Hindawi
Journal of Mathematics
Volume 2023, Article ID 5459916, 8 pages
https://doi.org/10.1155/2023/5459916

Research Article

On Fixed Point Convergence Results for a General Class of Nonlinear Mappings with a Supportive Application

Fayyaz Ahmad,[1] Kifayat Ullah ,[1] Junaid Ahmad ,[2] Om Kalthum S. K. Mohamed ,[3,4] and Awad A. Bakery [3,5]

[1]*Department of Mathematical Sciences, University of Lakki Marwat, Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan*
[2]*Department of Mathematics, International Islamic University, H-10, Islamabad 44000, Pakistan*
[3]*University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia*
[4]*Academy of Engineering and Medical Sciences, Department of Mathematics, Khartoum, Sudan*
[5]*Ain Shams University, Faculty of Science, Department of Mathematics, Cairo, Abbassia, Egypt*

Correspondence should be addressed to Om Kalthum S. K. Mohamed; om_kalsoom2020@yahoo.com

Received 30 December 2022; Revised 24 January 2023; Accepted 25 January 2023; Published 15 February 2023

Academic Editor: Valerii Obukhovskii

In this article, we considered the class of generalized (α, β)-nonexpansive (GABN) mappings that properly includes all nonexpansive, Suzuki nonexpansive (SN), generalized α-nonexpansive (GAN), and Reich–Suzuki nonexpansive (RSN) mappings. We used the iterative scheme JA for finding fixed points of these mappings in a Banach space setting. We provided both weak and strong convergence results under some mild conditions on the mapping, domain, and on the parameters involved in our iterative scheme. To support these results numerically, we constructed a new example of GABN mappings and proved that the JA iterative scheme converges to its fixed point. Moreover, we proved that JA iterative scheme converges faster to the fixed point corresponding to the some other iterative schemes of the literature. Eventually, we carried out an application of our main outcome to solve a split feasibility of problems (SFPs) in the setting of GABN mappings. Thus, our results were new in the literature and improved well-known results of the literature.

1. Introduction

We know that some time many analytical methods fail to find exact solution of the problems, therefore, fixed point theory recommends some alternate techniques for solving these problems. First, we expressed the solution of the problem in the fixed point of a certain map (the map may be contraction, nonexpansive, or generalized nonexpansive [1]). In this situation, an existence of a solution and the existence of a fixed point have the same meanings. We used some suitable iterative methods to find approximate unique fixed point (see, e.g., [2] and others). The Banach Contraction Principle (BCP) indicates, among other things, that if the operator is a contraction and the subset is a closed subset of a Banach space, there may be a single fixed point. As an operator P on a subset V of a Banach space B is called a contraction if, for all $v, w \in V$, we have the following equation:

$$\|Pv - Pw\| \leq \alpha \|v - w\|, \tag{1}$$

where $\alpha \in [0, 1)$.

A nonexpansive mapping P is a mapping that satisfies equation (1) for $\alpha = 1$. Actually, the Picard iteration method ($v_{r+1} = Pv_r$) was suggested by the BCP proof to determine the estimated value of the unique fixed point of the contraction P. If V is closed bounded and convex and B is a uniformly convex Banach space (UCBS), then P has a fixed point (see, e.g., Kirk [3], Browder [4], and Gohde [5]). In applied sciences, nonexpansive mappings have a key role for solving fixed points problems. Therefore, we tried to use some extensions of these mappings.

Suppose P is a self-map, that is, $P\colon V \longrightarrow V$ and $v, w \in V$, where V is any subset of a Banach space, then P is called as follows:

(a) SN [6] if $(1/2)\|v - Pv\| \leq \|v - w\| \Rightarrow \|Pv - Pw\| \leq \|v - w\|$

(b) GAN [7] (GAN, for short) if $(1/2)\|v - Pv\| \leq \|v - w\| \Rightarrow \|Pv - Pw\| \leq \alpha\|w - Pv\| + \alpha\|v - Pw\| + (1 - 2\alpha)\|v - w\|$

(c) RSN [8] (RSN, for short) if $(1/2)\|v - Pv\| \leq \|v - w\| \Rightarrow \|Pv - Pw\| \leq \beta\|v - Pv\| + \beta\|w - Pw\| + (1 - 2\beta)\|v - w\|$

The classes of RSN and GAN self-maps properly include the class of SN self-maps. It is very natural to ask that whether there exists any class of mappings that includes all the classes of mappings mentioned above.

To answer the above question in affirmative, Ullah et al. [1] presented a new class of nonlinear mappings that includes all the above mappings.

Definition 1 (see [1]). A self-map P on a subset V of a Banach space is said to be GABN when, for all $v, w \in V$, one can find two real constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ such that

$$\left(\frac{1}{2}\right)\|v - Pv\| \leq \|v - w\| \Rightarrow \|Pv - Pw\| \leq \alpha\|v - Pw\| + \alpha\|w - Pv\| + \beta\|v - Pv\| + \beta\|w - Pw\| + (1 - 2\alpha - 2\beta)\|v - w\|. \quad (2)$$

The class of GABN mappings includes all these mappings, and thus, the concept of GABN mappings is more difficult but more important than the other mappings mentioned above. The purpose of this work is to carry out some new fixed point convergence results under an effective iterative scheme. Using these results, we show that a SFP can be solved. Eventually, our main results will be supported by a numerical example. Consequently, our results were new in the fixed point literature and improved the well-known results of many authors.

2. Preliminaries

The purpose of this section is to present some earlier results and definitions that will be used in our main outcome.

Definition 2 (see [9]). If $\{v_r\}$ denotes any weakly convergent sequence in B, then a Banach space B is said to be with Opial's condition if, for all $v' \in B - \{v\}$, we have the following equation:

$$\liminf_{r\longrightarrow\infty}\left\|v_r - v\right\| < \liminf_{r\longrightarrow\infty}\left\|v_r - v'\right\|, \quad (3)$$

where v is the weak limit of V.

Definition 3 (see [10, 11]). Suppose $\{v_r\}$ denotes any bounded sequence in a closed convex subset V of a UCBS B. In this case, one denotes and defines the asymptotic radius of $\{v_r\}$ on the set V by $r(V, \{v_r\}) = \inf\left\{\limsup_{r\longrightarrow\infty}\left\|v_r - v\colon v \in V\right\}$, while the asymptotic center of $\{v_r\}$ on the set V is denoted and defined as $A(V, v_r\}) = \{v \in V\colon \limsup_{r\longrightarrow\infty}\|v_r - v\| = r(V, v_r)\}$. Moreover, the set $A(V, \{v_r\})$ contains only one element.

Now, we present some propositions and lemmas to understand the characterize of GABN maps.

Proposition 1. *Let V be a nonempty subset of a Banach space B and P be a self-map. Then,*

(a) When P is SN, then it follows that P is generalized (0,0)-nonexpansive

(b) When P is GAN, then it follows that P is generalized $(\alpha, 0)$-nonexpansive

(c) When P is β–RSN, then it follows that P is generalized $(0, \beta)$-nonexpansive

The following key results are from [1].

Lemma 1. *Suppose that P form a GABN self-map on a subset V of a Banach space B. Then,*

(i) If P admits at least one fixed point, then $\|Pq - Pv\| \leq \|q - v\|$ for all $q \in F(P)$ and for all $v \in V$.

(ii) By choosing v and w in V, the following estimate holds:

$$\|v - Pw\| \leq \left(\frac{3 + \alpha + \beta}{1 - \alpha - \beta}\right)\|v - Pv\| + \|v - w\|. \quad (4)$$

(iii) If B satisfies Opial's condition, then one has the following equation:

$$\{v_r\} \subseteq V, \quad v_r \rightharpoonup q, \ \left\|v_r - Pv_r\right\| \longrightarrow 0 \Rightarrow Pq = q. \quad (5)$$

The following lemma is well-known in the literature and can be found in [12].

Lemma 2. *Suppose B be a UCBS and $0 < a \leq k_r \leq b < 1$ for all $m \geq 1$. Considering $\{w_r\}$ and $\{s_r\}$ sequences in UCBS Bsatisfying the conditions $\limsup_{r\longrightarrow\infty}\|w_r\| \leq \gamma$, $\limsup_{r\longrightarrow\infty}\|s_r\| \leq \gamma$, and $\lim_{r\longrightarrow\infty}\|k_r w_r + (1 - k_r)s_r\| = \gamma$ for any real number $\gamma \geq 0$, we have $\lim_{r\longrightarrow\infty}\|w_r - s_r\| = 0$.*

3. Convergence Theorems in UCBS

Unlike contractions, the Picard iteration is not suitable for nonexpansive mappings [13]. There are simple examples in the literature that shows that if a nonexpansive mapping has a unique fixed point, then it is possible that the Picard may not converge to the fixed point. Thus, in the case of nonexpansive mappings, we studied iterative schemes that are more general than the Picard iteration. This section thus employs the JA iterative scheme of Abdeljawad et al. [14] to provide certain weak and strong convergence results for the class of generalized (α,β)-nonexpansive mapping in the linear setting of a UCBS. For $\alpha_r,\beta_r,\gamma_r\in(0,1)$, we collect some iterative processes of the literature as follows.

In 1953, Mann [15] suggested the following iteration:

$$\begin{cases} v_1 = v\in V,\\ v_{r+1}=(1-\alpha_r)v_r+\alpha_r P v_r. \end{cases}\tag{6}$$

In 1974, Ishikawa [16] generalized Mann iteration (6) to the setting of two steps as follows:

$$\begin{cases} v_1 = v\in V,\\ w_r=(1-\beta_r)v_r+\beta_r P v_r,\\ v_{r+1}=(1-\alpha_r)v_r+\alpha_r P w_r. \end{cases}\tag{7}$$

In 2000, Noor [16] generalized Ishikawa iteration (7) to the setting of three steps as follows:

$$\begin{cases} v_1 = v\in V,\\ s_r=(1-\gamma_r)v_r+\gamma_r P v_r,\\ w_r=(1-\beta_r)v_r+\beta_r P s_r,\\ v_{r+1}=(1-\alpha_r)v_r+\alpha_r P w_r. \end{cases}\tag{8}$$

In 2007, Agarwal et al. [17] suggested a new two-step iteration called S-iteration process as follows:

$$\begin{cases} v_1 = v\in V,\\ w_r=(1-\beta_r)v_r+\beta_r P v_r,\\ v_{r+1}=(1-\alpha_r)P v_r+\alpha_r P w_m. \end{cases}\tag{9}$$

In 2014, Abbas and Nazir [18] introduced the following new three-step iteration as follows:

$$\begin{cases} v_1 = v\in V,\\ s_r=(1-\gamma_r)v_r+\gamma_r P v_r,\\ w_r=(1-\beta_r)P v_r+\beta_r P w_r,\\ v_{r+1}=(1-\alpha_r)P w_r+\alpha_r P s_r. \end{cases}\tag{10}$$

In 2016, Thakur et al. [19] proposed the following new iteration:

$$\begin{cases} v_1 = v\in V,\\ s_r=(1-\beta_r)v_r+\beta_r P v_r,\\ w_r=P\big((1-\alpha_r)v_r+\alpha_r s_r\big),\\ v_{r+1}=P w_r. \end{cases}\tag{11}$$

In 2018, Ullah and Arshad [20] suggested M-iteration as follows:

$$\begin{cases} v_1 = v\in V,\\ s_r=(1-\alpha_r)v_r+\alpha_r P v_r,\\ w_r=P s_r,\\ v_{r+1}=P w_r. \end{cases}\tag{12}$$

Recently, Abdeljawad et al. [14] constructed a novel iteration as follows:

$$\begin{cases} v_1 = v\in V,\\ s_r=(1-\beta_r)v_r+\beta_r P v_r,\\ w_r=P s_r,\\ v_{r+1}=P\big((1-\alpha_r)P v_r+\alpha_r P w_r\big). \end{cases}\tag{13}$$

When we compute a fixed point of a certain nonlinear self-map, we try to use a faster iterative scheme in order to obtain a high accurate result in fewer steps of iterations. Agarwal et al. [17] used iterative scheme (9) for contractions and nonexpansive mappings and proved scheme (9) in this case is better than the Mann iteration (6). However, we note that Agarwal iteration (9) is a two-step iteration and it is known that three-step iteration is better than the one- and two-step iterations. Thus, in [18], Abbas and Nazir constructed a three-step iteration and proved that this new three-step iterative scheme converges better from the Agarwal iteration for nonexpansive mappings. After this, Thakur et al. [19] constructed iterative scheme (11) and proved that this scheme is better than the Abbas iteration for SN mappings. Thakur et al. [19] results were recently improved by Ullahn and Arshad [20] and Abdeljwad et al. [14] by constructing iterative schemes (12) and (13), respectively. In this article, we improved the results in [14, 19, 20] to the general context of GABN mappings.

Lemma 3. *Suppose that P is a GABN self-map on a convex closed subset V of a UCBS. If the fixed point set $F(P)$ is nonempty and $\{v_r\}$ is produced from JA iteration (13), then, subsequently, $\lim_{r\longrightarrow\infty}\|v_r-q\|$ exists for each choice of q in $F(P)$.*

Proof. Fixing any point $q\in F(P)$ and then applying Lemma 1 (i), one has the following equation:

$$\begin{aligned} \|s_r-q\| &= \|((1-\beta_r)v_r+\beta_r P v_r)-q\|\\ &\le \|(1-\beta_r)v_r+\beta_r P v_r-q\|\\ &\le (1-\beta_r)\|v_r-q\|+\beta_r\|P v_r-q\|\\ &\le (1-\beta_r)\|v_r-q\|+\beta_r\|v_r-q\|\\ &\le \|v_r-q\|,\\ \|w_r-q\| &= \|P s_r-q\|\\ &\le \|s_r-q\|, \end{aligned}\tag{14}$$

which implies that

$$\begin{aligned}\|v_{r+1}-q\| &= \|P((1-\alpha_r)Pv_r+\alpha_r Pw_r)-q\|\\ &\le \|(1-\alpha_r)Pv_r+\alpha_r Pw_r-q\|\\ &\le (1-\alpha_r)\|Pv_r-q\|+\alpha_r\|Pw_r-q\|\\ &\le (1-\alpha_r)\|v_r-q\|+\alpha_r\|w_r-q\|\\ &\le (1-\alpha_r)\|v_r-q\|+\alpha_r\|s_r-q\|\\ &\le (1-\alpha_r)\|v_r-q\|+\alpha_r\|v_r-q\|\\ &\le \|v_r-q\|.\end{aligned} \tag{15}$$

Hence, $\{\|v_r-q\|\}$ is bounded below and nonincreasing, so $\lim_{m\longrightarrow\infty}\|v_r-q\|$ exists for every $q\in F(P)$. □

Theorem 1. *Suppose that P is a GABN self-map on a convex closed subset V of a UCBS. Let $\{v_r\}$ be produced from the JA iteration (13). Then, subsequently, the fixed point set $F(P)$ is nonempty if and only if $\{v_r\}$ is bounded in V and satisfies the equation $\lim_{r\longrightarrow\infty}\|P(v_r)-v_r\|=0$.*

Proof. Let $F(P)\neq\emptyset$ and $q\in F(P)$. Then, by Lemma 3, $\lim_{r\longrightarrow\infty}\|v_r-q\|$ exists and $\{v_r\}$ is bounded. We put

$$\lim_{r\longrightarrow\infty}\|v_r-q\|=\gamma. \tag{16}$$

By the proof of Lemma 3 together with equation (16), we obtained the following equation:

$$\limsup_{r\longrightarrow\infty}\|s_r-q\|\le\limsup_{r\longrightarrow\infty}\|v_r-q\|=\gamma. \tag{17}$$

By Lemma 1 (i), we get the following equation:

$$\limsup_{r\longrightarrow\infty}\|Pv_r-q\|\le\limsup_{r\longrightarrow\infty}\|v_r-q\|=\gamma. \tag{18}$$

Also, by the proof of Lemma 3, we get the following equation:

$$\|v_{r+1}-q\|\le(1-\alpha_r)\|v_r-q\|+\alpha_r\|s_r-q\|. \tag{19}$$

Thus,

$$\begin{aligned}\|v_{r+1}-q\|-\|v_r-q\| &\le \frac{\|v_{r+1}-q\|-\|v_r-q\|}{\alpha_r}\\ &\le \|s_r-q\|-\|v_r-q\|.\end{aligned} \tag{20}$$

So, we obtained $\|v_{r+1}-q\|\le\|s_r-q\|$. Therefore,

$$\gamma\le\liminf_{r\longrightarrow\infty}\|s_r-q\|. \tag{21}$$

From equations (19) and (21), we get the following equation:

$$\gamma=\lim_{r\longrightarrow\infty}\|s_r-q\|. \tag{22}$$

From (22), we get the following equation:

$$\begin{aligned}\gamma &= \lim_{r\longrightarrow\infty}\|s_r-q\|\\ &= \lim_{r\longrightarrow\infty}\|((1-\beta_r)v_r+\beta_r Pv_r)-q\|\\ &\le \lim_{r\longrightarrow\infty}\|(1-\beta_r)v_r+\beta_r Pv_r-q\|\\ &= \lim_{r\longrightarrow\infty}\|(1-\beta_r)(v_r-q)+\beta_r(Pv_r-q)\|\\ &\le \lim_{r\longrightarrow\infty}(1-\beta_r)\|v_r-q\|+\lim_{r\longrightarrow\infty}\beta_r\|Pv_r-q\|\\ &\le \lim_{r\longrightarrow\infty}(1-\beta_r)\|v_r-q\|+\lim_{r\longrightarrow\infty}\beta_r\|v_r-q\|\le\gamma.\end{aligned} \tag{23}$$

Thus,

$$\begin{aligned}\gamma &= \lim_{r\longrightarrow\infty}\|s_r-q\|\\ &= \lim_{r\longrightarrow\infty}\|(1-\beta_r)(v_r-q)+\beta_r(Pv_r-q)\|.\end{aligned} \tag{24}$$

Since $0<\beta_r<1$ for all $r\ge1$, so using equation (16), (18), and (24) together with Lemma 1, we get the following equation:

$$\lim_{r\longrightarrow\infty}\|Pv_r-v_r\|=0. \tag{25}$$

Conversely, suppose that $\{v_r\}$ is bounded and $\lim_{r\longrightarrow\infty}\|Pv_r-v_r\|=0$ and $q\in A(V,\{v_r\})$. By Lemma 1 (ii), we have the following equation:

$$\begin{aligned}r(P(q),\{v_r\}) &= \limsup_{r\longrightarrow\infty}\|v_r-P(q)\|\\ &\le\left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)\limsup_{r\longrightarrow\infty}\|Pv_r-v_r\|\\ &+\limsup_{r\longrightarrow\infty}\|v_r-q\|\\ &=\limsup_{r\longrightarrow\infty}\|v_r-q\|\\ &= r(q,\{v_r\}).\end{aligned} \tag{26}$$

So, $Pq\in A(V,\{v_r\})$. As B is a UCBS, therefore, $A(V,\{v_r\})$ consists of a single point. Hence, $Pq=q$, that is, the fixed point set of P is nonempty. □

Theorem 2. *Suppose that P is a GABN self-map on a convex subset V of a UCBS. If the fixed point set $F(P)$ is nonempty and $\{v_r\}$ is produced from JA iteration (13), then, subsequently, $\{v_k\}$ converges (strongly) to some fixed point of P if the set V is compact.*

Proof. Since the subset V is convex and compact, we have a subsequence of the iterative sequence $\{v_r\}$ that we may denote by $\{v_{r_k}\}$ such that $v_{r_k}\longrightarrow q$, where q is a point in V. We prove that q is a fixed point of P and strong limit of the iterative sequence $\{v_r\}$ and that $v_{r_k}\longrightarrow Tq$. For this, we used

Theorem 1 and, hence, we get $\lim_{r\longrightarrow\infty}\|Pv_{r_k} - v_{r_k}\| = 0$. Hence, applying Lemma 1 (ii), one has the following equation:

$$\left\|v_{r_k} - Pq\right\| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)\left\|v_{r_k} - Pv_{r_k}\right\| + \left\|v_{r_k} - q\right\| \longrightarrow 0. \tag{27}$$

Thus, $q = Pq$, so q become a fixed point of P. Using Lemma 3, $\lim_{r\longrightarrow\infty}\|v_r - q\|$ exists. It now clear that q is the strong limit of the iterative sequence $\{v_r\}$. Hence, the proof is finished.

The next result does not require the compactness condition. This result is valid in general Banach space setting. □

Theorem 3. *Suppose that P is a GABN self-map on a convex closed subset V of a Banach space. If the fixed point set $F(P)$ is nonempty and $\{v_r\}$ is produced from JA iteration (13), then, subsequently, $\{v_k\}$ converges (strongly) to some fixed point of P if* $\liminf_{r\longrightarrow\infty} dist(v_r, F(P)) = 0$.

Definition 4 (see [21]). Suppose that P is a self-map on a convex closed subset V of a UCBS. Then, P is said to be with condition (I) if there is a function K with $K(0) = 0$ and $K(u) > 0$ for all $u \in (0,\infty)$ and all $v \in V$ imply that $\|v - P(v)\| \geq K(\mathrm{dist}(v, F(P)))$ for all $v \in V$.

Now, the strong convergence using condition (I) of P is as follows.

Theorem 4. *Suppose that P is a GABN self-map on a convex closed subset V of a UCBS. If the fixed point set $F(P)$ is nonempty and $\{v_r\}$ is produced from the JA iteration (13), then, subsequently, $\{v_k\}$ converges (strongly) to some fixed point of P if P is with condition (I).*

Proof. It follows from Theorem 1 that

$$\liminf_{r\longrightarrow\infty}\left\|Pv_r - v_r\right\| = 0. \tag{28}$$

As P satisfies condition I, so

$$\left\|v_r - Pv_r\right\| \geq k\left(\mathrm{dist}\left(v_r, F(P)\right)\right). \tag{29}$$

From equation (28), we get the following equation:

$$\liminf_{r\longrightarrow\infty} K\left(\mathrm{dist}\left(v_r, F(P)\right)\right) = 0. \tag{30}$$

Using the properties of K, we get the following equation:

$$\liminf_{r\longrightarrow\infty} \mathrm{dist}\left(v_r, F(P)\right) = 0. \tag{31}$$

Subsequently, we carry out all the requirements for Theorem 3, and hence, $\{v_r\}$ converges strong to some fixed point of P.

The last result of this section is the following weak convergence that is obtained under Opial's condition of the underlying domain. □

Theorem 5. *Suppose that P form a GABN self-map on a convex closed subset V of a UCBS B. If the fixed point set $F(P)$ is nonempty and $\{v_r\}$ is produced from JA iteration (13), then, subsequently, $\{v_k\}$ converges (weakly) to some fixed point of P if B is with Opial's condition.*

Proof. We notice that B is reflexive because UCBS is always reflexive. Now, since the sequence of iterates $\{v_r\}$ is bounded in V and so it has a convergent (weakly) subsequence, we denote it here by $\left\{v_{r_j}\right\}$ and its weak limit is denoted here by v_1. However, using Theorem 1, one has $\lim_{j\longrightarrow\infty}\|Pv_{r_j} - v_{r_j}\| = 0$. Thus, applying Lemma 1 (iii), the point v_1 becomes the fixed point for P. We now show that v_1 is a weak limit for $\{v_k\}$ and this will finish the proof. For this purpose, we assume on the contrary that v_1 is not a weak limit for $\{v_k\}$ and so we can find a new subsequence, namely, $\left\{v_{r_k}\right\}$ of $\{v_r\}$ such that $\left\{v_{r_k}\right\}$ is a weak convergent to some v_2 that is different from v_1. As shown earlier, we can show that v_2 is a fixed point of P. Now, using Opial's condition of P, it follows that

$$\begin{aligned}
\lim_{r\longrightarrow\infty}\left\|v_r - v_1\right\| &= \lim_{j\longrightarrow\infty}\left\|v_{r_j} - v_1\right\| < \lim_{j\longrightarrow\infty}\left\|v_{r_j} - v_2\right\| \\
&= \lim_{r\longrightarrow\infty}\left\|v_r - v_2\right\| = \lim_{k\longrightarrow\infty}\left\|v_{r_k} - v_2\right\| \\
&< \lim_{k\longrightarrow\infty}\left\|v_{r_k} - v_1\right\| = \lim_{r\longrightarrow\infty}\left\|v_r - v_1\right\|.
\end{aligned} \tag{32}$$

The above strict inequality suggests a contradiction because $v_2 \neq v_1$. Thus, we must accept that the sequence of iterates $\{v_r\}$ converges weakly to v_1. □

4. Example

Now, we offer a numerical example of GABN self-map as follows.

Example 1. We construct a self-map $P\colon [0,\infty) \longrightarrow [0,\infty)$ using the rule $Pv = (2v/3)$ if $(1/3) < v < \infty$ and $Pv = 0$ when $0 \leq v \leq (1/3)$. The first target is to show that P is GABN on its domain. We set $\alpha = \beta = (1/4)$. Then, we have three cases.

If $0 \leq v, w \leq (1/3)$, then we have the following equation:

$$\begin{aligned}
\frac{1}{4}|v - Pw| + \frac{1}{4}|w - Pv| + \frac{1}{4}|v - Pv| + \frac{1}{4}|w - Pw| &\geq 0 \\
&= |Pv - Pw|.
\end{aligned} \tag{33}$$

If $(1/3) < v, w < \infty$, then we have the following equation:

$$\begin{aligned}\frac{1}{4}|v-Pw|+\frac{1}{4}|w-Pv|+\frac{1}{4}|v-Pv|+\frac{1}{4}|w-Pw| &= \frac{1}{4}\left|v-\frac{2w}{3}\right|+\frac{1}{4}\left|w-\frac{2v}{3}\right| \\ &\quad+\frac{1}{4}\left|v-\frac{2v}{3}\right|+\frac{1}{4}\left|w-\frac{2w}{3}\right| \\ &\geq \frac{1}{4}\left|\frac{7v}{3}-\frac{7w}{3}\right|+\frac{1}{4}\left|\frac{v}{3}-\frac{w}{3}\right| \\ &\geq \frac{1}{4}\left|\frac{8v}{3}-\frac{8w}{3}\right| \\ &= \frac{2}{3}|v-w| \\ &= |Pv-Pw|.\end{aligned} \tag{34}$$

If $(1/3)<v<\infty$ and $0\leq w\leq(1/3)$, then we have the following equation:

$$\begin{aligned}\frac{1}{4}|v-Pw|+\frac{1}{4}|w-Pv|+\frac{1}{4}|v-Pv|+\frac{1}{4}|w-Pw| &= \frac{1}{4}|v|+\frac{1}{4}\left|w-\frac{2v}{3}\right|+\frac{1}{4}\left|v-\frac{2v}{3}\right|+\frac{1}{4}|w| \\ &= \frac{1}{4}|v|+\frac{1}{4}\left|w-\frac{2v}{3}\right|+\frac{1}{4}\left|\frac{v}{3}\right|+\frac{1}{4}|w| \\ &\geq \frac{1}{4}\left|\frac{8v}{3}\right| = \frac{2}{3}|v| = |Pv-Pw|.\end{aligned} \tag{35}$$

Hence, P is GABN with $\alpha=\beta=(1/4)$. However, for $v=(1/3)$ and $w=(2/3)$, we have $(1/2)|v-Pv|<|v-w|$. However,

(i) $|Pv-Pw|>|v-w|$

(ii) $|Pv-Pw|>(1/4)|v-Pw|+(1/4)|w-Pv|+(1-2(1/4))|v-w|$

(iii) $|Pv-Pw|>(1/4)|v-Pv|+(1/4)|w-Pw|+(1-2(1/4))|v-w|$

From (i)–(iii), we see that P does not belong to the classes of SN, GAN, and RSN mappings. We now set $\alpha_r=0.90$ and $\beta_r=0.65$ and compare the high accuracy our JA iteration in Table 1 and Figure 1. Clearly, our JA iterative scheme requires less steps to reach the fixed point.

5. Application to Split Feasibility Problems

Now, we consider Hilbert spaces, namely, B_1 and B_2 with $C\subseteq B_1$ and $Q\subseteq B_2$, compact and convex. If the operator $\mathscr{Y}\colon \mathscr{B}_1\longrightarrow\mathscr{B}_2$ bounded and linear, the SFP [22] is defined as follows:

$$\text{Search}\, q^{*}\in C\colon \mathscr{Y}q^{*}\in Q. \tag{36}$$

In our study, we target SFP (36) and assume that its solution set S is nonempty. In [23], it is shown that any point $q^{*}\in C$ solves our SFP (36) iff q^{*} solves essentially the following equation:

$$v=P_C\big(I-\eta\mathscr{Y}^{*}\big(I-P_Q\big)\mathscr{Y}\big)v, \tag{37}$$

where the set $P_{\mathscr{C}}$ (resp. the set $P_{\mathscr{Q}}$) is well-known in the literature and denotes the nearest point projection onto the set C (resp. onto the set Q) and $\eta>0$ and the mapping $\mathscr{Y}^{*}$ denotes the adjoint operator of $\mathscr{Y}$. Using the concept of nonexpansive self-maps, Byrne [24] was the first who proved that if $\eta>0$ with $0<\eta<(2/\xi)$, then the self-map

$$P=P_C\big(I-\eta\mathscr{Y}^{*}\big(I-P_Q\big)\mathscr{Y}\big), \tag{38}$$

is eventually nonexpansive, and its CQ iterative scheme is given by the following formula:

$$v_{r+1}=P_C\big(I-\eta\mathscr{Y}^{*}\big(I-P_Q\big)\mathscr{Y}\big)v_r, r\geq 0, \tag{39}$$

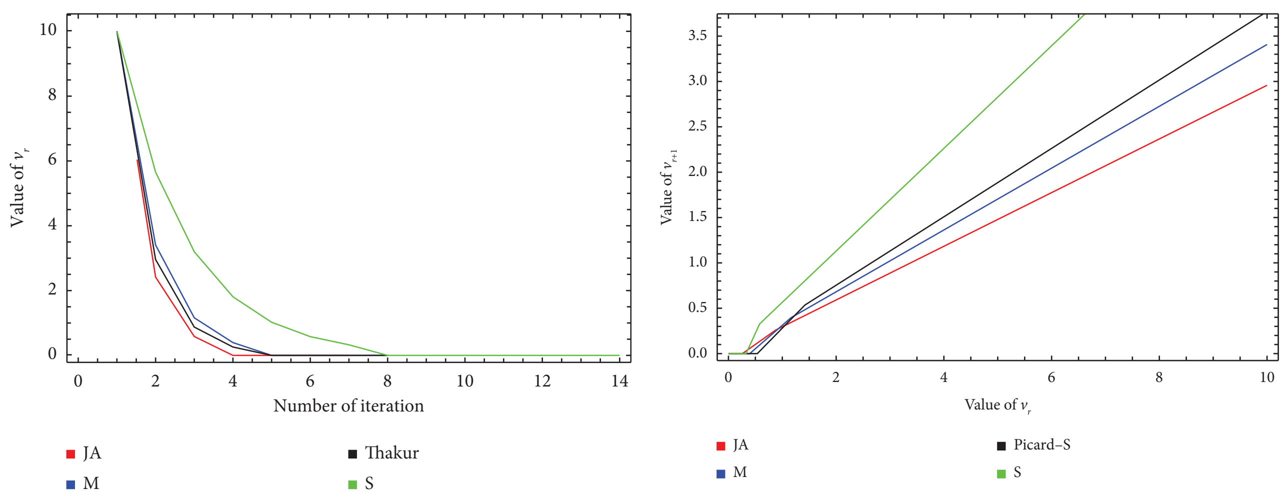
which converges (weakly) to some point of S.

It is not always an easy task to improve a weak convergence result to the setting of strong convergence. In previous literatures, authors used nonexpansive mappings for establishing a weak converges for SFPs. Our alternative approach in this article is to provide a strong convergence for SFPs using the concept of GABN mappings and new general projection type iterative scheme.

Theorem 6. *We now assume SFP (36) such that its solution set is nonempty and* $0<\eta<(2/\xi)$ *and* $P_C(I-\eta\mathscr{Y}^{*}(I-P_Q)\mathscr{Y})$ *is a GABN self-map. In this case, the sequence*

Table 1: Sequences generated by various iterations for Example 1.

r	JA (13)	M (12)	Thakur (11)	S (9)
1	10	10	10	10
2	2.4164609054	3.4074074074	2.9580246914	3.862500000
3	0.5839283307	1.1610425240	0.8749910075	5.655555555
4	0	0.3956144896	0.2588245004	3.198530864
5	0	0	0	1.808946899
6	0	0	0	1.023059968
7	0	0	0	0.578597249
8	0	0	0	0.327228888
9	0	0	0	0
10	0	0	0	0
11	0	0	0	0
12	0	0	0	0



(a) (b)

Figure 1: Various iteration schemes vs. our JA iterative scheme.

of iterates $\{v_r\}$ *(13) converges (strongly) to some solution of SFP (36).*

Proof. We take $P = P_C(I - \eta\mathscr{Y}^*(I - P_Q)\mathscr{Y})$, that is, the map P is generalized (α, β)-nonexpansive. In the view of Theorem 2, $\{v_r\}$ converges in the strong sense to some point of $F(P)$. Since $F(P) = S$, it follows that $\{v_r\}$ converges strongly to some solution of SFP (36). This finishes the proof. □

6. Conclusions

The following new results are obtained in this research:

(a) The JA iterative scheme is used for finding fixed points of GABN mappings

(b) Both weak and strong convergence results are obtained under possible mild assumptions

(c) We constructed a new example of GABN mappings

(d) We proved that JA iterative scheme is more effective than the other schemes of the literature

(e) We carried out an application of our work to solve a SFP in the setting of GABN mappings

(f) Accordingly, we improved and extended some recent and old results due to various authors in [14, 19, 20] from nonexpansive, SN, and GAN mappings to the general class of GABN mappings [25–27].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally.

Acknowledgments

This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant no. UJ-22-DR-66. The authors, therefore, acknowledge the University of Jeddah for its technical and financial support.

References

[1] K. Ullah, J. Ahmad, and M. d. l. Sen, "On generalized non-expansive maps in Banach Spaces," *Computation*, vol. 8, no. 3, p. 61, 2020.
[2] D. Ali, A. Hussain, E. Karapinar, and P. Cholamjiak, "Efficient fixed-point iteration for generalized nonexpansive mappings and its stability in Banach spaces," *Open Mathematics*, vol. 20, no. 1, pp. 1753–1769, 2022.
[3] W. A. Kirk, "A fixed point theorem for mappings which do not increase distance," *The American Mathematical Monthly*, vol. 72, pp. 1004–1006, 1965.
[4] F. E. Browder, "Nonexpansive nonlinear operators in a Banach space," *Proceedings of the National Academy of Sciences*, vol. 54, no. 4, pp. 1041–1044, 1965.
[5] D. Gohde, "Zum prinzip der Kontraktiven abbildung," *Mathematische Nachrichten*, vol. 30, no. 3-4, pp. 251–258, 1965.
[6] T. Suzuki, "Fixed point theorems and convergence theorems for some generalized nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 2, pp. 1088–1095, 2008.
[7] R. Pant and R. Shukla, "Approximating fixed points of generalized α-non-expansive mappings in Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 38, pp. 248–266, 2017.
[8] R. Pant and R. Pandey, "Existence and convergence results for a class of non-expansive type mappings in hyperbolic spaces," *Applied General Topology*, vol. 20, no. 1, pp. 281–295, 2019.
[9] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, no. 4, pp. 591–597, 1967.
[10] W. Takahashi, *Nonlinear Functional Analysis*, Yokohoma Publishers, Yokohoma, Japan, 2000.
[11] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications Series; Topological Fixed Point Theory and Its Applications*, Springer, New York, NY, USA, 2009.
[12] J. Schu, "Weak and strong convergence to fixed points of asymptotically non-expansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.
[13] A. Hussain, D. Ali, and E. Karapinar, "Stability Data Dependency and Errors Estimation for a General Iteration Method," *Alexandria Engineering Journal*, vol. 60, 2021.
[14] T. Abdeljawad, K. Ullah, and J. Ahmad, "Iterative algorithm for mappings satisfying $(B_{\gamma,\mu})$ condition," *Journal of Function Spaces*, vol. 2021, Article ID 3492549, 2021
[15] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, no. 3, pp. 506–510, 1953.
[16] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, no. 1, pp. 147–150, 1974.
[17] R. P. Agarwal, D. O'Regon, and D. R. Sahu, "Iterative construction of fixed points of nearly asymtotically non-expansive mappings," *J. Nonlinear Convex Anal.*vol. 8, pp. 61–79, 2007.
[18] M. Abbas and T. Nazir, "A new faster iteration process applied to constrained minimization and feasibility problems," *Matematicki Vesnik*, vol. 66, pp. 223–234, 2014.
[19] B. S. Thakur, D. Thakur, and M. Postolache, "A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized non-expansive mappings," *Applied Mathematics and Computation*, vol. 275, pp. 147–155, 2016.
[20] K. Ullah and M. Arshad, "Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process," *Filomat*, vol. 32, no. 1, pp. 187–196, 2018.
[21] H. F. Senter and W. G. Dotson, "Approximating fixed points of nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 44, no. 2, pp. 375–380, 1974.
[22] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2, pp. 221–239, 1994.
[23] G. Lopez, V. m. Martin-Ma Themrquez, and H. K. Xu, "Halpern's iteration for nonexpansive mappings," in *Proceedings of the Nonlinear Analysis and Optimization I. Nonlinear Analysis. A Conference In Celebration Of Alex Ioffe'S 70th and Simeon Reich's 60th Birthdays*, pp. 211–231, Haifa, Israel, June 2008.
[24] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.
[25] K. Aoyama and F. Kohsaka, "Fixed point theorem for α-nonexpansive mappings in Banach spaces," *Nonlinear Anal*, vol. 74, pp. 4387–4391, 2011.
[26] J. A. Clarkson, "Uniformly convex spaces," *Transactions of the American Mathematical Society*, vol. 40, no. 3, pp. 396–414, 1936.
[27] M. A. Noor, "New approximation schemes for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 217–229, 2000.