2.4948	0.3121	0.3868	0.4612	0.6756	0.8084	0.9364
	0.3118	0.3865	0.4609	0.6754	0.8083	0.9364
	0.3114	0.3861	0.4605	0.6751	0.8083	0.9364
5.0019	0.4462	0.5070	0.5666	0.7364	0.8432	0.9478
	0.4460	0.5068	0.5664	0.7363	0.8431	0.9478
	0.4458	0.5066	0.5662	0.7362	0.8431	0.9478
10.4311	0.6210	0.6610	0.7003	0.8145	0.8889	0.9630
	0.6208	0.6610	0.7003	0.8145	0.8889	0.9630
	0.6208	0.6609	0.7003	0.8145	0.8889	0.9630
15.4110	0.7075	0.7375	0.7671	0.8549	0.9129	0.9710
	0.7075	0.7375	0.7671	0.8549	0.9129	0.9710
	0.7075	0.7375	0.7671	0.8549	0.9129	0.9710
20.0000	0.7581	0.7825	0.8068	0.8793	0.9276	0.9759
	0.7581	0.7825	0.8068	0.8793	0.9276	0.9759
	0.7581	0.7825	0.8068	0.8793	0.9276	0.9759

Table 3: Values of risk ratios $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ (top) and $R_\omega(T^{(4)}_{\widehat{\delta},JS+}(\|Y\|^2),\nu)/R_\omega(Y,\nu)$ (bottom) for $d=10$ and different values of ω and $\lambda=\|\nu\|^2$.

λ	ω					
	0.0	0.1	0.2	0.5	0.7	0.9
1.2418	0.2252	0.3073	0.3905	0.6342	0.7852	0.9289
	0.2246	0.3068	0.3900	0.6340	0.7851	0.9289
2.4948	0.3114	0.3861	0.4605	0.6751	0.8083	0.9364
	0.3110	0.3857	0.4602	0.6750	0.8082	0.9364
5.0019	0.4458	0.5066	0.5662	0.7362	0.8431	0.9478
	0.4456	0.5046	0.5660	0.7361	0.8431	0.9478
10.4311	0.6208	0.6609	0.7003	0.8145	0.8889	0.9630
	0.6208	0.6609	0.7002	0.8144	0.8889	0.9630
15.4110	0.7075	0.7375	0.7671	0.8549	0.9129	0.9710
	0.7075	0.7375	0.7671	0.8548	0.9129	0.9710
20.0000	0.7581	0.7825	0.8068	0.8793	0.9276	0.9759
	0.7581	0.7825	0.8068	0.8793	0.9276	0.9759

Y for the different values of d and ω, and thus they are minimax. We remark that $T^{(2)}_{\widehat{\beta},JS+}(Y)$ dominates $T_{JS+}(Y)$ and $T^{(3)}_{\widehat{\gamma},JS+}(Y)$ dominates $T^{(2)}_{\widehat{\beta},JS+}(Y)$ for the chosen values of d and ω. We also note that the improvement increases when ω value is close to zero and decreases as ω approaches one. Tables 1 and 2 confirm this remark. In these tables, we started with chosen values of d and ω to compute $R_\omega(T_{JS+}(Y),\nu)/R_\omega(Y,\nu)$, $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$, and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$ at different values of λ. So, when the values of ω and $\lambda=\|\nu\|^2$ are small, we have got significant improvement of $R_\omega(T_{JS+}(Y),\nu)/R_\omega(Y,\nu)$, $R_\omega(T^{(2)}_{\widehat{\beta},JS+}(Y),\nu)/R_\omega(Y,\nu)$, and $R_\omega(T^{(3)}_{\widehat{\gamma},JS+}(Y),\nu)/R_\omega(Y,\nu)$. As ω and λ increase, the improvement decreases towards zero, and then a small improvement is obtained. For fixed value of ω, an indication of better improvement is deduced when the value of d increases. We conclude that the improvement of the estimators can be significant when the value of d is large, λ is small, and ω tends to be close to zero. Therefore, the improvement of the risks ratios is clearly affected by the combination of the different values of d, ω, and λ.

Tables 3 and 4 show the risk ratios $R_\omega(T^{(3)}_{\widehat{\gamma}}(Y),\nu)/R_\omega(Y,\nu)$ and $R_\omega(T^{(4)}_{\widehat{\delta}}(Y),\nu)/R_\omega(Y,\nu)$ for the selected values of d and ω at different values of λ. In these tables, we observe small improvement of $T^{(4)}_{\widehat{\delta},JS+}(Y)$ to $T^{(3)}_{\widehat{\gamma},JS+}(Y)$ in comparison

Table 4: Values of risk ratios $R_{\omega}(T^{(3)}_{\hat{\gamma},JS+}(Y),\nu)/R_{\omega}(Y,\nu)$ (top) and $R_{\omega}(T^{(4)}_{\hat{\delta},JS+}(\|Y\|^2),\nu)/R_{\omega}(Y,\nu)$ (bottom) for $d = 12$ and different values of ω and $\lambda = \|\nu\|^2$.

λ	ω					
	0.0	0.1	0.2	0.5	0.7	0.9
1.2418	0.1893	0.2751	0.3619	0.6157	0.7729	0.9246
	0.1891	0.2749	0.3618	0.6156	0.7729	0.9246
2.4948	0.2658	0.3450	0.4241	0.6518	0.7935	0.9313
	0.2656	0.3449	0.4240	0.6517	0.7934	0.9313
5.0019	0.3893	0.4560	0.5214	0.7079	0.8258	0.9420
	0.3893	0.4560	0.5214	0.7079	0.8258	0.9420
10.4311	0.5612	0.6077	0.6531	0.7850	0.8712	0.9571
	0.5612	0.6077	0.6531	0.7850	0.8712	0.9571
15.4110	0.6526	0.6883	0.7235	0.8276	0.8966	0.9655
	0.6526	0.6883	0.7235	0.8276	0.8966	0.9655
20.0000	0.7082	0.7377	0.7670	0.8545	0.9127	0.9709
	0.7082	0.7377	0.7670	0.8545	0.9127	0.9709

with the improvement of $T^{(2)}_{\beta,JS+}(Y)$ to $T_{JS+}(Y)$ or the improvement of $T^{(3)}_{\hat{\gamma},JS+}(Y)$ to $T^{(2)}_{\hat{\beta},JS+}(Y)$ that appeared in Tables 1 and 2. We also notice that d, ω, and λ have similar effect to the risks ratios as in Tables 1 and 2.

5. Conclusion

In this article, we investigated the estimation of the mean ν of the random vector $Y \sim N_d(\nu, I_d)$. The risk associated to the BLF is the adopted criterion to determine the quality of the considered estimators. We introduced a class of estimators $T^{(2)}_{\beta,JS+}(Y) = T_{JS+}(Y) + \beta(1/\|Y\|^2)^2 Y \mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}$. We gave a sufficient condition on β, so that $\mathrm{T}^{(2)}_{\beta,JS+}(\|Y\|^2)$ dominates $\mathrm{T}_{JS+}(\|Y\|^2)(\|Y\|^2)$. Then, we suggested the estimators of polynomial type with the indeterminate $(1/\|Y\|^2)\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}$. That is, we added recursively the term $\gamma(1/\|Y\|^2)^m \mathbb{1}_{(\alpha/\|Y\|^2)\leq 1} Y$. Then, at each time, we got estimators that improve those estimators defined previously. Therefore, we obtained a series of polynomial form's estimators with the indeterminate $(1/\|Y\|^2)\mathbb{1}_{(\alpha/\|Y\|^2)\leq 1}$ and proved that if we increase the degree of the polynomial, we can build a best estimator from the one given previously. A point that should be considered is that increasing the degree of the of the polynomial has to accompany with having large dimension space of the parameter in order to satisfy the domination conditions. However, more difficult computation of the risk of the estimators can be observed which can lead to difficulties in determining the sufficiency conditions of the domination. Further investigation of this point can be considered as future work to determine the optimal degree of the polynomial form that provides the ultimate best estimator.

As an extension of this work, we can look for analogous results and examine the performance of estimators of the type $\mathrm{T}_{JS+}(Y) + \beta(1/\|Y\|^2)^r \mathbb{1}_{(\alpha/\|Y\|^2)\leq 1} Y$, using the general BLF $\ell_{\omega,\rho}(T,\nu) = \omega\rho(\|T - T_0\|^2) + (1-\omega)\rho(\|T-\nu\|^2)$, $\quad 0 \leq \omega < 1$, where $\rho(\cdot)$ is an arbitrary positive real function. This work can also be investigated under the Bayesian framework.

Data Availability

The numerical dataset used to support the findings of this study is available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] C. Stein, "Inadmissibilty of the usual estimator for the mean of a multivariate normal distribution," in *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley, CA, USA, 1956.

[2] W. James and C. Stein, "Estimation with quadratic loss," in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, Berkeley, CA, USA, 1961.

[3] D. Lindley, "Discussion on professor Stein's paper," *Journal of the Royal Statistical Society: Series B*, vol. 24, pp. 285–287, 1962.

[4] P. K. Bhattacharya, "Estimating the mean of a multivariate normal population with general quadratic loss function," *The Annals of Mathematical Statistics*, vol. 37, no. 6, pp. 1819–1824, 1966.

[5] J. O. Berger, "Admissible minimax estimation of a multivariate normal mean with arbitrary quadratic loss," *Annals of Statistics*, vol. 4, no. 1, pp. 223–226, 1976.

[6] C. M. Stein, "Estimation of the mean of a multivariate normal distribution," *Annals of Statistics*, vol. 9, no. 6, pp. 1135–1151, 1981.

[7] M. Norouzirad and M. Arashi, "Preliminary test and Stein-type shrinkage ridge estimators in robust regression," *Statistical Papers*, vol. 60, no. 6, pp. 1849–1882, 2019.

[8] C. L. Cheng, A. Chaturvedi, and A. Chaturvedi, "Goodness of fit for generalized shrinkage estimation," *Theory of Probability and Mathematical Statistics*, vol. 100, pp. 191–214, 2020.

[9] M. Kashani, MR. Rabiei, and M. Arashi, "An integrated shrinkage strategy for improving efficiency in fuzzy regression modeling," *Soft Computing*, vol. 25, no. 13, pp. 8095–8107, 2021.

[10] W. E. Strawderman, "Proper Bayes minimax estimators of the multivariate normal mean," *The Annals of Mathematical Statistics*, vol. 42, no. 1, pp. 385–388, 1971.

[11] D. V. Lindley and A. F. M. Smith, "Bayes estimates for the linear model," *Journal of the Royal Statistical Society: Series B*, vol. 34, pp. 1–18, 1972.

[12] B. Efron and C. N. Morris, "Stein's estimation rule and its competitors: an empirical Bayes approach," *Journal of the American Statistical Association*, vol. 68, no. 341, p. 117, 1973.

[13] H. M. Hudson, *Empirical Bayes Estimation*, Stanford University, Stanford, CA, USA, 1974.

[14] A. Hamdaoui, A. Benkhaled, and N. Mezouar, "Minimaxity and limits of risks ratios of shrinkage estimators of a multivariate normal mean in the Bayesian case," *Statistics, Optimization & Information Computing*, vol. 8, no. 2, pp. 507–520, 2020.

[15] A. J. Baranchik, *Multiple Regression and Estimation of the Mean of A Multivariate Normal Distribution*, Stanford University, Stanford, CA, USA, 1964.

[16] A. Zellner, "Bayesian and non-Bayesian estimation using balanced loss functions," in *Statistical Decision Theory and Methods*, J. O. Berger and S. S. Gupta, Eds., pp. 337–390, Springer, Berlin, Germany, 1994.

[17] N. S. Farsipour and A. Asgharzadeh, "Estimation of a normal mean relative to balanced loss functions," *Statistical Papers*, vol. 45, no. 2, pp. 279–286, 2004.

[18] S. Kaçıranlar and I. Dawoud, "The optimal extended balanced loss function estimators," *Journal of Computational and Applied Mathematics*, vol. 345, pp. 86–98, 2019.

[19] N. Özbay and S. Kaçıranlar, "Risk performance of some shrinkage estimators," *Communications in Statistics - Simulation and Computation*, vol. 50, no. 2, pp. 323–342, 2021.

[20] H. Lahoucine, M. Eric, and O. Idir, "On shrinkage estimation of a spherically symetric distribution for balanced loss function," *Journal of Multivariate Analysis*, vol. 186, pp. 1–11, 2021.

[21] H. Karamikabir and M. Afsahri, "Generalized Bayesian shrinkage and wavelet estimation of location parameter for spherical distribution under balanced-type loss: minimaxity and admissibility," *Journal of Multivariate Analysis*, vol. 177, pp. 110–120, 2020.

[22] H. Karamikabir, M. Afshari, and F. Lak, "Wavelet threshold based on Stein's unbiased risk estimators of restricted location parameter in multivariate normal," *Journal of Applied Statistics*, vol. 48, no. 10, pp. 1712–1729, 2021.

[23] D. Benmansour and A. Hamdaoui, "Limit of the ratio of risks of James-Stein estimators with unknown variance," *Far East Journal of Theoretical Statistics*, vol. 36, pp. 31–53, 2011.

[24] A. Hamdaoui and D. Benmansour, "Asymptotic properties of risks ratios of shrinkage estimators," *Hacettepe Journal of Mathematics and Statistics*, vol. 44, pp. 1181–1195, 2015.

[25] A. Hamdaoui, "On shrinkage estimators improving the positive part of James-Stein estimator," *Demonstratio Mathematica*, vol. 54, no. 1, pp. 462–473, 2021.

[26] A. Hamdaoui, W. Almutiry, M. Terbeche, and A. Benkhaled, "Comparison of risk ratios of shrinkage estimators in high dimensions," *Mathematics*, vol. 52, no. 10, pp. 1–14, 2022.

[27] P. Shao and W. E. Strawderman, "Improving on the James-Stein positive-part estimator of the multivariate normal mean vector for the case of common unknown variances," *Annals of Statistics*, vol. 22, pp. 1517–1539, 1994.