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*Research Article*

# The Distribution Properties of Consecutive Quadratic Residue Sequences

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We consider any prime number $p$. Let $k, s$ be two positive integers. We are interested in the arithmetic progressions (sequences) with the common difference $s$ and length $k$, where the sequence entries are from the set of quadratic residue modulo $p$ or the set of quadratic nonresidue modulo $p$. The numbers of such sequences are denoted as $N_p(k,s)$ and $N'_p(k,s)$, respectively. In this paper, we apply analytic number theory methods, in particular, properties of Legendre's symbol modulo $p$ and character sums, to study the numbers $N_p(k,s)$ and $N'_p(k,s)$. Exact formulas are given for certain values of $k$ and $s$ under some restrictions. In addition, estimation formulas in other cases are given.

## 1. Introduction

Let $p$ be an odd prime. For any integer $a$ with $(a,p)=1$, namely, $a$ coprime $p$, if there exists an integer $b$ such that $b^2 \equiv a \pmod p$, then $a$ is called a quadratic residue modulo $p$. Otherwise, $a$ is called a quadratic nonresidue modulo $p$. We only concern those quadratic residues or nonresidues in the finite field $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$. Thus, throughout the paper, when we talk about a sequence of quadratic residues mod $p$, it means that the entries are the least positive quadratic residues mod $p$. It is the same for sequences of quadratic nonresidue mod $p$. In the study of quadratic residue modulo $p$, Legendre first introduced the *Legendre symbol*, a character function whose values determine the status of an integer being a quadratic residue modulo $p$ or not. For any integer $x$, the definition of the Legendre symbol $(x/p)$ is given as follows:

$$\left(\frac{x}{p}\right) = \begin{cases} 1, & \text{if x is a quadratic residue modulo } p, \\ -1, & \text{if x is a quadratic non-residue modulo } p, \\ 0, & \text{if } p \mid x. \end{cases} \tag{1}$$

The following properties of the Legendre symbol are well known:

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}, \qquad \left(\frac{2}{p}\right) = (-1)^{p^2-1/8}, \tag{2}$$

and for any two different odd prime numbers $p$ and $q$,

$$\left(\frac{q}{p}\right)\cdot\left(\frac{p}{q}\right) = (-1)^{(p-1)(q-1)/4} \text{(the quadratic reciprocity law)}. \tag{3}$$

Quadratic residues play important roles in solving many classical number theory problems. The study of quadratic residues brings not only profound theoretical significance but also a wide range of applications. This paper focuses on the distribution properties of arithmetic progression of quadratic residues and quadratic nonresidues. Many scholars have conducted in-depth research on similar problems and obtained many important results, some of which are useful for our project. For example, in 1956, Carlitz [1] estimated the number $N(p,s)$ of $s$ consecutive quadratic residues or $s$ consecutive quadratic nonresidues in the finite field $\mathbb{F}_p$:

$$N(p,s) = \frac{p}{2^s} + O\left(sp^{\theta}\right), \quad (\theta < 1). \tag{4}$$

Based on the improved Vinogradov estimation, Burgess [2] proved in 1957 that, if $\delta$ and $\varepsilon$ are arbitrarily fixed positive numbers, for sufficiently large odd prime numbers $p$ and any integer $N$, if $H > p^{1/4+\delta}$, then

$$\left|\sum_{n=N+1}^{N+H}\left(\frac{n}{p}\right)\right| < \varepsilon H. \tag{5}$$

This indicates that the maximum number of consecutive quadratic residues or consecutive quadratic nonresidue modulo of a sufficiently large prime $p$ is $O(p^{1/4+\delta})$. This result was further improved and can be referenced in literature [3–6]. In 1992, Peralta [7] improved the error term $E(p,s)$ in (4) and proved that

$$E(p,s) = \pm(s+1)(3+\sqrt{p}). \tag{6}$$

In 2020, Carella [8] used the method of exponential sums to improve the above results. Let $p$ be a sufficiently large prime number and $s$ be any positive integer satisfying $s = O(\log p)$, and the number $N(p,s)$ satisfies:

(i) $N(p,s) = p/2^s(1-1/p)^s(1+O(1/p))$

(ii) $N(p,s) = p/2^s + O(s^2)$

In 2020, Wang and Lv [9] studied the number of integers $1 \le a \le p-1$ such that $a$, $a+\overline{a}$, $a-\overline{a}$ are all quadratic residues and quadratic non-residues modulo $p$, where $\overline{a} \in \mathbb{F}_p$ satisfying $a\overline{a} = 1$. In 2022, the first author of this paper and Li [10] studied distribution properties of triples of consecutive quadratic residues (named 3-CQR) and consecutive quadratic nonresidue (3-CQN) modulo $p$ and provided exact formulas for the numbers $S_1(p)$ and $S_2(p)$ of 3-CQRs and 3-CQNs. Other interesting properties of quadratic residues can be referred to [11].

We start with the following definitions.

*Definition 1.* Let $p$ be an odd prime number and $k$ and $s$ be two positive integers.

(1) *We define $N_p(k,s)$ to be the number of arithmetic sequences of quadratic residues with the common difference $s$ and length $k$*

(2) *Correspondingly, $N'_p(k,s)$ denotes the number of arithmetic sequences, where the entries are all quadratic nonresidues, with the common difference $s$ and length $k$*

In this paper, we use analytic methods, combined with the properties of complete residue system modulo $p$, to study $N_p(k,s)$ and $N'_p(k,s)$, defined as above. We give exact formulas for $N_p(k,s)$ and $N'_p(k,s)$ under some restrictions, when $k,s$ are small. Estimation formulas in other cases are given as well.

The structure of the paper is as follows. In Section 2, we give some preliminary results stated as lemmas which are useful for developing the results in the later sections. In Section 3, we give the exact formulas on $N_p(k,s)$ and $N'_p(k,s)$ in the case $p \equiv 3 \pmod 4$ for certain $k=3$ and $s \in \{3,4\}$. In Section 4, we discuss the upper and lower bounds of $N_p(k,s)$ and $N'_p(k,s)$ in the case $p \equiv 3 \pmod 4$ and when $k \in \{4,5\}$ and $s \le 2$. Conclusions and future directions are given in Section 5.

## 2. Preliminary Lemmas

In order to prove the main results, we first develop two preliminary lemmas.

**Lemma 2.** *Let $p$ be an odd prime number, and for any integer $k$ with $(k,p)=1$, then*

$$\sum_{a=0}^{p-1}\left(\frac{a(a+k)}{p}\right) = -1. \tag{7}$$

*Proof.* From the properties of complete residue system modulo $p$, we have

$$\sum_{a=0}^{p-1}\left(\frac{1+ka}{p}\right) = 0. \tag{8}$$

Applying the properties of Legendre's symbol modulo $p$, we have

$$\begin{aligned}\sum_{a=0}^{p-1}\left(\frac{a(a+k)}{p}\right) &= \sum_{a=1}^{p-1}\left(\frac{a(a+k)}{p}\right) = \sum_{a=1}^{p-1}\left(\frac{a^2}{p}\right)\left(\frac{1+k\overline{a}}{p}\right) = \sum_{a=1}^{p-1}\left(\frac{1+k\overline{a}}{p}\right)\\ &= \sum_{a=1}^{p-1}\left(\frac{1+ka}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{1+ka}{p}\right) - \left(\frac{1}{p}\right) = -1.\end{aligned} \tag{9}$$

□

**Lemma 3.** *Let $p$ be an odd prime number, and for any integer $k$ with $(k, p) = 1$, then*

$$\sum_{a=1}^{p-1}\left(\frac{a^2+k}{p}\right) = -1 - \left(\frac{k}{p}\right). \tag{10}$$

*Proof.* From the properties of complete residue system modulo $p$, we have

$$\begin{aligned}\sum_{a=1}^{p-1}\left(\frac{a^2+k}{p}\right) &= \sum_{a=1}^{p-1}\left(1+\left(\frac{a}{p}\right)\right)\left(\frac{a+k}{p}\right) = \sum_{a=1}^{p-1}\left(\frac{a+k}{p}\right) + \sum_{a=1}^{p-1}\left(\frac{a(a+k)}{p}\right)\\ &= \sum_{a=0}^{p-1}\left(\frac{a+k}{p}\right) - \left(\frac{k}{p}\right) + \sum_{a=0}^{p-1}\left(\frac{a(a+k)}{p}\right) = -1 - \left(\frac{k}{p}\right).\end{aligned} \tag{11}$$

Next, we list several results from [10, 12–15], which are useful the proofs of later theorems. □

**Lemma 4 [10].** *Let $p$ be an odd prime number with $p \equiv 3 \pmod 4$. Then, the number of 3 consecutive quadratic residues is the same as the number of 3 consecutive quadratic nonresidues:*

$$N_p(3,1) = N_p'(3,1) = \begin{cases} \dfrac{1}{8}(p-3), & \text{if } p \equiv 3 \pmod 8, \\[2ex] \dfrac{1}{8}(p-7), & \text{if } p \equiv 7 \pmod 8. \end{cases} \tag{12}$$

**Lemma 5 [12].** *Assume $b \in \mathbb{Z}$ and $b^2 \not\equiv 1 \pmod p$, then*

$$\sum_{a=1}^{p-1}\left(\frac{a^2-b^2}{p}\right)\left(\frac{a^2-1}{p}\right) \le 3\sqrt{p}. \tag{13}$$

**Lemma 6 [13, 14].** *Let $k$ be a positive integer and $h(x) = (x-a_1)\cdots(x-a_k)$, where $a_1 \ldots, a_k$ are integers not pairwise congruent to each other modulo $p$, and for every $a \in \mathbb{F}_p$, then*

$$\sum_{x=0}^{p-1}\left(\frac{h(x)}{p}\right)e^{(2\pi i a x)/p} \le k\sqrt{p}. \tag{14}$$

**Lemma 7 [15].** *Let $N, M$ be any integers with $0 \le M \le p-1$, and we have*

$$\left|\sum_{x=N}^{N+M}\left(\frac{h(x)}{p}\right)\right| \le 2k\sqrt{p}\log p. \tag{15}$$

## 3. Formulas for $N_p(k,s)$ and $N_p'(k,s)$ When $k = = 3$ and $s \in \{3,4\}$

In this section, we discuss the enumeration of the arithmetic progressions of quadratic residues (or nonresidues) of length 3 with the common distances 3 or 4. Using the properties of Legendre's symbol modulo $p$, the properties of the complete residue system and reduced residue system, and the preliminary lemmas from Section 2, we give the exact formulas for $N_p(3,3)$, $N_p'(3,3)$, $N_p(3,4)$, and $N_p'(3,4)$ for certain prime numbers $p$.

**Theorem 8.** *Let $p$ be an odd prime number with $p \equiv 3 \pmod 4$ and $p > 3$, $s \in \{3,4\}$, and then*

*(1)* $N_p(3,3) = N_p'(3,3) = \begin{cases} 1/8(p-7), & \text{if } p \equiv 7 \pmod 8; \\ 1/8(p-15) + 1/2(5/p), & \text{if } p \equiv 3 \pmod 8. \end{cases}$

*(2)* $N_p(3,4) = N_p'(3,4)$

$$= \begin{cases} \dfrac{1}{8}(p-3), & \text{if } p \equiv 11 \pmod{24}, \\[2ex] \dfrac{1}{8}(p-7), & \text{if } p \equiv 23 \pmod{24}, \\[2ex] \dfrac{1}{8}(p-11) - \dfrac{1}{2}\left(\dfrac{5}{p}\right) - \dfrac{1}{2}\left(\dfrac{7}{p}\right), & \text{if } p \equiv 19 \pmod{24}, \\[2ex] \dfrac{1}{8}(p-15) - \dfrac{1}{2}\left(\dfrac{5}{p}\right) - \dfrac{1}{2}\left(\dfrac{7}{p}\right), & \text{if } p \equiv 7 \pmod{24}. \end{cases} \tag{16}$$

*Proof*

(1) From the definition of $N_p(3,3)$ and the properties of Legendre's symbol modulo $p$, we have

$$\begin{aligned}
N_p(3,3) &= \frac{1}{8}\sum_{a=4}^{p-4}\left(1+\left(\frac{a-3}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+3}{p}\right)\right) \\
&= \frac{1}{8}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-3}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+3}{p}\right)\right) - \frac{1}{8}\left(1+\left(\frac{-3}{p}\right)\right) \\
&\quad \cdot \left(1+\left(\frac{3}{p}\right)\right) - \frac{1}{2}\left(1+\left(\frac{-2}{p}\right)\right) - \frac{1}{8}\left(1+\left(\frac{3}{p}\right)\right)\left(1+\left(\frac{6}{p}\right)\right) \\
&\quad - \frac{1}{8}\left(1+\left(\frac{-6}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right) - \frac{1}{4}\left(1+\left(\frac{-5}{p}\right)\right)\left(1+\left(\frac{-2}{p}\right)\right) \\
&= \frac{1}{8}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-3}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+3}{p}\right)\right) - \frac{1}{4}\left(4-2\left(\frac{2}{p}\right)-\left(\frac{5}{p}\right)+\left(\frac{10}{p}\right)\right).
\end{aligned} \tag{17}$$

As $a$ passes through a reduced residue system mod $p$, $3a$ passes through a reduced residue system mod $p$ as well. By replacing $a$ by $3a$, we have

$$\begin{aligned}
&\frac{1}{8}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-3}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+3}{p}\right)\right) \\
&\quad = \frac{1}{8}\sum_{a=0}^{p-1}\left(1+\left(\frac{3a-3}{p}\right)\right)\left(1+\left(\frac{3a}{p}\right)\right)\left(1+\left(\frac{3a+3}{p}\right)\right) \\
&\quad = \frac{1}{8}\sum_{a=0}^{p-1}\left(1+\left(\frac{3}{p}\right)\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{3}{p}\right)\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{3}{p}\right)\left(\frac{a+1}{p}\right)\right).
\end{aligned} \tag{18}$$

Note that $(3/p) = -1$ if $p \equiv 7 \pmod{12}$. From Lemma 4, we have

$$\begin{aligned}
N_p(3,3) &= N_p'(3,1) + \frac{1}{4}\left(1+\left(\frac{2}{p}\right)\right) - \frac{1}{4}\left(4-2\left(\frac{2}{p}\right)-\left(\frac{5}{p}\right)+\left(\frac{10}{p}\right)\right) \\
&= N_p'(3,1) + \frac{1}{4}\left(-3+3\left(\frac{2}{p}\right)+\left(\frac{5}{p}\right)-\left(\frac{10}{p}\right)\right) \\
&= \begin{cases} \dfrac{1}{8}(p-7), & \text{if } p \equiv 7 \pmod 8, \\[2ex] \dfrac{1}{8}(p-15)+\dfrac{1}{2}\left(\dfrac{5}{p}\right), & \text{if } p \equiv 3 \pmod 8. \end{cases}
\end{aligned} \tag{19}$$

If $p \equiv 11 \pmod{12}$, $(3/p) = 1$. Then,

$$\begin{aligned} N_p(3,3) &= N_p'(3,1) + \frac{1}{4}\left(1 + \left(\frac{2}{p}\right)\right) - \frac{1}{4}\left(4 - 2\left(\frac{2}{p}\right) - \left(\frac{5}{p}\right) + \left(\frac{10}{p}\right)\right) \\ &= N_p'(3,1) + \frac{1}{4}\left(-3 + 3\left(\frac{2}{p}\right) + \left(\frac{5}{p}\right) - \left(\frac{10}{p}\right)\right) \\ &= \begin{cases} \frac{1}{8}(p-7), & \text{if } p \equiv 7 \pmod 8, \\ \frac{1}{8}(p-15) + \frac{1}{2}\left(\frac{5}{p}\right), & \text{if } p \equiv 3 \pmod 8. \end{cases} \end{aligned} \tag{20}$$

Combining (19) and (20), we have

$$N_p(3,3) = \begin{cases} \frac{1}{8}(p-7), & \text{if } p \equiv 7 \pmod 8, \\ \frac{1}{8}(p-15) + \frac{1}{2}\left(\frac{5}{p}\right), & \text{if } p \equiv 3 \pmod 8. \end{cases} \tag{21}$$

Similarly,

$$N_p'(3,3) = \begin{cases} \frac{1}{8}(p-7), & \text{if } p \equiv 7 \pmod 8, \\ \frac{1}{8}(p-15) + \frac{1}{2}\left(\frac{5}{p}\right), & \text{if } p \equiv 3 \pmod 8. \end{cases} \tag{22}$$

(2) From the definition of $N_p(3,4)$ and the properties of Legendre's symbol modulo $p$, we have

$$\begin{aligned} N_p(3,4) &= \frac{1}{8}\sum_{a=5}^{p-5}\left(1 + \left(\frac{a-4}{p}\right)\right)\left(1 + \left(\frac{a}{p}\right)\right)\left(1 + \left(\frac{a+4}{p}\right)\right) \\ &= \frac{1}{8}\sum_{a=0}^{p-1}\left(1 + \left(\frac{a-4}{p}\right)\right)\left(1 + \left(\frac{a}{p}\right)\right)\left(1 + \left(\frac{a+4}{p}\right)\right) - \frac{1}{8}\left(1 + \left(\frac{-4}{p}\right)\right)\left(1 + \left(\frac{4}{p}\right)\right) \\ &\quad - \frac{1}{4}\left(1 + \left(\frac{-3}{p}\right)\right)\left(1 + \left(\frac{5}{p}\right)\right) - \frac{1}{8}\left(1 + \left(\frac{-2}{p}\right)\right)\left(1 + \left(\frac{2}{p}\right)\right)\left(1 + \left(\frac{6}{p}\right)\right) \\ &\quad - \frac{1}{4}\left(1 + \left(\frac{2}{p}\right)\right) - \frac{1}{4}\left(1 + \left(\frac{-7}{p}\right)\right)\left(1 + \left(\frac{-3}{p}\right)\right) \\ &\quad - \left(1 + \left(\frac{-6}{p}\right)\right)\left(1 + \left(\frac{-2}{p}\right)\right)\left(1 + \left(\frac{2}{p}\right)\right) \\ &= \frac{1}{8}\sum_{a=0}^{p-1}\left(1 + \left(\frac{a-4}{p}\right)\right)\left(1 + \left(\frac{a}{p}\right)\right)\left(1 + \left(\frac{a+4}{p}\right)\right) \\ &\quad - \frac{1}{4}\left(3 + \left(\frac{2}{p}\right) - 2\left(\frac{3}{p}\right) + \left(\frac{5}{p}\right) - \left(\frac{7}{p}\right) - \left(\frac{15}{p}\right) + \left(\frac{21}{p}\right)\right). \end{aligned} \tag{23}$$

We replace $a$ by $4a$, and we have

$$\begin{aligned}
&\frac{1}{8}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-4}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+4}{p}\right)\right)\\
&=\frac{1}{8}\sum_{a=0}^{p-1}\left(1+\left(\frac{4a-4}{p}\right)\right)\left(1+\left(\frac{4a}{p}\right)\right)\left(1+\left(\frac{4a+4}{p}\right)\right)\\
&=\frac{1}{8}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+1}{p}\right)\right).
\end{aligned}\tag{24}$$

Noting that if $p\equiv 7\pmod{12}$, then $(3/p)=-1$. From Lemma 4, we have

$$\begin{aligned}
N_p(3,4)&=N_p(3,1)+\frac{1}{4}\left(1+\left(\frac{2}{p}\right)\right)-\frac{1}{4}\left(5+\left(\frac{2}{p}\right)+2\left(\frac{5}{p}\right)-2\left(\frac{7}{p}\right)\right)\\
&=N_p(3,1)+\frac{1}{4}\left(-4-2\left(\frac{5}{p}\right)-2\left(\frac{7}{p}\right)\right)\\
&=\begin{cases}\frac{1}{8}(p-11)-\frac{1}{2}\left(\frac{5}{p}\right)-\frac{1}{2}\left(\frac{7}{p}\right), & \text{if } p\equiv 3\pmod 8,\\ \frac{1}{8}(p-15)-\frac{1}{2}\left(\frac{5}{p}\right)-\frac{1}{2}\left(\frac{7}{p}\right), & \text{if } p\equiv 7\pmod 8,\end{cases}
\end{aligned}\tag{25}$$

and if $p\equiv 11\pmod{12}$, then $(3/p)=1$. We have $(5/p)-(7/p)-(15/p)+(21/p)=0$, and then,

$$\begin{aligned}
N_p(3,4)&=N_p(3,1)+\frac{1}{4}\left(1+\left(\frac{2}{p}\right)\right)-\frac{1}{4}\left(3+\left(\frac{2}{p}\right)-2\left(\frac{3}{p}\right)\right)\\
&=N_p(3,1)=\begin{cases}\frac{1}{8}(p-3), & \text{if } p\equiv 3\pmod 8,\\ \frac{1}{8}(p-7), & \text{if } p\equiv 7\pmod 8.\end{cases}
\end{aligned}\tag{26}$$

Combining (25) and (26), we obtain the desired formula for $N_p(3,4)$. Furthermore, by similar calculations, we can show that $N'_p(3,4)=N_p(3,4)$. □

## 4. Formulas for $N_p(k,s)$ and $N_p^{'}(k,s)$ When $k\in\{4,5\}$ and $s\leq 2$

In this section, we focus on the cases when $k\in\{4,5\}$ and $s\leq 2$. For convenience, we introduce some new notations.

*Definition 9.* For any integer $a$ and prime number $p$, let

$$A(i,j,\ldots,l)=\sum_{a=0}^{p-1}\left(\frac{a+i}{p}\right)\left(\frac{a+j}{p}\right)\cdots\left(\frac{a+l}{p}\right),\tag{27}$$

where $i,j,l$ are integers.

Note that

$$\begin{aligned}
&A(i)=0,\\
&A(i,j)=\sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a+j-i}{p}\right)=-1,\quad \text{where}\,(j-i,p)=1.
\end{aligned}\tag{28}$$

Now, we give upper and lower bounds for $N_p(k,s)$ and $N'_p(k,s)$ when $k\in\{4,5\}$ and $s\leq 2$. We start with $k=4$.

**Theorem 10.** *Let $p$ be a prime with $p\equiv 3\pmod 4$. Then,*

$$\begin{aligned}
&N_p(4,1)=N_p(4,2)=N'_p(4,1)=N'_p(4,2),\\
&\frac{p-7-6\sqrt{p}\log p}{16}-A_0\leq N_p(4,2)\leq\frac{p-7+6\sqrt{p}\log p}{16}-A_0,
\end{aligned}\tag{29}$$

*where*$A_0=1/8(1+(2/p))(1+(3/p))$.

*Proof*

(1) Note that $N_p(4,1)$ is identical with the following sum:

$$\begin{aligned}
&\frac{1}{16}\sum_{a=2}^{p-3}\left(1+\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+1}{p}\right)\right)\left(1+\left(\frac{a+2}{p}\right)\right)\\
&=\frac{1}{16}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+1}{p}\right)\right)\left(1+\left(\frac{a+2}{p}\right)\right)\\
&\quad-\frac{1}{8}\left(1+\left(\frac{2}{p}\right)\right)\left(1+\left(\frac{3}{p}\right)\right).
\end{aligned}\tag{30}$$

Let $A_0 = 1/8(1 + (2/p))(1 + (3/p))$. We can write

$$\begin{aligned} N_p(4,1) = \frac{1}{16}(p &+ A(-1) + A(0) + A(1) + A(2) + A(-1,0) + A(-1,1) \\ &+ A(-1,2) + A(0,1) + A(0,2) + A(1,2) + A(-1,0,1) \\ &+ A(0,1,2) + A(-1,0,2) + A(-1,1,2) + A(-1,0,1,2)) - A_0. \end{aligned} \tag{31}$$

Next, we calculate several summands of $N_p(4,1)$ as follows:

$$\begin{aligned} A(-1,0,1) &= \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^2-1}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{-a}{p}\right)\left(\frac{a^2-1}{p}\right) \\ &= -\sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^2-1}{p}\right) = 0, \\ A(0,1,2) &= \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right) \\ &= A(-1,0,1) = 0, \\ A(-1,0,2) &= \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+2}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{-a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a-2}{p}\right) \\ &= -\sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a-2}{p}\right) = -\sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) \\ &= -A(-1,1,2). \end{aligned} \tag{32}$$

The above implies

$$\begin{aligned} A(-1,0,1,2) &= \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{2a-1}{p}\right)\left(\frac{2a}{p}\right)\left(\frac{2a+1}{p}\right)\left(\frac{2a+2}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{4a^2-1}{p}\right) = \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{4a^2-1}{p}\right) \\ &= \sum_{a=1}^{p-1}\left(\frac{\overline{a}+1}{p}\right)\left(\frac{4-\overline{a}^2}{p}\right) = -\sum_{a=1}^{p-1}\left(\frac{a+1}{p}\right)\left(\frac{a^2-4}{p}\right) \\ &= -A(-2,1,2) - 1. \end{aligned} \tag{33}$$

Inserting these formulas into the previous formula for $N_p(4,1)$, we obtain

$$N_p(4,1) = \frac{1}{16}(p - 6 + A(-1,0,1,2)) - A_0 = \frac{1}{16}(p - 7 - A(-2,1,2)) - A_0. \tag{34}$$

By Lemma 7, $|A(-2,1,2)| \le 6\sqrt{p}\log p$, which gives an upper bound and a lower bound for $N_p(4,1)$:

$$\frac{p - 7 - 6\sqrt{p}\log p}{16} - A_0 \le N_p(4,1) \le \frac{p - 7 + 6\sqrt{p}\log p}{16} - A_0. \tag{35}$$

(2) Similarly, $N_p(4,2)$ is identical with the following sum:

$$\begin{aligned}
&\frac{1}{16}\sum_{a=3}^{p-5}\left(1+\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+2}{p}\right)\right)\left(1+\left(\frac{a+4}{p}\right)\right)\\
&= \frac{1}{16}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+2}{p}\right)\right)\left(1+\left(\frac{a+4}{p}\right)\right)\\
&\quad -\frac{1}{8}\left(1+\left(\frac{2}{p}\right)\right)\left(1+\left(\frac{6}{p}\right)\right).
\end{aligned} \tag{36}$$

Replacing $a$ by $2a$, we obtain

$$\begin{aligned}
&\frac{1}{16}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+2}{p}\right)\right)\left(1+\left(\frac{a+4}{p}\right)\right)\\
&= \frac{1}{16}\sum_{a=0}^{p-1}\left(1+\left(\frac{2a-2}{p}\right)\right)\left(1+\left(\frac{2a}{p}\right)\right)\left(1+\left(\frac{2a+2}{p}\right)\right)\left(1+\left(\frac{2a+4}{p}\right)\right)\\
&= \frac{1}{16}\sum_{a=0}^{p-1}\left(1+\left(\frac{2}{p}\right)\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{2}{p}\right)\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{2}{p}\right)\left(\frac{a+1}{p}\right)\right)\cdot\left(1+\left(\frac{2}{p}\right)\left(\frac{a+2}{p}\right)\right).
\end{aligned} \tag{37}$$

When $p \equiv 3 \pmod 4$, we have $N_p(4,1) = N'_p(4,1)$, and thus,

$$\begin{aligned}
&\frac{1}{16}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+1}{p}\right)\right)\left(1+\left(\frac{a+2}{p}\right)\right)\\
&= \frac{1}{16}\sum_{a=0}^{p-1}\left(1-\left(\frac{a-1}{p}\right)\right)\left(1-\left(\frac{a}{p}\right)\right)\left(1-\left(\frac{a+1}{p}\right)\right)\left(1-\left(\frac{a+2}{p}\right)\right).
\end{aligned} \tag{38}$$

Therefore,

$$\begin{aligned} N_p(4,2) &= N'_p(4,2) \\ &= N_p(4,1) + \frac{1}{8}\left(1+\left(\frac{2}{p}\right)\right)\left(1+\left(\frac{3}{p}\right)\right) - \frac{1}{8}\left(1+\left(\frac{2}{p}\right)\right)\left(1+\left(\frac{6}{p}\right)\right) \\ &= N_p(4,1) + \frac{1}{8}\left(\frac{3}{p}\right)\left(1+\left(\frac{2}{p}\right)\right)\left(1-\left(\frac{2}{p}\right)\right) = N_p(4,1). \end{aligned} \tag{39}$$

It proves Theorem 10.

The following example verified the results given in Theorem 10. □

*Example 1*

$$\begin{aligned} &p = 7, A(-1,0,1,2) = -1, A(-2,1,2) = 0, N_p(4,1) = 0, \\ &p = 11, A(-1,0,1,2) = -5, A(-2,1,2) = 4, N_p(4,1) = 0, \\ &p = 19, A(-1,0,1,2) = 3, A(-2,1,2) = -4, N_p(4,1) = 1, \\ &p = 23, A(-1,0,1,2) = 7, A(-2,1,2) = -8, N_p(4,1) = 1, \\ &p = 31, A(-1,0,1,2) = -9, A(-2,1,2) = 8, N_p(4,1) = 1. \end{aligned} \tag{40}$$

In cases $k = 5$ and $s = 1$ or 2, a similar result is presented below.

**Theorem 11.** *Let $p$ be a prime with $p \equiv 3 \pmod 4$. Then,*

$$\begin{aligned} &N_p(5,1) = N_p(5,2) = N'_p(5,1) = N'_p(5,2), \\ &\frac{p - 14 - 32\sqrt{p}\log p}{32} - A_0 \le N_p(5,2) \\ &\qquad \le \frac{p - 14 + \sqrt{p}(3 + 24\log p)}{32} - A_0, \end{aligned} \tag{41}$$

*where $A_0 = 1/8(1 + (2/p))(1 + (3/p))$.*

*Proof*

(1) If $k = 5$ and $s = 1$, from the definition of $N_p(5,1)$ and the properties of Legendre's symbol modulo $p$, we have

$$\begin{aligned} &\frac{1}{32}\sum_{a=3}^{p-3}\left(1+\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+1}{p}\right)\right)\cdot\left(1+\left(\frac{a+2}{p}\right)\right) \\ &= \frac{1}{32}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+1}{p}\right)\right)\cdot\left(1+\left(\frac{a+2}{p}\right)\right) - A_0. \end{aligned} \tag{42}$$

Let $A_0 = 1/8(1 + (2/p))(1 + (3/p))$, and we have

$$\begin{aligned} N_p(5,1) = \frac{1}{32}(&p + A(-2) + A(-1) + A(0) + A(1) + A(2) + A(-2,-1) \\ &+ A(-2,0) + A(-2,1) + A(-2,2) + A(-1,0) + A(-1,1) + A(-1,2) + A(0,1) \\ &+ A(0,2) + A(1,2) + A(-2,-1,0) + A(-1,0,1) + A(0,1,2) + A(-2,0,2) \\ &+ A(-2,-1,1) + A(-2,0,1) + A(-1,0,2) + A(-1,1,2) + A(-2,-1,2) \\ &+ A(-2,1,2) + A(-2,-1,0,1) + A(-2,-1,0,2) + A(-2,-1,1,2) \\ &+ A(-2,0,1,2) + A(-1,0,1,2) + A(-2,-1,0,1,2)) - A_0. \end{aligned} \tag{43}$$

From the properties of $A(i, j, \ldots, l)$, we have

$$\begin{aligned}
A(-1,0,1) &= \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^2-1}{p}\right)\\
&= \sum_{a=0}^{p-1}\left(\frac{-a}{p}\right)\left(\frac{a^2-1}{p}\right) = -\sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^2-1}{p}\right) = 0,\\
A(0,1,2) &= \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\\
&= A(-1,0,1) = 0,\\
A(-2,-1,0) &= \sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\\
&= A(-1,0,1) = 0,\\
A(-2,0,2) &= \sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+2}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^2-4}{p}\right)\\
&= \sum_{a=0}^{p-1}\left(\frac{-a}{p}\right)\left(\frac{a^2-4}{p}\right) = -\sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^2-4}{p}\right) = 0,\\
A(-2,-1,1) &= \sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a-1}{p}\right)\left(\frac{a+1}{p}\right)\\
&= \sum_{a=0}^{p-1}\left(\frac{-a-2}{p}\right)\left(\frac{-a-1}{p}\right)\left(\frac{-a+1}{p}\right)\\
&= -\sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) = -A(-1,1,2).
\end{aligned} \tag{44}$$

Then, $A(-2,-1,1) + A(-1,1,2) = 0$.

$$\begin{aligned}
A(-1,0,2) &= \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+2}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{-a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a-2}{p}\right)\\
&= -\sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right) = -A(-2,0,1).
\end{aligned} \tag{45}$$

Then, $A(-1,0,2)+A(-2,0,1)=0$.

$$\begin{aligned} A(-2,-1,2) &= \sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a-1}{p}\right)\left(\frac{a+2}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{-a-2}{p}\right)\left(\frac{-a-1}{p}\right)\left(\frac{-a+2}{p}\right) \\ &= -\sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) = -A(-2,1,2). \end{aligned} \tag{46}$$

Then, $A(-2,-1,2)+A(-2,1,2)=0$.

$$\begin{aligned} A(-1,0,1,2) &= \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{-a-1}{p}\right)\left(\frac{-a}{p}\right)\left(\frac{-a+1}{p}\right)\left(\frac{-a+2}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right) = A(-2,-1,0,1), \\ A(-2,0,1,2) &= \sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{-a-2}{p}\right)\left(\frac{-a}{p}\right)\left(\frac{-a+1}{p}\right)\left(\frac{-a+2}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+2}{p}\right) = A(-2,-1,0,2), \\ A(-1,0,1,2) &= \sum_{a=0}^{p-1}\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{2a-1}{p}\right)\left(\frac{2a}{p}\right)\left(\frac{2a+1}{p}\right)\left(\frac{2a+2}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{4a^2-1}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{\overline{a}}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{\overline{a}^2}{p}\right)\left(\frac{4a^2-1}{p}\right) \\ &= \sum_{a=0}^{p-1}\left(\frac{\overline{a}+1}{p}\right)\left(\frac{4-\overline{a}^2}{p}\right) = -\sum_{a=0}^{p-1}\left(\frac{a+1}{p}\right)\left(\frac{a^2-4}{p}\right) \\ &= -A(-2,1,2)-1, \\ A(-2,0,1,2) &= \sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a^2-4}{p}\right) = \sum_{a=0}^{p-1}\left(\frac{2a}{p}\right)\left(\frac{2a+1}{p}\right)\left(\frac{4a^2-4}{p}\right)\\
&= \sum_{a=0}^{p-1}\left(\frac{2\overline{a}}{p}\right)\left(\frac{2a+1}{p}\right)\left(\frac{\overline{a}^2}{p}\right)\left(\frac{a^2-1}{p}\right)\\
&= \sum_{a=0}^{p-1}\left(\frac{2}{p}\right)\left(\frac{2+\overline{a}}{p}\right)\left(\frac{1-\overline{a}^2}{p}\right)\\
&= -\sum_{a=0}^{p-1}\left(\frac{2}{p}\right)\left(\frac{a+2}{p}\right)\left(\frac{a^2-1}{p}\right) = -\left(\frac{2}{p}\right)A(-1,1,2)-1,\\
A(-2,-1,0,1,2) &= \sum_{a=0}^{p-1}\left(\frac{a-2}{p}\right)\left(\frac{a-1}{p}\right)\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right)\left(\frac{a+2}{p}\right)\\
&= \sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^2-1}{p}\right)\left(\frac{a^2-4}{p}\right)\\
&= \sum_{a=0}^{p-1}\left(\frac{-a}{p}\right)\left(\frac{a^2-1}{p}\right)\left(\frac{a^2-4}{p}\right)\\
&= -\sum_{a=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{a^2-1}{p}\right)\left(\frac{a^2-1}{p}\right) = 0.
\end{aligned}
\tag{47}
$$

We can immediately deduce that

$$
\begin{aligned}
N_p(5,1) &= \frac{1}{32}\left(p-10+A(-2,-1,1,2)+2A(-1,0,1,2)+2A(-2,0,1,2)\right)-A_0\\
&= \frac{1}{32}\left(p-14+A(-2,-1,1,2)-2A(-2,1,2)-2\left(\frac{2}{p}\right)A(-1,1,2)\right)-A_0.
\end{aligned}
\tag{48}
$$

From Lemma 5, we have

$$
A(-2,-1,1,2) = \sum_{a=0}^{p-1}\left(\frac{a^2-1}{p}\right)\left(\frac{a^2-4}{p}\right) \le 3\sqrt{p}. \tag{49}
$$

From Lemma 7, $|A(-2,-1,1,2)| \le 8\sqrt{p}\log p$, $|A(-2,1,2)| \le 6\sqrt{p}\log p$, and $|A(-1,1,2)| \le 6\sqrt{p}\log p$. Then,

$$
\frac{p-14-32\sqrt{p}\log p}{32} - A_0 \le N_p(5,1) \le \frac{p-14+\sqrt{p}(3+24\log p)}{32} - A_0. \tag{50}
$$

(2) Similarly, we can give the estimate for $N_p(5,2)$:

$$
\begin{aligned}
&\frac{1}{32}\sum_{a=5}^{p-5}\left(1+\left(\frac{a-4}{p}\right)\right)\left(1+\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+2}{p}\right)\right)\cdot\left(1+\left(\frac{a+4}{p}\right)\right)\\
&= \frac{1}{32}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-4}{p}\right)\right)\left(1+\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+2}{p}\right)\right)\cdot\left(1+\left(\frac{a+4}{p}\right)\right)-\frac{1}{8}\left(1+\left(\frac{2}{p}\right)\right)\left(1+\left(\frac{6}{p}\right)\right).
\end{aligned}
\tag{51}
$$

We replace $a$ by $2a$, and then,

$$\begin{aligned}
&\frac{1}{32}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-4}{p}\right)\right)\left(1+\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+2}{p}\right)\right)\cdot\left(1+\left(\frac{a+4}{p}\right)\right)\\
&=\frac{1}{32}\sum_{a=0}^{p-1}\left(1+\left(\frac{2a-4}{p}\right)\right)\left(1+\left(\frac{2a-2}{p}\right)\right)\left(1+\left(\frac{2a}{p}\right)\right)\left(1+\left(\frac{2a+2}{p}\right)\right)\cdot\left(1+\left(\frac{2a+4}{p}\right)\right)\\
&=\frac{1}{32}\sum_{a=0}^{p-1}\left(1+\left(\frac{2}{p}\right)\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{2}{p}\right)\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{2}{p}\right)\left(\frac{a}{p}\right)\right)\cdot\left(1+\left(\frac{2}{p}\right)\left(\frac{a+1}{p}\right)\right)\left(1+\left(\frac{2}{p}\right)\left(\frac{a+2}{p}\right)\right).
\end{aligned}\tag{52}$$

Since $p\equiv 3\ (\mathrm{mod}\,4)$, $N_p(5,1)=N_p'(5,1)$, which implies

$$\begin{aligned}
&\frac{1}{32}\sum_{a=0}^{p-1}\left(1+\left(\frac{a-2}{p}\right)\right)\left(1+\left(\frac{a-1}{p}\right)\right)\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{a+1}{p}\right)\right)\cdot\left(1+\left(\frac{a+2}{p}\right)\right)\\
&=\frac{1}{32}\sum_{a=0}^{p-1}\left(1-\left(\frac{a-2}{p}\right)\right)\left(1-\left(\frac{a-1}{p}\right)\right)\left(1-\left(\frac{a}{p}\right)\right)\left(1-\left(\frac{a+1}{p}\right)\right)\cdot\left(1-\left(\frac{a+2}{p}\right)\right).
\end{aligned}\tag{53}$$

We can immediately obtain that

$$\begin{aligned}
N_p(5,2)=N_p'(5,2)&=N_p(5,1)+\frac{1}{8}\left(1+\left(\frac{2}{p}\right)\right)\left(1+\left(\frac{3}{p}\right)\right)-\frac{1}{8}\left(1+\left(\frac{2}{p}\right)\right)\left(1+\left(\frac{6}{p}\right)\right)\\
&=N_p(5,1)+\frac{1}{8}\left(\frac{3}{p}\right)\left(1+\left(\frac{2}{p}\right)\right)\left(1-\left(\frac{2}{p}\right)\right)=N_p(5,1).
\end{aligned}\tag{54}$$

□

**Proposition 12.** *Let $p$ be a prime with $p\equiv 3\ (\mathrm{mod}\,4)$. Then, $N_p(k,1)=N_p(k,2)$, when $2\le k\le p-1$.*

*Proof.* From the definition of $N_p(k,1)$ and $N_p(k,2)$, we can obtain

$$\begin{aligned}
N_p(k,1)&=\frac{1}{2^k}\sum_{a=1}^{p-k}\prod_{i=1}^{k}\left(1+\left(\frac{a+(i-1)}{p}\right)\right),\\
N_p(k,2)&=\frac{1}{2^k}\sum_{a=1}^{p-2k+1}\prod_{j=1}^{k}\left(1+\left(\frac{a+2(j-1)}{p}\right)\right).
\end{aligned}\tag{55}$$

Furthermore,

$$N_p(k,1) = \frac{1}{2^k}\sum_{a=1}^{p-1}\prod_{i=1}^{k}\left(1+\left(\frac{a+(i-1)}{p}\right)\right) - \frac{1}{2^k}\sum_{a=p-k+1}^{p-1}\prod_{i=1}^{k}\left(1+\left(\frac{a+(i-1)}{p}\right)\right),$$
$$N_p(k,2) = \frac{1}{2^k}\sum_{a=1}^{p-1}\prod_{j=1}^{k}\left(1+\left(\frac{a+2(j-1)}{p}\right)\right) - \frac{1}{2^k}\sum_{a=p-2k+2}^{p-1}\prod_{j=1}^{k}\left(1+\left(\frac{a+2(j-1)}{p}\right)\right). \tag{56}$$

For convenience, we assume

$$N_{11} = \frac{1}{2^k}\sum_{a=1}^{p-1}\prod_{i=1}^{k}\left(1+\left(\frac{a+(i-1)}{p}\right)\right),$$
$$N_{12} = \frac{1}{2^k}\sum_{a=p-k+1}^{p-1}\prod_{i=1}^{k}\left(1+\left(\frac{a+(i-1)}{p}\right)\right),$$
$$N_{21} = \frac{1}{2^k}\sum_{a=1}^{p-1}\prod_{j=1}^{k}\left(1+\left(\frac{a+2(j-1)}{p}\right)\right),$$
$$N_{22} = \frac{1}{2^k}\sum_{a=p-2k+2}^{p-1}\prod_{j=1}^{k}\left(1+\left(\frac{a+2(j-1)}{p}\right)\right). \tag{57}$$

From the above identities, we have

$$N_p(k,1) - N_p(k,2) = N_{11} - N_{21} + N_{22} - N_{12}. \tag{58}$$

From the properties of character sums, we have

$$\frac{1}{2^k}\sum_{a=1}^{p-1}\prod_{i=1}^{k}\left(1+\left(\frac{a+(i-1)}{p}\right)\right) = \frac{1}{2^k}\sum_{a=1}^{p-1}\prod_{j=1}^{k}\left(1+\left(\frac{a+2(j-1)}{p}\right)\right),$$
$$\frac{1}{2^k}\sum_{a=p-k+1}^{p-1}\prod_{i=1}^{k}\left(1+\left(\frac{a+(i-1)}{p}\right)\right) = \frac{1}{2^k}\sum_{a=p-2k+2}^{p-1}\prod_{j=1}^{k}\left(1+\left(\frac{a+2(j-1)}{p}\right)\right). \tag{59}$$

Combining the above, the proposition is proved. □

## 5. Conclusions

Counting the number of special sequences made of quadratic residues or nonresidues has been of common interests of many researchers for many years. Among those sequences of interests are those made of consecutive quadratic residues or nonresidues, which belong to the family of arithmetic sequences with common difference 1. In this paper, we apply analytic number theory methods, in particular, properties of Legendre's symbol and character sum modulo of a prime number $p$, to enumerate arithmetic sequences of quadratic residues (or nonresidues) with common difference $s > 1$ and of length $k$. The corresponding numbers are denoted as $N_p(k,s)$ and $N'_p(k,s)$.

The main results of this paper are stated in Theorems 8, 10, and 11. Theorem 8 gives exact formulas for $N_p(3,3)$, $N'_p(3,3)$, $N_p(3,4)$, and $N'_p(3,4)$. Theorems 10 and 11 claim that

$$N_p(k,1) = N_p(k,2) = N'_p(k,1) = N'_p(k,2) \quad \text{for } k = 4, 5, \tag{60}$$

and provide an upper bound and a lower bound for each of $N_p(4,2)$ and $N_p(5,2)$.

Further work includes the study of $N_p(k,s)$ and $N'_p(k,s)$ for larger values of $k$ and $s$.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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