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Research Article

Bayesian and E-Bayesian Estimation for the Generalized Rayleigh Distribution under Different Forms of Loss Functions with Real Data Application

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This paper investigates the estimation of an unknown shape parameter of the generalized Rayleigh distribution using Bayesian and expected Bayesian estimation techniques based on type-II censoring data. Subsequently, these estimators are obtained using four different loss functions: the linear exponential loss function, the weighted linear exponential loss function, the compound linear exponential loss function, and the weighted compound linear exponential loss function. The weighted compound linear exponential loss function is a novel suggested loss function generated by combining weights with the compound linear exponential loss function. We use the gamma distribution as a prior distribution. In addition, the expected Bayesian estimator is obtained through three different prior distributions of the hyperparameters. Moreover, depending on the four distinct forms of loss functions, Bayesian and expected Bayesian estimation techniques are performed using Monte Carlo simulations to verify the effectiveness of the suggested loss function and to compare Bayesian and expected Bayesian estimation methods. Furthermore, the simulation results indicate that, depending on the minimum mean squared error, the Bayesian and expected Bayesian estimations corresponding to the weighted compound linear exponential loss function suggested in this paper have significantly better performance compared to other loss functions, and the expected Bayesian estimator also performs better than the Bayesian estimator. Finally, the proposed techniques are demonstrated using a set of real data from the medical field to clarify the applicability of the suggested estimators to real phenomena and to show that the discussed weighted compound linear exponential loss function is efficient and can be applied in a real-life scenario.

1. Introduction

The expected Bayesian (E-Bayesian) estimation is a novel technique for estimating unknown parameters. It presents an expectancy of the Bayesian estimator based on the hyperparameters' distributions as prior distributions and was first proposed by Han [1]. Many researchers have used the E-Bayesian method with different lifetime distributions. For example, Reyad and Othman [2] estimated the parameters of a two-component mixture of the inverse Lomax distribution by E-Bayesian and Bayesian estimation, depending on the squared error loss function (SELF), the linear exponential loss function (LINEXLF), and the entropy loss function (ELF) under type-I censoring data. Okasha [3] proposed the E-Bayesian approach to estimate the parameter of the exponential distribution and the reliability function based on higher-recorded statistical data. Using E-Bayesian estimation, Liu and Yin [4] evaluated the reliability function for the geometric distribution based on the scaled SELF. Okasha [5] also determined the Lomax distribution's parameters and the reliability function while considering the balanced SELF with type-II censored data.

In the study of progressive type-II censored samples, Okasha et al. [6] focused on the Bayesian and E-Bayesian estimation of the Weibull distribution's scale parameter, reliability, and hazard rate functions. Algarni and Almarashi [7] used a similar methodology to estimate the parameters of bathtub-shaped lifespan distributions and determine the reliability functions that depended on type-II censoring samples. Athirakrishnan and Abdul-Sathar [8] introduced E-Bayesian estimation techniques for evaluating the scale parameters for the inverse Rayleigh distribution depending on SELF and precautionary loss functions. Heidari et al. [9] presented Bayesian and E-Bayesian approaches to conclude the Rayleigh distribution's parameter, considering SELF, depending on type-II censored samples.

Recently, Athirakrishnan and Abdul-Sathar [10] estimated the scale parameter of the Chen distribution using E-Bayesian and Bayesian methods under Type II censoring samples. Based on different loss functions (LFs) such as SELF, ELF, and LINEXLF, Wang et al. [11] used E-Bayesian and Bayesian estimation for a simple step-stress model under progressively Type-II censoring. Iqbal and Yousuf Shad [12] used the E-Bayesian technique to estimate the parameters of the inverse gamma distribution. Basheer et al. [13] presented hierarchical and E-Bayesian estimations for the inverse Weibull model and concluded that the E-Bayesian was better than the other compared methods. Nassar et al. [14] used the E-Bayesian and Bayesian estimation approaches to estimate the generalized inverted exponential distribution parameters under type-II censored data. Mohie El-din et al. [15] estimated the parameters of the Gompertz distribution under different types of LFs using E-Bayesian estimation. Algarni et al. [16] estimated the scale parameter of the Chen distribution via an E-Bayesian approach based on a type-I censoring scheme. Prabhu [17] used E-Bayesian techniques to assess the shape parameter of the Lomax distribution under different types of LFs. Liu and Zhang [18] estimated the parameter of the Lomax distribution using E-Bayesian estimation under generalized Type-I hybrid censoring and agreed that E-Bayesian estimation is the best and most effective method.

In past years, Hosseini et al. [19] estimated the scale parameter of a two-parameter exponential distribution using Bayesian and Bayesian shrinkage estimations under the SELF and Al-Bayyati LFs based on right-censored data. Amirzadi et al. [20] used maximum likelihood and Bayesian estimations to evaluate the shape parameter of the inverse generalized Weibull distribution and reliability function based on several LFs such as the general entropy, squared log error, weight squared error, and a new loss function (LF). Moreover, Hosseini et al. [21] used a novel lifetime distribution called the Exponential-Weibull logarithmic transformation, and they estimated its parameter through maximum likelihood, Bayesian, and Bayesian shrinkage estimations that depended on the right censored scheme.

More recently, a three-parameter bounded beta distribution was a new model introduced by Althubyani et al. [22] and MLE and Bayesian estimation under LINEXLF and squared error were used to estimate the distribution's unknown parameters. Atchadé et al. [23] estimated the unknown parameters of the new power Topp–Leone generated distribution using MLE. Rasekhi et al. [24] presented a simple method that performs better than MLE, using the approximation of the likelihood equations, to estimate the scale and location parameters of a generalized Gudermannian distribution. Alghamdi et al. [25] presented a new distribution called the half-logistic modified Kies exponential distribution and estimated its parameters using eight distinct estimation methods to determine the most accurate methodology by comparing the results. Almuqrin et al. [26] presented the extended reduced Kies distribution and estimated its parameters employing eight different schemes.

The generalized Rayleigh (GR) distribution is a continuous distribution presented by Surles and Padgett [27, 28]. Several authors have used various estimation techniques to estimate the parameters and the reliability functions of the GR distribution, as detailed in [29–34].

The GR's probability density function (pdf) has the following form:

$$f(x;\vartheta,\nu) = 2\vartheta\nu^2 x e^{-(\nu x)^2}\left(1-e^{-(\nu x)^2}\right)^{\vartheta-1},\ x>0;\quad \vartheta,\nu>0. \tag{1}$$

The cumulative distribution function (CDF), survival function (SF), and hazard (failure) rate function (HARF) are obtained as follows:

$$\begin{aligned} F(x;\vartheta,\nu) &= \left(1-e^{-(\nu x)^2}\right)^{\vartheta},\ x>0;\quad \vartheta,\nu>0,\\ R(x) &= 1-\left(1-e^{-(\nu x)^2}\right)^{\vartheta},\\ h(x;\vartheta,\nu) &= \frac{2\vartheta\nu^2 x e^{-(\nu x)^2}\left(1-e^{-(\nu x)^2}\right)^{\vartheta-1}}{1-\left(1-e^{-(\nu x)^2}\right)^{\vartheta}},\ x>0;\quad \vartheta,\nu>0, \end{aligned} \tag{2}$$

where ϑ, ν, and x are the shape parameter, the scale parameter, and the random variable, respectively.

The GR distribution is a significant continuous life distribution widely used to analyze skewed data, conduct survival analysis, and construct lifetime models in various fields, including medical, industrial, and life sciences. In this paper, we used actual data on bladder cancer to determine a suitable statistical model for this data. It is significant to analyze all distributions that are suitable for use as lifetime distributions. After applying a precise identity to this distribution, we found that the GR distribution fits these data well, as medical studies often exhibit a right-skewed distribution. Positively skewed distributions, such as the GR distribution, play a crucial role in decision-making when using Bayesian estimators, especially in medicine and public health. For treatment diagnosis, doctors can use Bayesian estimation to study the results of patients with specified diseases by considering prior information.

Figure 1 illustrates various representative plots of $f(x)$, $F(x)$, $R(x) = 1 - F(x)$, and $h(x)$ for the GR distribution with $\nu = 1$. The shape of both curves for the GR functions, the failure rate (FR) function, and the survival function (SF) depends on the value of the shape parameter ϑ. The FR increases and concaves up when $\vartheta > 1$, while when $\vartheta \le 1$, the FR reduces to a straight line. Due to the flexibility of the FR, the GR distribution has been suggested in statistical

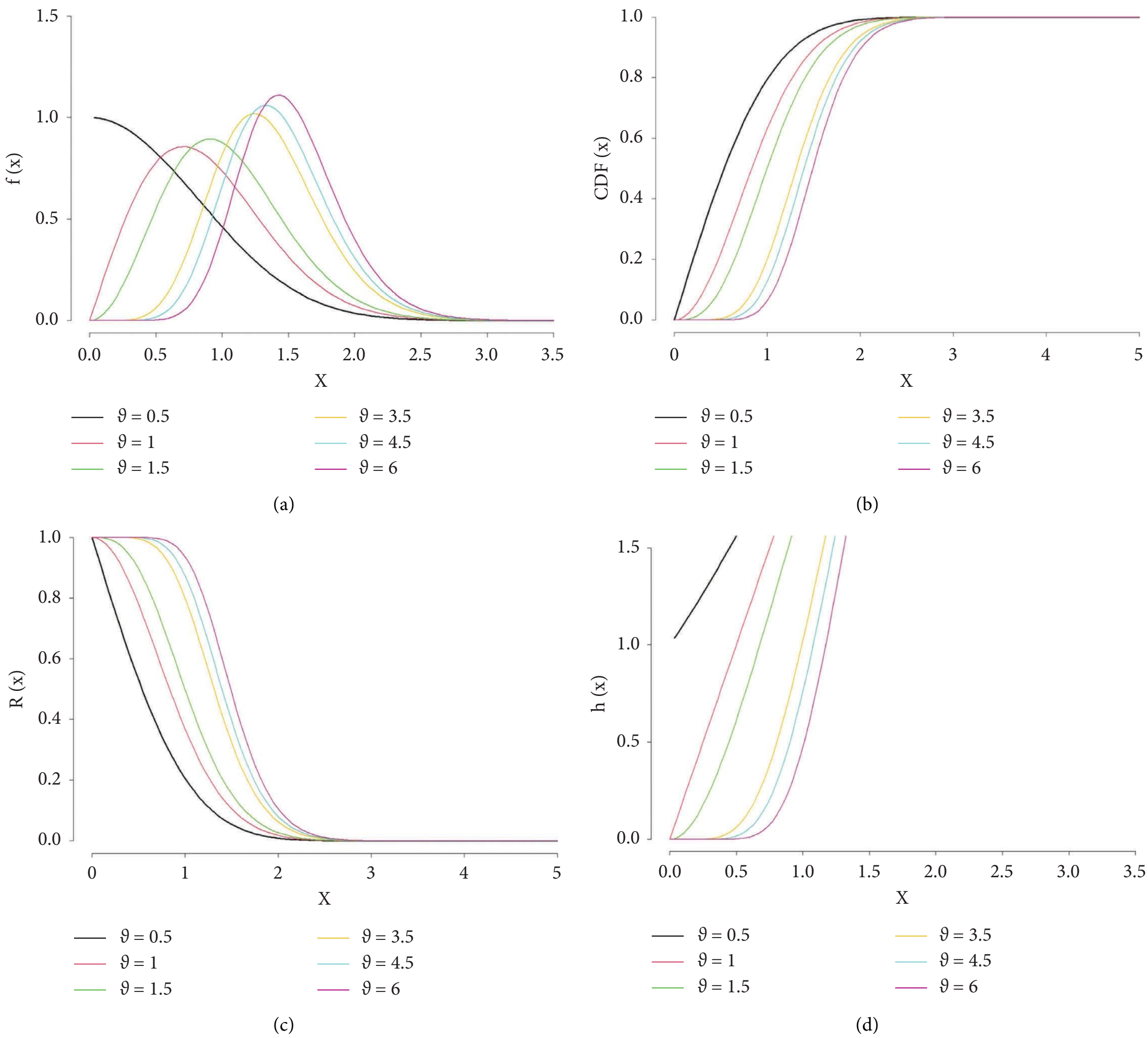


Figure 1: Plots of pdf, CDF, SF, and HARF for GR distribution for different values of ϑ for given $\nu = 1$.

literature for analyzing real-world applications in medical reliability inference. On the other hand, the SF decreases and concaves up when $\vartheta < 1$, and when $\vartheta \geq 1$, the SF declines and concaves down.

We estimate the shape parameter and set the scale to one because the shape parameter of the asymmetric probability distribution is often more difficult to assess and more accurate than the location and scale parameters. This paper suggests an intuitive but innovative estimation method for this parameter. Due to the changing value of the shape parameter, the GR distribution enables the modeling of various physical and medical applications, as shown in Figure 1.

The purpose of this research paper is to estimate an unknown shape parameter of the GR distribution under type-II censoring data through Bayesian and E-Bayesian estimation, depending on a category of informative prior (IP) based on the hypotheses of four LFs, presented throughout the paper. We aim to determine which one of the LFs is the best in terms of the estimation approach. We used an innovative novel LF called the weighted compound linear exponential loss function, which produced impressive results in the estimation compared to other LF techniques. This research is unique because no attempts have been made to evaluate all these E-Bayesian and Bayesian estimation techniques for the shape parameter of the GR distribution using type-II censoring data.

The remainder of this paper is organized as follows. Section 2 presents the GR's derivation using various LFs. Section 3 discusses the Bayesian estimation for GR's unknown shape parameter. The E-Bayesian method has been used in Section 4 to estimate the unknown parameter. In Section 5, the Monte Carlo simulation is described, and the results of the simulations are discussed. In Section 6, we used real data on bladder cancer to demonstrate the proposed inference. Finally, in Section 7, the study's conclusions are presented.

2. Loss Function (LF)

The LF is essential to Bayesian analysis and decision theory because of its crucial use in describing underestimation and overestimation in analysis. Square and LINEX are the two most used LFs. The LINEXLF gives different weights to underestimation and overestimation, and it is a symmetrical generalization of the square loss function. In contrast, the square loss function treats underestimation and overestimation equally. The LINEXLF is more practical and advantageous in real-world applications than the square loss. For instance, overestimating the reliability function or average failure time in reliability and survival analysis is typically more serious than underestimating the reliability function or mean failure time. In comparison, underestimating the failure rate has more negative consequences than overestimating. Moreover, numerous other authors have discovered that the frequently used square loss function may not be appropriate in real applications. These authors include Zellner [35], Huang and Chang [36], and Matin and Khatun [37]. As a result, in this paper, we used four types of LINEXLF and got the minimum MSE under the suggested WCLINEXLF according to E-Bayesian and Bayesian estimations.

This section spotlights four main categories of LF: LINEXLF, CLINEXLF, WLINXLF, and WCLINEXLF.

2.1. Linear Exponential Loss Function (LINEXLF). The LINEXLF for the parameter ψ is evaluated by the following [38, 39]:

$$L(\hat{\vartheta},\vartheta) \propto [\exp(\psi(\hat{\vartheta}-\vartheta)) - \psi(\hat{\vartheta}-\vartheta) - 1]; \psi \neq 0, \tag{3}$$

where $\hat{\vartheta}$ is an estimator for ϑ. The Bayesian estimator, depending on LINEXLF, is defined as follows:

$$\hat{\vartheta}_{BL} = \frac{-1}{\psi}\ln\left(E_{\vartheta}(\exp(-\psi\vartheta) \mid x)\right), \tag{4}$$

where E_{ϑ} is the expectation of posterior and $E_{\vartheta}(\exp(-\psi\vartheta))$ is finite and exists.

2.2. Weighted Linear Exponential Loss Function (WLINEXLF). Al-Duais [40, 41] and Al-Duais and Hmood [42] presented this loss function.

$$\begin{aligned} L_w(\hat{\vartheta}-\vartheta) &= w(\vartheta)[\exp(\psi(\hat{\vartheta}-\vartheta)) - \psi(\hat{\vartheta}-\vartheta) - 1], \\ w(\vartheta) &= \exp(-w\vartheta), \end{aligned} \tag{5}$$

where $w(\vartheta)$ is denoted as the proposed weighted function.

The value of $\hat{\vartheta}$ is the Bayesian estimator of ϑ under WLINEXLF, which minimizes equation (5) and is defined as follows:

$$\hat{\vartheta}_{WBL} = \frac{1}{\psi}\ln\left(\frac{E_{\vartheta}(\exp(-w\vartheta) \mid x)}{E_{\vartheta}(\exp(-\vartheta(w+\psi)) \mid x)}\right), \tag{6}$$

where $E_{\vartheta}(\exp(-w\vartheta))$ and $E_{\vartheta}(\exp(-\vartheta(w+\psi)))$ are finite and exist.

2.3. Composite Linear Exponential Loss Function (CLINEXLF). The CLINEXLF is formed as follows [43, 44]:

$$\begin{aligned} L(\hat{\vartheta},\vartheta) &= L_{\psi}(\hat{\vartheta},\vartheta) + L_{-\psi}(\hat{\vartheta},\vartheta) \\ &= [\exp(-\psi(\hat{\vartheta}-\vartheta)) + \exp(\psi(\hat{\vartheta}-\vartheta)) - 2]; \ \psi > 0. \end{aligned} \tag{7}$$

The Bayesian estimator of ϑ, depending on CLINEXLF, is obtained as follows:

$$\hat{\vartheta}_{CBL} = \frac{1}{2\psi}\ln\left(\frac{E_{\vartheta}(\exp(\psi\vartheta) \mid x)}{E_{\vartheta}(\exp(-\psi\vartheta) \mid x)}\right). \tag{8}$$

2.4. Weighted Composite Linear Exponential Loss Function (WCLINEXLF). Al-Bossly [45] suggested WCLINEXLF, which is based on the weighting for CLINEXLF. The form of WCLINEXLF is given by the following equation:

$$\begin{aligned} L_w(\hat{\vartheta},\vartheta) &= w(\vartheta)L(\hat{\vartheta},\vartheta) = w(\vartheta)L_{\psi}(\hat{\vartheta},\vartheta) = w(\vartheta)L_{-\psi}(\hat{\vartheta},\vartheta) \\ &= [w(\vartheta)\exp(-\psi(\hat{\vartheta},\vartheta)) + w(\vartheta)\exp(\psi(\hat{\vartheta},\vartheta)) \\ &\quad - 2]; \ \psi > 0, \end{aligned} \tag{9}$$

where $w(\vartheta) = \exp(-w\vartheta)$ is denoted as the proposed weighted function.

The Bayesian estimator of ϑ under WCLINEXLF is obtained as follows:

$$\hat{\vartheta}_{WCBL} = \frac{1}{2\psi}\ln\left(\frac{E_{\vartheta}(\exp(-\vartheta(w-\psi)) \mid x)}{E_{\vartheta}(\exp(-\vartheta(w+\psi)) \mid x)}\right). \tag{10}$$

Note: when $w = 0$ in equation (10), the CLINEXLF will be the special case of WCLINEXLF. It implies that the WCLINEXLF will represent the general form of the CLINEXLF.

3. Bayesian Estimation

In this section, we obtained the Bayesian estimation for the shape parameter ϑ by considering different types of LFs, including LINEXLF, WLINEXLF, CLINEXLF, and WCLINEXLF. The likelihood function under the type-II censoring sample of size s obtained from n items of a life test from the GR is expressed as follows:

$$\begin{aligned} L(\vartheta,\nu\mid x) &= \frac{n!}{(n-s)!}\times\prod_{i=1}^{s}2\vartheta\nu^2x_i\exp\left(-(\nu x_i)^2\right)\left(1-\exp\left(-(\nu x_i)^2\right)\right)^{\vartheta-1}\times\left(1-\left(1-\exp\left(-(\nu x_s)^2\right)\right)^{\vartheta}\right)^{n-s}\\ &= \frac{n!}{(n-s)!}2^s\vartheta^s\nu^{2s}\left(\prod_{i=1}^{s}x_i\frac{\exp\left(-(\nu x_i)^2\right)}{1-\exp\left(-(\nu x_i)^2\right)}\right)\exp(-\vartheta\varepsilon+\xi), \end{aligned} \tag{11}$$

where $\varepsilon = -[\sum_{i=1}^{s}\ln(1-\exp(-(\nu x_i)^2)) - \sum_{k=1}^{\infty}\ln(1-\exp(-(\nu x_s)^2))^k]$;

$$\xi = \sum_{k=1}^{\infty}\ln\left(\frac{n-s}{k}\right). \tag{12}$$

More illustrations for equation (11) were placed in Appendix A.

The gamma distribution (φ,ϕ) used as a prior distribution of ϑ and its pdf is given by

$$u(\vartheta\mid\varphi,\phi) = \frac{\phi^{\varphi}}{\Gamma(\varphi)}\vartheta^{\varphi-1}\exp(-\phi\vartheta);\ \varphi,\phi>0. \tag{13}$$

By substituting the likelihood function mentioned in equation (11) and the prior distribution for ϑ mentioned in equation (13), the posterior distribution for ϑ is given by

$$\begin{aligned} \pi(\vartheta\mid x) &= \frac{L(\vartheta,\nu\mid x)u(\vartheta)}{\int_0^{\infty}L(\vartheta,\nu\mid x)u(\vartheta)d\vartheta}\\ &= \frac{\vartheta^{\varphi+s-1}\exp(-\vartheta(\varepsilon+\phi))(\phi+\varepsilon)^{\varphi+s}}{\Gamma(\varphi+s)};\ \vartheta>0. \end{aligned} \tag{14}$$

3.1. Bayesian Estimation for ϑ Depending on LINEXLF. The Bayesian estimator of ϑ based on LINEXLF, which is symbolized by $\widehat{\vartheta}_{BL}$, can be expressed as follows:

$$\widehat{\vartheta}_{BL} = \frac{\varphi+s}{\psi}\ln\left(1+\frac{\psi}{\phi+\varepsilon}\right). \tag{15}$$

We have included additional illustrations for equation (15) in Appendix B.

3.2. Bayesian Estimation for ϑ Depending on CLINEXLF. The Bayesian estimator of ϑ based on CLINEXLF, which is symbolized by $\widehat{\vartheta}_{CBL}$, can be shown as follows:

$$\widehat{\vartheta}_{CBL} = \frac{-(\varphi+s)}{2\psi}\ln\left(\frac{1-\psi/\varepsilon+\phi}{1+\psi/\varepsilon+\phi}\right). \tag{16}$$

More illustrations for equation (16) were placed in Appendix C.

3.3. Bayesian Estimation for ϑ Depending on WLINEXLF. The Bayesian estimator of ϑ based on WLINEXLF, which is symbolized by $\widehat{\vartheta}_{WBL}$, can be obtained as follows:

$$\widehat{\vartheta}_{WBL} = \frac{-(\varphi+s)}{\psi}\ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right). \tag{17}$$

We have included additional illustrations for equation (17) in Appendix D.

3.4. Bayesian Estimation for ϑ Depending on WCLINEXLF. The Bayesian estimator of ϑ based on WCLINEXLF, which is symbolized by $\widehat{\vartheta}_{WCBL}$, can be given as follows:

$$\widehat{\vartheta}_{WCBL} = \frac{-(\varphi+s)}{2\psi}\ln\left(\frac{1+w-\psi/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right). \tag{18}$$

More illustrations for equation (18) were placed in Appendix E.

4. E-Bayesian Estimation

Throughout this section, we evaluated the E-Bayesian estimation for the shape parameter ϑ by considering different types of LFs, including LINEXLF, WLINEXLF, CLINEXLF, and WCLINEXLF. As stated by [46], the prior parameters φ and ϕ should be chosen to ensure that the former $u(\vartheta\mid\varphi,\phi)$ in equation (13) decreases as a function of ϑ. The derivative of $u(\vartheta\mid\varphi,\phi)$ according to ϑ is expressed as follows:

$$\frac{\partial u(\vartheta\mid\varphi,\phi)}{\partial\vartheta} = \frac{\vartheta^{\varphi-2}\exp(-\vartheta\phi)(\phi)^{\varphi}}{\Gamma(\varphi)}[(\varphi-1)-\phi\vartheta];\ \phi>0, 0<\varphi<1. \tag{19}$$

Therefore, for $\phi>0, \varphi>0,$ and $\vartheta>0$ in equation (13), it follows $\phi>0$ and $0<\varphi<1$ due to $\partial u(\vartheta\mid\varphi,\phi)/\partial\vartheta<0$, and the prior $u(\vartheta\mid\varphi,\phi)$ decreases as a function of ϑ. Let the hyperparameters φ and ϕ be independent random variables with pdfs $\pi_1(\varphi)$ and $\pi_2(\phi)$. The joint bivariate probability density function of φ and ϕ is given by

$$\pi(\varphi,\phi) = \pi_1(\varphi)\pi_2(\phi). \tag{20}$$

Thus, the expectation of Bayesian estimation of φ is obtained as follows:

$$\widehat{\vartheta}_{EB} = E(\widehat{\vartheta} \mid x) = \int\int_{\forall\Omega} \widehat{\vartheta}_B(\varphi,\phi)\pi(\varphi,\phi)d\varphi\, d\phi. \tag{21}$$

Such Ω refers to the set of all values of φ and ϕ. $\widehat{\vartheta}_B(\varphi,\phi)$ is the Bayesian estimation for ϑ by equations (15)–(18). Therefore, ϕ should be smaller than an upper bound G, where $G > 0$ is a constant. Accordingly, hyperparameters φ and ϕ should be selected with the restriction of $0 < \varphi < 1$ and $0 < \phi < G$. The E-Bayesian estimation for ϑ is determined using three distributions for hyperparameters φ and ϕ. These distributions are used to study the impact of the various prior distributions on the E-Bayesian estimation for ϑ. The distributions for φ and ϕ are written as follows:

$$\pi_1(\varphi,\phi) = \frac{2(G-\phi)}{G^2};\ 0 < \varphi < 1, 0 < \phi < G, \tag{22}$$

$$\pi_2(\varphi,\phi) = \frac{1}{G};\ 0 < \varphi < 1, 0 < \phi < G, \tag{23}$$

$$\pi_3(\varphi,\phi) = \frac{2\phi}{G^2};\ 0 < \varphi < 1, 0 < \phi < G. \tag{24}$$

4.1. E-Bayesian Estimation for ϑ under LINEXLF. The E-Bayesian estimator for ϑ based on LINEXLF with $\pi_1(\varphi,\phi)$ can be obtained by solving equations (15), (21), and (22) as follows:

$$\begin{aligned}
\widehat{\vartheta}_{EBL1} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{BL}\, \pi_1(\varphi,\phi)d\varphi\, d\phi \\
&= \int_0^1\int_0^G \left(\frac{2(G-\phi)}{G^2}\right) \frac{\varphi+s}{\psi} \ln\left(1+\frac{\psi}{\phi+\varepsilon}\right) d\phi\, d\varphi \\
&= \frac{2}{\psi G^2}\int_0^G (G-\phi)\ln\left(1+\frac{\psi}{\phi+\varepsilon}\right)\int_0^1 (\varphi+s)d\varphi\, d\phi \\
&= \frac{2s+1}{\psi G^2}\left[G\int_0^G \ln\left(1+\frac{\psi}{\phi+\varepsilon}\right)d\phi - \int_0^G \phi\ln\left(1+\frac{\psi}{\phi+\varepsilon}\right)d\phi\right] \\
&= \frac{2s+1}{\psi G^2}\left[GI_7 - I_8\right],
\end{aligned} \tag{25}$$

such that

$$\begin{aligned}
I_7 &= \int_0^G \ln\left(1+\frac{\psi}{\phi+\varepsilon}\right)d\phi \\
&= \{(G+\varepsilon+\psi)\ln(G+\varepsilon+\psi) - (\varepsilon+\psi)\ln(\varepsilon+\psi) - (G+\varepsilon)\ln(G+\varepsilon) + (\varepsilon)\ln(\varepsilon)\}.
\end{aligned} \tag{26}$$

Also, we have

$$\begin{aligned}
I_8 &= \int_0^G \phi\ln\left(1+\frac{\psi}{\phi+\varepsilon}\right)d\phi \\
&= \left\{\left(\frac{1}{2}\right)\left[\left(\varepsilon^2 - G^2\right)\ln(G+\varepsilon) - \left(\varepsilon^2+G^2+\psi^2+2\psi\varepsilon\right)\ln(\varepsilon+\psi+G) + \left(\varepsilon^2+\psi^2+2\psi\varepsilon\right)\ln(\psi+\varepsilon) - \left(\varepsilon^2\right)\ln(\varepsilon) + G\psi\right]\right\}.
\end{aligned} \tag{27}$$

Therefore, the E-Bayesian estimation for ϑ is given by

$$\widehat{\vartheta}_{EBL1} = \frac{2s+1}{\psi G^2}\left[G I_7 - I_8\right]. \tag{28}$$

Furthermore, we drive the E-Bayesian estimation for ϑ according to LINEXLF under $\pi_2(\varphi,\phi)$ and $\pi_3(\varphi,\phi)$ by solving equations (15), (23), and (24). It can be written as follows:

$$\begin{aligned}
\widehat{\vartheta}_{EBL2} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{BL}\pi_2(\varphi,\phi) d\varphi\, d\phi \\
&= \int_0^1\int_0^G \left(\frac{1}{G}\right) \frac{\varphi+s}{\psi}\ln\left(1+\frac{\psi}{\phi+\varepsilon}\right) d\phi\, d\varphi \\
&= \frac{2s+1}{2\psi G}\left[\int_0^G \ln\left(1+\frac{\psi}{\phi+\varepsilon}\right) d\phi\right] \\
&= \frac{2s+1}{2\psi G}\left[I_7\right] \\
&= \frac{2s+1}{2\psi G}\left[(G+\varepsilon+\psi)\ln(G+\varepsilon+\psi) - (\varepsilon+\psi)\ln(\varepsilon+\psi) - (G+\varepsilon)\ln(G+\varepsilon) + (\varepsilon)\ln(\varepsilon)\right].
\end{aligned} \tag{29}$$

Also, we have

$$\begin{aligned}
\widehat{\vartheta}_{EBL3} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{BL}\pi_3(\varphi,\phi) d\varphi\, d\phi \\
&= \int_0^1\int_0^G \left(\frac{2\phi}{G^2}\right) \frac{\varphi+s}{\psi}\ln\left(1+\frac{\psi}{\phi+\varepsilon}\right) d\phi\, d\varphi \\
&= \frac{2s+1}{\psi G^2}\left[\int_0^G \phi\ln\left(1+\frac{\psi}{\phi+\varepsilon}\right) d\phi\right] \\
&= \frac{2s+1}{\psi G^2}\left[I_8\right] \\
&= \frac{2s+1}{\psi G^2}\left[\left(\frac{1}{2}\right)\left[\left(\varepsilon^2 - G^2\right)\ln(G+\varepsilon) - \left(\varepsilon^2 + G^2 + \psi^2 + 2\psi\varepsilon\right)\ln(\varepsilon+\psi+G) + \left(\varepsilon^2+\psi^2+2\psi\varepsilon\right)\ln(\psi+\varepsilon) - \left(\varepsilon^2\right)\ln(\varepsilon) + G\psi\right]\right].
\end{aligned} \tag{30}$$

4.2. E-Bayesian Estimation for ϑ under WLINEXLF. The E-Bayesian estimator for ϑ based on WLINEXLF with $\pi_1(\varphi,\phi)$ can be obtained by solving equations (17), (21), and (22) as follows:

$$\begin{aligned}
\widehat{\vartheta}_{EWBL1} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{WBL} \pi_1(\varphi,\phi) d\varphi\, d\phi \\
&= \int_0^1 \int_0^G \left(\frac{2(G-\phi)}{G^2}\right)\left(\frac{-(\varphi+s)}{\psi}\right) \ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right) d\phi\, d\varphi \\
&= \frac{-2}{\psi G^2} \int_0^G (G-\phi) \ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right) \int_0^1 (\varphi+s) d\varphi\, d\phi \\
&= \frac{-(2s+1)}{\psi G^2}\left[G\int_0^G \ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right) d\phi - \int_0^G \phi \ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right) d\phi\right] \\
&= \frac{-(2s+1)}{\psi G^2}\left[G I_9 - I_{10}\right],
\end{aligned} \tag{31}$$

such that

$$\begin{aligned}
I_9 &= \int_0^G \ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right) d\phi \\
&= \{(G+\varepsilon+w)\ln(G+\varepsilon+w) - (G+\varepsilon+\psi+w)\ln(G+\varepsilon+\psi+w) - (w+\varepsilon)\ln(w+\varepsilon) + (\varepsilon+w+\psi)\ln(\varepsilon+w+\psi)\}.
\end{aligned} \tag{32}$$

Also, we have

$$\begin{aligned}
I_{10} &= \int_0^G \phi \ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right) d\phi \\
&= \int_0^G \phi \ln(\varepsilon+\phi+w) d\phi - \int_0^G \phi \ln(\varepsilon+\phi+w+\psi) d\phi \\
&= A_1 - A_2,
\end{aligned} \tag{33}$$

where A_1 is expressed as follows:

$$\begin{aligned}
A_1 &= \int_0^G \phi \ln(\varepsilon+\phi+w) d\phi \\
&= \left\{\left(G^2 - (-\varepsilon-w)^2\right)\left(\frac{\ln(G+w+\varepsilon)}{2}\right) + (-\varepsilon-w)^2\left(\frac{\ln(w+\varepsilon)}{2}\right) - \frac{\left(G^2+2G(\varepsilon+w)\right)}{4} - (G-w-\varepsilon)\right\}.
\end{aligned} \tag{34}$$

Also, A_2 is expressed as follows:

$$A_2 = \int_0^G \phi \ln(\varepsilon + \phi + w + \psi)d\phi$$

$$= \left\{ \left(G^2 + (w + \varepsilon + \psi)^2\right)\left(\frac{\ln(G + w + \varepsilon + \psi)}{2}\right) + (w + \varepsilon + \psi)^2\left(\frac{\ln(w + \varepsilon + \psi)}{2}\right) - \frac{\left(G^2 + 2G(\varepsilon + w + \psi)\right)}{4} + G(w + \varepsilon + \psi) \right\}. \tag{35}$$

Thus, the E-Bayesian estimation for ϑ is given by

$$\widehat{\vartheta}_{EWBL1} = \frac{-(2s+1)}{\psi G^2}\left[GI_9 - I_{10}\right]. \tag{36}$$

Furthermore, we drive the E-Bayesian estimation for ϑ according to WLINEXLF under $\pi_2(\varphi, \phi)$ and $\pi_3(\varphi, \phi)$ by solving equations (17), (23), and (24). It can be written as follows:

$$\begin{aligned}
\widehat{\vartheta}_{EWBL2} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{WBL}\pi_2(\varphi, \phi)d\varphi\, d\phi \\
&= \int_0^1\int_0^G \left(\frac{1}{G}\right)\left(\frac{-(\varphi + s)}{\psi}\right)\ln\left(\frac{1 + w/\varepsilon + \phi}{1 + w + \psi/\varepsilon + \phi}\right)d\phi\, d\varphi \\
&= \frac{-(2s+1)}{2\psi G}\left[\int_0^G \ln\left(\frac{1 + w/\varepsilon + \phi}{1 + w + \psi/\varepsilon + \phi}\right)d\phi\right] \\
&= \frac{-(2s+1)}{2\psi G}\left[I_9\right] \\
&= \frac{2s+1}{2\psi G}\{(G + \varepsilon + w)\ln(G + \varepsilon + w) - (G + \varepsilon + \psi + w)\ln(G + \varepsilon + \psi + w) - (w + \varepsilon)\ln(w + \varepsilon) \\
&\quad + (\varepsilon + w + \psi)\ln(\varepsilon + w + \psi)\}.
\end{aligned} \tag{37}$$

Also, we have

$$\begin{aligned}
\widehat{\vartheta}_{EWBL3} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{WBL} \pi_3(\varphi,\phi) d\varphi\, d\phi \\
&= \int_0^1 \int_0^G \left(\frac{2\phi}{G^2}\right)\left(\frac{-(\varphi+s)}{\psi}\right) \ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right) d\phi\, d\varphi \\
&= \frac{-(2s+1)}{\psi G^2}\left[\int_0^G \phi \ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right) d\phi\right] \\
&= \frac{-(2s+1)}{\psi G^2}\left[I_{10}\right].
\end{aligned} \tag{38}$$

4.3. E-Bayesian Estimation for ϑ under CLINEXLF. The E-Bayesian estimator for ϑ based on CLINEXLF with $\pi_1(\varphi,\phi)$ is obtained by solving equations (16), (21), and (22) as follows:

$$\begin{aligned}
\widehat{\vartheta}_{ECBL1} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{CBL}\, \pi_1(\varphi,\phi) d\varphi\, d\phi \\
&= \int_0^1 \int_0^G \left(\frac{2(G-\phi)}{G^2}\right) \frac{-(\varphi+s)}{2\psi} \ln\left(\frac{1-\psi/\varepsilon+\phi}{1+\psi/\varepsilon+\phi}\right) d\phi\, d\varphi \\
&= \frac{-(2s+1)}{\psi G^2}\left[G\int_0^G \ln\left(\frac{1-\psi/\varepsilon+\phi}{1+\psi/\varepsilon+\phi}\right) d\phi - \int_0^G \phi \ln\left(\frac{1-\psi/\varepsilon+\phi}{1+\psi/\varepsilon+\phi}\right) d\phi\right] \\
&= \frac{-(2s+1)}{\psi G^2}\left[G I_{11} - I_{12}\right],
\end{aligned} \tag{39}$$

such that

$$\begin{aligned}
I_{11} &= \int_0^G \ln\left(\frac{1-\psi/\varepsilon+\phi}{1+\psi/\varepsilon+\phi}\right) d\phi \\
&= \left\{(G+\varepsilon-\psi)\ln(G+\varepsilon-\psi) - (G+\varepsilon+\psi)\ln(G+\varepsilon+\psi)\right\}.
\end{aligned} \tag{40}$$

Also, we have

$$\begin{aligned}
I_{12} &= \int_0^G \phi \ln\left(\frac{1-\psi/\varepsilon+\phi}{1+\psi/\varepsilon+\phi}\right) d\phi \\
&= \int_0^G \phi \ln(\varepsilon+\phi-\psi) d\phi - \int_0^G \phi \ln(\varepsilon+\phi+\psi) d\phi \\
&= A_3 - A_4,
\end{aligned} \tag{41}$$

where A_3 is expressed as follows:

$$\begin{aligned}
A_3 &= \int_0^G \phi \ln(\varepsilon+\phi-\psi) d\phi \\
&= \left\{\left(G^2\right)\left(\frac{\ln(G-\psi+\varepsilon)}{2}\right) + \left(\frac{(\psi-\varepsilon)^2}{2}\right)\left[\ln(G-\psi+\varepsilon) - \ln(\varepsilon-\psi)\right] - \frac{\left(G^2+2G(\varepsilon-\psi)\right)}{4} - G(\psi-\varepsilon)\right\}.
\end{aligned} \tag{42}$$

Also, A_4 is expressed as follows:

$$\begin{aligned} A_4 &= \int_0^G \phi \ln(\varepsilon + \phi + \psi) d\phi \\ &= \left\{ \left(G^2\right)\left(\frac{\ln(G + \varepsilon + \psi)}{2}\right) - \frac{(\varepsilon + \psi)^2}{2}\left[\ln(G + \varepsilon + \psi) - \ln(\varepsilon + \psi)\right] - \frac{\left(G^2 + 2G(\varepsilon + \psi)\right)}{4} + G(\varepsilon + \psi)^2 \right\}. \end{aligned} \tag{43}$$

Therefore, the E-Bayesian estimation for ϑ is given by

$$\widehat{\vartheta}_{ECBL1} = \frac{-(2s+1)}{\psi G^2}\left[G I_{11} - I_{12}\right]. \tag{44}$$

Furthermore, we drive the E-Bayesian estimation for ϑ according to CLINEXLF under $\pi_2(\varphi, \phi)$ and $\pi_3(\varphi, \phi)$ by solving equations (16), (23), and (24). It can be written as follows:

$$\begin{aligned} \widehat{\vartheta}_{ECBL2} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{CBL} \pi_2(\varphi, \phi) d\varphi\, d\phi \\ &= \int_0^1 \int_0^G \left(\frac{1}{G}\right) \frac{-(\varphi + s)}{2\psi} \ln\left(\frac{1 - \psi/\varepsilon + \phi}{1 + \psi/\varepsilon + \phi}\right) d\phi\, d\varphi \\ &= \frac{-(2s+1)}{4\psi G}\left[\int_0^G \ln\left(\frac{1 - \psi/\varepsilon + \phi}{1 + \psi/\varepsilon + \phi}\right) d\phi\right] \\ &= \frac{-(2s+1)}{4\psi G}\left[I_{11}\right] \\ &= \frac{-(2s+1)}{4\psi G}\{(G + \varepsilon - \psi)\ln(G + \varepsilon - \psi) \\ &\quad -(G + \varepsilon + \psi)\ln(G + \varepsilon + \psi)\}. \end{aligned} \tag{45}$$

Also, we have

$$\begin{aligned} \widehat{\vartheta}_{ECBL3} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{CBL} \pi_3(\varphi, \phi) d\varphi\, d\phi \\ &= \int_0^1 \int_0^G \left(\frac{2\phi}{G^2}\right) \frac{-(\varphi + s)}{2\psi} \ln\left(\frac{1 - \psi/\varepsilon + \phi}{1 + \psi/\varepsilon + \phi}\right) d\phi\, d\varphi \\ &= \frac{-(2s+1)}{2\psi G^2}\left[\int_0^G \phi \ln\left(\frac{1 - \psi/\varepsilon + \phi}{1 + \psi/\varepsilon + \phi}\right) d\phi\right] \\ &= \frac{-(2s+1)}{2\psi G^2}\left[I_{12}\right]. \end{aligned} \tag{46}$$

4.4. E-Bayesian Estimation for ϑ under WCLINEXLF.

The E-Bayesian estimator for ϑ based on WCLINEXLF with $\pi_1(\varphi, \phi)$ is obtained by solving equations (18), (21), and (22) as follows:

$$\begin{aligned} \widehat{\vartheta}_{EWCBL1} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{WCBL}\, \pi_1(\varphi, \phi) d\varphi\, d\phi \\ &= \int_0^1 \int_0^G \left(\frac{2(G - \phi)}{G^2}\right) \frac{-(\varphi + s)}{2\psi} \ln\left(\frac{(1 + w - \psi/\varepsilon + \phi)}{1 + w + \psi/\varepsilon + \phi}\right) \phi d\varphi \\ &= \frac{-(2s+1)}{\psi G^2}\left[G \int_0^G \ln\left(\frac{(1 + w - \psi/\varepsilon + \phi)}{1 + w + \psi/\varepsilon + \phi}\right) d\phi - \int_0^G \phi \ln\left(\frac{(1 + w - \psi/\varepsilon + \phi)}{1 + w + \psi/\varepsilon + \phi}\right) d\phi\right] \\ &= \frac{-(2s+1)}{2\psi G^2}\left[G I_{13} - I_{14}\right]; \end{aligned} \tag{47}$$

Such that

$$I_{13}=\int_0^G \ln\left(\frac{(1+w-\psi/\varepsilon+\phi)}{1+w+\psi/\varepsilon+\phi}\right)d\phi$$
$$=\{(G)[\ln(\phi+\varepsilon+w-\psi)-\ln(\phi+\varepsilon+w+\psi)]-(w+\varepsilon+\psi)[\ln(G+\varepsilon+w+\psi)-\ln(w+\varepsilon+\psi)]$$
$$+(w+\varepsilon-\psi)[\ln(G+\varepsilon+w-\psi)-\ln(w+\varepsilon-\psi)]\}. \tag{48}$$

Also, we have

$$I_{14}=\int_0^G \phi\ln\left(\frac{(1+w-\psi/\varepsilon+\phi)}{1+w+\psi/\varepsilon+\phi}\right)d\phi$$
$$=\int_0^G \phi\ln(\varepsilon+\phi+w-\psi)d\phi-\int_0^G \phi\ln(\varepsilon+\phi+w+\psi)d\phi$$
$$=A_5-A_6, \tag{49}$$

where A_5 is expressed as follows:

$$A_5=\int_0^G \phi\ln(\varepsilon+\phi+w-\psi)d\phi$$
$$=\left\{\left(G^2\right)\left(\frac{\ln(G-\psi+\varepsilon+w)}{2}\right)-\left(\frac{(\varepsilon+w-\psi)^2}{2}\right)[\ln(G-\psi+\varepsilon+w)-\ln(\varepsilon-\psi+w)]\right.$$
$$\left.-\frac{\left(G^2-2G(\varepsilon-\psi+w)\right)}{4}+G(w-\psi+\varepsilon)\right\}. \tag{50}$$

Also, A_6 is expressed as follows:

$$A_6=\int_0^G \phi\ln(\varepsilon+\phi+w+\psi)d\phi$$
$$=\left\{\left(G^2\right)\left(\frac{\ln(G+\varepsilon+w+\psi)}{2}\right)-\frac{(w+\varepsilon+\psi)^2}{2}[\ln(w+G+\varepsilon+\psi)-\ln(w+\varepsilon+\psi)]\right.$$
$$\left.-\frac{\left(G^2-2G(\varepsilon+w+\psi)\right)}{4}+G(\varepsilon+\psi+w)^2\right\}. \tag{51}$$

Thus, the E-Bayesian estimation for ϑ is given by

$$\widehat{\vartheta}_{EWCBL1}=\frac{-(2s+1)}{2\psi G^2}[GI_{13}-I_{14}]. \tag{52}$$

Furthermore, we drive the E-Bayesian estimation for ϑ according to WCLINEXLF under $\pi_2(\varphi,\phi)$ and $\pi_3(\varphi,\phi)$ by solving equations (18), (23), and (24). It can be written as follows:

$$\begin{aligned}
\widehat{\vartheta}_{EWCBL2} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{WCBL}\pi_2(\varphi,\phi)d\varphi\, d\phi \\
&= \int_0^1\int_0^G \left(\frac{1}{G}\right)\frac{-(\varphi+s)}{2\psi}\ln\left(\frac{(1+w-\psi/\varepsilon+\phi)}{1+w+\psi/\varepsilon+\phi}\right)d\phi\, d\varphi \\
&= \frac{(2s+1)}{4\psi G}\left[\int_0^G \ln\left(\frac{(1+w-\psi/\varepsilon+\phi)}{1+w+\psi/\varepsilon+\phi}\right)d\phi\right] \\
&= \frac{(2s+1)}{4\psi G}\left[I_{13}\right] \\
&= \frac{(2s+1)}{4\psi G}\{(G)[\ln(\phi+\varepsilon+w-\psi)-\ln(\phi+\varepsilon+w+\psi)]-(w+\varepsilon+\psi)[\ln(G+\varepsilon+w+\psi) \\
&\quad -\ln(w+\varepsilon+\psi)]+(w+\varepsilon-\psi)[\ln(G+\varepsilon+w-\psi)-\ln(w+\varepsilon-\psi)]\}.
\end{aligned} \tag{53}$$

Also, we have

$$\begin{aligned}
\widehat{\vartheta}_{EWCBL3} &= \int\int_{\forall\Omega} \widehat{\vartheta}_{WCBL}\pi_3(\varphi,\phi)d\varphi\, d\phi \\
&= \int_0^1\int_0^G \left(\frac{2\phi}{G^2}\right)\frac{-(\varphi+s)}{2\psi}\ln\left(\frac{(1+w-\psi/\varepsilon+\phi)}{1+w+\psi/\varepsilon+\phi}\right)d\phi\, d\varphi \\
&= \frac{-(2s+1)}{2\psi G}\left[\int_0^G \phi\ln\left(\frac{(1+w-\psi/\varepsilon+\phi)}{1+w+\psi/\varepsilon+\phi}\right)d\phi\right] \\
&= \frac{-(2s+1)}{2\psi G}\left[I_{14}\right].
\end{aligned} \tag{54}$$

The hyperparameters are selected in Bayesian and E-Bayesian models based on prior knowledge (informative prior) of the data and are randomly sampled from predefined distributions over the hyperparameter space. The model is evaluated for each sample, and a random search is conducted to identify the best combination of hyperparameters. The methodological approach was extensively used in the studies cited in the references [10, 18].

5. Simulation Study

The behavior of Bayesian and E-Bayesian estimators for the GR distribution's shape parameter has been evaluated and examined using a Monte Carlo simulation study. The following procedures in simulation analysis have been carried out through R software:

(1) By considering different censoring schemes, simulations are run at $n = 30, 50, \ldots, 110$, $s = 75\%$, 50%, 100%, $\nu = 2$, $w = 0.5$, $\psi = 2, -1, 1$, and $G = 2$.

(2) Determined values of φ and ϕ are 0.3 and 0.7, respectively.

(3) Generate the value of ϑ from the pdf of the gamma distribution given in equation (13).

(4) For n, we generate censoring samples type-II from GR (ϑ, ν) with a known value ν by applying the following scheme:

 (a) Generate u from a uniform distribution on the interval $(0, 1)$.

 (b) Apply the inverse transform sampling method as follows:

$$X_i = F^{-1}(u_i) = \frac{1}{\nu}\left[-\ln\left(1-(u_i)^{\frac{1}{\vartheta}}\right)\right]^{1/2};\ i = 1, \ldots, n. \tag{55}$$

(5) The estimates $\widehat{\vartheta}_{BL}$, $\widehat{\vartheta}_{EBL1}$, $\widehat{\vartheta}_{EBL2}$, and $\widehat{\vartheta}_{EBL3}$ of ϑ under LINEXLF are evaluated from equations (15), (28), (29), and (30), respectively.

(6) The estimates $\widehat{\vartheta}_{WBL}$, $\widehat{\vartheta}_{EWBL1}$, $\widehat{\vartheta}_{EWBL2}$, and $\widehat{\vartheta}_{EWBL3}$ of ϑ under WLINEXLF are evaluated from equations (17), (36), (37), and (38), respectively.

(7) The estimates $\widehat{\vartheta}_{CBL}$, $\widehat{\vartheta}_{ECBL1}$, $\widehat{\vartheta}_{ECBL2}$, and $\widehat{\vartheta}_{ECBL3}$ of ϑ under CLINEXLF are evaluated from equations (16), (44), (45), and (46), respectively.

(8) The estimates $\widehat{\vartheta}_{WCBL}$, $\widehat{\vartheta}_{EWCBL1}$, $\widehat{\vartheta}_{EWCBL2}$, and $\widehat{\vartheta}_{EWCBL3}$ of ϑ under WCLINEXLF are evaluated from equations (18), (52), (53), and (54), respectively.

(9) We repeat the above steps 10000 times. The mean square errors (MSE) for each estimate $\widehat{\vartheta}$ are then calculated as follows:

$$MSE(\widehat{\vartheta}) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\vartheta - \widehat{\vartheta}_i\right)^2, \tag{56}$$

where $\widehat{\vartheta}_i$ is denoted as the estimate at i^{th} run and $\vartheta = 1.5386$.

Table 1: The values of MSEs for the Bayesian estimation for ϑ.

n	s		$\widehat{\vartheta}_{BL}$			$\widehat{\vartheta}_{WBL}$			$\widehat{\vartheta}_{CBL}$			$\widehat{\vartheta}_{WCBL}$	
		$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$
30	15	0.2548	0.1999	0.1948	0.1521	0.1412	0.1311	0.1958	0.1812	0.1615	0.1594	0.1301	0.1310
	22	0.1506	0.1397	0.1226	0.1198	0.1078	0.1008	0.1676	0.1201	0.1151	0.1205	0.0909	0.0926
	30	0.1321	0.1231	0.1105	0.1056	0.1048	0.0995	0.1451	0.1069	0.1014	0.0958	0.0901	0.0909
50	25	0.2465	0.1936	0.1232	0.1432	0.1210	0.1080	0.1319	0.1216	0.1015	0.1315	0.1005	0.0981
	37	0.2387	0.1910	0.0943	0.1352	0.0911	0.0945	0.1248	0.0956	0.0921	0.1009	0.0907	0.0899
	50	0.1337	0.0974	0.0899	0.1099	0.0905	0.0934	0.0961	0.0901	0.0897	0.0900	0.0844	0.0891
70	35	0.1329	0.0886	0.0853	0.1054	0.0968	0.0812	0.1145	0.1009	0.1000	0.1054	0.0789	0.0798
	52	0.1222	0.0831	0.0821	0.0956	0.0875	0.0745	0.0991	0.0985	0.0914	0.0800	0.0701	0.0645
	70	0.1189	0.0787	0.0709	0.0850	0.0795	0.0694	0.0876	0.0856	0.0807	0.0641	0.0681	0.0542
90	45	0.1129	0.0840	0.0811	0.0954	0.0940	0.0756	0.0992	0.0977	0.0968	0.0861	0.0709	0.0683
	67	0.1107	0.0760	0.0713	0.0845	0.0845	0.0698	0.0934	0.0864	0.0795	0.0753	0.0614	0.0415
	90	0.1102	0.0752	0.0721	0.0802	0.0756	0.0584	0.0719	0.0798	0.0741	0.0617	0.0518	0.0318
110	55	0.1122	0.0725	0.0526	0.0891	0.0712	0.0642	0.0807	0.0894	0.0801	0.0689	0.0689	0.0546
	82	0.0986	0.0719	0.0523	0.0650	0.0618	0.0601	0.0656	0.0764	0.0684	0.0602	0.0502	0.0489
	110	0.0973	0.0689	0.0501	0.0512	0.0602	0.0542	0.0541	0.0645	0.0548	0.0482	0.0458	0.0405

Table 2: The values of MSEs for the E-Bayesian estimation for ϑ under $\pi_1(\varphi, \phi)$.

n	s		$\widehat{\vartheta}_{EBL1}$			$\widehat{\vartheta}_{EWBL1}$			$\widehat{\vartheta}_{ECBL1}$			$\widehat{\vartheta}_{EWCBL1}$	
		$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$
30	15	0.2335	0.1798	0.1745	0.1475	0.1384	0.1247	0.1841	0.1774	0.1602	0.1387	0.1284	0.1282
	22	0.1417	0.1302	0.1212	0.1149	0.1067	0.0994	0.1607	0.1182	0.1083	0.0964	0.0867	0.0921
	30	0.1241	0.1207	0.1049	0.1014	0.1011	0.0968	0.1389	0.1043	0.1008	0.0900	0.0853	0.0884
50	25	0.2265	0.1856	0.1194	0.1404	0.1302	0.1052	0.1012	0.1184	0.0997	0.0907	0.0985	0.0962
	37	0.2199	0.1605	0.0913	0.1338	0.0907	0.0931	0.0918	0.0949	0.0916	0.0852	0.0902	0.0846
	50	0.1298	0.0897	0.0801	0.1064	0.0894	0.0914	0.0913	0.0872	0.0877	0.0794	0.0819	0.0817
70	35	0.1289	0.0804	0.0814	0.1013	0.0876	0.0796	0.0875	0.0982	0.0973	0.0813	0.0774	0.0712
	52	0.1147	0.0789	0.0807	0.0939	0.0842	0.0731	0.0743	0.0968	0.0819	0.0700	0.0697	0.0619
	70	0.1101	0.0702	0.0700	0.0816	0.0784	0.0682	0.0712	0.0813	0.0784	0.0678	0.0675	0.0532
90	45	0.1065	0.0826	0.0799	0.0923	0.0936	0.0731	0.0861	0.0961	0.0947	0.0808	0.0681	0.0674
	67	0.1009	0.0715	0.0709	0.0814	0.0829	0.0662	0.0756	0.0838	0.0768	0.0617	0.0609	0.0407
	90	0.1000	0.0689	0.0701	0.0795	0.0748	0.0541	0.0630	0.0764	0.0712	0.0548	0.0507	0.0312
110	55	0.1081	0.0712	0.0520	0.0776	0.0708	0.0642	0.0701	0.0872	0.0768	0.0601	0.0672	0.0537
	82	0.0809	0.0701	0.0517	0.0615	0.0611	0.0590	0.0654	0.0745	0.0613	0.0512	0.0498	0.0472
	110	0.0801	0.0654	0.0495	0.0499	0.0587	0.0512	0.0531	0.0598	0.0519	0.0404	0.0441	0.0394

Table 3: The values of MSEs for the E-Bayesian estimation for ϑ under $\pi_2(\varphi, \phi)$.

n	s		$\widehat{\vartheta}_{EBL2}$			$\widehat{\vartheta}_{EWBL2}$			$\widehat{\vartheta}_{ECBL2}$			$\widehat{\vartheta}_{EWCBL2}$	
		$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$
30	15	0.1987	0.1618	0.1652	0.1468	0.1287	0.1199	0.1706	0.1624	0.1581	0.1254	0.1141	0.1102
	22	0.1394	0.1294	0.1120	0.1108	0.1001	0.0896	0.1581	0.1152	0.0917	0.0879	0.0807	0.0894
	30	0.1237	0.1123	0.1027	0.0982	0.0983	0.0843	0.1295	0.1007	0.0908	0.0851	0.0812	0.0798
50	25	0.1962	0.1590	0.1175	0.1312	0.1217	0.0982	0.1189	0.1037	0.0971	0.0900	0.0917	0.0892
	37	0.1948	0.1548	0.0874	0.1296	0.0864	0.0927	0.0821	0.0931	0.0892	0.0819	0.0852	0.0806
	50	0.1245	0.0867	0.0768	0.0967	0.0827	0.0909	0.0813	0.0827	0.0818	0.0709	0.0749	0.0737
70	35	0.1271	0.0794	0.0717	0.0984	0.0782	0.0783	0.0854	0.0974	0.0945	0.0713	0.0701	0.0698
	52	0.1132	0.0751	0.0709	0.0924	0.0747	0.0704	0.0819	0.0824	0.0807	0.0700	0.0681	0.0609
	70	0.1079	0.0689	0.0668	0.0802	0.0720	0.0672	0.0817	0.0807	0.0719	0.0670	0.0657	0.0523
90	45	0.0984	0.0818	0.0710	0.0876	0.0794	0.0713	0.0807	0.0902	0.0913	0.0647	0.0618	0.0647
	67	0.0972	0.0704	0.0672	0.0784	0.0738	0.0608	0.0749	0.0817	0.0747	0.0554	0.0587	0.0398
	90	0.0819	0.0637	0.0658	0.0682	0.0704	0.0527	0.0662	0.0752	0.0681	0.0432	0.0492	0.0261
110	55	0.0967	0.0694	0.0449	0.0612	0.0692	0.0613	0.0686	0.0794	0.0702	0.0608	0.0512	0.0514
	82	0.0771	0.0642	0.0408	0.0598	0.0607	0.0571	0.0649	0.0672	0.0611	0.0400	0.0381	0.0370
	110	0.0719	0.0607	0.0397	0.0417	0.0543	0.0484	0.0528	0.0517	0.0412	0.0357	0.0321	0.0301

Table 4: The values of MSEs for the E-Bayesian estimation for ϑ under $\pi_3(\varphi, \phi)$.

n	s	$\hat{\vartheta}_{EBL3}$			$\hat{\vartheta}_{EWBL3}$			$\hat{\vartheta}_{ECBL3}$			$\hat{\vartheta}_{EWCBL3}$		
		$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$	$\psi=-1$	$\psi=1$	$\psi=2$
	15	0.1712	0.1547	0.1427	0.1378	0.1124	0.1098	0.1549	0.1524	0.1491	0.1103	0.1042	0.0954
30	22	0.1254	0.1192	0.1091	0.1012	0.0981	0.0749	0.1205	0.1018	0.0815	0.0804	0.0794	0.0701
	30	0.1148	0.1101	0.0941	0.0817	0.0972	0.0715	0.0984	0.0976	0.0801	0.0775	0.0712	0.0684
	25	0.1514	0.1437	0.1012	0.1197	0.1182	0.0908	0.1119	0.1012	0.0894	0.0813	0.0783	0.0714
50	37	0.1502	0.1409	0.0718	0.1104	0.0798	0.0819	0.0987	0.0871	0.0862	0.0750	0.0764	0.0672
	50	0.1497	0.0841	0.0684	0.0916	0.0746	0.0807	0.0956	0.0814	0.0806	0.0714	0.0708	0.0636
	35	0.1113	0.0774	0.0682	0.0892	0.0713	0.0719	0.0850	0.0891	0.0873	0.0744	0.0684	0.0674
70	52	0.1097	0.0711	0.0674	0.0871	0.0681	0.0684	0.0800	0.0805	0.0749	0.0612	0.0652	0.0583
	70	0.1007	0.0640	0.0619	0.0794	0.0679	0.0653	0.0715	0.0976	0.0707	0.0607	0.0638	0.0492
	45	0.0974	0.0782	0.0657	0.0796	0.0681	0.0819	0.0735	0.0814	0.0864	0.0594	0.0586	0.0537
90	67	0.0962	0.0614	0.0613	0.0763	0.0674	0.0557	0.0702	0.0807	0.0684	0.0430	0.0542	0.0308
	90	0.0767	0.0607	0.0594	0.0614	0.0612	0.0512	0.0617	0.0734	0.0664	0.0308	0.0461	0.0167
	55	0.0910	0.0581	0.0412	0.0597	0.0594	0.0584	0.0718	0.0772	0.0608	0.0412	0.0475	0.0468
110	82	0.0641	0.0574	0.0381	0.0519	0.0526	0.0536	0.0640	0.0651	0.0587	0.0364	0.0356	0.0248
	110	0.0634	0.0551	0.0372	0.0408	0.0508	0.0474	0.0516	0.0503	0.0319	0.0315	0.0274	0.0201

The simulation results are shown in Tables 1–4. We conclude the following from the results:

(1) According to the lowest MSE, the E-Bayesian estimation for ϑ surpasses the Bayesian estimation in all cases

(2) The Bayesian estimation for ϑ under the WCLINEXLF surpasses the Bayesian estimations under LINEXLF, CLINEXLF, and WLINEXLF because of the smallest value of MSE

(3) The E-Bayesian estimation for ϑ under WCLINEXLF with $\pi_1(\varphi, \phi)$ and $\pi_2(\varphi, \phi)$ has the least MSE regarding all other estimations

(4) The E-Bayesian estimation for ϑ under WLINEXLF with $\pi_3(\varphi, \phi)$ has the least MSE regarding all other estimations

(5) The Bayesian and E-Bayesian estimation methods for ϑ under the suggested WCLINEXLF outperform the Bayesian and E-Bayesian estimation methods under CLINEXLF because of the minimum MSE

(6) The values of MSE under Bayesian and E-Bayesian estimation techniques are decreasing when the sample size n and s are increasing

Therefore, we suggest using the E-Bayesian approach for estimating the parameter ϑ. In addition, applying the type-II censoring data for the life test owes to its better performance than other estimates.

6. Applications on Real Data

A real data set is applied to illustrate Bayesian and E-Bayesian estimation. The data set, first published by Wang and Lee [47], includes the remission times (months) for a random sample of 137 patients with bladder cancer, which are displayed in Table 5.

According to the type-II censoring scheme, we assume 137 independent patients (n) are placed on a life test with the corresponding remission times (months) $x_1, x_2, \ldots, x_{137} = 0.08, 0.2, \ldots, 79.05$, which are displayed in Table 5. It is supposed that these variables are identical and independent with pdf in equation (1). We determine $s = 25$ on the condition that $s < n$. The experiment was terminated after the 25$^{\text{th}}$ independent patient was observed, by the time that $x_{25} = 2.64$. This decision was made based on a predetermined number of failures that had occurred in the experiment up to that point. In addition, the remission time of the surviving units $(n - s)$ is removed from the test.

Regarding these data, we calculated the Kolmogorov–Smirnov (KS) distance (D) between the empirical distribution function and its fitted function to be 0.1910, and the p value is 0.6385. The Bayesian estimation for ϑ and the standard error (St. E) under type II censored data for bladder cancer data according to LINEXLF, CLINEXLF, WLINEXLF, and WCLINEXLF are shown in Table 6. The E-Bayesian estimation for ϑ under $\pi_1(\varphi, \phi)$ and St. E under type II censored data for bladder cancer data according to LINEXLF, CLINEXLF, WLINEXLF, and WCLINEXLF are shown in Table 7. The E-Bayesian estimation for ϑ under $\pi_2(\varphi, \phi)$ and St. E under type II censored data for bladder cancer data according to LINEXLF, CLINEXLF, WLINEXLF, and WCLINEXLF are shown in Table 8. The E-Bayesian estimation for ϑ based on $\pi_3(\varphi, \phi)$ and St. E under type II censored data for bladder cancer data according to LINEXLF, CLINEXLF, WLINEXLF, and WCLINEXLF are shown in Table 9. The results are shown in Tables 6–9. Therefore, we can obtain the following results (see Figure 2):

(1) Due to the lowest value of St. E, the E-Bayesian estimation for ϑ transcends the Bayesian estimations in all cases.

Table 5: Remission times (months) for a random sample of 137 patients with bladder cancer.

8.65	2.23	32.15	4.87	5.71	7.59	3.02	4.51	1.05	9.47
6.54	4.23	3.48	2.46	22.69	3.82	26.31	4.65	5.41	4.34
79.05	2.02	4.26	11.25	10.34	10.66	12.03	2.64	14.76	1.19
8.66	14.83	5.62	18.1	25.74	17.36	1.35	9.02	6.94	7.26
4.7	3.7	3.64	3.57	11.64	6.25	25.82	3.88	3.02	19.36
20.28	46.12	5.17	0.2	36.66	10.06	4.98	5.06	16.62	12.07
6.97	0.08	1.4	2.75	7.32	1.26	6.76	8.6	7.62	3.52
9.74	0.4	5.41	2.54	2.69	8.26	0.5	5.32	5.09	2.09
7.93	12.02	13.8	5.85	7.09	5.32	4.33	2.83	8.37	14.77
8.53	11.98	1.76	4.4	34.26	2.07	17.12	12.63	7.66	4.18
13.29	23.63	3.25	7.63	2.87	3.31	2.26	2.69	11.79	5.34
24.8	10.86	17.14	15.96	7.28	4.33	7.39	13.11	10.75	6.93
2.62	0.9	21.73	0.87	0.51	3.36	43.01	0.81	3.36	1.46
4.5	19.13	14.24	7.87	5.49	2.02	9.22			

Table 6: The Bayesian estimation of ϑ and St. E under type II censored data for bladder cancer data.

$s = 25, \nu = 0.7565, w = 0.5, \psi = -1,$ *and* $G = 2$							
$\widehat{\vartheta}_{BL}$	St. E	$\widehat{\vartheta}_{WBL}$	St. E	$\widehat{\vartheta}_{CBL}$	St. E	$\widehat{\vartheta}_{WCBL}$	St. E
3.6573	0.3645	3.4853	0.3467	3.1721	0.3301	2.9681	0.2970

Table 7: The E-Bayesian estimation for ϑ under $\pi_1(\varphi, \phi)$ and St. E under type II censored data for bladder cancer data.

$s = 25, \nu = 0.7565, w = 0.5, \psi = -1,$ *and* $G = 2$							
$\widehat{\vartheta}_{EBL1}$	St. E	$\widehat{\vartheta}_{EWBL1}$	St. E	$\widehat{\vartheta}_{ECBL1}$	St. E	$\widehat{\vartheta}_{EWCBL1}$	St. E
2.8721	0.2776	2.4687	0.2504	1.9896	0.2276	1.7592	0.1843

Table 8: The E-Bayesian estimation for ϑ under $\pi_2(\varphi, \phi)$ and St. E under type II censored data for bladder cancer data.

$s = 25, \nu = 0.7565, w = 0.5, \psi = -1,$ *and* $G = 2$							
$\widehat{\vartheta}_{EBL2}$	St. E	$\widehat{\vartheta}_{EWBL2}$	St. E	$\widehat{\vartheta}_{ECBL2}$	St. E	$\widehat{\vartheta}_{EWCBL2}$	St. E
1.5367	0.1690	1.3964	0.1320	1.1943	0.1112	1.0872	0.1080

Table 9: The E-Bayesian estimation for ϑ under $\pi_3(\varphi, \phi)$ and St. E under type II censored data for bladder cancer data.

$s = 25, \nu = 0.7565, w = 0.5, \psi = -1,$ *and* $G = 2$							
$\widehat{\vartheta}_{EBL3}$	St. E	$\widehat{\vartheta}_{EWBL3}$	St. E	$\widehat{\vartheta}_{ECBL3}$	St. E	$\widehat{\vartheta}_{EWCBL3}$	St. E
0.9589	0.0756	0.7963	0.0479	0.6279	0.0195	0.5196	0.0002

(2) The Bayesian estimation for ϑ under WCLINEXLF surpasses the Bayesian estimations under LINEXLF, CLINEXLF, and WLINEXLF because of the smallest value of St. E.

(3) The E-Bayesian estimation for ϑ under WCLINEXLF with $\pi_1(\varphi, \phi)$, $\pi_2(\varphi, \phi)$, and $\pi_3(\varphi, \phi)$ has the least St. E regarding all other estimators.

(4) The Bayesian and E-Bayesian estimation of ϑ under the suggested WCLINEXLF transcends the Bayesian and E-Bayesian estimation under CLINEXLF because of the minimum St. E.

Therefore, we suggest using the E-Bayesian approach under the suggested WCLINEXLF to estimate the parameter ϑ depending on the type-II censoring scheme due to its better performance than other estimators.

The histogram plot, MCMC convergence, and approximate marginal posterior density of ϑ are represented in Figures 3–6.

Figure 2 illustrates the plots between the empirical and its fitted function under the CDF curve, the histogram, the P-P plot, and the Q-Q plot for GR, resulting in the GR fitting the bladder cancer data set.

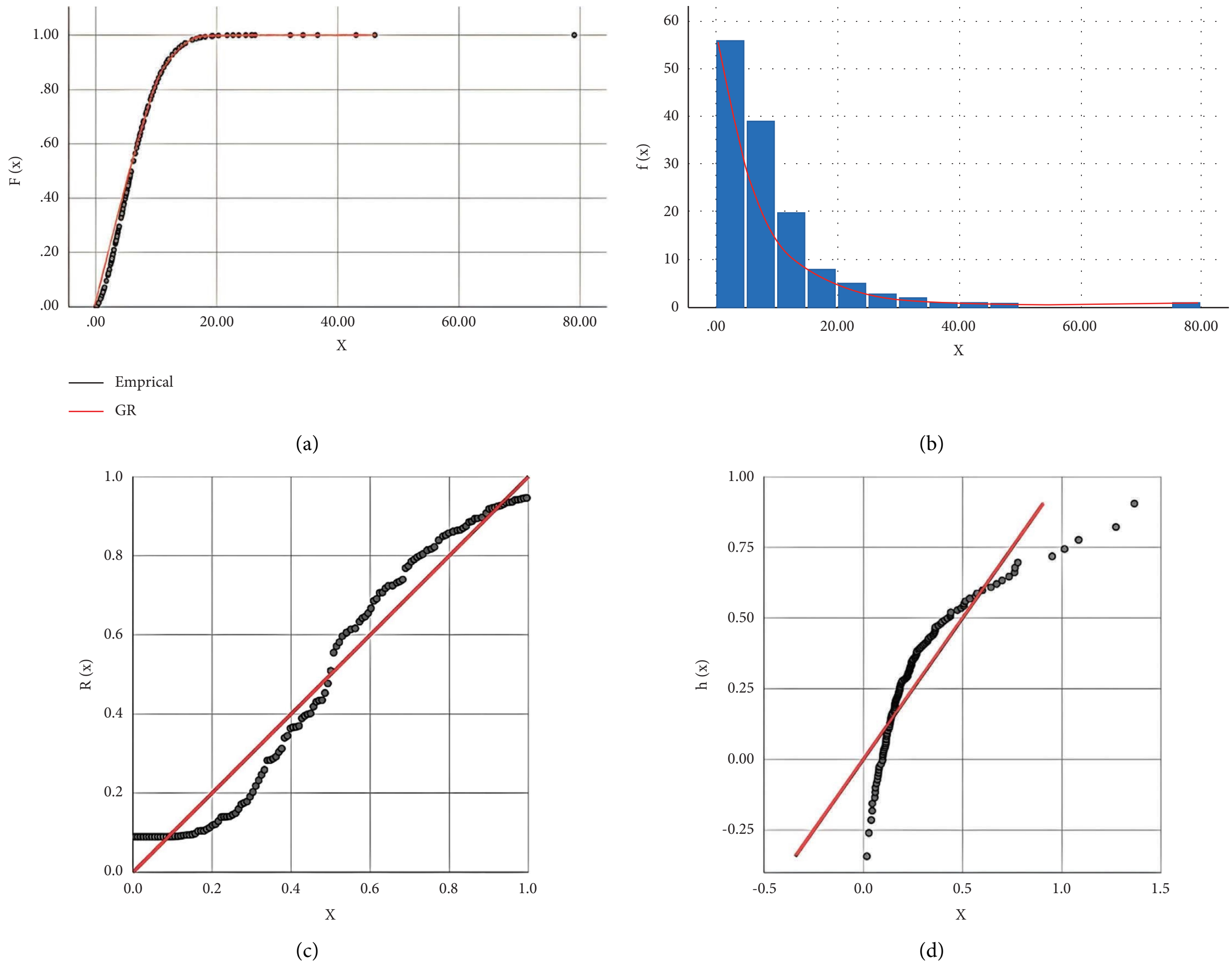


FIGURE 2: The plot of the maximum distance between two CDF curves, histogram, P-P plot, and Q-Q plot of GR for bladder cancer data.

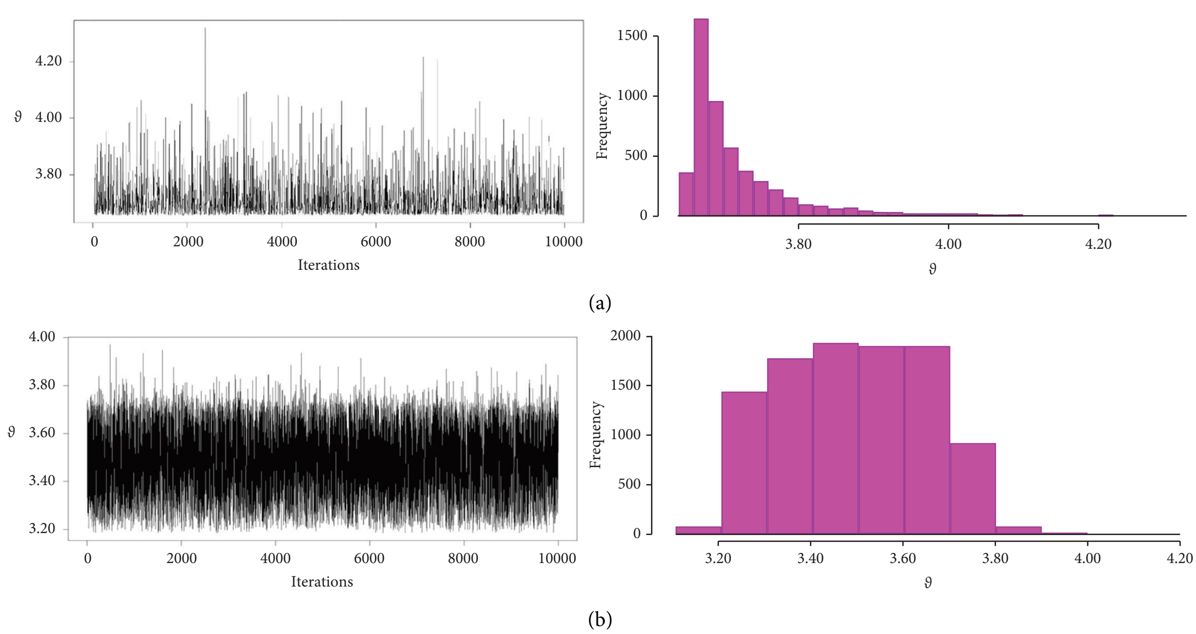
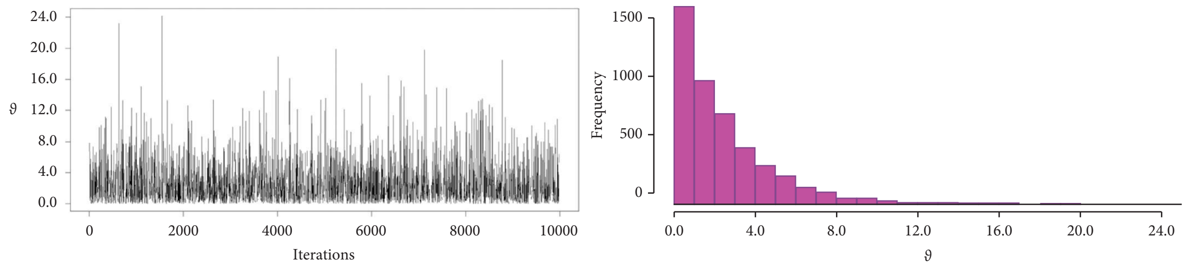
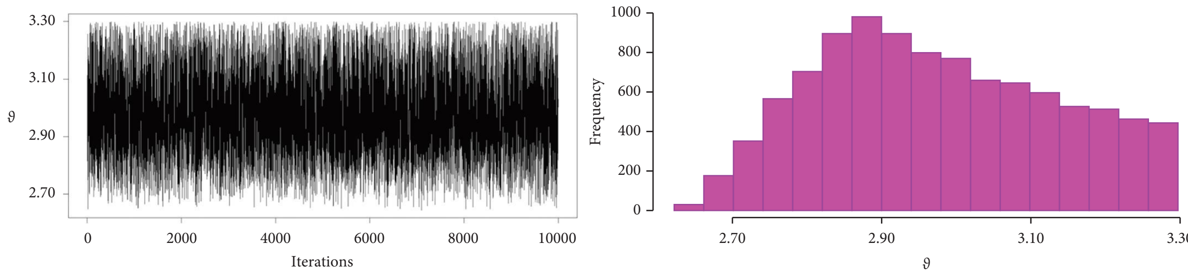


FIGURE 3: Continued.

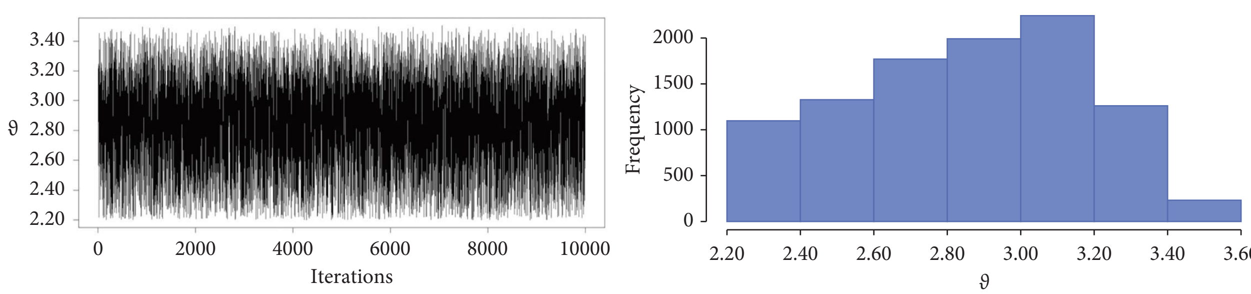


(c)

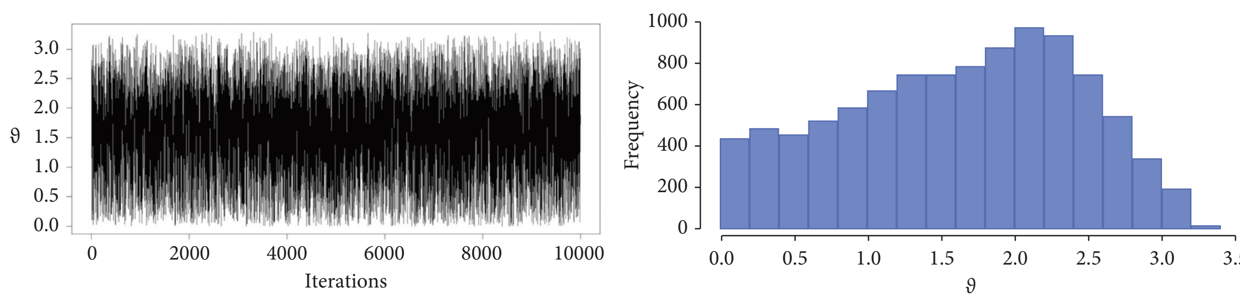


(d)

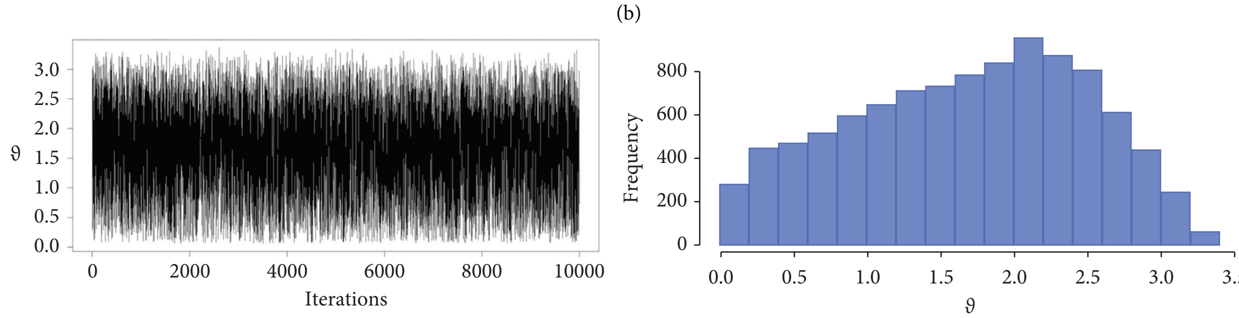
Figure 3: The MCMC plots of the Bayesian estimation for ϑ according to LINEXLF, CLINEXLF, WLINEXLF, and WCLINEXLF, respectively, under type II censored data of GR for bladder cancer data.



(a)

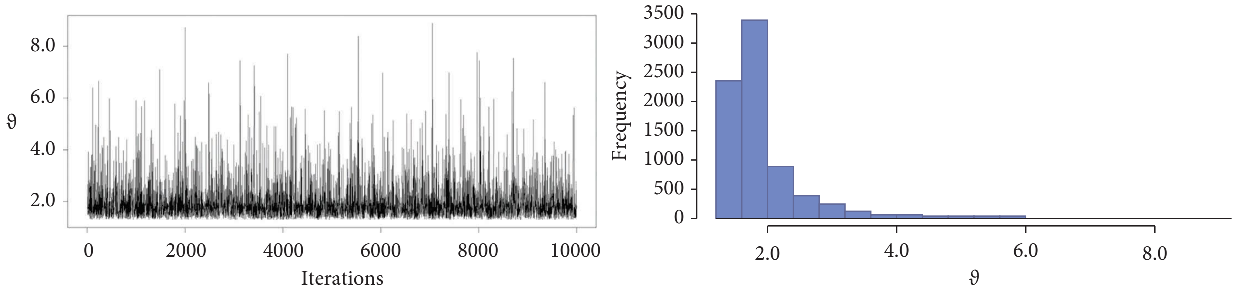


(b)



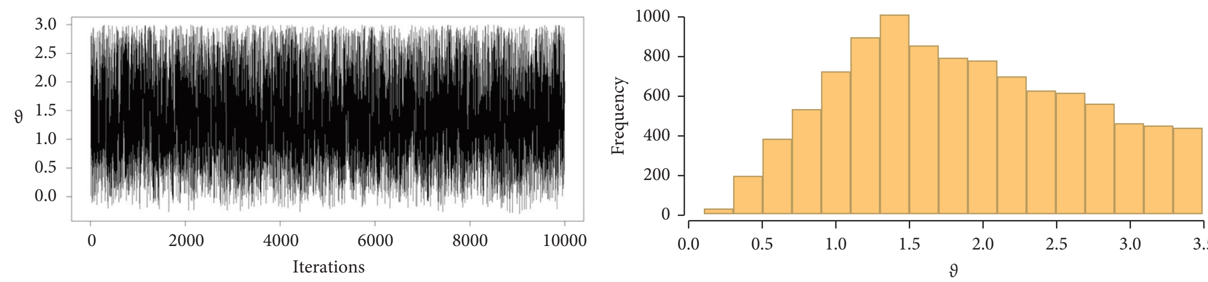
(c)

Figure 4: Continued.

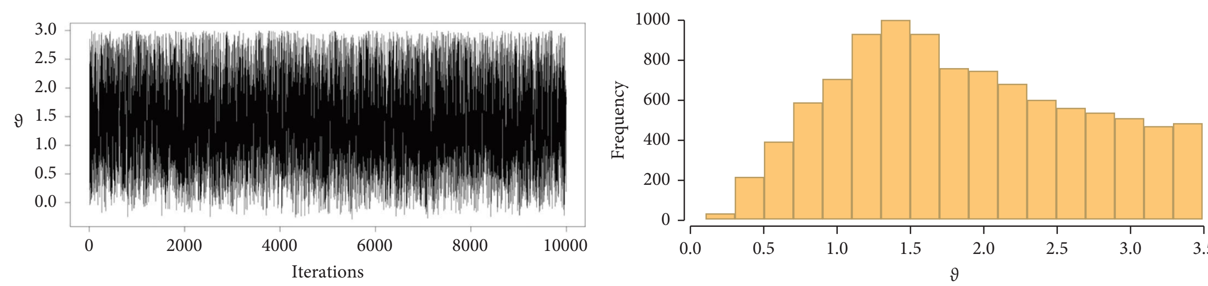


(d)

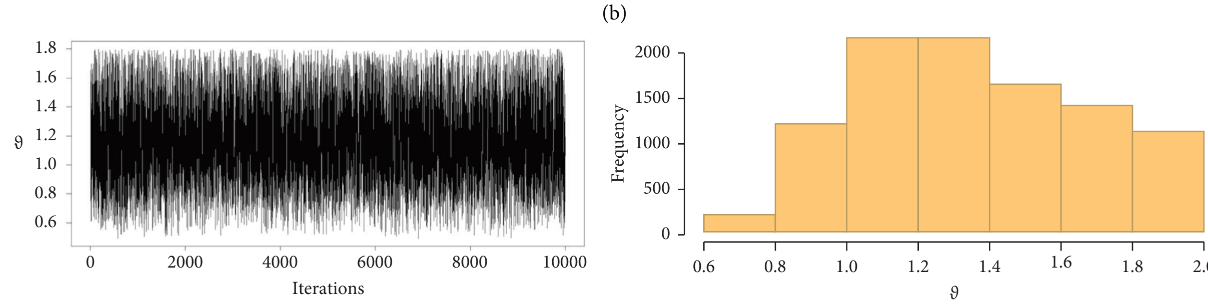
Figure 4: The MCMC plots of the E-Bayesian estimation for ϑ under $\pi_1(\varphi, \phi)$ according to LINEXLF, CLINEXLF, WLINEXLF, and WCLINEXLF, respectively, under type II censored data of GR for bladder cancer data.



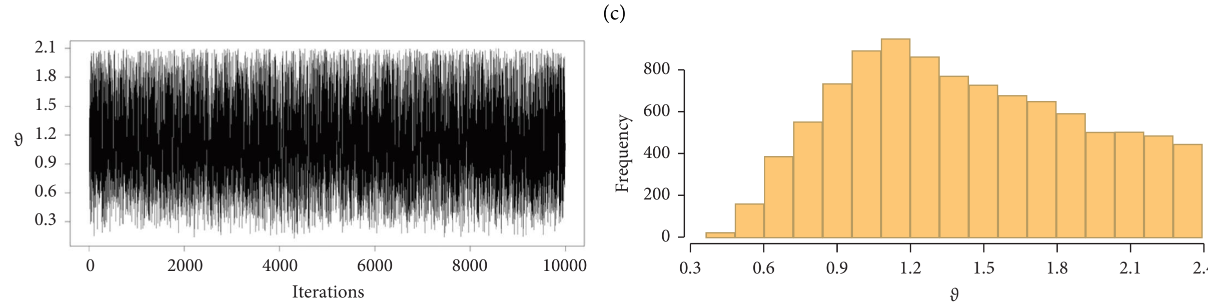
(a)



(b)

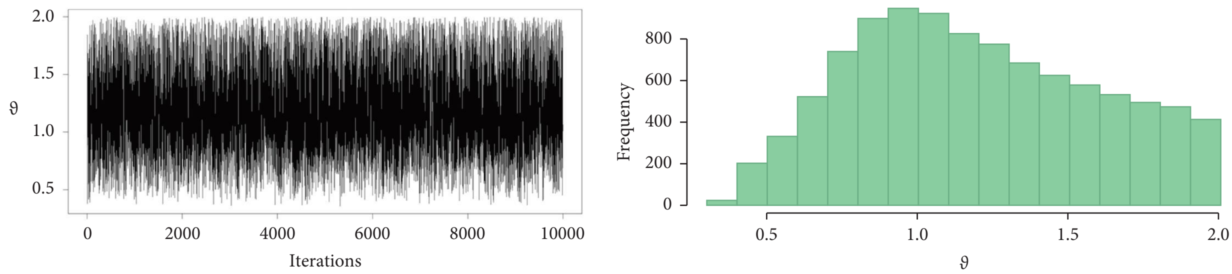


(c)

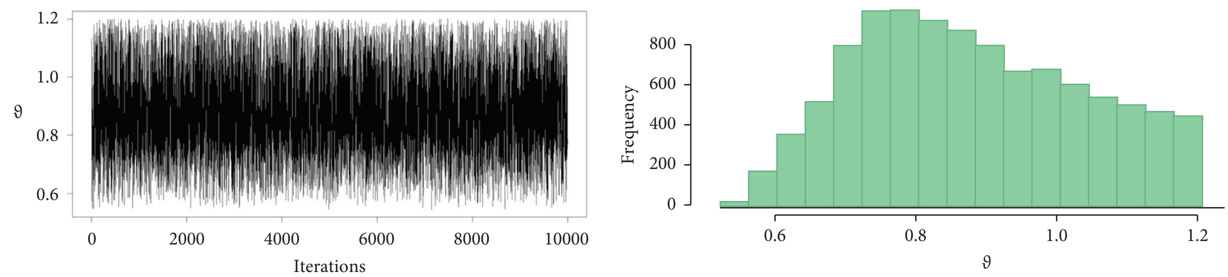


(d)

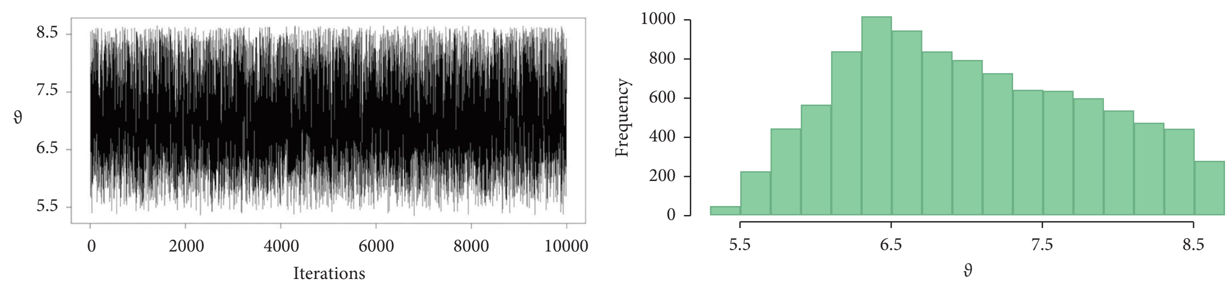
Figure 5: The MCMC plots of the E-Bayesian estimation for ϑ under $\pi_2(\varphi, \phi)$ according to LINEXLF, CLINEXLF, WLINEXLF, and WCLINEXLF, respectively, under type II censored data of GR for bladder cancer data.



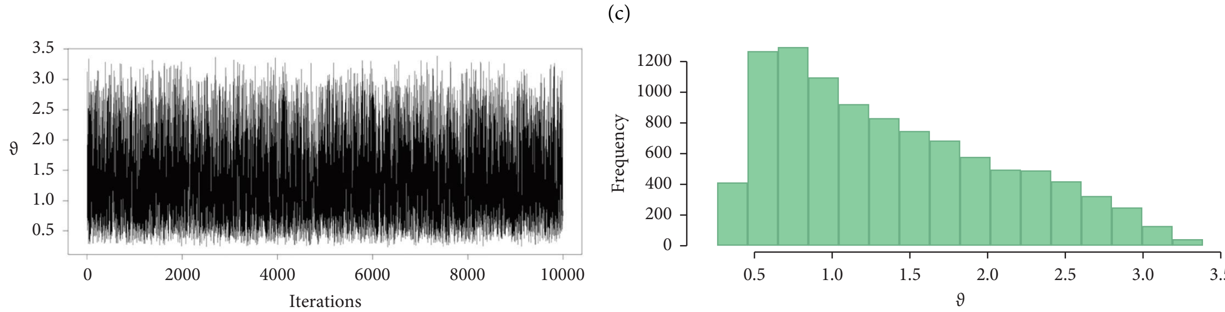
(a)



(b)



(c)



(d)

FIGURE 6: The MCMC plots of the E-Bayesian estimation for ϑ under $\pi_3(\varphi, \phi)$ according to LINEXLF, CLINEXLF, WLINEXLF, and WCLINEXLF, respectively, under type II censored data of GR for bladder cancer data.

Figures 3–6 show the trace plots of 10,000 MCMC samples and histogram plots of generated ϑ under Bayesian estimation and E-Bayesian estimation according to $\pi_1(\varphi, \phi), \pi_2(\varphi, \phi)$, and $\pi_3(\varphi, \phi)$ based on type II censored data of GR for bladder cancer data according to LINEXLF, CLINEXLF, WLINEXLF, and WCLINEXLF, respectively. In the simulation study, we found that Bayesian and E-Bayesian estimation methods of ϑ under the suggested WCLINEXLF transcend other methods.

7. Conclusion

The current paper focuses on the Bayesian and E-Bayesian estimation procedures of an unknown shape parameter of the GR distribution based on type-II censoring data. A precise procedure for the Bayesian and E-Bayesian estimators has been proposed using several LF methods. The gamma distribution is used as a conjugate prior for GR's parameter, and the various LFs, including LINEXLF, WLINEXLF, CLINEXLF, and a new suggested LF called WCLINEXLF, are used to derive the E-Bayesian and Bayesian estimators. In addition, the E-Bayesian estimator is derived using three prior hyperparameter distributions.

Under various prior assumptions and several LFs, the accuracy of Bayesian and E-Bayesian estimators for an unknown shape parameter of the GR distribution has been investigated via a Monte Carlo technique. The results of the simulation indicated that each of the prior distributions under the shape parameter had performed brilliantly and efficiently according to the suggested WCLINEXLF, as the Bayesian and E-Bayesian estimation methods under WCLINEXLF outperformed the Bayesian and E-Bayesian estimation schemes under the rest of the loss functions, particularly when it comes to the least MSE. Furthermore, it has been recommended that a novel LF called WCLINEXLF be used while estimating hyperparameters. According to simulation results, the E-Bayesian estimator performs better than the Bayesian estimator based on the minimum MSE. In addition, the results of the real data on bladder cancer clarify how to get the suggested estimators in real life and show that the E-Bayesian and Bayesian estimation techniques according to WCLINEXLF surpass other loss functions due to the minimum value of St. E, which means that these results of this analysis of the application accord with the simulation results. Based on the simulation and real data results, we recommend using the E-Bayesian method according to WCLINXLF to estimate an unknown shape parameter of the GR distribution under type-II censoring data.

It is essential to remember that this paper assumes a known prior parameter. If the prior parameter is unknown, this paper can be improved by evolving empirical Bayesian estimators. Another improvement of this work is developing an estimation scheme under the multivariate structure of the GR model. Moreover, given the failure data that follows various probability distributions, comparisons between the researchers' suggested formula and other approaches may be established. The proposed formula may be improved, and comparisons to other theoretical probability distributions can be made theoretically and practically in medical applications.

Appendix

A. Derivation of the Likelihood Function under the Type-II Censoring Sample in Equation (11)

The purpose of this derivation is to explain the steps involved in arriving at the likelihood function under the type-II censoring sample in equation (11) for the GR distribution. This derivation is based on the following assumptions:

$$\begin{aligned} L(\vartheta,\nu \mid x) &= \frac{n!}{(n-s)!}\times\prod_{i=1}^{s} 2\vartheta\nu^2 x_i \exp\left(-(\nu x_i)^2\right)\left(1-\exp\left(-(\nu x_i)^2\right)\right)^{\vartheta-1}\times\left(1-\left(1-\exp\left(-(\nu x_s)^2\right)\right)^{\vartheta}\right)^{n-s}\\ &= \frac{n!}{(n-s)!}2^s\vartheta^s\nu^{2s}\times\left(\prod_{i=1}^{s} x_i\frac{\exp\left(-(\nu x_i)^2\right)}{1-\exp\left(-(\nu x_i)^2\right)}\right)\left(\prod_{i=1}^{s}\left(1-\exp\left(-(\nu x_i)^2\right)\right)^{\vartheta}\right)\times\left(\left(1-\left(1-\exp\left(-(\nu x_s)^2\right)\right)^{\vartheta}\right)^{n-s}\right). \end{aligned} \tag{A.1}$$

First, we assume $F=\prod_{i=1}^{s}(1-\exp(-(\nu x_i)^2))^{\vartheta}$; using some properties of logarithms and exponential functions, we conclude that $F=\exp\ (\vartheta.\sum_{i=1}^{s}\ln(1-\exp(-(\nu x_i)^2))$ and we also assume $Z=(1-(1-\exp(-(\nu x_s)^2))^{\vartheta})^{n-s}$; using some properties of logarithms, exponential functions, and Taylor expansion $\ln(1-x)=\sum_{k=1}^{\infty}-(x)^k/k$, we conclude that $Z=\exp(\sum_{k=1}^{\infty}\ln(n-s/k)-\vartheta\sum_{k=1}^{\infty}\ln\ (1-\ \exp(-\ (\nu x_s)^2))^k)$. Finally, we substitute their values into the equation.

$$\begin{aligned} &= \frac{n!}{(n-s)!}2^s\vartheta^s\nu^{2s}\left(\prod_{i=1}^{s} x_i\frac{\exp\left(-(\nu x_i)^2\right)}{1-\exp\left(-(\nu x_i)^2\right)}\right)\left(\exp\ \ (\vartheta.\sum_{i=1}^{s}\ln\left(1-\exp\left(-(\nu x_i)^2\right)\right)\right)\\ &\times\left(\exp\left(\sum_{k=1}^{\infty}\ln\left(\frac{n-s}{k}\right)-\vartheta\sum_{k=1}^{\infty}\ln\left(1-\ \exp\left(-\left(\nu x_s\right)^2\right)\right)^k\right)\right) \end{aligned} \tag{A.2}$$

$$= \frac{n!}{(n-s)!}2^s\vartheta^s\nu^{2s}\left(\prod_{i=1}^{s} x_i\frac{\exp\left(-(\nu x_i)^2\right)}{1-\exp\left(-(\nu x_i)^2\right)}\right)\exp(-\vartheta\varepsilon+\xi),$$

where $\varepsilon=-[\sum_{i=1}^{s}\ln(1-\exp(-(\nu x_i)^2))-\sum_{k=1}^{\infty}\ln\ (1-\ \exp\ (-\ (\nu x_s)^2))^k]$;

$$\xi=\sum_{k=1}^{\infty}\ln\left(\frac{n-s}{k}\right). \tag{A.3}$$

B. Derivation of the Bayesian Estimator of ϑ Based on LINEXLF in Equation (15)

$$\begin{aligned}\widehat{\vartheta}_{BL} &= \frac{-1}{\psi}\ln\left[E_{\vartheta}(\exp(-\psi\vartheta)\,|\,x)\right] = \frac{-1}{\psi}\ln\int_0^{\infty}\exp(-\psi\vartheta)\pi(\vartheta\,|\,x)d\vartheta\\ &= \frac{-1}{\psi}\ln\int_0^{\infty}\exp(-\psi\vartheta)\frac{\vartheta^{\varphi+s-1}\exp(-\vartheta(\varepsilon+\phi))(\phi+\varepsilon)^{\varphi+s}}{\Gamma(\varphi+s)}d\vartheta\\ &= \frac{\varphi+s}{\psi}\ln\left(1+\frac{\psi}{\phi+\varepsilon}\right).\end{aligned} \tag{B.1}$$

C. Derivation of the Bayesian Estimator of ϑ Based on CLINEXLF in Equation (16)

$$\widehat{\vartheta}_{CBL} = \frac{1}{2\psi}\ln\left(\frac{E_{\vartheta}(\exp(\psi\vartheta)\,|\,x)}{E_{\vartheta}(\exp(-\psi\vartheta)\,|\,x)}\right) = \frac{1}{2\psi}\ln\left(\frac{I_1}{I_2}\right), \tag{C.1}$$

such that

$$\begin{aligned}I_1 &= E_{\vartheta}(\exp(\psi\vartheta)\,|\,x) = \int_0^{\infty}\exp(\psi\vartheta)\pi(\vartheta\,|\,x)d\vartheta\\ &= \int_0^{\infty}\exp(\psi\vartheta)\frac{\vartheta^{\varphi+s-1}\exp(-\vartheta(\varepsilon+\phi))(\phi+\varepsilon)^{\varphi+s}}{\Gamma(\varphi+s)}d\vartheta\\ &= \left(1-\frac{\psi}{\varepsilon+\phi}\right)^{-(\varphi+s)}.\end{aligned} \tag{C.2}$$

Also, we have

$$\begin{aligned}I_2 &= E_{\vartheta}(\exp(-\psi\vartheta)\,|\,x) = \int_0^{\infty}\exp(-\psi\vartheta)\pi(\vartheta\,|\,x)d\vartheta\\ &= \int_0^{\infty}\exp(-\psi\vartheta)\frac{\vartheta^{\varphi+s-1}\exp(-\vartheta(\varepsilon+\phi))(\phi+\varepsilon)^{\varphi+s}}{\Gamma(\varphi+s)}d\vartheta\\ &= \left(1+\frac{\psi}{\varepsilon+\phi}\right)^{-(\varphi+s)}.\end{aligned} \tag{C.3}$$

Then, the Bayesian estimator of ϑ has the following form:

$$\begin{aligned}\widehat{\vartheta}_{CBL} &= \frac{1}{2\psi}\ln\left(\frac{(1-\psi/\varepsilon+\phi)^{-(\varphi+s)}}{(1+\psi/\varepsilon+\phi)^{-(\varphi+s)}}\right)\\ &= \frac{-(\varphi+s)}{2\psi}\ln\left(\frac{1-\psi/\varepsilon+\phi}{1+\psi/\varepsilon+\phi}\right).\end{aligned} \tag{C.4}$$

D. Derivation of the Bayesian Estimator of ϑ Based on WLINEXLF in Equation (17)

$$\widehat{\vartheta}_{WBL} = \frac{1}{\psi}\ln\left(\frac{E_{\vartheta}(\exp(-w\vartheta)\,|\,x)}{E_{\vartheta}(\exp(-\vartheta(w+\psi))\,|\,x)}\right) = \frac{1}{\psi}\ln\left(\frac{I_3}{I_4}\right), \tag{D.1}$$

such that

$$\begin{aligned}I_3 &= E_{\vartheta}(\exp(-w\vartheta)\,|\,x) = \int_0^{\infty}\exp(-w\vartheta)\pi(\vartheta\,|\,x)d\vartheta\\ &= \int_0^{\infty}\exp(-w\vartheta)\frac{\vartheta^{\varphi+s-1}\exp(-\vartheta(\varepsilon+\phi))(\phi+\varepsilon)^{\varphi+s}}{\Gamma(\varphi+s)}d\vartheta\\ &= \left(1+\frac{w}{\varepsilon+\phi}\right)^{-(\varphi+s)}.\end{aligned} \tag{D.2}$$

Also, we have

$$I_4 = E_\vartheta(\exp(-\vartheta(w+\psi))\,|\,x) = \int_0^\infty \exp(-\vartheta(w+\psi))\pi(\vartheta\,|\,x)d\vartheta$$
$$= \int_0^\infty \exp(-\vartheta(w+\psi))\frac{\vartheta^{\varphi+s-1}\exp(-\vartheta(\varepsilon+\phi))(\phi+\varepsilon)^{\varphi+s}}{\Gamma(\varphi+s)}d\vartheta \tag{D.3}$$
$$= \left(1+\frac{w+\psi}{\varepsilon+\phi}\right)^{-(\varphi+s)}.$$

Therefore, the Bayesian estimator of ϑ is expressed as follows:

$$\widehat{\vartheta}_{WBL} = \frac{1}{\psi}\ln\left(\frac{(1+w/\varepsilon+\phi)^{-(\varphi+s)}}{(1+w+\psi/\varepsilon+\phi)^{-(\varphi+s)}}\right)$$
$$= \frac{-(\varphi+s)}{\psi}\ln\left(\frac{1+w/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right). \tag{D.4}$$

E. Derivation of the Bayesian Estimator of ϑ Based on WCLINEXLF in Equation (18)

$$\widehat{\vartheta}_{WCBL} = \frac{1}{2\psi}\ln\left(\frac{E_\vartheta(\exp(-\vartheta(w-\psi))\,|\,x)}{E_\vartheta(\exp(-\vartheta(w+\psi))\,|\,x)}\right)$$
$$= \frac{1}{2\psi}\ln\left(\frac{I_5}{I_6}\right), \tag{E.1}$$

such that

$$I_5 = E_\vartheta(\exp(-\vartheta(w-\psi))\,|\,x) = \int_0^\infty \exp(-\vartheta(w-\psi))\pi(\vartheta\,|\,x)d\vartheta$$
$$= \int_0^\infty \exp(-\vartheta(w-\psi))\frac{\vartheta^{\varphi+s-1}\exp(-\vartheta(\varepsilon+\phi))(\phi+\varepsilon)^{\varphi+s}}{\Gamma(\varphi+s)}d\vartheta \tag{E.2}$$
$$= \left(1+\frac{w-\psi}{\varepsilon+\phi}\right)^{-(\varphi+s)}.$$

Also, we have

$$I_6 = E_\vartheta(\exp(-\vartheta(w+\psi))\,|\,x) = \int_0^\infty \exp(-\vartheta(w+\psi))\pi(\vartheta\,|\,x)d\vartheta$$
$$= \int_0^\infty \exp(-\vartheta(w+\psi))\frac{\vartheta^{\varphi+s-1}\exp(-\vartheta(\varepsilon+\phi))(\phi+\varepsilon)^{\varphi+s}}{\Gamma(\varphi+s)}d\vartheta \tag{E.3}$$
$$= \left(1+\frac{w+\psi}{\varepsilon+\phi}\right)^{-(\varphi+s)}.$$

Then, the Bayesian estimator of ϑ is expressed as follows:

$$\widehat{\vartheta}_{WCBL} = \frac{1}{2\psi}\ln\left(\frac{(1+w-\psi/\varepsilon+\phi)^{-(\varphi+s)}}{(1+w+\psi/\varepsilon+\phi)^{-(\varphi+s)}}\right)$$
$$= \frac{-(\varphi+s)}{2\psi}\ln\left(\frac{1+w-\psi/\varepsilon+\phi}{1+w+\psi/\varepsilon+\phi}\right). \tag{E.4}$$

Data Availability

The data considered were obtained from the study by Lee and Wang [47].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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