Hindawi
Journal of Mathematics
Volume 2023, Article ID 5467353, 25 pages
https://doi.org/10.1155/2023/5467353

Research Article

Applications on Bipolar Vague Soft Sets

Hanan H. Sakr,[1] **Abdisalam Hassan Muse**,[2] **Mohamed S. Mohamed**,[3] **and Saieed F. Ateya**[3]

[1]*Mathematics Department, Faculty of Education, Ain Shams University, Cairo 11341, Egypt*
[2]*Mathematics (Statistics Option) Program, Pan African University, Institute for Basic Science, Technology and Innovation (PAUSTI), Nairobi 6200-00200, Kenya*
[3]*Department of Mathematics & Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia*

Correspondence should be addressed to Hanan H. Sakr; hananhassan@edu.asu.edu.eg and Abdisalam Hassan Muse; abdisalam.hassan@amoud.edu.so

Received 15 November 2021; Revised 31 January 2022; Accepted 18 February 2022; Published 31 March 2023

Academic Editor: Jun Ye

The purpose of this research is to interpolate bipolarity into the definition of the vague soft set. This gives a new more applicable, flexible, and generalized extension of the soft set, the fuzzy soft set, or even the vague soft set, which is the bipolar vague soft set. In addition, types of bipolar vague soft sets, as well as some new related concepts and operations are established with examples. Moreover, properties of bipolar vague soft sets including absorption, commutative, associative, distributive, and De Morgan's laws are discussed in detail. Furthermore, a bipolar vague soft set-designed decision-making algorithm is provided generalizing Roy and Maji method. This allows making more effective decisions to choose the optimal alternative. Finally, an applied problem is introduced with a comparative analysis to illustrate how the proposed algorithm works more successfully than the previous models for problems that contain uncertain ambiguous data.

1. Motivation and Introduction

Decision-making is a technique to identify and select alternatives, based on individual preferences that are mostly used at the manager level of any business. Every decision environment is described as a combination of data, replacement options, values, and choices available at the moment of the choice. Because both information and its replacements are limited by the work and time required to gather statistics or find alternatives, any conclusions reached must be made within such a restricted context.

Nowadays, because of its close link to success and effectiveness, decision-making has become one of the most crucial components of life and work. Effective and efficient decision-making is how successful people attain their life and career goals. Individual views, values, and attitudes, as well as concepts, are frequently used to govern decision-making. While a person can make judgments based on a variety of concepts, they should be very careful to choose one that is effective and adds to great achievement. Nonetheless, these ideas exist to assist a person in becoming a better decision-maker in the world. In recent years, the decision-making problem in an uncertain environment has gained prominence.

Uncertainty and ambiguity are the most common sources of complexity, while trying to make decisions, in the real world. In many essential applications, such as economics, corporate management, engineering, medical science, environmental research, sociology, and many more, uncertain data are inherent and widespread, especially when applying decision-making. This uncertain data is caused by delayed data updates, information incompleteness, data randomness, measuring instruments limitations, etc.

In literature, there is a large number of studies and applications about many special mathematical tools including probability theory, fuzzy sets [1], intuitionistic fuzzy set theory [2], vague set theory [3], soft set theory [4], and other mathematical tools which are useful approaches to model uncertain data and make effective useful decisions. However, all of them have their own difficulties in dealing with uncertainty. The probability theory is an old and effective technique for dealing with uncertainty, but it can only be applied to situations with random processes, that is, processes in which the occurrence of events is solely governed by chance.

In 1937, Black [5] introduced the vagueness of a term as it is shown by producing "borderline cases" to model uncertain data in many important applications. For example, we may imagine a set of "chairs" with minor differences in quality on display at some implausible museum of applied logic. A Chippendale chair may be at one end of a long line with thousands of exhibits, while a little, unnoticed lump of wood could be at the other. Any typical viewer viewing the series will have a difficult time distinguishing between a chair and anything else. Indeed, the demand that this operation is performed is deemed in principle inappropriate; "chair is not the kind of word that admits of such a sharp distinction," is the kind of response that is given, and if it were, if we were forbidden to use it for any object that differed even slightly from the limiting term, it would not be as useful to us as it is. This is a sensible method, however, it causes logical complications.

After that, Zadeh [1] established the theory of the fuzzy sets, in 1965, as an extension of crisp sets for expressing and coping with uncertainty. A fuzzy set is a collection of items that has a range of membership grades. A membership (characteristic) function that assigns a grade of membership to each object ranging from zero to one characterizes such a set. To overcome the difficulty caused by fuzzy set theory, Gau and Buehrer [3] introduced a new extension of the set theory called vague set theory in 1993, based on Black's definition of vagueness. Instead of a single value, they assigned each element a membership grade that is a subinterval of $[0, 1]$. This subinterval keeps track of both the supporting and opposing evidence.

However, all of the aforementioned theories have inherent difficulties due to the inadequacy of the theories' parametrization tools as stated in [4]. So, the softness concept, initiated then in 1999 by Molodtsov [4], has been considered for a long time, as an effective new mathematical tool that facilitates treating uncertainties in decision-making problems. Although of this progress, it has been discovered that applying the soft sets in representing parameters' vagueness, in problems, is difficult. For this reason, many efforts have been made to develop decision-making techniques using hybrid sets' extensions like fuzzy soft sets (FSSs) [6], vague soft sets (VSSs) [7] and many others. Roy and Maji [6] have developed an effective method for determining the best item to purchase from a large number of options using fuzzy soft theory. For more details about those efforts and many others not stated in this section, one can refer to [8–10], and [11].

On the other hand, the human mind's inclination to reason and make judgments based on positive and negative influences causes bipolarity. It expresses the fact that, in addition to ranking pieces of information or actions in terms of plausibility, utility, and so on, the human mind relies on absolute landmarks with positive and negative flavor, as well as a third landmark expressing neutrality or indifference, which corresponds to the positive-negative zone's boundary. People, for example, assess the benefits and drawbacks of several options while making decisions. Then they make a decision based on whether the good or bad sides are stronger. The importance of bipolar reasoning in human cognitive activities has been highlighted by cognitive psychology research. Positive and negative emotions do not appear to be processed in the same area of the brain.

So that, it has been clarified that the theory of fuzzy set, vague set, or soft set is no longer a suitable instrument for treating bipolarity, just like a meal that is not sweet does not have to be sour. To overcome these difficulties, the concept of bipolar fuzzy sets was developed by Lee [12], in 2000, as a new generalization of the fuzzy sets. After that, the theory of bipolar fuzzy soft sets was proposed by Abdullah et al. [13] as an effective mathematical tool for dealing with bipolarity and data fuzziness together. But almost all the time, the vagueness of the related parameters cannot be described by the idea of the bipolar fuzzy soft set. From this point arose the need for a more generalized concept which is the bipolar vague soft set.

The chief motivation of this study is to overcome the limitations of the previous tools in decision-making problems. So, in this article, the bipolar vague soft set (BVSS), its types, operations, properties and applications are introduced with an illustrative example on each. A bipolar vague soft sets-based decision-making technique is designed, which extends Roy and Maji strategy and helps us to make more successful conclusions when picking the correct choice. An actual issue is shown along with a comparison study to show how the suggested technique improves existing models for situations with uncertain data.

The rest of the article is constructed as follows: Section 2 gives a short overview of some related works. Section 3 is set up to give the basic important definitions, concepts and preliminaries. After that, in Section 4, the bipolar vague soft set is defined with its types and some new related concepts and operations with examples. Furthermore, the aim of Section 5 is to present general properties, absorption properties, commutative properties, associative properties, distributive laws and De Morgan's laws with proof on each property. Moreover, the aim of Section 6 is to establish a generalized algorithm for Roy and Maji method based on the bipolar vague soft sets to determine the optimal alternative among others given in a decision-making problem. Finally, Section 7 gives open questions for further investigations and concluding remarks.

2. Related Work

This section provides a brief overview of the proposed three basic set theory's extensions; fuzzy sets, soft sets and vague sets, and a few selected hybrid sets that occur as a result of combining two or more extensions of sets, as well as some of their potential present applications.

2.1. Fuzzy Sets (FSs). In 1965, Zadeh [1] introduced an extension of ordinary (crisp) sets for describing uncertainty and dealing with it, namely the theory of the fuzzy sets (FSs). Just as an ordinary set on the initial universal set X is determined by its membership function from X to $\{0, 1\}$, in fuzzy set theory, an element's membership degree is determined by its membership percentage (the characteristic function from the domain X to the interval $[0,1]$).Inclusion, intersection, union, complement, convexity, relation, and other ideas are extended to such new sets, and various features of these notions are demonstrated in the context of fuzzy sets. In fact, the concept of a fuzzy set is entirely non-statistical. Although the fuzzy set theory is a valuable mathematical tool for dealing with uncertainty, this single value (membership degree) combines evidence for and against element belonging without showing how much of each there is, i.e., the single number tells us nothing about its accuracy.

2.2. Intuitionistic Fuzzy Sets (IFSs). In 1986, Atanassov [2] gave the notion "intuitionistic fuzzy set" (IFS), which is a generalization of the word "fuzzy set", along with an example. The properties of various operations and relations over sets, as well as modal and topological operators defined over the set of intuitionistic fuzzy sets, are proved.

2.3. Vague Sets (VSs). In 1993, Gau and Buehrer [3] developed a new extension of the set theory called vague set (VS) theory, based on Black's idea of vagueness in 1937 [5], to overcome the difficulties posed by fuzzy set theory. Rather than a single value, they assigned each element a membership grade that is a subinterval of $[0,1]$. This subinterval maintains track of both the favoring and the opposing evidence. However, due to the inadequacies of the theories' parametrization tool, all of the aforementioned theories have intrinsic issues. In fact, the concept of vague sets is actually an extension or a generalization of the concept of fuzzy sets and intuitionistic fuzzy sets.

2.4. Soft Sets (SSs. In 1999, Molodtsov [4] suggested that one of the reasons for the above difficulties may be the parametrization tools' inadequacy of the above theories, so he introduced the softness concept. The softness concept or the soft set (SS) concept is a mathematical tool, free from those above difficulties, for dealing easier with uncertainties. After that, in 2002, Maji et al. ([14,15]) considered and studied the theory of soft set initiated by Molodtsov. They discussed many notions in soft set theory, made a clear theoretical survey on soft sets in more detail and applied it in a decision-making problem.

Many efforts made to formulate crisp concepts in soft set settings as follows: Ali et al. [16] introduced many new definitions and concepts in the soft set theory. In addition, Sezgin and Atag *ü* n [17] established several novel theoretical operations in the soft set theory. Furthermore, Majumdar and Samanta [18] investigated the concept of soft mappings and Choudhure et al. worked on soft relation concept, and consequently used it for solving various decision-making issues. Moreover, Aktas and Ca $\breve{g}$ man [19] extended softness concept to group theory and defined the soft group concept. In addition, Feng et al. [20] applied and extended the soft set concept to semirings, Acar [21] initiated soft rings and Jun et al. extended softness concept to BCK/BCI-algebras ([22–24]). Also, Sezgin and Atag *ü* n [25] introduced the normalistic soft groups concept, Zhan et al. [26] defined the concept of soft ideal of BL-algebras and Kazancı et al. [27] applied softness concept to BCH-algebras. Moreover, Sezgin et al. [28] worked on soft near-rings and Ca $\breve{g}$ man et al. [29] defined group soft union and group soft intersection of a group (for more details, one can refer also to [30]).

In addition, many other researchers introduced new extended concepts based on soft sets in recent years, providing examples and studying their properties, such as: soft point [31], soft real numbers ([32,33]), soft complex numbers ([34]), soft metric spaces [35], soft normed spaces [36], soft inner product spaces [37] and soft Hilbert spaces [38]. Finally, to make it easier in dealing with soft sets, Ça$\breve{g}$ man et al. [39] established soft matrix theory and organized a model of soft decision-making.

2.5. Bipolar Fuzzy Sets (BFSs). In 2000, Lee [12] introduced the bipolar fuzzy set (BFS) concept as a novel extension or generalization of the fuzzy set concept. The membership degree range of an object, in this case, is expanded from the interval $[0,1]$ to the interval $[-1,1]$. In a bipolar valued fuzzy set, membership degree 0 means that elements are irrelevant to the corresponding property, membership degree $(0,1]$ means that elements partially satisfy the property, and membership degree $[-1,0)$ means that elements partially satisfy the implicit counter property. The appearance of the bipolar fuzzy sets and the intuitionistic fuzzy sets is similar. They are, nonetheless, distinct from one another.

2.6. Fuzzy Soft Sets (FSSs). In 2001, Maji et al. [40] defined fuzzy soft set (FSS) theory by entering the fuzzy sets ideas. In addition, Roy and Maji [6] proposed a decision-making technique to determine the optimal (best) choice of an object to buy among many objects based on fuzzy soft set. After that, Yang et al. [41] introduced a fuzzy soft set matrix representation and Ça $\breve{g}$ man et al. [42] studied the fuzzy soft matrices (FSMs), several algebraic operations and made a theoretical study in fuzzy soft settings.Later on, Basu et al. [43] and Kumar and Kaur [44] studied fuzzy soft matrices and established some new notions and operations on them. Finally, Faried et al. developed a fuzzy soft version of functional analysis by introducing a series of results as follows: FS inner product space [45], FS Hilbert space [46], FS linear operators [47], and FS spectral theory [48–51].

2.7. Intuitionistic Fuzzy Soft Sets (IFSSs). In 2004, Maji et al. [52] introduced intuitionistic fuzzy soft sets (IFSSs) as a new extension of soft sets. In addition, new operations on intuitionistic fuzzy soft sets were introduced, and some features of these operations were established. Furthermore, as an illustration of how this mathematical tool can be used, a basic example was provided. Then, Chetia et al. [53] initiated the intuitionistic fuzzy soft matrices (IFSMs) concepts to represent the intuitionistic fuzzy soft sets easily. They, also, defined their more functional operations in order to conduct theoretical research in intuitionistic fuzzy soft set theory, and established some results.

2.8. Vague Soft Sets (VSSs). In real-world, the mapping may be too vague, so that the fuzzy soft set concept or the intuitionistic fuzzy soft set concept fails in dealing with it, so we need a more general extension. In 2010, Xu et al. [54] proposed the vague soft set (VSS) concept and presented its general properties. In fact, vague soft set theory enables object world descriptions more accurate, practical and realistic making it a versatile tool at least in some cases. Recently, Wang [55] introduced many results on vague soft set theory and studied its associated properties and potential applications.

Moreover, Alhazaymeh and Hassan [56] initiated the vague soft set relations and functions concepts. In addition, Varol et al. [57] defined vague soft groups and Yin et al. [58] studied vague soft hemirings. Furthermore, Selvachandran and Salleh ([59,60]) introduced rings and ideals in the vague soft sets settings and established some algebraic hyperstructures of the vague soft set theory related to hyper-rings and hyper-ideals. At present, Inthumathi and Pavithra [61] and Faried et al. [7] established vague soft matrix notion, investigated its general properties, and discussed its novel applications.

2.9. Bipolar Fuzzy Soft Sets (BFSSs). In 2014, Abdullah et al. [13] investigated the bipolar fuzzy soft set (BFSS) concept and introduced their basic characteristics. In addition, basic operations on bipolar fuzzy soft sets were examined. Furthermore, they used the bipolar fuzzy soft set to overcome decision-making difficulties. It may appear that the bipolar fuzzy soft sets and the intuitionistic fuzzy soft sets are similar, but, in fact, there is a difference between them.

3. Definitions and Preliminaries

In this section, main notations, definitions, preliminaries and lemmas, which are needed in the sequel new results, are introduced.

Definition 1 (see [1]). Let Π be a universal set (space of points or objects). A fuzzy set (class) F over Π is a set characterized by a function $\eta_1\colon \Pi \longrightarrow [0,1]$. η_F is called the membership, characteristic or indicator function of the fuzzy set F and the value $\eta_F(u)$ is called the grade of membership of $\pi \in \Pi$ in $\mathscr{F}$. A fuzzy set $\mathscr{F}$ over a universal set Π can be represented by $F = \{(\eta_F(\pi)/\pi)\colon \pi \in \Pi, \eta_{\mathscr{F}}(\pi) \in [0,1]\}$, or $F = \{(\pi, \eta_F(\pi))\colon \pi \in \Pi, \eta_{\mathscr{F}}(\pi) \in [0,1]\}$.

Definition 2 (see [12]). A bipolar fuzzy set $\mathscr{B}$ over Π is defined by $\mathscr{B} = \{(\pi, \eta^+_{\mathscr{B}}(\pi), \eta^-_{\mathscr{B}}(\pi))\colon \pi \in \Pi\}$, where $\eta^+_{\mathscr{B}}\colon \Pi \longrightarrow [0,1]$ is the positive membership degree denotes the satisfaction degree of u to the property corresponding to $\mathscr{B}$, and $\eta^-_{\mathscr{B}}\colon \Pi \longrightarrow [-1,0]$ is the negative membership degree denotes the satisfaction degree of u to some implicit counter-property of $\mathscr{B}$.

Definition 3 (see [3]). Given the universal set $\Pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$, a vague set V over Π is a set determined by a truth membership function τ_V and a false membership function η_V. The exact grade of membership of $\pi \in \Pi(\mu_V(\pi))$ belongs to an interval $[\tau_V(\pi), 1-\eta_V(\pi)] \subseteq [0,1]$, i.e., $\mu_V(\pi)$ may be unknown, but it is bounded by $\tau_V(\pi) \leq \mu_V(\pi) \leq 1-\eta_V(\pi)$, where $\tau_V(\pi) + \eta_V(\pi) \leq 1$ and $\tau_V, \eta_V\colon \Pi \longrightarrow [0,1]$. We can represent V as $V = \{(\tau_V(\pi), 1-\eta_V(\pi)/\pi)\colon \pi \in \Pi, \tau_V(\pi), \eta_V(\pi) \in [0,1]\}$, or $V = \{\pi[\tau_V(\pi), 1-\eta_V(\pi)]\colon \pi \in \Pi, \tau_V(\pi), \eta_V(\pi) \in [0,1]\}$.

Definition 4 (see [62]). A bipolar vague set $\mathscr{V}$ over the universal set Π is defined by $\mathscr{V} = \{(\pi, [\tau^+_{\mathscr{V}}(\pi), 1-\eta^+_{\mathscr{V}}(\pi)], [-1-\eta^-_{\mathscr{V}}(\pi), \tau^-_{\mathscr{V}}(\pi)])\colon \pi \in \Pi\}$, where $\tau^+_{\mathscr{V}}, \eta^+_{\mathscr{V}}\colon \Pi \longrightarrow [0,1]$ are the positive truth and false membership functions denote the satisfaction degree of an element u to the property corresponding to $\mathscr{V}$, such that $\tau^+_{\mathscr{V}} + \eta^+_{\mathscr{V}} \leq 1$, and $\tau^-_{\mathscr{V}}, \eta^-_{\mathscr{V}}\colon \Pi \longrightarrow [-1,0]$ are the negative truth and false membership functions denote the satisfaction degree of π to some implicit counter-property of $\mathscr{V}$, such that $\tau^-_{\mathscr{V}} + \eta^-_{\mathscr{V}} \geq -1$, i.e., the intervals $[\tau^+_{\mathscr{V}}(\pi), 1-\eta^+_{\mathscr{V}}(\pi)]$ and $[-1-\eta^-_{\mathscr{V}}(\pi), \tau^-_{\mathscr{V}}(\pi)]$ denote the satisfaction region of u to the property corresponding to $\mathscr{V}$ and to some implicit counter-property of $\mathscr{V}$, respectively.

Definition 5 (see [4]). Let Π be a universal set, Υ be a set of parameters (or attributes), and $\Lambda \subseteq \Upsilon$. The power set of Π is defined by $P(\Pi) = 2^{\Pi}$. A pair (Γ, Λ) or Γ_Λ is called a soft set over Π, where Γ is a mapping given by $\Gamma\colon \Lambda \longrightarrow P(\Pi)$. Also, Γ_Λ may be written as a set of ordered pairs $\Gamma_\Lambda = \{(\lambda, \Gamma_\Lambda(\lambda))\colon \lambda \in \Lambda, \Gamma_\Lambda(\lambda) \in P(\Pi)\}$. Λ is called the support of Γ_Λ and we have $\Gamma_\Lambda(\lambda) \neq \phi$ for all $\lambda \in \Lambda$ and $\Gamma_\Lambda(\lambda) = \phi$ for all $\lambda \notin \Lambda$. In other words, a soft set (Γ, Λ) over Π is a parameterized family of subsets of the set Π.

Lemma 1. *If $\mathscr{V}(\Pi)$ is the set of all vague sets over the universe Π, then, for all $V \in \mathscr{V}(\Pi)$, $\pi \in \Pi$ and its vague value $[\tau_V(\pi), 1-\eta_V(\pi)]$, the fuzzy membership function $\eta_{V^F}(\pi)$, where V^F is the fuzzy set corresponding to the vague set V, is defined by:*

3.1. Method (1) [63].

$$\eta_{V^F}(\pi) = \frac{1+\tau_V(\pi)-\eta_V(\pi)}{2}. \tag{1}$$

The median idea is used in the development of Method (1). Calculating the corresponding median membership value of the corresponding true and false membership values can be used to determine the fuzzy set membership value (the corresponding vague set membership value). That is, the fuzzy value is defined as the whole quantity of evidence included in a vague value, which is represented by the median membership value.

3.2. Method (2) [63].

$$\eta_{V^F}(\pi) = \frac{\tau_V(\pi)}{\tau_V(\pi) + \eta_V(\pi)}. \quad (2)$$

The defuzzification function is used to derive Method (2). Calculating the associated defuzzification value of the respective true and false membership values can yield the fuzzy set membership value.

3.3. Method (3) [64].

$$\eta_{V^F}(\pi) = \tau_V(\pi) + \frac{1}{2} \times \left[1 + \frac{\tau_V(\pi) - \eta_V(\pi)}{\tau_V(\pi) + \eta_V(\pi) + 2}\right] \cdot \left[1 - \tau_V(\pi) - \eta_V(\pi)\right]. \quad (3)$$

The idea in Method (3) is to examine the mapping between the elements of vague sets and points on a plane. The transformation of a vague set into a fuzzy set is found to be a many-to-one mapping relation. It is also discovered to be a generic transformation model for turning vague set membership values to fuzzy set membership values.

For more details about the difference between Method (1), Method (2) and Method (3), and how they signify, one can refer to [63–65].

Lemma 2. *For any two real numbers α and β, we have that:*

(1) $\max\{\alpha, \min\{\alpha, \beta\}\} = \alpha$,

(2) $\min\{\alpha, \min\{\alpha, \beta\}\} = \alpha$

Proof. (1) and (2) are satisfied for all three possible cases ($\alpha > \beta$, $\alpha < \beta$, and $\alpha = \beta$).

Lemma 3. *For any three real numbers α, β and γ, the following results are satisfied:*

(1) $\max\{\alpha, \min\{\beta, \gamma\}\} = \min\{\max\{\alpha, \beta\}, \max\{\alpha, \gamma\}\}$

(2) $\min\{\alpha, \max\{\beta, \gamma\}\} = \max\{\min\{\alpha, \beta\}, \min\{\alpha, \gamma\}\}$

Proof. By checking satisfaction of the two parts of the Lemma in all possible cases ($\alpha > \beta > \gamma$, $\alpha > \gamma > \beta$, $\beta > \alpha > \gamma$, $\gamma > \alpha > \beta$, $\beta > \gamma > \alpha$, $\gamma > \beta > \alpha$, $\beta = \alpha > \gamma$, $\beta = \alpha < \gamma$, $\alpha > \beta = \gamma$, $\alpha < \beta = \gamma$, and $\alpha = \beta = \gamma$), it is found that this Lemma is true.

4. Bipolar Vague Soft Sets and Operations on Them

The purpose of this section is to introduce the definition of the bipolar vague soft sets and many related new concepts and operations on them with illustrative examples on each item.

Definition 6. Assume that Π is a universal set, Υ is a parameter set and $\Lambda \subseteq \Upsilon$. Then, a pair $(\sqsupset, \Lambda)$ given by $\sqsupset_\Lambda = \{(\lambda, \sqsupset_\Lambda(\lambda)) \colon \lambda \in \Lambda, \sqsupset_\Lambda(\lambda) \in \mathscr{BV}(\Pi)\}$ is said to be a bipolar vague soft set over the universal set Π, where $\sqsupset$ is a mapping defined as $\sqsupset\colon \Lambda \longrightarrow \mathscr{BV}(\Pi)$, $\mathscr{BV}(\Pi)$ is the power set of bipolar vague sets on Π (i.e., the family of all bipolar vague subsets of Π) and the bipolar vague subset of Π is the same as stated previously in Definition 4.

Definition 7. Any bipolar vague soft set $(\sqsupset, \Lambda)$ on the universal set Π is called a complete (or an absolute) bipolar vague soft set, stand for $\mathscr{C}_\Lambda$, if for all $\lambda \in \Lambda$, we have $\sqsupset_\Lambda(\lambda) = \mathscr{BV}(\Pi)$. That is to say that $\tau^+_{\sqsupset_\Lambda(\lambda)}(\pi) = 1$, $1 - \eta^+_{\sqsupset_\Lambda(\lambda)}(\pi) = 1$ (i.e., $\eta^+_{\sqsupset_\Lambda(\lambda)}(\pi) = 0$), $-1 - \eta^-_{\sqsupset_\Lambda(\lambda)}(\pi) = -1$ (i.e., $\eta^-_{\sqsupset_\Lambda(\lambda)}(\pi) = 0$) and $\tau^-_{\sqsupset_\Lambda(\lambda)}(\pi) = -1$, for all $\lambda \in \Lambda$ and for all $\pi \in \Pi$. According to those assumptions, we have $\mathscr{C}_\Lambda = \{(\lambda, \{(\pi, [1,1], [-1,-1])\}) \colon \lambda \in \Lambda, \pi \in \Pi\}$.

Definition 8. Any bipolar vague soft set $(\sqsupset, \Lambda)$ on the universal set Π is called a null (or an empty) bipolar vague soft set, stand for ϕ_Λ, if for all $\lambda \in \Lambda$, we have $\sqsupset_\Lambda(\lambda) = \phi$. That is to say that $\tau^+_{\sqsupset_\Lambda(\lambda)}(\pi) = 0$, $1 - \eta^+_{\sqsupset_\Lambda(\lambda)}(\pi) = 0$ (i.e., $\eta^+_{\sqsupset_\Lambda(\lambda)}(\pi) = 1$), $-1 - \eta^-_{\sqsupset_\Lambda(\lambda)}(\pi) = 0$ (i.e., $\eta^-_{\sqsupset_\Lambda(\lambda)}(\pi) = -1$) and $\tau^-_{\sqsupset_\Lambda(\lambda)}(\pi) = 0$, for all $\lambda \in \Lambda$ and for all $\pi \in \Pi$. According to those assumptions, we have $\phi_\Lambda = \{(\lambda, \{(\pi, [0,0], [0,0])\}) \colon \lambda \in \Lambda, \pi \in \Pi\}$.

Example 1. Assume that we have the bipolar vague soft set $(\sqsupset, \Upsilon)$ which describes the "attractiveness of cars" under the consideration of a decision maker Mr. X to purchase. Suppose that there are three cars to be considered in the universal set Π, denoted by π_1, π_2, π_3, i.e., $\Pi = \{\pi_1, \pi_2, \pi_3\}$. Let the two sets of attributes be $\Upsilon = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, $\lambda_i (i = 1, 2, 3, 4)$ and its opposite (counter) set $-\Upsilon = \{-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4\}$, $-\lambda_i (i = 1, 2, 3, 4)$ stand for the properties and the counter-properties that describe the cars. For instance, let each property and its counter-property $(\lambda_i, -\lambda_i)$, $(i = 1, 2, 3, 4)$ be in a word of: ("expensive," "cheap"), ("beautiful," "ugly"), ("up-to-date-technology," "classical-technology"), and ("in a good repair," "in a bad repair"), respectively. Thus, we express the bipolar vague soft set $(\sqsupset, \Upsilon)$ on Π by

$$\begin{aligned}
(\sqsupset, \Upsilon) &= \left(\lambda_1\{(\pi_1, [0.4, 0.5], [-0.8, -0.6]), (\pi_2, [0.2, 0.3], [-1, -0.7]), (\pi_3, [0, 0], [-1, -1])\}\right),\\
&= \left(\lambda_2\{(\pi_1, [0.5, 0.6], [-0.7, -0.4]), (\pi_2, [1, 1], [-1, -1]), (\pi_3, [0.9, 1], [-0.3, -0.1])\}\right),\\
&= \left(\lambda_3\{(\pi_1, [0, 0.1], [-0.1, 0]), (\pi_2, [0.4, 0.6], [-0.9, -0.8]), (\pi_3, [0.5, 0.7], [-0.2, -0.1])\}\right),\\
&= \left(\lambda_4\{[0.8, 0.9], [-0.3, -0.2], (\pi_2, [0, 0], [0, 0]), (\pi_3, [0.6, 0.9], [-1, -0.8])\}\right).
\end{aligned} \quad (4)$$

Example 2. Under bipolarity absence (i.e., if we do not consider the opposite set of attributes $-\Upsilon$) in Example 1, then the bipolar vague soft set is reduced to the following vague soft set $(\sqsupset, \Upsilon)$, defined by Xu et al. [54]:

$$(\sqsupset, \Upsilon) = \left\{ \begin{array}{c} (\lambda_1, \{(\pi_1, [0.4, 0.5]), (\pi_2, [0.2, 0.3]), (\pi_3, [0, 0])\}) \\ (\lambda_2, \{(\pi_1, [0.5, 0.6]), (\pi_2, [1, 1]), (\pi_3, [0.9, 1])\}) \\ (\lambda_3, \{(\pi_1, [0, 0.1]), (\pi_2, [0.4, 0.6]), (\pi_3, [0.5, 0.7])\}) \\ (\lambda_4, \{(\pi_1, [0.8, 0.9]), (\pi_2, [0, 0]), (\pi_3, [0.6, 0.9])\}) \end{array} \right\}. \tag{5}$$

Definition 9. The complement of a bipolar vague soft set $(\sqsupset, \Lambda)$ is defined by $(\sqsupset, \Lambda)^c = (\sqsupset^c, \Lambda)$, where $\sqsupset^c\colon \Lambda \longrightarrow \mathscr{BV}(\Pi)$ is a mapping given by $\tau^+_{\sqsupset^c(\lambda)}(\pi) = \eta^+_{\sqsupset(\lambda)}(\pi)$, $\eta^+_{\sqsupset^c(\lambda)}(\pi) = \tau^+_{\sqsupset(\lambda)}(\pi)$, $\tau^-_{\sqsupset^c(\lambda)}(\pi) = \eta^-_{\sqsupset(\lambda)}(\pi)$ and $\eta^-_{\sqsupset^c(\lambda)}(\pi) = \tau^-_{\sqsupset(\lambda)}(\pi)$, for all $\lambda \in \Lambda$ and for all $\pi \in \Pi$. According to those assumptions, we have $\sqsupset^c_\Lambda = \{(\lambda, \{(\pi, [\eta^+_{\sqsupset_\Lambda(\lambda)}(\pi), 1 - \tau_{\sqsupset_\Lambda}(\lambda)^+(\pi)], [-1 - \tau^-_{\sqsupset_\Lambda(\lambda)}(\pi), \eta^-_{\sqsupset_\Lambda(\lambda)}(\pi)])\})\colon \lambda \in \Lambda, \pi \in \Pi\}$.

Example 3. The complement of $(\sqsupset, \Upsilon)$ in Example 1 is as follows:

$$(\sqsupset^c, \Upsilon) = \left\{ \begin{array}{c} (\lambda_1, \{(\pi_1, [0.5, 0.6], [-0.4, -0.2]), (\pi_2, [0.7, 0.8], [-0.3, 0]), (\pi_3, [1, 1], [0, 0])\}) \\ (\lambda_2, \{(\pi_1, [0.4, 0.5], [-0.6, -0.3]), (\pi_2, [0, 0], [0, 0]), (\pi_3, [0, 0.1], [-0.9, -0.7])\}) \\ (\lambda_3, \{(\pi_1, [0.9, 1], [-1, -0.9]), (\pi_2, [0.4, 0.6], [-0.2, -0.1]), (\pi_3, [0.3, 0.5], [-0.9, -0.8])\}) \\ (\lambda_4, \{(\pi_1, [0.1, 0.2], [-0.8, -0.7]), (\pi_2, [1, 1], [-1, -1]), (\pi_3, [0.1, 0.4], [-0.2, 0])\}) \end{array} \right\}. \tag{6}$$

Definition 10. Assume that (Γ, Λ) and (Ψ, Θ) are two bipolar vague soft sets on a universal set Π. (Γ, Λ) is called a bipolar vague soft subset of (Ψ, Θ) if $\Lambda \subseteq \Theta$, and $\Gamma(\lambda) \subseteq \Psi(\lambda)$ for all $\lambda \in \Lambda$, that is to say that $\tau^+_{\Gamma(\lambda)}(\pi) \le \tau^+_{\Psi(\lambda)}(\pi)$, $1 - \eta^+_{\Gamma(\lambda)}(\pi) \le 1 - \eta^+_{\Psi(\lambda)}(\pi)$, $-1 - \eta^-_{\Gamma(\lambda)}(\pi) \ge -1 - \eta^-_{\Psi(\lambda)}(\pi)$ and $\tau^-_{\Gamma(\lambda)}(\pi) \ge \tau^-_{\Psi(\lambda)}(\pi)$, i.e., we have $\tau^+_{\Gamma(\lambda)}(\pi) \le \tau^+_{\Psi(\lambda)}(\pi)$, $\eta^+_{\Gamma(\lambda)}(\pi) \ge \eta^+_{\Psi(\lambda)}(\pi)$, $\eta^-_{\Gamma(\lambda)}(\pi) \le \eta^-_{\Psi(\lambda)}(\pi)$ and $\tau^-_{\Gamma(\lambda)}(\pi) \ge \tau^-_{\Psi(\lambda)}(\pi)$, for all $\lambda \in \Lambda$ and for all $\pi \in \Pi$. One can write $(\Gamma, \Lambda) \widetilde{\subseteq} (\Psi, \Theta)$. In this case, (Ψ, Θ) is called a bipolar vague soft superset of (Γ, Λ), denoted by $(\Psi, \Theta) \widetilde{\supseteq} (\Gamma, \Lambda)$.

Definition 11. Two bipolar vague soft sets (Γ, Λ) and (Ψ, Θ) on a common universal set Π are called bipolar vague soft equal if they are bipolar vague soft subsets of each other, i.e., $(\Gamma, \Lambda) \widetilde{\subseteq} (\Psi, \Theta)$ and $(\Psi, \Theta) \widetilde{\supseteq} (\Gamma, \Lambda)$.

Example 4. Consider the same Π and Υ in Example 1, and let $\Lambda = \{\lambda_1, \lambda_2\}$ and $\Theta = \{\lambda_1, \lambda_2, \lambda_3\}$ be two subsets of Υ. Then, we can define two bipolar vague soft sets (Γ, Λ) and (Ψ, Θ) on Π, respectively, as follows:

$$\begin{aligned} (\Gamma, \Lambda) &= \left\{ \begin{array}{c} (\lambda_1, \{(\pi_1, [0.4, 0.5], [-0.7, -0.5]), (\pi_2, [0.2, 0.3], [-0.9, -0.6]), (\pi_3, [0, 0], [-1, -1])\}), \\ (\lambda_2, \{(\pi_1, [0.5, 0.6], [-0.3, -0.2]), (\pi_2, [1, 1], [-1, -1]), (\pi_3, [0.9, 1], [0, 0])\}), \end{array} \right\}, \\ (\Psi, \Theta) &= \left\{ \begin{array}{c} (\lambda_1, \{(\pi_1, [0.5, 0.7], [-0.8, -0.6]), (\pi_2, [0.2, 0.3], [-1, -0.7]), (\pi_3, [0, 0], [-1, -1])\}), \\ (\lambda_2, \{(\pi_1, [0.5, 0.6], [-0.3, -0.2]), (\pi_2, [1, 1], [-1, -1]), (\pi_3, [1, 1], [-0.3, -0.1])\}), \\ (\lambda_3, \{(\pi_1, [0, 0.1], [-0.1, 0]), (\pi_2, [0.4, 0.6], [-0.9, -0.8]), (\pi_3, [0.5, 0.7], [-0.2, -0.1])\}), \end{array} \right\}. \end{aligned} \tag{7}$$

Since we have $\Lambda \subseteq \Theta$, and for all $\lambda \in \Lambda$ and for all $\pi \in \Pi$, we obtain that $\tau^+_{\Gamma(\lambda)}(\pi) \le \tau^+_{\Psi(\lambda)}(\pi)$, $\eta^+_{\Gamma(\lambda)}(\pi) \ge \eta^+_{\Psi(\lambda)}(\pi)$, $\eta^-_{\Gamma(\lambda)}(\pi) \le \eta^-_{\Psi(\lambda)}(\pi)$ and $\tau^-_{\Gamma(\lambda)}(\pi) \ge \tau^-_{\Psi(\lambda)}(\pi)$, i.e, $\Gamma(\lambda) \subseteq \Psi(\lambda)$ for all $\lambda \in \Lambda$. Then, $(\Gamma, \Lambda) \widetilde{\subseteq} (\Psi, \Theta)$.

Definition 12. The union of two bipolar vague soft sets (Γ, Λ) and (Ψ, Θ) on a common universal set Π is a bipolar vague soft set $(\sqsupset, \Delta)$, written as $(\sqsupset, \Delta) = (\Gamma, \Lambda) \widetilde{\cup} (\Psi, \Theta)$, where $\Delta = \Lambda \cup \Theta$ and for all $\delta \in \Delta$:

$$(\sqsupset,\Delta)=\begin{cases}=\left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\Gamma(\delta)}(\pi),1-\eta^{+}_{\Gamma(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\Gamma(\delta)}(\pi),\tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right),\pi\in\Pi\right\}, & if\ \delta\in\Lambda-\Theta,\\ =\left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\Psi(\delta)}(\pi),1-\eta^{+}_{\Psi(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\Psi(\delta)}(\pi),\tau^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right),\pi\in\Pi\right\}, & if\ \delta\in\Theta-\Lambda,\\ =\left\{\delta,\left(\begin{array}{l}\pi,\left[\max\left\{\tau^{+}_{\Gamma(\delta)}(\pi),\tau^{+}_{\Psi(\delta)}(\pi)\right\},\max\left\{1-\eta^{+}_{\Gamma(\delta)}(\pi),1-\eta^{+}_{\Psi(\delta)}(\pi)\right\}\right],\\ \left[\min\left\{-1-\eta^{-}_{\Gamma(\delta)}(\pi),-1-\eta^{-}_{\Psi(\delta)}(\pi)\right\},\min\left\{\tau^{-}_{\Gamma(\delta)}(\pi),\tau^{-}_{\Psi(\delta)}(\pi)\right\}\right],\pi\in\Pi\end{array}\right.\right\} & if\ \delta\in\Lambda\cap\Theta.\end{cases} \tag{8}$$

Example 5. Under assumptions of Example 4, we have the union bipolar vague soft set $(\sqsupset,\Delta)=(\Gamma,\Lambda)\tilde{\cup}(\Psi,\Theta)$, where $\Delta=\Lambda\cup\Theta=\{\lambda_1,\lambda_2,\lambda_3\}$ as follows:

$$\begin{aligned}(\sqsupset,\Delta)=&\{(\lambda_1,\{(\pi_1,[0.5,0.7],[-0.8,-0.6]),(\pi_2,[0.2,0.3],[-1,-0.7])\}),(\pi_3,[0,0],[-1,-1])\}\\ &((\lambda_2,\{(\pi_1,[0.8,0.9],[-0.7,-0.4]),(\pi_2,[1,1],[-1,-1])\}),(\pi_3,[-0.3,-0.1]))\\ &((\lambda_3,\{(\pi_1,[0,0.1],[-0.1,0]),(\pi_2,[0.4,0.6],[-0.9,-0.8])\}),(\pi_3,[0.5,0.7],[-0.2,-0.1])).\end{aligned} \tag{9}$$

Definition 13. The union of a family $\{(\Gamma_i,\Lambda_i)\colon i\in I\}$ of bipolar vague soft sets over a universal set Π is a bipolar vague soft set $(\sqsupset,\Delta)$, written as $\sqsupset_\Delta=(\sqsupset,\Delta)=\tilde{\cup}(\Gamma_i,\Lambda_i)$, where $\Delta=\cup\Lambda_i$, for all $i\in I$ defined as follows, for all $\delta\in\Delta$:

$$\sqsupset_\Delta(\delta)=\begin{cases}\Gamma_i(\delta),\ if\ \delta\in\Lambda_i-\underset{j\neq i}{\cup}\Lambda_j, & \text{for all } i\in I,\\ \underset{i\in I}{\cup}\Gamma_i(\delta),\ if\ \delta\in\underset{i\in I}{\cap}\Lambda_i.\end{cases} \tag{10}$$

Example 6. Suppose that $\Pi=\{\pi_1,\pi_2,\pi_3\}$ is a universal set, where π_1, π_2 and π_3 are three cars, and $\Upsilon=\{\lambda_1=$ "expensive", $\lambda_2=$ "beautiful", $\lambda_3=$ "up-to-date-technology", $\lambda_4=$ "in a good repair" $\}$ is the set of attributes. Let $\Lambda_1=\{\lambda_1,\lambda_2,\lambda_3\}$, $\Lambda_2=\{\lambda_1,\lambda_2\}$, $\Lambda_3=\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ and $\Lambda_4=\{\lambda_1\}$ be four subsets of Υ. Then, we can define four bipolar vague soft sets (Γ_1,Λ_1), (Γ_2,Λ_2), (Γ_3,Λ_3) and (Γ_4,Λ_4) on Π, respectively, as follows:

$$\begin{aligned}(\Gamma_1,\Lambda_1)&=\left\{\begin{array}{c}(\lambda_1,\{\pi_1,[0.5,0.7],[-0.7,-0.5],(\pi_2,[0.2,0.3],[-0.9,-0.6]),(\pi_3,[0,0],[-1,-1])\})\\ (\lambda_2,\{\pi_1,[0.8,0.9],[-0.3,-0.2],(\pi_2,[1,1],[-1,-1]),(\pi_3,[1,1],[0,0])\})\\ (\lambda_3,\{\pi_1,[0,0.1],[-0.1,0],(\pi_2,[0.4,0.6],[-0.9,-0.8]),(\pi_3,[0.5,0.7],[-0.2,-0.1])\})\end{array}\right\},\\ (\Gamma_2,\Lambda_2)&=\left\{\begin{array}{c}(\lambda_1,\{\pi_1,[0.4,0.5],[-0.8,-0.6],(\pi_2,[0.2,0.3],[-1,-0.7]),(\pi_3,[0.3,0.4],[-0.9,-0.8])\})\\ (\lambda_2,\{\pi_1,[0.5,0.6],[-0.7,-0.4],(\pi_2,[1,1],[-1,-1]),(\pi_3,[0.9,1],[-0.3,-0.1])\})\end{array}\right\},\\ (\Gamma_3,\Lambda_3)&=\left\{\begin{array}{c}(\lambda_1,\{\pi_1,[0.8,0.9],[-0.5,-0.4],(\pi_2,[0,0.1],[-0.4,-0.3]),(\pi_3,[0,0],[-1,-1])\})\\ (\lambda_2,\{\pi_1,[0,0],[-0.2,-0.1],(\pi_2,[0.1,0.2],[-1,-1]),(\pi_3,[1,1],[0,0])\})\\ (\lambda_3,\{\pi_1,[0.8,0.9],[-0.2,-0.1],(\pi_2,[0.2,0.3],[-0.4,-0.3]),(\pi_3,[0.9,1],[-0.3,-0.2])\})\\ (\lambda_4,\{\pi_1,[0,0],[0,0],(\pi_2,[0.7,0.8],[-0.7,-0.6]),(\pi_3,[0.5,0.6],[-0.3,-0.2])\})\end{array}\right\},\\ (\Gamma_4,\Lambda_4)&=\{(\lambda_1,\{(\pi_1,[0.1,0.2],[-0.4,-0.3]),(\pi_2,[0.9,1],[-0.1,0]),(\pi_3,[0.7,0.8],[-0.8,-0.7])\})\}.\end{aligned} \tag{11}$$

Then, we have the union bipolar vague soft set $(\sqsupset,\Delta)=\tilde{\cup}(\Gamma_i,\Lambda_i)$, where $\Delta=\cup\Lambda_i=\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, for $i=1,2,3,4$ as follows:

$$\begin{aligned}(\sqsupset,\Delta)&=\tilde{\cup}(\Gamma_i,\Lambda_i),i=1,2,3,4\\ &=(\Gamma_1,\Lambda_1)\tilde{\cup}(\Gamma_2,\Lambda_2)\tilde{\cup}(\Gamma_3,\Lambda_3)\tilde{\cup}(\Gamma_4,\Lambda_4)\\ &=\left\{\begin{array}{c}(\lambda_1,\{(\pi_1,[0.8,0.9],[-0.8,-0.6]),t(\pi_2,[0.9,1],[-1,-0.7])n,q(\pi_3,[0.7,t0.8],[-1,-1])\})\\ (\lambda_2,\{(\pi_1,[0.8,0.9],[-0.7,-0.4]),(\pi_2,[1,1],[-1,-1]),(\pi_3,[1,1],[-0.3,-0.1])\})\\ (\lambda_3,\{(\pi_1,[0.8,0.9],[-0.2,-0.1]),(\pi_2[0.4,0.6][-0.9,-0.8]),(\pi_3,[0.9,1],[-0.3,-0.2])\})\\ (\lambda_4,\{(\pi_1,[0,0],[0,0]),(\pi_2,[0.7,0.8],[-0.7,-0.6]),(\pi_3,[0.5,0.6],[-0.3,-0.2])\})\end{array}\right\}.\end{aligned} \tag{12}$$

Definition 14. The restricted union of two bipolar vague soft sets (Γ, Λ) and (Ψ, Θ) on a common universal set Π is a bipolar vague soft set $(\sqsupset, \Delta)$, written as $(\sqsupset, \Delta) = (\Gamma, \Lambda) \tilde{\cup}_R (\Psi, \Theta)$, where $\Delta = \Lambda \cap \Theta \neq \phi$ and for all $\delta \in \Delta$,

$$(\sqsupset, \Delta) = \left\{ \left(\begin{array}{c} \delta, \left\{ \left(\pi, \left[\max\left\{ \tau^{+}_{\Gamma(\delta)}(\pi), \tau^{+}_{\Psi(\delta)}(\pi) \right\}, \max\left\{ 1 - \eta^{+}_{\Gamma(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi) \right\} \right], \right. \right. \\ 18pt \left[\min\left\{ -1 - \eta^{-}_{\Gamma(\delta)}(\pi), -1 - \eta^{-}_{\Psi(\delta)}(\pi) \right\}, \min\left\{ \tau^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi) \right\} \right] : \delta \in \Lambda \cap \Theta, \pi \in \Pi \end{array} \right) \right\}. \tag{13}$$

Example 7. Under assumptions of Example 4, we have the restricted union bipolar vague soft set $(\sqsupset, \Delta) = (\Gamma, \Lambda) \tilde{\cup}_R (\Psi, \Theta)$, where $\Delta = \Lambda \cap \Theta = \{\lambda_1, \lambda_2\}$ as follows:

$$(\sqsupset, \Delta) = \left\{ \begin{array}{c} (\lambda_1, \{(\pi_1, [0.5, 0.7], [-0.8, -0.6]), (\pi_2, [0.2, 0.3], [-1, -0.7]), (\pi_3, [0, 0], [-1, -1])\}) \\ (\lambda_2, \{(\pi_1, [0.8, 0.9], [-0.7, -0.4]), (\pi_2, [1, 1], [-1, -1]), (\pi_3, [-0.3, -0.1])\}) \end{array} \right\}. \tag{14}$$

Definition 15. The restricted union of a family $\{(\Gamma_i, \Lambda_i): i \in I\}$ of bipolar vague soft sets over a universal set Π is a bipolar vague soft set $(\sqsupset, \Delta)$, written as $\sqsupset_\Delta = (\sqsupset, \Delta) = \tilde{\cup}_R (\Gamma_i, \Lambda_i)$, where $\Delta = \cap \Lambda_i$, for all $i \in I$ defined by $\sqsupset_\Delta(\delta) = \cup \Gamma_i(\delta)$, $\delta \in \Delta = \cap \Lambda_i$.

Example 8. Under assumptions of Example 6, we have the restricted union bipolar vague soft set $(\sqsupset, \Delta) = \tilde{\cup}_R (\Gamma_i, \Lambda_i)$, where $\Delta = \cap \Lambda_i = \{\lambda_1\}$, for $i = 1, 2, 3, 4$ as follows:

$$\begin{aligned} (\sqsupset, \Delta) &= \tilde{\cup}_R (\Gamma_i, \Lambda_i), i = 1, 2, 3, 4 \\ &= (\Gamma_1, \Lambda_1) \tilde{\cup}_R (\Gamma_2, \Lambda_2) \tilde{\cup}_R (\Gamma_3, \Lambda_3) \tilde{\cup}_R (\Gamma_4, \Lambda_4) \\ &= \{(\lambda_1, \{(\pi_1, [0.8, 0.9], [-0.8, -0.6]), (\pi_2, [0.9, 1], [-1, -0.7])\} (\pi_3, [0.7, 0.8], [-1, -1]))\}. \end{aligned} \tag{15}$$

Definition 16. The intersection of two bipolar vague soft sets (Γ, Λ) and (Ψ, Θ) over a common universal set Π is a bipolar vague soft set $(\sqsupset, \Delta)$, written as $(\sqsupset, \Delta) = (\Gamma, \Lambda) \tilde{\cap} (\Psi, \Theta)$, where $\Delta = \Lambda \cup \Theta$ and for all $\delta \in \Delta$:

$$(\sqsupset, t\Delta) = \left\{ \begin{array}{l} = \left\{ \left(\delta, \left\{ \left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi) t, n1q - h\eta^{+}_{\Gamma(\delta)}(\pi) \right], \left[-1t - n\eta^{-}_{\Gamma(\delta)} q(\pi) h, \tau^{-}_{\Gamma(\delta)} x(\pi) \right] \right) \right\} \right), \pi \in \Pi \right\}, \\ if\ \delta \in \Lambda - \Theta, \\ = \left\{ \left(\delta, \left\{ \left(\pi, \left[\tau^{+}_{\Psi(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi) \right], \left[-1 - \eta^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi) \right] \right) \right\} \right), \pi \in \Pi \right\}, \\ if\ \delta \in \Theta - \Lambda, \\ = \left\{ \left(\delta, \left\{ \begin{array}{c} \left(\pi, \left[\min\left\{ \tau^{+}_{\Gamma(\delta)}(\pi), \tau^{+}_{\Psi(\delta)}(\pi) \right\}, \min\left\{ 1 - \eta^{+}_{\Gamma(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi) \right\} \right], \right) \\ \left[\max\left\{ -1 - \eta^{-}_{\Gamma(\delta)}(\pi), -1 - \eta^{-}_{\Psi(\delta)}(\pi) \right\}, \max\left\{ \tau^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi) \right\} \right] \end{array} \right\} \right), \pi \in \Pi \right\} \\ if\ \delta \in \Lambda \cap \Theta. \end{array} \right. \tag{16}$$

Example 9. Under assumptions of Example 4, we have the intersection bipolar vague soft set $(\sqsupset, \Delta) = (\Gamma, \Lambda) \tilde{\cap} (\Psi, \Theta)$, where $\Delta = \Lambda \cup \Theta = \{\lambda_1, \lambda_2, \lambda_3\}$ as follows:

$$(\sqsupset, \Delta) = \left\{ \begin{array}{c} (\lambda_1, \{(\pi_1, [0.4, 0.5], [-0.7, -0.5]), (\pi_2, [0.2, 0.3], [-0.9, -0.6]), (\pi_3, [0, 0], [-1, -1])\}) \\ (\lambda_2, \{(\pi_1, [0.5, 0.6], [-0.3, -0.2]), (\pi_2, [1, 1], [-1, -1]), (\pi_3, [0.9, 1], [0, 0])\}) \\ (\lambda_3, \{(\pi_1, [0, 0.1], [-0.1, 0]), (\pi_2, [0.4, 0.6], [-0.9, -0.8]), (\pi_3, [0.5, 0.7], [-0.2, -0.1])\}) \end{array} \right\}. \tag{17}$$

Definition 17. The intersection of a family $\{(\Gamma_i, \Lambda_i): i \in I\}$ of bipolar vague soft sets over a universal set Π is a bipolar vague soft set $(\sqsupset, \Delta)$, written as $\sqsupset_\Delta = (\sqsupset, \Delta) = \widetilde{\cap}\,(\Gamma_i, \Lambda_i)$, where $\Delta = \cup \Lambda_i$, for all $i \in I$ defined as follows, for all $\delta \in \Delta$:

$$\sqsupset_\Delta(\delta) = \begin{cases} \Gamma_i(\delta), \; if \; \delta \in \Lambda_i - \bigcup_{j \neq i} \Lambda_j, & \text{for all } i \in I, \\ \bigcap_{i \in I} \Gamma_i(\delta), \; if \; \delta \in \bigcap_{i \in I} \Lambda_i. \end{cases} \tag{18}$$

Example 10. Under assumptions of Example 6, we have the intersection bipolar vague soft set $(\sqsupset, \Delta) = \widetilde{\cap}\,(\Gamma_i, \Lambda_i)$, where $\Delta = \cup \Lambda_i = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, for $i = 1, 2, 3, 4$ as follows:

$$\begin{aligned} (\sqsupset, \Delta) &= \widetilde{\cap}\,(\Gamma_i, \Lambda_i), \; i = 1, 2, 3, 4 \\ &= (\Gamma_1, \Lambda_1) \widetilde{\cap} (\Gamma_2, \Lambda_2) \widetilde{\cap} (\Gamma_3, \Lambda_3) \widetilde{\cap} (\Gamma_4, \Lambda_4) \\ &= \left\{ \begin{array}{c} (\lambda_1, \{(\pi_1, [0.1, 0.2], [-0.4, -0.3]), (\pi_2, [0, 0.1], [-0.1, 0]), (\pi_3, [0, 0], [-0.8, -0.7])\}) \\ (\lambda_2, \{(\pi_1, [0, 0], [-0.2, -0.1]), (\pi_2, [0.1, 0.2], [-1, -1]), (\pi_3, [0.9, 1], [0, 0])\}) \\ (\lambda_3, \{(\pi_1, [0, 0.1], [-0.1, 0]), (\pi_2, [0.2, 0.3], [-0.4, -0.3]), (\pi_3, [0.5, 0.7], [-0.2, -0.1])\}) \\ (\lambda_4, \{(\pi_1, [0, 0], [0, 0]), (\pi_2, [0.7, 0.8], [-0.7, -0.6]), (\pi_3, [0.5, 0.6], [-0.3, -0.2])\}) \end{array} \right\}. \end{aligned} \tag{19}$$

Definition 18. The restricted intersection of two bipolar vague soft sets (Γ, Λ) and (Ψ, Θ) on a common universal set Π is a bipolar vague soft set $(\sqsupset, \Delta)$, written as $(\sqsupset, \Delta) = (\Gamma, \Lambda) \widetilde{\cap}_R (\Psi, \Theta)$, where $\Delta = \Lambda \cap \Theta \neq \phi$ and for all $\delta \in \Delta$,

$$(\sqsupset, \Delta) = \left\{ \left(\delta, \left(\begin{array}{c} \pi, \left[\min\left\{\tau^+_{\Gamma(\delta)}(\pi), \tau^+_{\Psi(\delta)}(\pi)\right\}, \min\left\{1 - \eta^+_{\Gamma(\delta)}(\pi), 1 - \eta^+_{\Psi(\delta)}(\pi)\right\}\right], \\ \left[\max\left\{-1 - \eta^-_{\Gamma(\delta)}(\pi), -1 - \eta^-_{\Psi(\delta)}(\pi)\right\}, \max\left\{\tau^-_{\Gamma(\delta)}(\pi), \tau^-_{\Psi(\delta)}(\pi)\right\}\right]: \delta \in \Lambda \cap \Theta, \pi \in \Pi \end{array} \right) \right) \right\}. \tag{20}$$

Example 11. Under assumptions of Example 4, we have the restricted intersection bipolar vague soft set $(\sqsupset, \Delta) = (\Gamma, \Lambda) \widetilde{\cap}_R (\Psi, \Theta)$, where $\Delta = \Lambda \cap \Theta = \{\lambda_1, \lambda_2\}$ as follows:

$$(\sqsupset, \Delta) = \left\{ \begin{array}{c} (\lambda_1, \{(\pi_1, [0.4, 0.5], [-0.7, -0.5]), (\pi_2, [0.2, 0.3], [-0.9, -0.6]), (\pi_3, [0, 0], [-1, -1])\}) \\ (\lambda_2, \{(\pi_1, [0.5, 0.6], [-0.3, -0.2]), (\pi_2, [1, 1], [-1, -1]), (\pi_3, [0.9, 1], [0, 0])\}) \end{array} \right\}. \tag{21}$$

Definition 19. The restricted intersection of a family $\{(\Gamma_i, \Lambda_i): i \in I\}$ of bipolar vague soft sets over a universal set Π is a bipolar vague soft set $(\sqsupset, \Delta)$, written as $\sqsupset_\Delta = (\sqsupset, \Delta) = \widetilde{\cap}_R(\Gamma_i, \Lambda_i)$, where $\Delta = \cap \Lambda_i$, for all $i \in I$ defined by $\sqsupset_\Delta(\delta) = \cap \Gamma_i(\delta)$, $\delta \in \Delta = \cap \Lambda_i$.

Example 12. Under assumptions of Example 6, the restricted intersection bipolar vague soft set $(\sqsupset, \Delta) = \widetilde{\cap}_R(\Gamma_i, \Lambda_i)$, where $\Delta = \cap \Lambda_i = \{\lambda_1\}$, for $i = 1, 2, 3, 4$ is:

$$\begin{aligned} (\sqsupset, \Delta) &= \widetilde{\cap}_R(\Gamma_i, \Lambda_i), \; i = 1, 2, 3, 4 \\ &= (\Gamma_1, \Lambda_1) \widetilde{\cap}_R (\Gamma_2, \Lambda_2) \widetilde{\cap}_R (\Gamma_3, \Lambda_3) \widetilde{\cap}_R (\Gamma_4, \Lambda_4) \\ &= \{(\lambda_1, \{(\pi_1, [0.8, 0.9], [-0.8, -0.6]), (\pi_2, [0.9, 1], [-1, -0.7]), (\pi_3, [0.7, 0.8], [-1, -1])\})\}. \end{aligned} \tag{22}$$

5. Properties of Bipolar Vague Soft Sets

In this section, some important properties of bipolar vague soft sets are listed such as absorption properties, commutative properties, distributive laws, associative properties and De Morgan's laws. Furthermore, the proof for each property is introduced. Throughout the rest of the paper, unless otherwise stated, suppose that $(\Gamma, \Lambda) = \{(\lambda, \{(\pi, [\tau^+_{\Gamma(\lambda)}(\pi), 1 - \eta_\Gamma(\lambda)^+(\pi)],$

$[-1-\eta^{-}_{\Gamma(\lambda)}(\pi), \tau^{-}_{\Gamma(\lambda)}(\pi)]\})\colon \lambda \in \Lambda, \pi \in \Pi\}$, $(\Psi, \Theta) = \{(\theta, \{(\pi, [\tau^{+}_{\Psi(\theta)}(\pi), 1-\eta^{+}_{\Psi(\theta)}(\pi)], [-1-\eta^{-}_{\Psi(\theta)}(\pi), \tau^{-}_{\Psi(\theta)}(\pi)]\})\colon \theta \in \Theta, \pi \in \Pi\}$, and $(\Sigma, \Omega) = \{(\omega, \{(\pi, [\tau^{+}_{\Sigma(\omega)}(\pi), 1-\eta^{+}_{\Sigma(\omega)}(\pi)], [-1-\eta^{-}_{\Sigma(\omega)}(\pi), \tau^{-}_{\Sigma(\omega)}(\pi)]\})\}\colon \omega \in \Omega, \pi \in \Pi$.

Theorem 1. *Let (Γ, Λ) be any bipolar vague soft set, $\mathscr{C}_{\Lambda}$ be the complete bipolar vague soft set and ϕ_{Λ} be the null bipolar vague soft set on a common universal set Π, then*

$$
\begin{aligned}
&(1)\ (\Gamma, \Lambda) \tilde{\cup} (\Gamma, \Lambda) = (\Gamma, \Lambda),\ 18pt\\
&(2)\ (\Gamma, \Lambda) \tilde{\cap} (\Gamma, \Lambda) = (\Gamma, \Lambda), 18pt\\
&(3)\ (\Gamma, \Lambda) \tilde{\cup} \phi_{\Lambda} = (\Gamma, \Lambda), 18pt18pt\\
&(4)\ (\Gamma, \Lambda) \tilde{\cap} \phi_{\Lambda} = \phi_{\Lambda},\ 18pt\\
&(5)\ (\Gamma, \Lambda) \tilde{\cup} \mathscr{C}_{\Lambda} = \mathscr{C}_{\Lambda}, 18pt\\
&(6)\ (\Gamma, \Lambda) \tilde{\cap} \mathscr{C}_{\Lambda} = (\Gamma, \Lambda),\\
&(7)\ \mathscr{C}_{\Lambda} \tilde{\cup} \phi_{\Lambda} = \mathscr{C}_{\Lambda}\\
&(8)\ \mathscr{C}_{\Lambda} \tilde{\cap} \phi_{\Lambda} = \phi_{\Lambda}.
\end{aligned}
\tag{23}
$$

Proof. We prove only one result and the rest of the results follow similarly. Now, one can prove (8) as follows. From Definitions 2 and 3, $\mathscr{C}_{\Lambda} = \{(\lambda, \{(\pi, [1,1], [-1,-1]\})\colon \lambda \in \Lambda, \pi \in \Pi\}$ and $\phi_{\Lambda} = \{(\lambda, \{(\pi, [0,0], [0,0]\})\colon \lambda \in \Lambda, \pi \in \Pi\}$, respectively. Suppose that: $(\sqsupset, \Delta) = \mathscr{C}_{\Lambda} \tilde{\cap} \phi_{\Lambda} = \{(\delta, \{(\pi, [\tau^{+}_{\sqsupset(\delta)}(\pi), 1-\eta^{+}_{\sqsupset(\delta)}(\pi)], [-1-\eta^{-}_{\sqsupset(\delta)}(\pi), \tau^{-}_{\sqsupset(\delta)}(\pi)]\})\colon \delta \in \Delta, \pi \in \Pi\}$, where $\Delta = \Lambda \cup \Lambda = \Lambda$. We have to prove that $\mathscr{C}_{\Lambda} \tilde{\cap} \phi_{\Lambda} = \phi_{\Lambda}$ for all three cases according to the Definition 11 of intersection of two bipolar vague soft sets. For the first and second case, we have no parameters, since $\delta \in \Lambda - \Lambda = \phi$. For the third case, if $\delta \in \Lambda \cap \Lambda = \Lambda$, then we get from Intersection Definition 11 that:

$$
\begin{aligned}
\mathscr{C}_{\Lambda} \tilde{\cap} \phi_{\Lambda} &= (\sqsupset, \Delta)\\
&\quad \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\sqsupset(\delta)}(\pi), 1-\eta^{+}_{\sqsupset(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\sqsupset(\delta)}(\pi), \tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta \in \Lambda, \pi \in \Pi\right\}\\
&= \{(\delta, \{(\pi, [\min\{1,0\}, \min\{1,0\}], [\max\{-1,0\}, \max\{-1,0\}])\})\colon \delta \in \Lambda, \pi \in \Pi\}\\
&= \{(\delta, \{(\pi, [0,0], [0,0])\})\colon \delta \in \Lambda, \pi \in \Pi\} = \phi_{\Lambda}.
\end{aligned}
\tag{24}
$$

Theorem 2. *Assume that (Γ, Λ) and (Ψ, Θ) are two bipolar vague soft sets on a universal set Π; then, we have the absorption properties satisfied for them as follows:*

$$
\begin{aligned}
&(1)\ (\Gamma, \Lambda) \tilde{\cup} \Big((\Gamma, \Lambda) \tilde{\cap}_{R} (\Psi, \Theta)\Big) = (\Gamma, \Lambda),\ 18pt\\
&(2)\ (\Gamma, \Lambda) \tilde{\cap}_{R} ((\Gamma, \Lambda) \tilde{\cup} (\Psi, \Theta)) = (\Gamma, \Lambda).
\end{aligned}
\tag{25}
$$

Proof. We just prove the first result and the second result can be proved by following the same steps. Now, we prove (1) as follows. Let

$$
\begin{aligned}
(\Phi, \Xi) &= (\Gamma, \Lambda) \tilde{\cap}_{R} (\Psi, \Theta),\ \Xi = \Lambda \cap \Theta\\
&= \left\{\left(\xi, \left\{\left(\pi, \left[\tau^{+}_{\Phi(\xi)}(\pi), 1-\eta^{+}_{\Phi(\xi)}(\pi)\right], \left[-1-\eta^{-}_{\Phi(\xi)}(\pi), \tau^{-}_{\Phi(\xi)}(\pi)\right]\right\}\right)\colon \xi \in \Xi, \pi \in \Pi\right\},\\
(\sqsupset, \Delta) &= (\Gamma, \Lambda) \tilde{\cup} (\Phi, \Xi),\ \Delta = \Lambda \cup \Xi\\
&= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\sqsupset(\delta)}(\pi), 1-\eta^{+}_{\sqsupset(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\sqsupset(\delta)}(\pi), \tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta \in \Delta, \pi \in \Pi\right\}.
\end{aligned}
\tag{26}
$$

According to Definition 7 of union of two bipolar vague soft sets, we have to show that (1) holds for all following three cases:

(1) If $\delta \in \Lambda - \Theta$, then we obtain from Definition 13 of restricted intersection of two bipolar vague soft sets that

$$
\begin{aligned}
(\Phi, \Xi) &= (\Gamma, \Lambda) \tilde{\cap}_{R} (\Psi, \Theta)\\
&= \left\{\left(\xi, \left\{\left(\pi, \left[-1-\eta^{-}_{\Phi(\xi)}(\pi), \tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi \in \Lambda - \Theta, \pi \in \Pi\right\} = \phi.
\end{aligned}
\tag{27}
$$

Then, by using (3) from Theorem 1, we have

$$(\sqsupset, \Delta) = (\Gamma, \Lambda)\,\tilde{\cup}\,(\Phi, \Xi) = (\Gamma, \Lambda)\,\tilde{\cup}\,\phi = (\Gamma, \Lambda). \tag{28}$$

(2) If $\delta \in \Theta - \Lambda$, then we obtain from Definition 13 of restricted intersection of two bipolar vague soft sets that:

$$\begin{aligned}(\Phi, \Xi)' &= (\Gamma, \Lambda)\,\tilde{\cap}_R\,(\Psi, \Theta) \\ ' &= \left\{\left(\xi, \left\{\left(\pi, \left[\tau^{+}_{\Phi(\xi)}(\pi), 1 - \eta^{+}_{\Phi(\xi)}(\pi)\right], \left[-1 - \eta^{-}_{\Phi(\xi)}(\pi), \tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right) : \xi \in \Theta - \Lambda, \pi \in \Pi\right\} = \phi.\end{aligned} \tag{29}$$

Then, by using (3) from Theorem 1, we have

$$(\sqsupset, \Delta) = (\Gamma, \Lambda)\,\tilde{\cup}\,(\Phi, \Xi) = (\Gamma, \Lambda)\,\tilde{\cup}\,\phi = (\Gamma, \Lambda). \tag{30}$$

(3) If $\delta \in \Lambda \cap \Theta$, then we obtain from Definition 13 of restricted intersection of two bipolar vague soft sets that

$$\begin{aligned}(\Phi, \Xi) &= (\Gamma, \Lambda)\,\tilde{\cap}_R\,(\Psi, \Theta) \\ &= \left\{\left(\xi, \left\{\left(\pi, \left[\tau^{+}_{\Phi(\xi)}(\pi), 1 - \eta^{+}_{\Phi(\xi)}(\pi)\right], \left[-1 - \eta^{-}_{\Phi(\xi)}(\pi), \tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right) : \xi \in \Lambda \cap \Theta, \pi \in \Pi\right\} \\ &= \left\{\left(\xi, \left\{\left(\begin{array}{c}\pi, \left[\min\left\{\tau^{+}_{\Gamma(\xi)}(\pi), \tau^{+}_{\Psi(\xi)}(\pi)\right\}, \min\left\{1 - \eta^{+}_{\Gamma(\xi)}(\pi), 1 - \eta^{+}_{\Psi(\xi)}(\pi)\right\}\right], \\ \left[\max\left\{-1 - \eta^{-}_{\Gamma(\xi)}(\pi), -1 - \eta^{-}_{\Psi(\xi)}(\pi)\right\}, \max\left\{\tau^{-}_{\Gamma(\xi)}(\pi), \tau^{-}_{\Psi(\xi)}(\pi)\right\}\right]\end{array}\right)\right\}\right) : \xi \in \Lambda \cap \Theta, \pi \in \Pi\right\}.\end{aligned} \tag{31}$$

Therefore, we have from Definition 7 of union of two bipolar vague soft sets that

$$\begin{aligned}(\sqsupset, t\Delta) &= (\Gamma, t\Lambda)\,\tilde{\cup}\,(\Phi, t\Xi) \\ &= \left\{\left(\delta, \left\{\left[\pi, \left[\tau^{+}_{\sqsupset(\delta)}(\pi), 1 - \eta^{+}_{\sqsupset(\delta)}(\pi)\right], \left[-1 - \eta^{-}_{\sqsupset(\delta)}(\pi), \tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right]\right\}\right) : \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\} \\ &= \left\{\left(\delta, \left\{\pi, \left(\begin{array}{c}\left[\begin{array}{c}\max\left\{\tau^{+}_{\Gamma(\delta)}(\pi), \min\left\{\tau^{+}_{\Gamma(\delta)}(\pi), \tau^{+}_{\Psi(\delta)}(\pi)\right\}\right\}, \\ \max\left\{1 - \eta^{+}_{\Gamma(\delta)}(\pi), \min\left\{1 - \eta^{+}_{\Gamma(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi)\right\}\right\}\end{array}\right] \\ \left[\begin{array}{c}\min\left\{-1 - \eta^{-}_{\Gamma(\delta)}(\pi), \max\left\{-1 - \eta^{-}_{\Gamma(\delta)}(\pi), -1 - \eta^{-}_{\Psi(\delta)}(\pi)\right\}\right\}, \\ \min\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \max\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right\}\right\}\end{array}\right]\end{array}\right)\right\}\right) : \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\}.\end{aligned} \tag{32}$$

Then, by applying (1) and (2) from Lemma 2, we get that $\max\left\{\tau^{+}_{\Gamma(\delta)}(\pi), \min\left\{\tau^{+}_{\Gamma(\delta)}(\pi), \tau^{+}_{\Psi(\delta)}(\pi)\right\}\right\} = \tau^{+}_{\Gamma(\delta)}(\pi)$, $\min\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \max\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right\}\right\} = \tau^{-}_{\Gamma(\delta)}(\pi)$, $\max\{1 - \eta^{+}_{\Gamma(\delta)}(\pi), \min\left\{1 - \eta^{+}_{\Gamma(\delta)}\right.(\pi)), 1 - \eta^{+}_{\Psi(\delta)}(\pi)\}\} = 1 - \eta^{+}_{\Gamma(\delta)}(\pi)$, and $\min\left\{-1 - \eta^{-}_{\Gamma(\delta)}(\pi), \max\{-1- \eta^{-}_{\Gamma(\delta)}(\pi), -1 - \eta^{-}_{\Psi(\delta)}(\pi)\}\right\} = -1 - \eta^{-}_{\Gamma(\delta)}(\pi)$. Thus, by substituting from the above four equations in (32), we have

$$(\sqsupset, \Delta) = \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi), 1 - \eta^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1 - \eta^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right) : \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\} = (\Gamma, \Lambda). \tag{33}$$

From (28)–(33), then (1) holds for the first, second, and third cases, respectively.

Corollary 1. *For two bipolar vague soft sets on a common universal set* (Γ, Λ) *and* (Ψ, Θ),

$$(\Gamma, \Lambda)\,\tilde{\cup}\left((\Gamma, \Lambda)\,\tilde{\cap}_R\,(\Psi, \Theta)\right) = (\Gamma, \Lambda)\,\tilde{\cap}_R\left((\Gamma, \Lambda)\,\tilde{\cup}\,(\Psi, \Theta)\right) = (\Gamma, \Lambda). \tag{34}$$

Proof. Direct from Theorem 2.

Theorem 3. *Suppose that* (Γ, Λ) *and* (Ψ, Θ) *are two bipolar vague soft sets on a universal set* Π*; then, we have the commutative (abelian) property satisfied for them as follows:*

$$\begin{aligned}&(1)\ (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta) = (\Psi,\Theta)\,\tilde{\cap}\,(\Gamma,\Lambda)18pt\\ &(2)\ (\Gamma,\Lambda)\,\tilde{\cup}\,(\Psi,\Theta) = (\Psi,\Theta)\,\tilde{\cup}\,(\Gamma,\Lambda).\end{aligned} \tag{35}$$

Proof. We just prove the first result and by following similar steps, and the second result can be proved. Now, one can prove (1) as follows. Suppose that

$$\begin{aligned}(\sqsupset,\Delta) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta), \Delta = \Lambda\cup\Theta\\ &= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\sqsupset(\delta)}(\pi), 1-\eta^{+}_{\sqsupset(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\sqsupset(\delta)}(\pi), \tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right)\right\}\right): \delta\in\Delta, \pi\in\Pi\right\},\\ (\Sigma,\Omega) &= (\Psi,\Theta)\,\tilde{\cap}\,(\Gamma,\Lambda), \Omega = \Theta\cup\Lambda\\ &= \left\{\left(\omega, \left\{\left(\pi, \left[\tau^{+}_{\Sigma(\omega)}(\pi), 1-\eta^{+}_{\Sigma(\omega)}(\pi)\right], \left[-1-\eta^{-}_{\Sigma(\omega)}(\pi), \tau^{-}_{\Sigma(\omega)}(\pi)\right]\right\}\right): \omega\in\Omega, \pi\in\Pi\right\}.\end{aligned} \tag{36}$$

According to Intersection Definition 11 of two bipolar vague soft sets, we must find $(\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta)$ for all the following three cases:

(1) If $\delta\in\Lambda-\Theta$, then we get from intersection Definition 11 that

$$\begin{aligned}(\sqsupset,\Delta) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta)\\ &= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi), 1-\eta^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right): \delta\in\Lambda, \pi\in\Pi\right\} = (\Gamma,\Lambda).\end{aligned} \tag{37}$$

(2)If $\delta\in\Theta-\Lambda$, then we get from Intersection Definition 11 that:

$$\begin{aligned}(\sqsupset,\Delta) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta)\\ &= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Psi(\delta)}(\pi), 1-\eta^{+}_{\Psi(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right): \delta\in\Theta, \pi\in\Pi\right\} = (\Psi,\Theta).\end{aligned} \tag{38}$$

(3)If $\delta\in\Lambda\cap\Theta$, then we get from Intersection Definition 11 that

$$\begin{aligned}(\sqsupset,\Delta) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta)\\ &= \left\{\begin{pmatrix}\delta, \left\{\left(\pi, \left[\min\left\{\tau^{+}_{\Gamma(\delta)}(\pi), \tau^{+}_{\Psi(\delta)}(\pi)\right\}, \min\left\{1-\eta^{+}_{\Gamma(\delta)}(\pi), 1-\eta^{+}_{\Psi(\delta)}(\pi)\right\}\right],\right.\right.\\ \left[\max\left\{-1-\eta^{-}_{\Gamma(\delta)}(\pi), -1-\eta^{-}_{\Psi(\delta)}(\pi)\right\}, \max\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right\}\right]\end{pmatrix}: \delta\in\Lambda\cap\Theta, \pi\in\Pi\right\}\\ &= \left\{\begin{pmatrix}\delta, \left\{\left(\pi, \left[\min\left\{\tau^{+}_{\Psi(\delta)}(\pi), \tau^{+}_{\Gamma(\delta)}(\pi)\right\}, \min\left\{1-\eta^{+}_{\Psi(\delta)}(\pi), 1-\eta^{+}_{\Gamma(\delta)}(\pi)\right\}\right],\right.\right.\\ \left[\max\left\{-1-\eta^{-}_{\Psi(\delta)}(\pi), -1-\eta^{-}_{\Gamma(\delta)}(\pi)\right\}, \max\left\{\tau^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right\}\right]\end{pmatrix}: \delta\in\Theta\cap\Lambda, \pi\in\Pi\right\}\\ &= (\Psi,\Theta)\,\tilde{\cap}\,(\Gamma,\Lambda).\end{aligned} \tag{39}$$

By combining obtained equations from (1), (2), and (3), we have, for $\delta \in \Delta$,

$$(\sqsupset, \Delta) = \begin{cases} (\Psi, \Theta), \; if \; \delta \in \Theta - \Lambda \\ (\Gamma, \Lambda), \; if \; \delta \in \Lambda - \Theta \\ (\Psi, \Theta) \tilde{\cap} (\Gamma, \Lambda), \; if \; \delta \in \Theta \cap \Lambda, \end{cases} \quad (40)$$

where $\Delta = \Lambda \cup \Theta = \Theta \cup \Lambda$, since the crisp (ordinary) union is commutative. Thus, the above obtained Form (40) of $(\sqsupset, \Delta)$ coincides with the well-known form of (Σ, Ω). Therefore, $(\sqsupset, \Delta) = (\Sigma, \Omega)$, where $\Delta = \Theta \cup \Lambda = \Omega$. Hence, $(\Gamma, \Lambda) \tilde{\cap} (\Psi, \Theta) = (\Psi, \Theta) \tilde{\cap} (\Gamma, \Lambda)$. □

Proposition 1. *If* (Γ, Λ) *and* (Ψ, Θ) *are two bipolar vague soft sets,* $(\Gamma, \Lambda) \tilde{\subseteq} (\Psi, \Theta)$, *then*

$$\begin{aligned} &(1)\ (\Gamma, \Lambda) \tilde{\cap}_R (\Psi, \Theta) = (\Gamma, \Lambda),\ 18pt \\ &(2)\ (\Gamma, \Lambda) \tilde{\cup} (\Psi, \Theta) = (\Psi, \Theta). \end{aligned} \quad (41)$$

Proof. We just prove the first result and to prove the second result, one can follow the same steps. Now, we prove (1) as follows. Assume that

$$\begin{aligned} (\sqsupset, \Delta) &= (\Gamma, \Lambda) \tilde{\cap}_R (\Psi, \Theta),\ \Delta = \Lambda \cap \Theta \\ &= \left\{ \left(\delta, \left\{ \left(\pi, \left[\tau^{+}_{\sqsupset(\delta)}(\pi), 1 - \eta^{+}_{\sqsupset(\delta)}(\pi) \right], \left[-1 - \eta^{-}_{\sqsupset(\delta)}(\pi), \tau^{-}_{\sqsupset(\delta)}(\pi) \right] \right) \right\} \right) : \delta \in \Delta, \pi \in \Pi \right\}. \end{aligned} \quad (42)$$

Then, from Definition 13 of restricted intersection of two bipolar vague soft sets, we get

$$\begin{aligned} (\sqsupset, \Delta) &= \left\{ \left(\delta, \left\{ \left(\pi, \left[\tau^{+}_{\sqsupset(\delta)}(\pi), 1 - \eta^{+}_{\sqsupset(\delta)}(\pi) \right], \left[-1 - \eta^{-}_{\sqsupset(\delta)}(\pi), \tau^{-}_{\sqsupset(\delta)}(\pi) \right] \right) \right\} \right) \right\} \\ &= \left\{ \begin{pmatrix} \delta, \left\{ \left(\pi, \left[\min\left\{ \tau^{+}_{\Gamma(\delta)}(\pi), \tau^{+}_{\Psi(\delta)}(\pi) \right\}, \min\left\{ 1 - \eta^{+}_{\Gamma(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi) \right\} \right], \right. \right. \\ \left. \left. \left[\max\left\{ -1 - \eta^{-}_{\Gamma(\delta)}(\pi), -1 - \eta^{-}_{\Psi(\delta)}(\pi) \right\}, \max\left\{ \tau^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi) \right\} \right] \right. \right. \end{pmatrix} : \delta \in \Lambda \cap \Theta, \pi \in \Pi \right\}. \end{aligned} \quad (43)$$

But since $(\Gamma, \Lambda) \tilde{\subseteq} (\Psi, \Theta)$, then we get from Definition 5 of bipolar vague soft containment that $\Lambda \subseteq \Theta$, i.e., $\Lambda \cap \Theta = \Lambda$ and $\Gamma(\delta) \subseteq \Psi(\delta)$ for all $\delta \in \Delta$, i.e., $\tau^{+}_{\Gamma(\delta)}(\pi) \leq \tau^{+}_{\Psi(\delta)}(\pi)$, $1 - \eta^{+}_{\Gamma(\delta)}(\pi) \leq 1 - \eta^{+}_{\Psi(\delta)}(\pi)$, $-1 - \eta^{-}_{\Gamma(\delta)}(\pi) \geq -1 - \eta^{-}_{\Psi(\delta)}(\pi)$ and $\tau^{-}_{\Gamma(\delta)}(\pi) \geq \tau^{-}_{\Psi(\delta)}(\pi)$, for all $\delta \in \Delta$ and for all $\pi \in \Pi$. Therefore, we obtain that

$$(\sqsupset, \Delta) = \left\{ \left(\delta, \left\{ \left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi), 1 - \eta^{+}_{\Gamma(\delta)}(\pi) \right], \left[-1 - \eta^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi) \right] \right\} \right) : \delta \in \Lambda, \pi \in \Pi \right\} = (\Gamma, \Lambda). \quad (44)$$

□

Theorem 4. *Suppose that* (Γ, Λ), (Ψ, Θ) *and* (Σ, Ω) *are bipolar vague soft sets on a common universal set* Π, *then we have distributive laws are satisfied for them as the following:*

(1) $(\Gamma, \Lambda) \tilde{\cap} ((\Psi, \Theta) \tilde{\cup} (\Sigma, \Omega)) = ((\Gamma, \Lambda) \tilde{\cap} (\Psi, \Theta)) \tilde{\cup} ((\Gamma, \Lambda) \tilde{\cap} (\Sigma, \Omega))$

(2) $(\Gamma, \Lambda) \tilde{\cup} ((\Psi, \Theta) \tilde{\cap} (\Sigma, \Omega)) = ((\Gamma, \Lambda) \tilde{\cup} (\Psi, \Theta)) \tilde{\cap} ((\Gamma, \Lambda) \tilde{\cup} (\Sigma, \Omega))$

Proof. We just prove the first result. The second result can be proved by following similar steps performed in the first result. Now, we prove (1) as follows. According to Intersection Definition 11 and Union Definition 7, we have to show that (1) hold for all three cases.

(*a*) To find the L.H.S. of (1) in all three cases, suppose that

$$\begin{aligned} (\Phi, \Xi) &= (\Psi, \Theta) \tilde{\cup} (\Sigma, \Omega),\ \Xi = \Theta \cup \Omega \\ &= \left\{ \left(\xi, \left\{ \left(\pi, \left[\tau^{+}_{\Phi(\xi)}(\pi), 1 - \eta^{+}_{\Phi(\xi)}(\pi) \right], \left[-1 - \eta^{-}_{\Phi(\xi)}(\pi), \tau^{-}_{\Phi(\xi)}(\pi) \right] \right) \right\} \right) : \xi \in \Xi, \pi \in \Pi \right\}, \\ (\sqsupset, \Delta) &= (\Gamma, \Lambda) \tilde{\cap} (\Phi, \Xi),\ \Delta = \Lambda \cup \Xi \\ &= \left\{ \left(\delta, \left\{ \left(\pi, \left[\tau^{+}_{\sqsupset(\delta)}(\pi), 1 - \eta^{+}_{\sqsupset(\delta)}(\pi) \right], \left[-1 - \eta^{-}_{\sqsupset(\delta)}(\pi), \tau^{-}_{\sqsupset(\delta)}(\pi) \right] \right) \right\} \right) : \delta \in \Delta, \pi \in \Pi \right\}. \end{aligned} \quad (45)$$

Then, the L.H.S. of (1) is as follows in each case:

$(a)(i)$ We have from Union Definition 7 that

$$\begin{aligned}(\Phi,\Xi) &= (\Psi,\Theta)\,\widetilde{\cup}\,(\Sigma,\Omega)\\ &= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Phi(\xi)}(\pi),1-\eta^{+}_{\Phi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Phi(\xi)}(\pi),\tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Theta-\Omega,\pi\in\Pi\right\}\\ &= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Psi(\xi)}(\pi),1-\eta^{+}_{\Psi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Psi(\xi)}(\pi),\tau^{-}_{\Psi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Theta,\pi\in\Pi\right\} = (\Psi,\Theta).\end{aligned} \tag{46}$$

Therefore, by applying Intersection Definition 11, we obtain that

$$\begin{aligned}(\sqsupset,\Delta) &= (\Gamma,\Lambda)\,\widetilde{\cap}\,(\Phi,\Xi) = (\Gamma,\Lambda)\,\widetilde{\cap}\,(\Psi,\Theta)\\ &= \left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\sqsupset(\delta)}(\pi),1-\eta^{+}_{\sqsupset(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\sqsupset(\delta)}(\pi),\tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta\in\Lambda-\Theta,\pi\in\Pi\right\}\\ &= \left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\Gamma(\delta)}(\pi),1-\eta^{+}_{\Gamma(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\Gamma(\delta)}(\pi),\tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta\in\Lambda,\pi\in\Pi\right\} = (\Gamma,\Lambda).\end{aligned} \tag{47}$$

$(a)(ii)$ We have from Union Definition 7 that

$$\begin{aligned}(\Phi,\Xi) &= (\Psi,\Theta)\,\widetilde{\cup}\,(\Sigma,\Omega)\\ &= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Phi(\xi)}(\pi),1-\eta^{+}_{\Phi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Phi(\xi)}(\pi),\tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Omega-\Theta,\pi\in\Pi\right\}\\ &= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Sigma(\xi)}(\pi),1-\eta^{+}_{\Sigma(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Sigma(\xi)}(\pi),\tau^{-}_{\Sigma(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Omega,\pi\in\Pi\right\} = (\Sigma,\Omega).\end{aligned} \tag{48}$$

Therefore, by applying Intersection Definition 11, we obtain that

$$\begin{aligned}(\sqsupset,\Delta) &= (\Gamma,\Lambda)\,\widetilde{\cap}\,(\Phi,\Xi) = (\Gamma,\Lambda)\,\widetilde{\cap}\,(\Sigma,\Omega)\\ &= \left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\sqsupset(\delta)}(\pi),1-\eta^{+}_{\sqsupset(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\sqsupset(\delta)}(\pi),\tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta\in\Omega-\Lambda,\pi\in\Pi\right\}\\ &= \left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\Sigma(\delta)}(\pi),1-\eta^{+}_{\Sigma(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\Sigma(\delta)}(\pi),\tau^{-}_{\Sigma(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta\in\Omega,\pi\in\Pi\right\} = (\Sigma,\Omega).\end{aligned} \tag{49}$$

$(a)(iii)$ We have from Union Definition 7 that

$$\begin{aligned}(\Phi,\Xi) &= (\Psi,\Theta)\,\widetilde{\cup}\,(\Sigma,\Omega)\\ &= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Phi(\xi)}(\pi),1-\eta^{+}_{\Phi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Phi(\xi)}(\pi),\tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Theta\cap\Omega,\pi\in\Pi\right\}\\ &= \left\{\xi,\left\{\left(\begin{array}{l}\pi,\left[\max\left\{\tau^{+}_{\Psi(\xi)}(\pi),\tau^{+}_{\Sigma(\xi)}(\pi)\right\},\ \max\left\{1-\eta^{+}_{\Psi(\xi)}(\pi),1-\eta^{+}_{\Sigma(\xi)}(\pi)\right\}\right]\\ ,\left[\min\left\{-1-\eta^{-}_{\Psi(\xi)}(\pi),-1-\eta^{-}_{\Sigma(\xi)}(\pi)\right\},\ \min\left\{\tau^{-}_{\Psi(\xi)}(\pi),\tau^{-}_{\Sigma(\xi)}(\pi)\right\}\right]\end{array}\right)\colon \xi\in\Theta\cap\Omega,\pi\in\Pi\right\}\right\}.\end{aligned} \tag{50}$$

Therefore, by applying Intersection Definition 11, we obtain that

$$\begin{aligned}
(\sqsupset,\Delta)` &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Phi,\Xi)\\
` &= \left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\sqsupset(\delta)}(\pi),1-\eta^{+}_{\sqsupset(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\sqsupset(\delta)}(\pi),\tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta\in\Lambda\cap\Theta\cap\Omega,\pi\in\Pi\right\}\\
` &= \left\{\left(\delta,\left\{\left(\pi,\begin{bmatrix}\min\left\{\tau^{+}_{\Gamma(\delta)}(\pi),\ \max\left\{\tau^{+}_{\Psi(\delta)}(\pi),\tau^{+}_{\Sigma(\delta)}(\pi)\right\}\right\},\\ \min\left\{1-\eta^{+}_{\Gamma(\delta)}(\pi),\ \max\left\{1-\eta^{+}_{\Psi(\delta)}(\pi),1-\eta^{+}_{\Sigma(\delta)}(\pi)\right\}\right\},\\ \left[\max\left\{-1-\eta^{-}_{\Gamma(\delta)}(\pi),\ \min\left\{-1-\eta^{-}_{\Psi(\delta)}(\pi),-1-\eta^{-}_{\Sigma(\delta)}(\pi)\right\}\right\}\max\left\{\tau^{-}_{\Gamma(\delta)}(\pi),\ \min\left\{\tau^{-}_{\Psi(\delta)}(\pi),\tau^{-}_{\Sigma(\delta)}(\pi)\right\}\right\}\right]\end{bmatrix}\right)\right\}\right)\right\},\\
&\colon \delta`\in\Lambda\cap\Theta\cap\Omega,\pi\in\Pi.
\end{aligned}\tag{51}$$

(b) Now, to find the R.H.S. of (1) in all three cases, suppose that

$$\begin{aligned}
(\Phi,\Xi)` &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta),\ \Xi=\Lambda\cup\Theta\\
` &= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Phi(\xi)}(\pi),1-\eta^{+}_{\Phi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Phi(\xi)}(\pi),\tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Xi,\pi\in\Pi\right\},\\
(\urcorner,\Upsilon) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Sigma,\Omega),\ \Upsilon=\Lambda\cup\Omega\\
&= \left\{\left(\nu,\left\{\left(\pi,\left[\tau^{+}_{\urcorner(\epsilon)}(\pi),1-\eta^{+}_{\urcorner(\epsilon)}(\pi)\right],\left[-1-\eta^{-}_{\urcorner(\epsilon)}(\pi),\tau^{-}_{\urcorner(\epsilon)}(\pi)\right]\right)\right\}\right)\colon \epsilon\in\Upsilon,\pi\in\Pi\right\},\\
(\sqsupset,\Delta) &= (\Phi,\Xi)\,\tilde{\cup}\,(\urcorner,\Upsilon),\ \Delta=\Xi\cup\Upsilon\\
&= \left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\sqsupset(\delta)}(\pi),1-\eta^{+}_{\sqsupset(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\sqsupset(\delta)}(\pi),\tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta\in\Delta,\pi\in\Pi\right\}.
\end{aligned}\tag{52}$$

Then, the R.H.S. of (1) is as follows in each case:

$(b)(i)$ We have from Intersection Definition 11 that

$$\begin{aligned}
(\Phi,\Xi) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta)\\
&= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Phi(\xi)}(\pi)),1-\eta^{+}_{\Phi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Phi(\xi)}(\pi),\tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Lambda-\Theta,\pi\in\Pi\right\}\\
&= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Gamma(\xi)}(\pi),1-\eta^{+}_{\Gamma(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Gamma(\xi)}(\pi),\tau^{-}_{\Gamma(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Lambda,\pi\in\Pi\right\}=(\Gamma,t\Lambda),\\
(\urcorner,t\Upsilon) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta)\\
&= \left\{\left(\nu,\left\{\left(\pi,\left[\tau^{+}_{\urcorner(\epsilon)}(\pi),1-\eta^{+}_{\urcorner(\epsilon)}(\pi)\right],\left[-1-\eta^{-}_{\urcorner(\epsilon)}(\pi),\tau^{-}_{\urcorner(\epsilon)}(\pi)\right]\right)\right\}\right)\colon \epsilon\in\Lambda-\Omega,\pi\in\Pi\right\}\\
&= \left\{\left(\nu,\left\{\left(\pi,\left[\tau^{+}_{\Gamma(\epsilon)}(\pi),1-\eta^{+}_{\Gamma(\epsilon)}(\pi)\right],\left[-1-\eta^{-}_{\Gamma(\epsilon)}(\pi),\tau^{-}_{\Gamma(\epsilon)}(\pi)\right]\right)\right\}\right)\colon \epsilon\in\Lambda,\pi\in\Pi\right\}=(\Gamma,\Lambda).
\end{aligned}\tag{53}$$

Then, from the above two equations, and by using (1) from Theorem 1, we get that

$$(\sqsupset,\Delta)=(\Phi,\Xi)\,\tilde{\cup}\,(\urcorner,\Upsilon)=(\Gamma,\Lambda)\,\tilde{\cup}\,(\Gamma,\Lambda)=(\Gamma,\Lambda).\tag{54}$$

$(b)(ii)$ We have from Intersection Definition 11 that

$$\begin{aligned}
(\Phi,\Xi) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta)\\
&= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Phi(\xi)}(\pi),1-\eta^{+}_{\Phi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Phi(\xi)}(\pi),\tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Theta-\Lambda,\pi\in\Pi\right\}\\
&= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Psi(\xi)}(\pi),1-\eta^{+}_{\Psi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Psi(\xi)}(\pi),\tau^{-}_{\Psi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Theta,\pi\in\Pi\right\}=(\Psi,\Theta),\\
(\urcorner,\Upsilon) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Sigma,\Omega)\\
&= \left\{\left(\nu,\left\{\left(\pi,\left[\tau^{+}_{\urcorner(\epsilon)}(\pi),1-\eta^{+}_{\urcorner(\epsilon)}(\pi)\right],\left[-1-\eta^{-}_{\urcorner(\epsilon)}(\pi),\tau^{-}_{\urcorner(\epsilon)}(\pi)\right]\right)\right\}\right)\colon \epsilon\in\Omega-\Lambda,\pi\in\Pi\right\}\\
&= \left\{\left(\nu,\left\{\left(\pi,\left[\tau^{+}_{\Sigma(\epsilon)}(\pi),1-\eta^{+}_{\Sigma(\epsilon)}(\pi)\right],\left[-1-\eta^{-}_{\Sigma(\epsilon)}(\pi),\tau^{-}_{\Sigma(\epsilon)}(\pi)\right]\right)\right\}\right)\colon \epsilon\in\Omega,\pi\in\Pi\right\}=(\Sigma,\Omega).
\end{aligned}\tag{55}$$

Then, by applying Union Definition 7 on the above two equations, we get that

$$\begin{aligned}(\sqsupset,\Delta) &= (\Phi,\Xi)\,\tilde{\cup}\,(\urcorner,\Upsilon) = (\Psi,\Theta)\,\tilde{\cup}\,(\Sigma,\Omega)\\ &= \left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\sqsupset(\delta)}(\pi),1-\eta^{+}_{\sqsupset(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\sqsupset(\delta)}(\pi),\tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta\in\Omega-\Theta,\pi\in\Pi\right\}\\ &= \left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\Sigma(\delta)}(\pi),1-\eta^{+}_{\Sigma(\delta)}(\pi)\right],\left[-1-\eta^{-}_{\Sigma(\delta)}(\pi),\tau^{-}_{\Sigma(\delta)}(\pi)\right]\right)\right\}\right)\colon \delta\in\Omega,\pi\in\Pi\right\} = (\Sigma,\Omega).\end{aligned} \tag{56}$$

$(b)\,(iii)$ We have from Intersection Definition 11 that

$$\begin{aligned}(\Phi,\Xi) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Psi,\Theta)\\ &= \left\{\left(\xi,\left\{\left(\pi,\left[\tau^{+}_{\Phi(\xi)}(\pi),1-\eta^{+}_{\Phi(\xi)}(\pi)\right],\left[-1-\eta^{-}_{\Phi(\xi)}(\pi),\tau^{-}_{\Phi(\xi)}(\pi)\right]\right)\right\}\right)\colon \xi\in\Lambda\cap\Theta,\pi\in\Pi\right\}\\ &\left\{\left(\xi,\left\{\begin{pmatrix}\pi,\left[\min\left\{\tau^{+}_{\Gamma(\xi)}(\pi),\tau^{+}_{\Psi(\xi)}(\pi)\right\},\min\left\{1-\eta^{+}_{\Gamma(\xi)}(\pi),1-\eta^{+}_{\Psi(\xi)}(\pi)\right\}\right],\\ \left[\max\left\{-1-\eta^{-}_{\Gamma(\xi)}(\pi),-1-\eta^{-}_{\Psi(\xi)}(\pi)\right\},\max\left\{\tau^{-}_{\Gamma(\xi)}(\pi),\tau^{-}_{\Psi(\xi)}(\pi)\right\}\right]\end{pmatrix}\right\}\right.\right\}\colon \xi\in\Lambda\cap\Theta,\pi\in\Pi,\\ (\urcorner,\Upsilon) &= (\Gamma,\Lambda)\,\tilde{\cap}\,(\Sigma,\Omega)\\ &= \left\{\left(\nu,\left\{\left(\pi,\left[\tau^{+}_{\urcorner(\epsilon)}(\pi),1-\eta^{+}_{\urcorner(\epsilon)}(\pi)\right],\left[-1-\eta^{-}_{\urcorner(\epsilon)}(\pi),\tau^{-}_{\urcorner(\epsilon)}(\pi)\right]\right)\right\}\right)\colon \epsilon\in\Lambda\cap\Omega,\pi\in\Pi\right\}\\ &\left\{\left(\nu,\left\{\begin{pmatrix}\pi,\left[\min\left\{\tau^{+}_{\Gamma(\epsilon)}(\pi),\tau^{+}_{\Sigma(\epsilon)}(\pi)\right\},\min\left\{1-\eta^{+}_{\Gamma(\epsilon)}(\pi),1-\eta^{+}_{\Sigma(\epsilon)}(\pi)\right\}\right],\\ \left[\max\left\{-1-\eta^{-}_{\Gamma(\epsilon)}(\pi),-1-\eta^{-}_{\Sigma(\epsilon)}(\pi)\right\},\max\left\{\tau^{-}_{\Gamma(\epsilon)}(\pi),\tau^{-}_{\Sigma(\epsilon)}(\pi)\right\}\right]\end{pmatrix}\right\}\right)\colon \epsilon\in\Lambda\cap\Omega,\pi\in\Pi\right\}.\end{aligned} \tag{57}$$

Then, by applying Union Definition 7 on the above two equations, we get

$$\begin{aligned}(\sqsupset,\Delta) &= (\Phi,\Xi)\,\tilde{\cup}\,(\urcorner,\Upsilon)\\ &= \left\{\left(\delta,\left\{\left(\pi,\left[\tau^{+}_{\sqsupset(\delta)}(\pi),1-\eta^{+}_{\sqsupset(\delta)}(\pi)\right]\right),\left[-1-\eta^{-}_{\sqsupset(\delta)}(\pi),\tau^{-}_{\sqsupset(\delta)}(\pi)\right]\right\}\right)\colon \delta\in\Lambda\cap\Theta\cap\Omega,\pi\in\Pi\right\}.\end{aligned} \tag{58}$$

Therefore, we have

$$\begin{aligned}(\sqsupset,\Delta) &= (\Phi,\Xi)\,\tilde{\cup}\,(\urcorner,\Upsilon)\\ &= \left\{\left(\delta,\left\{\left(\begin{matrix}\pi,\begin{bmatrix}\max\left\{\min\left\{\tau^{+}_{\Gamma(\delta)}(\pi),\tau^{+}_{\Psi(\delta)}(\pi)\right\},\min\left\{\tau^{+}_{\Gamma(\epsilon)}(\pi),\tau^{+}_{\Sigma(\delta)}(\pi)\right\},\\ \max\left\{\min\left\{1-\eta^{+}_{\Gamma(\delta)}(\pi),1-\eta^{+}_{\Psi(\delta)}(\pi)\right\},\min\left\{1-\eta^{+}_{\Gamma(\epsilon)}(\pi),1-\eta^{+}_{\Sigma(\epsilon)}(\pi)\right\}\right\}\end{bmatrix}\\ \begin{bmatrix}\min\left\{\max\left\{-1-\eta^{-}_{\Gamma(\delta)}(\pi),-1-\eta^{-}_{\Psi(\delta)}(\pi)\right\},\max\left\{-1-\eta^{-}_{\Gamma(\epsilon)}(\pi),-1-\eta^{-}_{\Sigma(\delta)}(\pi)\right\}\right\},\\ \min\left\{\max\left\{\tau^{-}_{\Gamma(\delta)}(\pi),\tau^{-}_{\Psi(\delta)}(\pi)\right\},\max\left\{\tau^{-}_{\Gamma(\epsilon)}(\pi),\tau^{-}_{\Sigma(\epsilon)}(\pi)\right\}\right\}\end{bmatrix}\end{matrix}\right)\right\}\right)\colon \delta\in\Lambda\cap\Theta\cap\Omega,\pi\in\Pi\right\}.\end{aligned} \tag{59}$$

Then, by applying (1) and (2) from Lemma 3, we get that $\max\left\{\min\left\{\tau^{+}_{\Gamma(\delta)}(\pi),\tau^{+}_{\Psi(\delta)}(\pi)\right\},\min\left\{\tau^{+}_{\Gamma(\epsilon)}(\pi),\tau^{+}_{\Sigma(\delta)}(\epsilon)\right\}\right\} = \min\ \tau^{+}_{\Gamma(\delta)}(\pi),\max\left\{\tau^{+}_{\Psi(\delta)}(\pi),\ \tau^{+}_{\Sigma(\delta)}(\pi)\}\right\}$, $\max\left\{\min\left\{1-\eta^{+}_{\Gamma(\delta)}(\pi),1-\eta^{+}_{\Psi(\delta)}(\pi)\right\},\ \min\left\{1-\eta^{+}_{\Gamma(\epsilon)}(\pi),1-\eta^{+}_{\Sigma(\epsilon)}(\pi)\right\}\right\}= \min\ \left\{1-\eta^{+}_{\Gamma(\delta)}(\pi),\max\left\{1-\eta^{+}_{\Psi(\delta)}(\pi),1-\eta^{+}_{\Sigma(\delta)}(\pi)\right\}\right\}$, $\min\ \{\max\left\{-1-\eta^{-}_{\Gamma(\delta)}(\pi),-\ 1-\eta^{-}_{\Psi(\delta)}(\pi)\},\max\left\{-1-\eta^{-}_{\Gamma(\epsilon)}(\pi),-1-\eta^{-}_{\Sigma(\delta)}(\pi)\}\right\}= \max\left\{-1-\eta^{-}_{\Gamma(\delta)}(\pi),\ \min\left\{-1-\eta^{-}_{\Psi(\delta)}(\pi),\ -1-\eta^{-}_{\Sigma(\delta)}(\pi)\}\right\}\right.$, and $\min\left\{\max\left\{\tau^{-}_{\Gamma(\delta)}(\pi),\tau^{-}_{\Psi(\delta)}(\pi)\right\},\ \max\left\{\tau^{-}_{\Gamma(\epsilon)}(\pi),\right.\right.$

$\tau^{-}_{\Sigma(\epsilon)}(\pi)\} = \max\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \min\tau^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Sigma(\delta)}(\pi)\right\}\}$. Thus, by substituting from the above four equations in (59), we have

$$(\sqsupset, \Delta) = \left\{\left(\delta\left\{\left(\pi, \begin{bmatrix} \min\left\{\tau^{+}_{\Gamma(\delta)}(\pi), \max\left\{\tau^{+}_{\Psi(\delta)}(\pi), \tau^{+}_{\Sigma(\delta)}(\pi)\right\}\right\}, \\ \min\left\{1-\eta^{+}_{\Gamma(\delta)}(\pi), \max\left\{1-\eta^{+}_{\Psi(\delta)}(\pi), 1-\eta^{+}_{\Sigma(\delta)}(\pi)\right\}\right\}, \end{bmatrix} \begin{bmatrix} \max\left\{-1-\eta^{-}_{\Gamma(\delta)}(\pi), \min\left\{-1-\eta^{-}_{\Psi(\delta)}(\pi), -1-\eta^{-}_{\Sigma(\delta)}(\pi)\right\}\right\}, \\ \max\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \min\left\{\tau^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Sigma(\delta)}(\pi)\right\}\right\}, \end{bmatrix}\right)\right\}\right): \delta \in \Lambda \cap \Theta \cap \Omega, \pi \in \Pi\right\}. \tag{60}$$

Hence, from ((a) (i) and (b) (i)), ((a) (ii) and (b) (ii)), and ((a) (iii) and (b) (iii)), we have (1) holds for the first, second, and third case, respectively, which completes the proof.

Theorem 5. *Suppose that (Γ, Λ), (Ψ, Θ), and (Σ, Ω) are bipolar vague soft sets on a common universal set Π, then associative laws hold for them as follows:*

(1) $(\Gamma, \Lambda) \tilde{\cap} ((\Psi, \Theta) \tilde{\cap} (\Sigma, \Omega)) = ((\Gamma, \Lambda) \tilde{\cap} (\Psi, \Theta)) \tilde{\cap} (\Sigma, \Omega)$

(2) $(\Gamma, \Lambda) \tilde{\cup} ((\Psi, \Theta) \tilde{\cup} (\Sigma, \Omega)) = ((\Gamma, \Lambda) \tilde{\cup} (\Psi, \Theta)) \tilde{\cup} (\Sigma, \Omega)$

Proof. By following the same methodology performed in Theorem 4, one can prove this theorem.

Theorem 6. *De Morgan's laws are valid for any two bipolar vague soft sets (Γ, Λ) and (Ψ, Θ) on a common universal set Π as follows:*

$$\begin{aligned} &(1)\ ((\Gamma, \Lambda) \tilde{\cup} (\Psi, \Theta)) = (\Gamma, \Lambda)^c \tilde{\cap} (\Psi, \Theta)^c, \\ &(2)\ ((\Gamma, \Lambda) \tilde{\cap} (\Psi, \Theta)) (\Gamma, \Lambda)^c \widetilde{\cup} (\Psi, \Theta)^c. \end{aligned} \tag{61}$$

Proof. We just prove the first result. By performing the same steps of it, one can prove the second result. Now, (1) can be proved as follows. Let $\delta \in \Delta = \Lambda \cup \Theta$. We have to prove that (1) for all three cases according to Union (4.7) and Intersection (4.11) Definitions.

(1) If $\delta \in \Lambda - \Theta$, then we get from Union (Definition 7) and Intersection (Definition 11) definitions that

$$\begin{aligned} &((\Gamma, \Lambda) \tilde{\cup} (\Psi, \Theta))^c \\ &= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi), 1-\eta^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda, \pi \in \Pi\right\}^c \\ &= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi), 1-\eta^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda, \pi \in \Pi\right\} \\ &= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi), 1-\eta^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda, \pi \in \Pi\right\} \\ &\tilde{\cap} \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi), 1-\eta^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Theta, \pi \in \Pi\right\} \\ &= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi), 1-\eta^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda, \pi \in \Pi\right\}^c \\ &\tilde{\cap} \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Gamma(\delta)}(\pi), 1-\eta^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1-\eta^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Theta, \pi \in \Pi\right\}^c \\ &= (\Gamma, \Lambda)^c \tilde{\cap} (\Psi, \Theta)^c. \end{aligned} \tag{62}$$

(2)If $\delta \in \Theta - \Lambda$, then we get from Union (Definition 7) and Intersection (Definition 11) definitions that

$$
\begin{aligned}
&((\Gamma, \Lambda) \widetilde{\cup} (\Psi, \Theta))^{c} \\
&= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Psi(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi)\right], \left[-1 - \eta^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Theta, \pi \in \Pi\right\}^{c} \\
&= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Psi(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi)\right], \left[-1 - \eta^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Theta, \pi \in \Pi\right\} \\
&= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Psi(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi)\right], \left[-1 - \eta^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda, \pi \in \Pi\right\} \\
&\widetilde{\cap} \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Psi(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi)\right], \left[-1 - \eta^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Theta, \pi \in \Pi\right\} \\
&= \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Psi(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi)\right], \left[-1 - \eta^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda, \pi \in \Pi\right\}^{c} \\
&\widetilde{\cap} \left\{\left(\delta, \left\{\left(\pi, \left[\tau^{+}_{\Psi(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi)\right], \left[-1 - \eta^{-}_{\Psi(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Theta, \pi \in \Pi\right\}^{c} \\
&= (\Gamma, \Lambda)^{c} \widetilde{\cap} (\Psi, \Theta)^{c}.
\end{aligned}
\tag{63}
$$

(3)If $\delta \in \Lambda \cap \Theta$, then we get from Union (Definition 7) and Intersection (Definition 11) definitions that

$$
\begin{aligned}
&((\Gamma, \Lambda) \widetilde{\cup} (\Psi, \Theta))^{c} \\
&= \left\{\left(\begin{array}{c} \delta, \left\{\pi, \max\left\{\tau^{+}_{\Gamma(\delta)}(\pi), \tau^{+}_{\Psi(\delta)}(\pi)\right\}\right\}, \max\left\{1 - \eta^{+}_{\Gamma(\delta)}(\pi), 1 - \eta^{+}_{\Psi(\delta)}(\pi)\right\} \\ \left[\min\left\{-1 - \eta^{-}_{\Gamma(\delta)}(\pi), -1 - \eta^{-}_{\Psi(\delta)}(\pi)\right\}, \min\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right\}\right] \end{array}\right): \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\}^{c} \\
&= \left\{\left(\begin{array}{c} \delta, \pi, \left[\max\left\{\tau^{+}_{\Gamma(\delta)}(\pi), \tau^{+}_{\Psi(\delta)}(\pi)\right\}, 1 - \min\left\{\eta^{+}_{\Gamma(\delta)}(\pi), \eta^{+}_{\Psi(\delta)}(\pi)\right\}\right] \\ \left[-1 - \max\left\{\eta^{-}_{\Gamma(\delta)}(\pi), \eta^{-}_{\Psi(\delta)}(\pi)\right\}, \min\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right\}\right] \end{array}\right): \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\}^{c} \\
&= \left\{\left(\delta, \left(\begin{array}{c} \pi, \left[\min\left\{\eta^{+}_{\Gamma(\delta)}(\pi), \eta^{+}_{\Psi(\delta)}(\pi), 1 - \max\left\{\tau^{+}_{\Gamma(\delta)}(\pi), \tau^{+}_{\Psi(\delta)}(\pi)\right\}\right\}\right] \\ \left[-1 - \min\left\{\tau^{-}_{\Gamma(\delta)}(\pi), \tau^{-}_{\Psi(\delta)}(\pi)\right\}, \max\left\{\eta^{-}_{\Gamma(\delta)}(\pi), \eta^{-}_{\Psi(\delta)}(\pi)\right\}\right] \end{array}\right)\right): \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\}, \\
&= \left\{\left(\delta, \left\{\left(\pi, \left[\begin{array}{c} \min\left\{\eta^{+}_{\Gamma(\delta)}(\pi), \eta^{+}_{\Psi(\delta)}(\pi), \min\left\{1 - \tau^{+}_{\Gamma(\delta)}(\pi), 1 - \tau^{+}_{\Psi(\delta)}(\pi)\right\}\right\} \\ \left[\max\left\{-1 - \tau^{-}_{\Gamma(\delta)}(\pi), -1 - \tau^{-}_{\Psi(\delta)}(\pi)\right\}, \max\left\{\eta^{-}_{\Gamma(\delta)}(\pi), \eta^{-}_{\Psi(\delta)}(\pi)\right\}\right] \end{array}\right]\right)\right\}\right): \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\} \\
&= \left\{\left(\delta, \left\{\left(\pi, \left[\eta^{+}_{\Gamma(\delta)}(\pi), 1 - \tau^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1 - \tau^{-}_{\Gamma(\delta)}(\pi), \eta^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\} \\
&\widetilde{\cap} \left\{\left(\delta, \left\{\left(\pi, \left[\eta^{+}_{\Psi(\delta)}(\pi), 1 - \tau^{+}_{\Psi(\delta)}(\pi)\right], \left[-1 - \tau^{-}_{\Psi(\delta)}(\pi), \eta^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\} \\
&= \left\{\left(\delta, \left\{\left(\pi, \left[\eta^{+}_{\Gamma(\delta)}(\pi), 1 - \tau^{+}_{\Gamma(\delta)}(\pi)\right], \left[-1 - \tau^{-}_{\Gamma(\delta)}(\pi), \eta^{-}_{\Gamma(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\}^{c} \\
&\widetilde{\cap} \left\{\left(\delta, \left\{\left(\pi, \left[\eta^{+}_{\Psi(\delta)}(\pi), 1 - \tau^{+}_{\Psi(\delta)}(\pi)\right], \left[-1 - \tau^{-}_{\Psi(\delta)}(\pi), \eta^{-}_{\Psi(\delta)}(\pi)\right]\right)\right\}\right): \delta \in \Lambda \cap \Theta, \pi \in \Pi\right\}^{c} \\
&= (\Gamma, \Lambda)^{c} \widetilde{\cap} (\Psi, \Theta)^{c}.
\end{aligned}
\tag{64}
$$

□

6. Decision-Making Based on Bipolar Vague Soft Sets Using Roy and Maji Method

This section is devoted to discuss an applied real-life example for solving a socialistic decision-making problem. In fact, Roy and Maji [6] have given an effective technique to determine the optimal choice of an object to buy among many objects using the fuzzy soft theory. But it is very important to consider the bipolarity of knowledge in decision-making problems because it is a very useful factor when developing a mathematical framework for most situations in decision-making problems. For illustration, bipolarity denotes the favorable and unfavorable sides of any decision-making problem.

The idea beyond bipolarity is that bipolar subjective thoughts involve a wide range of human decision analyses. Each of happiness and sorrow, effects and side effects, sweet and sour, poverty and rich are some examples of different directions of decision analysis. The balance pretty mutual cohabitation of each of those two parts is regarded as a clue to a neutral social setting. The fuzzy set tool only, the vague set tool only, or the soft set tool only is insufficient to deal with this form of bipolarity; as an example, a medicine that is ineffective may not have any side effects.

At this destination, it is very useful to interpolate the concept of bipolarity into vague soft set theory. Generalizing Roy and Maji method, one can apply their technique in the bipolar vague soft set environment instead of the fuzzy soft set environment. Moreover, different methods to transform bipolar vague values into bipolar fuzzy values are used to make the decision. One can perform the decision-making using each method and then compare the results of all methods with the newly proposed method to show technique effectiveness. Furthermore, detailed comparative analysis, as well as a discussion of the results are provided at the end of this part.

Definition 20. (comparison table). The comparison table is a square table, its rows and columns are labeled by the object name of the universe such as $\pi_1, \pi_2, \pi_3, \ldots, \pi_n$ and the entries d_{ij}, where d_{ij} = the number of parameters for which, the value of d_i (the membership value of π_i) exceeds or equal to $(\geq)$ the value of d_j (the membership value of π_j).

Lemma 4. *Applying methods (1)–(3) stated, respectively, in Lemma 1, we can transform bipolar vague values into bipolar fuzzy values. The difference between the three methods is in how the relationship between bipolar fuzzy sets and bipolar vague sets is analyzed. The derivation methodology of the positive and negative fuzzy set membership values from the corresponding positive and negative vague set membership values, respectively, differs from one method to another.*

Let $\mathscr{BV}(\Pi)$ be the set of all bipolar vague sets of the universal set Π. Then, for all $B \in \mathscr{BV}(\Pi)$, $\pi \in \Pi$, its positive vague value $B^+ = [\tau_B^+(\pi), 1 - \eta_B^+(\pi)]$ and its negative vague value $B^- = [-1 - \eta_B^-(\pi), \tau_B^-(\pi)]$, the corresponding fuzzy positive membership function $\eta_{B^F}^+$ of u and the corresponding fuzzy negative membership function $\eta_{B^F}^-$ of u are defined by different three methods as the following:

$$\eta_{B^F}^+(\pi) = \frac{1 + \tau_B^+(\pi) - \eta_B^+(\pi)}{2}, \tag{65}$$

$$\eta_{B^F}^-(\pi) = \frac{-1 + \tau_B^-(\pi) - \eta_B^-(\pi)}{2}. \tag{66}$$

The derivation of Method (1) depends on the median concept. For illustration, one can find the positive and negative fuzzy set membership values by obtaining the corresponding positive and negative median membership values of the corresponding (positive and negative) true and false membership values (the corresponding positive and negative vague set membership values), respectively. That is to say that the bipolar fuzzy value is regarded as the total amount of evidence included in a bipolar vague value and is represented by the median membership value.

$$\eta_{B^F}^+(\pi) = \frac{\tau_B^+(\pi)}{\tau_B^+(\pi) + \eta_B^+(\pi)}, \tag{67}$$

$$\eta_{B^F}^-(\pi) = \frac{\tau_B^-(\pi)}{\tau_B^-(\pi) - \eta_B^-(\pi)}. \tag{68}$$

Method (2) derivation is based on the defuzzification function. For instance, the positive and negative fuzzy set membership values can be found by calculating the corresponding positive and negative defuzzification values of the corresponding positive and negative true and false membership values, respectively.

$$\eta_{B^F}^+(\pi) = \tau_B^+(\pi) + \frac{1}{2} \times \left[1 + \frac{\tau_B^+(\pi) - \eta_B^+(\pi)}{\tau_B^+(\pi) + \eta_B^+(\pi) + 2}\right] \left[1 - \tau_B^+(\pi) - \eta_B^+(\pi)\right], \tag{69}$$

$$\eta_{B^F}^-(\pi) = -1 - \eta_B^-(\pi) + \frac{1}{2} \times \left[1 + \frac{\tau_B^-(\pi) - \eta_B^-(\pi) - 2}{-\tau_B^-(\pi) - \eta_B^-(\pi) + 2}\right] \left[1 + \tau_B^-(\pi) + \eta_B^-(\pi)\right]. \tag{70}$$

Method (3) is relied on analyzing the mapping between the elements of bipolar vague sets and points on a plane. In Method (3), The transformation of either the positive or the negative vague set membership values into either the positive or the negative fuzzy set membership values is discovered to be a many-to-one mapping relation. Furthermore, Method (3) is found to be a general transformation model for converting bipolar vague set membership value to bipolar fuzzy set membership value.

6.1. Algorithm for Selecting the Best Alternative among the Others. The following steps can be followed by Mr. X to choose, correctly, the optimal alternative (object) among all alternatives and to rank those alternatives from the best to the worst:

Step 1: input the set $\Lambda \subseteq \Upsilon$ of preferring parameters for Mr. X.

Step 2: consider the bipolar vague soft subset according to $\Lambda \subseteq \Upsilon$.

Step 3: represent positive information functions (intervals) in a table.

Step 4: compute the comparison table of positive information functions, as stated in Definition 20.

Step 5: compute the positive information score $(\mathfrak{R}_1 - \mathfrak{C}_1)$ for each element π_i by subtracting its column sum

$\mathfrak{C}_1$ from its row sum $\mathfrak{R}_1$, then put these scores in a table called the positive membership score table.

Step 6: repeat steps (3), (4) and (5) but for negative information, regarding that $(\mathfrak{R}_2 - \mathfrak{C}_2)$ represents negative information score.

Step 7: compute the final score $(\mathfrak{R} - \mathfrak{C})$ for each element π_i by subtracting its negative information score $\mathfrak{C}$ from its positive information score $\mathfrak{R}$, then put these final scores in a table called the final score table.

Step 8: find k for which $\pi_k = \max \pi_i$, then the optimal (best) choice object (1^{st} alternative) for Mr. X to buy is π_k, followed by 2^{nd} alternative, scored by the element π_q having the second maximum score, and so on . . ..

Step 9: obtain the ranking of alternatives π_i by ranking their final scores in descending order (ordinal ranking).

Step 10: use the Methods (1), (2) and (3) stated in Lemma 4 to transform bipolar vague values into bipolar fuzzy values for positive information and negative information.

Step 11: repeat steps (4) and (5) for positive information and negative information, and then steps (8) and (9) to make the decision for each method.

Step 12: compare the decisions obtained from each method, and then make the final decision which is proposed by most of them (or all of them, if applicable).

Example 13. Consider the same Π and Υ in Example (4.1) and $(\sqsupset, \Upsilon)$ as follows:

$$
\begin{aligned}
(\sqsupset, \Upsilon) &= \{(\lambda_1, (\pi_1, [0.5, 0.7], [-0.7, -0.5]), (\pi_2, [0.2, 0.3], [-0.9, -0.6]), (\pi_3, [0, 0], [-1, -1]))\} \\
&= (\lambda_2, \{(\pi_1, [0.5, 0.6], [-0.7, -0.4], (\pi_2, [1, 1], [-1, -1]), (\pi_3, [0.9, 1], [-0.3, -0.1]))\}) \\
&\quad (\lambda_3, \{(\pi_1, [0.8, 0.9] [-0.3, -0.2], (\pi_2, [1, 1], [-1, -1]), (\pi_2, [1, 1], [0, 0]))\}) \\
&\quad (\lambda_4, \{(\pi_1, [0, 0.1], [-0.1, 0], (\pi_2, [0.4, 0.6], [-0.9, -0.8]), \pi_3, [0.5, 0.7], [-0.2, -0.1])\}).
\end{aligned} \tag{71}
$$

If Mr. X would like to purchase a car based on his own preferring parameters among those which are listed above (say, $\lambda_1, \lambda_3, \lambda_4$), then what will be the best car for him to buy according to the above bipolar vague soft set using Roy and Maji technique. ?

Solution. Our aim in this example is to follow up the steps of Roy and Maji algorithm for choosing, exactly, the best car for Mr. X to buy among the three given cars.

(1) Assume that $\{\lambda_1, \lambda_3, \lambda_4\} = \Lambda \subseteq \Upsilon$ is the preferring parameters subset for Mr. X.

(2) Consider the bipolar vague soft subset $(\Gamma, \Lambda) \widetilde{\subseteq} (\sqsupset, \Upsilon)$ as below according to Λ.

$$
(\Gamma, \Lambda) = \left\{ \begin{array}{c} (\lambda_1, \{(\pi_1, [0.5, 0.7], [-0.7, -0.5], (\pi_2, [0.2, 0.3], [-0.9, -0.6]), (\pi_3, [0, 0], [-1, -1]))\}) \\ (\lambda_3, \{(\pi_1, [0.8, 0.9], [-0.3, -0.2], (\pi_2, [1, 1], [-1, -1]), (\pi_3, [1, 1], [0, 0]))\}) \\ (\lambda_3, \{(\pi_1, [0, 0.1], [-0.1, 0], (\pi_2, [0.4, 0.6], [-0.9, -0.8]), (\pi_3, [0.5, 0.7], [-0.2, -0.1]))\}) \end{array} \right\}. \tag{72}
$$

(3) Represent the positive information membership intervals in Table 1.

(4) Compute all entries of the comparison table of Table 1 as stated in Definition 20 and represent them in a tabular form below in Table 2.

(5) Compute the sum of every row $(\mathfrak{R}_1)$ and the sum of every column $(\mathfrak{C}_1)$, then calculate their membership scores $(\mathfrak{R}_1$-$\mathfrak{C}_1)$ to put them in the positive membership score table (namely, Table 3).

(6) Repeat steps (3), (4) and (5) for negative information as follows. Represent the negative information membership intervals in Table 4. Then, compute all entries of the comparison table of Table 4 as stated in Definition 20 and represent them in Table 5. After that, compute the sum of every row $(\mathfrak{R}_2)$ and the sum of every column $(\mathfrak{C}_2)$, then calculate their nonmembership scores $(\mathfrak{R}_2 - \mathfrak{C}_2)$ to put them in the negative membership score table (namely, Table 6).

(7) Now, subtract each negative score (from Table 3) from its opposite positive score (from Table 6) to calculate entries of the final score table, then put all of them in Table 7.

(8) Since the first car π_1 has the maximum score (which is (2) among the others, then the decision is that the best car for Mr. X to buy among the three given cars is the first car π_1.

(9) Consequently, we have the ranking of the alternatives (cars) as follows: $\pi_1 > \pi_3 > \pi_2$.

Table 1: Tabular representation of positive information membership intervals.

Line Π	λ_1	λ_3	λ_4
Line π_1	[0.5, 0.7]	[0.8, 0.9]	[0, 0.1]
π_2	[0.2, 0.3]	[1, 1]	[0.4, 0.6]
π_3	[0, 0]	[13]	[0.5, 0.7]
Line			

Table 2: The comparison table of positive information represented in Table 1.

Line Π	π_1	π_2	π_3
Line π_1	3	1	1
π_2	2	3	2
π_3	2	2	3
Line			

Table 3: The positive membership score table.

Line Π	$\mathfrak{R}_1$	$\mathfrak{C}_1$	$\mathfrak{R}_1$-$\mathfrak{C}_1$
Line π_1	5	7	-2
π_2	7	6	1
π_3	7	6	1
Line			

Table 4: Negative information membership intervals.

Line Π	λ_1	λ_3	λ_4
Line π_1	[-0.7, -0.5]	[-0.3, -0.2]	[-0.1, 0]
π_2	[-0.9, -0.6]	[-1, -1]	[-0.9, -0.8]
π_3	[-1, -1]	[0, 0]	[-0.2, -0.1]
Line			

Table 5: The comparison table of negative information represented in Table 4.

Line Π	π_1	π_2	π_3
Line π_1	3	0	1
π_2	3	3	2
π_3	2	1	3
Line			

Table 6: The negative membership score table.

Line Π	$\mathfrak{R}_2$	$\mathfrak{C}_2$	$\mathfrak{R}_2 - \mathfrak{C}_2$
Line π_1	4	8	-4
π_2	8	4	4
π_3	6	6	0
Line			

(10) Use Methods (1), (2) and (3) stated in Lemma 4 as follows. For Method (1), convert all positive information membership intervals to membership functions (values) using Formula (65), then represent them in Table 8 and do the same with negative information using Formula (66), putting them in Table 9. For Method (2), do the same with positive and negative information using Formulas (67) and (68), putting them in Tables 10 and 11, respectively. For Method (3), do the same with positive and negative information using Formulas (69) and (70), putting them in Tables 12 and 13, respectively.

(11) By repeating steps (4) and (5) to compute the comparison tables of Tables 8, 10, and 12, we find that each of them coincides with the comparison Table 3. Also, by computing the comparison tables of Tables 9, 11, and 13, we find that each of them coincides with the comparison Table 5. Then, by repeating steps (8) and (9) for Methods (1), (2), and (3), we get the same result (decision) obtained in the vague case, since all the following steps depend on the two comparison tables of positive and negative information.

(12) Finally, we can compare the results of all used methods in Table 14. It is clear from Table 14 that there is an agreement in the decision. So, the final conclusion extracted from Table 14 is that the best car for Mr. X to buy among the three given cars is the first car π_1 followed by the third car π_3 followed by the second car π_2. That is to say that the three given cars (alternatives) are, finally, ranked in the following order: $\pi_1 > \pi_3 > \pi_2$.

Table 7: The final score table.

Line Π	Positive score ($\mathfrak{R}$)	Negative score ($\mathfrak{C}$)	Final score ($\mathfrak{R} - \mathfrak{C}$)
Line π_1	-2	-4	2
π_2	1	4	-3
π_3	1	0	1
Line			

6.2. Comparative Analysis. To compare the issue of decision-making under the bipolar vague soft set (BVSS) environment with previous different models or methods, a comparative analysis is conducted. We solve the same Example 13 under those different models and the results of this analysis are summarized as the following:

(1) If we make the decision under the algorithm steps using the vague soft set (VSS), presented by Faried et al. [7], then the results are provided as follows. The final scores of the three cars π_1, π_2 and π_3 are -2, 1 and 1, respectively. Therefore, the maximum score is 1 scored by both car π_2 and car π_3, so the decision is that the best car for Mr. X to buy among the three given cars is both the second car π_2 and the third car π_3. Hence, we have the ranking of the alternatives (cars) as follows: $\pi_2 = \pi_3 > \pi_1$.

(2) If the decision is made under the algorithm steps using the bipolar fuzzy soft set (BFSS), introduced by Abdullah et al. [13], then we obtain the results as

Table 8: Positive membership values obtained by Method (1).

Line Π	λ_1	λ_3	λ_4
Line π_1	0.6	0.85	0.05
π_2	0.25	1	0.5
π_3	0	1	0.6
Line			

Table 9: Negative membership values obtained by Method (1).

Line Π	λ_1	λ_3	λ_4
Line π_1	-0.6	-0.25	-0.05
π_2	-0.75	-1	-0.85
π_3	-1	0	-0.15
Line			

Table 10: Positive membership values obtained by Method (2).

Line Π	λ_1	λ_3	λ_4
Line π_1	0.625	0.8	0
π_2	0.2	1	0.5
π_3	0	1	0.625
Line			

Table 11: Negative membership values obtained by Method (2).

Line Π	λ_1	λ_3	λ_4
Line π_1	-0.416	-0.18	0
π_2	-0.461	-1	-0.72
π_3	-1	0	-0.09
Line			

Table 12: Positive membership values obtained by Method (3).

Line Π	λ_1	λ_3	λ_4
Line π_1	0.607	0.862	0.034
π_2	0.241	1	0.5
π_3	0	1	0.607
Line			

Table 13: Negative membership values obtained by Method (3).

Line Π	λ_1	λ_3	λ_4
Line π_1	-0.678	-0.275	-0.068
π_2	-0.8	-1	-0.896
π_3	-1	0	-0.172
Line			

follows. The final scores of the three cars π_1, π_2, and π_3 are 2, −3, and 1, respectively. Therefore, the top score is 2 obtained by car π_1, so the conclusion is that the best car for Mr. X to buy among the three given cars is the first car π_1. Hence, we have the ranking of the alternatives (cars) as follows: $\pi_1 > \pi_3 > \pi_2$.

(3) If one applies the decision-making process steps under the fuzzy soft set (FSS) environment, investigated by Roy and Maji [6], to make the decision, then one can obtain the following results. The final scores of the three cars π_1, π_2 and π_3 are −2, 1 and 1, respectively. Therefore, the maximum score is 1 scored by both car π_2 and car π_3, so the decision is that the best car for Mr. X to buy among the three given cars is both the second car π_2 and the third car π_3. Hence, we have the ranking of the alternatives (cars) as follows: $\pi_2 = \pi_3 > \pi_1$.

Table 14: Comparative results of all used methods.

Line alternatives' rank	All used methods			
	Bipolar vague method	Transformed bipolar fuzzy methods		
		Method (1)	*Method* (2)	*Method* (3)
Line 1^{st} alternative	π_1	π_1	π_1	π_1
Line 2^{nd} alternative	π_3	π_3	π_3	π_3
Line 3^{rd} alternative	π_2	π_2	π_2	π_2
Line				

Furthermore, we can put the final scores as well as the ranking order results of the three given objects (alternatives) in a comparative table, namely Table 15, to compare all used models as shown below.

6.3. Discussion. The results obtained utilizing the suggested technique are confirmed by those acquired by using several of the most extensively used and similar methodologies in this area, as seen by the comparative studies mentioned above. Although these alternative methods demonstrate that similar results can be obtained by utilizing many other hybrid models of vague or fuzzy sets, they, in addition, demonstrate that the decision-making procedure is less accurate in some of them and includes some problems, which makes it less efficient than the proposed model or method. Furthermore, models/methods like fuzzy soft model/method, bipolar fuzzy soft model/method, and vague soft model/method have proven to be quite effective in making accurate decisions when treating some complicated decision-making issues. They can only deal with the information supported by their particular structures; hence, they can only be used in their respective settings. This problem can be handled by combining two or more models, with the new hybrid model being created by combining its parent models, which are more general and reliable than the previous ones.

On the one hand, the fuzzy soft model is best for treating with fuzzy soft information; however, it fails for treating with bipolar fuzzy soft information. It is clear from the above comparative table (Table 15) that combining bipolarity with the vague soft set or even with the fuzzy soft set makes the decision more accurate and specified. Likewise, the vagueness of the related parameters cannot be described by the idea of the soft set only as shown in its definition or even by the fuzzy soft set definition. Consequently, we obtain that the conceptualization of the bipolar fuzzy soft set is good at dealing with bipolar fuzzy soft information, but not so good at dealing with too vague information seen in many real-life decision-making problems. This necessitates the use of the bipolar

Table 15: final scores and ranking order of alternatives using different models on Example 13.

Line models	π_1	π_2	π_3	Ranking order
Line roy and maji (FSS) [6]	-2	1	1	$\pi_2 = \pi_3 > \pi_1$
Abdullah et al. (BFSS) [13]	2	-3	1	$\pi_1 > \pi_3 > \pi_2$
Faried et al. (VSS) [7]	-2	1	1	$\pi_2 = \pi_3 > \pi_1$
Proposed BVSS Line	2	-3	1	$\pi_1 > \pi_3 > \pi_2$

vague soft sets model, which can handle bipolar vague soft information. As a result, the proposed bipolar vague soft model is so much more realistic and practical in handling such decision-making scenarios. This gives the proposed method its strength and uniqueness. On the other hand, in some circumstances, the proposed technique might also have natural weaknesses. While we have a big number of attributes or alternatives, one of these constraints arises, resulting in a high number of computations when using the proposed technique. Another weakness in the proposed model is it does not handle issues arising in a multipolar setting.

7. Concluding Remarks and Future Work

On the one hand, many mathematicians have investigated either fuzzy sets, vague sets or soft sets, separately (the basic well-known set theory's extensions), have studied their properties and have introduced their applications. On the other hand, combining any two basic extensions of them is not only more general and flexible than applying one only of them, but also gives more accurate, applicable and extended results. Some mathematicians have studied a few types of these combined extensions like fuzzy soft sets, bipolar fuzzy soft sets and vague soft sets, have introduced their properties and have applied them in many fields as an example representing real-life decision-making problems. Obviously, one can consider some of these extensions as extensions for each other, not for the set theory itself only.

In this paper, a new combinatorial set which is the bipolar vague soft set has been defined, their types and some new related concepts have been established, and operations on them have been investigated, illustrated by examples. Furthermore, absorption properties, commutative properties, associative properties, distributive laws and De Morgan's laws are introduced with proofs in detail. Moreover, Roy and Maji method has been generalized by using the concept of the bipolar vague soft set instead of the fuzzy soft set concept to make more effective decisions to choose the optimal object among others. In addition, an applicable real-life example has been introduced to explain and clarify the proposed method. And then, three methods have been used to convert bipolar vague values to bipolar fuzzy values to ensure that the proposed algorithm successfully works for problems that contain uncertain data. It occurs by comparing the final decisions obtained from each method with that of the proposed technique. Finally, a detailed comparative analysis between the proposed method and some of the previous ones, as well as a discussion of the results has been conducted. This investigation type completes some gaps in the literature.

The academic contribution of the proposed model is obvious because the vague soft model works well when dealing with vague soft data, but it fails to deal with problems that include bipolarity. Similarly, the fuzzy soft model is good when dealing with fuzzy soft data, but it cannot treat vagueness information observed in a variety of different real-life situations. That is to say that the vagueness of the associated parameters cannot be represented by the fuzzy soft set definition only. And so, mixing bipolarity with the vague soft set improves the accuracy and uniqueness of the decision. Hence, the advantage or strength of the proposed method is that it is more general, intelligent, and flexible than the previous methods because of applying bipolar vague soft sets rather than bipolar fuzzy soft sets or vague soft sets. The bipolar fuzzy soft sets and the vague soft sets can be considered as special cases of the bipolar vague soft sets.

Like any other method or model, the proposed method may have its inherent limitations, disadvantages, or weakness in some cases. One of these limitations occurs when we have a large number of parameters and/or objects, which leads to a big number of calculations when applying the proposed technique. To overcome this limitation, one can use various mathematical programs, like $Wolfram Mathematica^{®}$ and $MATLAB^{®}$, which can handle large data rapidly and effectively. Another disadvantage of the proposed model is that it fails to deal with problems that occur in a multipolar environment. In the future work, we can define the multipolar vague soft set to resolve this weakness and deal with those complicated problems. In addition, as future research work, several new results with some more generalized measures can be introduced using analogous methods in this article. Furthermore, our future research ideas can be extended to $m-$polar vague soft sets, spherical vague soft sets, and Pythagorean vague soft sets. Moreover, there are some other extensions of set theory (non-classical sets), have been defined by many authors other than those mentioned here such as rough sets, hard sets and multisets, etc. that could be also combined together. In many fields of science, one can make many other applications by applying any of those combined extensions.

Data Availability

All the data sets are provided within the main body of the paper.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

The authors are thankful to Taif University and Taif University researchers supporting project (number TURSP-2020/160), Taif University, Taif, Saudi Arabia.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[3] W.-L. Gau and D. J. Buehrer, "Vague sets," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 23, no. 2, pp. 610–614, 1993.

[4] D. Molodtsov, "Soft set theory-First results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[5] M. Black, "Vagueness. An exercise in logical analysis," *Philosophy of Science*, vol. 4, no. 4, pp. 427–455, 1937.

[6] A. R. Roy and P. K. Maji, "A fuzzy soft set theoretic approach to decision making problems," *Journal of Computational and Applied Mathematics*, vol. 203, no. 2, pp. 412–418, 2007.

[7] N. Faried, M. S. S. Ali, and H. H. Sakr, "Vague soft matrix-based decision making," *Global Journal of Pure and Applied Mathematics*, vol. 15, no. 5, pp. 755–780, 2019.

[8] L. Abualigah, M. A. Elaziz, P. Sumari, Z. W. Geem, and A. H. Gandomi, "Reptile search algorithm (RSA): a nature-inspired meta-heuristic optimizer," *Expert Systems with Applications*, vol. 191, Article ID 116158, 2022.

[9] L. Abualigah, A. Diabat, S. Mirjalili, M. Abd Elaziz, and A. H. Gandomi, "The arithmetic optimization algorithm," *Computer Methods in Applied Mechanics and Engineering*, vol. 376, Article ID 113609, 2021.

[10] L. Abualigah, A. Diabat, P. Sumari, and A. H. Gandomi, "Applications, deployments, and integration of internet of drones (IoD): a review," *IEEE Sensors Journal*, vol. 21, no. 22, pp. 25532–25546, 2021.

[11] L. Abualigah, D. Yousri, M. Abd Elaziz, A. A. Ewees, M. A. A. Al-qaness, and A. H. Gandomi, "Aquila optimizer: a novel meta-heuristic optimization algorithm," *Computers & Industrial Engineering*, vol. 157, Article ID 107250, 2021.

[12] K. M. Lee, "Bipolar-valued fuzzy sets and their operations," in *Proceedings of the International Conference of Intelligence Technology*, pp. 307–312, Bangkok, Thailand, April 2000.

[13] S. Abdullah, M. Aslam, and K. Ullah, "Bipolar fuzzy soft sets and its applications in decision making problem," *Journal of Intelligent and Fuzzy Systems*, vol. 27, no. 2, pp. 729–742, 2014.

[14] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.

[15] P. K. Maji, A. R. Roy, and R. Biswas, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[16] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir, "On some new operations in soft set theory," *Computers & Mathematics with Applications*, vol. 57, no. 9, pp. 1547–1553, 2009.

[17] A. Sezgin and A. O. Atagün, "On operations of soft sets," *Computers & Mathematics with Applications*, vol. 61, no. 5, pp. 1457–1467, 2011.

[18] P. Majumdar and S. K. Samanta, "On soft mappings," *Computers & Mathematics with Applications*, vol. 60, no. 9, pp. 2666–2672, 2010.

[19] H. Aktas and N. Cagman, "Soft sets and soft groups," *Information Science*, vol. 177, pp. 2726–2735, 2007.

[20] F. Feng, Y. B. Jun, and X. Zhao, "Soft semirings," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2621–2628, 2008.

[21] U. Acar, F. Koyuncu, and B. Tanay, "Soft sets and soft rings," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3458–3463, 2010.

[22] Y. B. Jun, "Soft BCK/BCI-algebras," *Computers & Mathematics with Applications*, vol. 56, no. 5, pp. 1408–1413, 2008.

[23] Y. B. Jun, K. J. Lee, and J. Zhan, "Soft p-ideals of soft BCI-algebras," *Computers & Mathematics with Applications*, vol. 58, pp. 2060–2068, 2009.

[24] Y. B. Jun and C. H. Park, "Applications of soft sets in ideal theory of BCK/BCI-algebras," *Information Science*, vol. 178, pp. 2466–2475, 2008.

[25] A. Sezgin and A. O. Atagün, "Soft groups and normalistic soft groups," *Computers & Mathematics with Applications*, vol. 62, no. 2, pp. 685–698, 2011.

[26] J. Zhan and Y. B. Jun, "Soft BL-algebras based on fuzzy sets," *Computers & Mathematics with Applications*, vol. 59, no. 6, pp. 2037–2046, 2010.

[27] O. Kazancı, S. Yilmaz, and S. Yamak, "Soft sets and soft BCH-algebras," *Hacet Journal Mathematical Statistics*, vol. 39, no. 2, pp. 205–217, 2010.

[28] A. Sezgin, O. Atagün, and E. Aygün, "A note on soft near-rings and idealistic soft near-rings," *Filomat*, vol. 25, no. 1, pp. 53–68, 2011.

[29] N. Çagman, F. Çitak, and H. Aktas, "Soft int-groups and its applications to group theory," *Neural Computing & Applications*, vol. 21, no. Suppl.1, pp. S151–S158, 2012.

[30] A. Sezgin, A. O. Atagun, and N. Cagman, "Soft intersection near-ring with its applications," *Neural Computing & Applications*, vol. 21, no. Suppl.1, pp. S221–S229, 2012.

[31] S. Bayramov and C. Gunduz, "Soft locally compact spaces and soft paracompact spaces," *Journal of Mathematics and System Science*, vol. 3, pp. 122–130, 2013.

[32] S. Das and S. K. Samanta, "Soft real sets, soft real numbers and their properties," *Journal of Fuzzy Mathematics*, vol. 20, no. 3, pp. 551–576, 2012.

[33] O. A. Tantawy and R. M. Hassan, "Soft real analysis," *Journal of Progression Research Mathematics*, vol. 8, no. 1, pp. 1207–1219, 2016.

[34] S. Das and S. K. Samanta, "On soft complex sets and soft complex numbers," *Journal of Fuzzy Mathematics*, vol. 21, no. 1, pp. 195–216, 2013.

[35] S. Das and S. K. Samanta, "Soft metric," *Annal Fuzzy Mathematical Information*, vol. 6, no. 1, pp. 77–94, 2013.

[36] M. I. Yazar, T. Bilgin, S. Bayramov, and C. Gunduz, "A new view on soft normed spaces," *International Mathematical Forum*, vol. 9, no. 24, pp. 1149–1159, 2014.

[37] S. Das and S. K. Samanta, "On soft inner product spaces," *Annal Fuzzy Mathematical Information*, vol. 6, no. 1, pp. 151–170, 2013.

[38] M. I. Yazar, C. G. Aras, and S. Bayramov, "Results on soft Hilbert spaces," *TWMS Journal of Applied Engineering Math*, vol. 9, no. 1, pp. 159–164, 2019.

[39] N. Çagman and S. Enginoglu, "Soft matrix theory and its decision-making," *Computers & Mathematics with Applications*, vol. 59, pp. 3308–3314, 2010.

[40] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft set," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 677–692, 2001.

[41] Y. Yang and C. Ji, "Fuzzy soft matrices and their applications," *Artificial Intelligence and Computational Intelligence*, pp. 618–627, 2011.

[42] N. Çagman and S. Enginoglu, "Fuzzy soft matrix theory and its application in decision making," *Iranian Journal of Fuzzy System*, vol. 9, no. 1, pp. 109–119, 2012.

[43] T. M. Basu, N. K. Mahapatra, and S. K. Mondal, "Different types of matrices in fuzzy soft set theory and their application in decision-making problems," *Engineering Science and Technology*, vol. 2, no. 3, pp. 389–398, 2012.
[44] A. Kumar and J. Kaur, "A new method for solving fuzzy linear programs with trapezoidal fuzzy numbers," *Journal of Fuzzy Set Valued Analysis*, vol. 2011, no. 1, pp. 1–12, 2011.
[45] N. Faried, M. S. S. Ali, and H. H. Sakr, "Fuzzy soft inner product spaces," *Applied Mathematical Information Science*, vol. 14, no. 4, pp. 709–720, 2020.
[46] N. Faried, M. S. S. Ali, and H. H. Sakr, "Fuzzy soft Hilbert spaces," *Journal of Mathematical and Computational Science*, vol. 22, no. 2021, pp. 142–157, 2020.
[47] N. Faried, M. S. S. Ali, and H. H. Sakr, "On fuzzy soft linear operators in fuzzy soft Hilbert spaces," *Abstract and Applied Analysis*, vol. 2020, Article ID 5804957, 13 pages, 2020.
[48] N. Faried, M. S. S. Ali, and H. H. Sakr, "Fuzzy soft symmetric operators," *Annal Fuzzy Mathematical Information*, vol. 19, no. 3, pp. 275–280, 2020.
[49] N. Faried, M. S.S. ALI, and H. Sakr, "Fuzzy soft hermitian operators," *Advances in Mathematics: Scientific Journal*, vol. 9, no. 1, pp. 73–82, 2020.
[50] N. Faried, M. Ali, and H. Sakr, "A note on FS isometry operators," *Mathematical Science Letter*, vol. 10, no. 1, pp. 1–3, 2021.
[51] N. Faried, M. Ali, and H. Sakr, "On FS normal operators," *Mathematical Science Letter*, vol. 10, no. 2, pp. 41–46, 2021.
[52] P. K. Maji, A. R. Roy, and R. Biswas, "On intuitionistic fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 12, no. 3, pp. 669–683, 2004.
[53] B. Chetia and P. K. Das, "Some results of intuitionistic fuzzy matrix theory," *Advances in Applied Science Research*, vol. 3, no. 1, pp. 421–423, 2012.
[54] W. Xu, J. Ma, S. Wang, and G. Hao, "Vague soft sets and their properties," *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 787–794, 2010.
[55] C. Wang and A.-J. Qu, "The applications of vague soft sets and generalized vague soft sets," *Acta Mathematicae Applicatae Sinica, English Series*, vol. 31, no. 4, pp. 977–990, 2015.
[56] K. Alhazaymeh and N. Hassan, "Vague soft set relations and functions," *Journal of Intelligent and Fuzzy Systems*, vol. 28, no. 3, pp. 1205–1212, 2015.
[57] B. P. Varol and H. Aygaun, "On vague soft groups," *Ann. Fuzzy Math. Inform.*vol. 7, no. 2, pp. 205–218, 2014.
[58] Y. Yin, Y. B. Jun, and J. Zhan, "Vague soft hemirings," *Computers & Mathematics with Applications*, vol. 62, no. 1, pp. 199–213, 2011.
[59] G. Selvachandran and A.-R. Salleh, "Algebraic hyperstructures of vague soft sets associated with hyperrings and hyperideals," *Science World Journal*, vol. 2015, Article ID 780121, 12 pages, 2015.
[60] G. Selvachandran and A.-R. Salleh, "Rings and ideals in a vague soft set setting," *Far East Journal of Mathematical Sciences*, vol. 99, no. 2, pp. 279–300, 2016.
[61] V. Inthumathi and M. Pavithra, "Operations on vague soft matrices and its applications in decision making," *Journal of Physics: Conference Series*, vol. 1139, p. 012037, 2018.
[62] I. Arockiarani and S. Cicily Flora, "Positive implicative bipolar vague ideals in BCK-algebras," *International Research Journal of pure Algebra*, vol. 6, no. 8, pp. 1–7, 2016.
[63] F. Li, Z. H. Lu, and L. J. Cai, "The entropy of vague sets based on fuzzy sets," *Journal of Huazhong University of Science and Technology*, vol. 1, no. 31, pp. 1–3, 2003, (in Chinese).
[64] Y. Liu, G. Wang, and L. Feng, "A general model for transforming vague sets into fuzzy sets," *Lecture Notes in Computer Science*, vol. 5150, pp. 133–144, 2008.
[65] A. Solairaju, P. J. Robinson, and T. Lenin, "Applications of transforming vague sets into fuzzy sets for knowledge management," *International Journal of Computation Algorithm*, vol. 2, pp. 430–439, 2013.